Axiomatic Generation of Nonstandard Naturals within PA

Abstract

We demonstrate several constructions of nonstandard models of arithmetic using only the axioms of Peano Arithmetic (PA), without relying on any external semantic interpretations or additional logical frameworks. Each model satisfies all PA axioms while exhibiting structural deviations from the standard model $\mathbb{N}$. Our aim is to illustrate that the existence of nonstandard naturals is not merely a semantic artifact but a syntactic consequence of the axioms themselves.

Contents

1 Introduction

The standard model of arithmetic $\mathbb{N}$ is often regarded as the canonical realization of the Peano axioms (PA), and while model theory assures the existence of nonstandard models, these are typically treated as abstract consequences of the compactness theorem or other semantic constructions. As a result, there has been little interest in constructing such models explicitly within the syntax of PA itself.

This lack of development may reflect an implicit assumption that the generation of nonstandard models from PA alone is infeasible. However, as we will demonstrate, such models can be constructed directly—and quite easily—without departing from the axioms.

The aim of this paper is to present several nonstandard models of natural numbers derived strictly from the Peano axioms, each constructed in a straightforward and transparent manner. These examples serve to illustrate the syntactic flexibility of PA and the surprising abundance of models it permits, even in the absence of any external interpretive apparatus.

2 Definitions

2.1 Peano Axioms

Let $\mathbb{N}$ be a set, and let $S : \mathbb{N} \to \mathbb{N}$ be a unary function called the *successor function*. The structure $(\mathbb{N}, 0, S)$ satisfies the Peano axioms[1] if the following hold:

1. $0 \in \mathbb{N}$
2. $\forall n \in \mathbb{N},\ S(n) \in \mathbb{N}$
3. $\forall n \in \mathbb{N},\ S(n) \neq 0$
4. $\forall m, n \in \mathbb{N},\ S(m) = S(n) \Rightarrow m = n$
5. **(Second-order Induction)** For every property $P \subseteq \mathbb{N}$, if:

$$P(0) \wedge \forall n \in \mathbb{N},\ P(n) \Rightarrow P(S(n)) \quad \Rightarrow \quad \forall n \in \mathbb{N},\ P(n)$$

[1] See e.g. Tristin Cleveland, *Number Theory*, p. 7; Mendelson, *Introduction to Mathematical Logic*, 4th ed., p. 153.

Alternatively, the fifth axiom may be expressed using the schema of first-order induction:

$$(\varphi(0) \land \forall n\,(\varphi(n) \to \varphi(S(n)))) \to \forall n\,\varphi(n)$$

for any first-order formula $\varphi(n)$.

2.2 Models of PA

A **model** of Peano Arithmetic (PA) is a structure

$$\mathcal{M} = (D, 0^{\mathcal{M}}, S^{\mathcal{M}})$$

where:

- D is a nonempty set (the domain),
- $0^{\mathcal{M}} \in D$ is a distinguished element,
- $S^{\mathcal{M}} : D \to D$ is a unary function (successor),

such that all Peano axioms hold in $\mathcal{M}$, under the usual interpretation of logical connectives and quantifiers.

The standard model is given by:

$$\mathbb{N} = (\mathbb{N}, 0, n \mapsto n + 1)$$

but this is only one among many possible models satisfying PA. The purpose of this paper is to explore alternative, nonstandard models that also satisfy all the axioms but differ structurally from $\mathbb{N}$.

2.3 The Standard Model

By the **standard model** of PA, we refer to the structure

$$\mathbb{N}_{\text{std}} = (\mathbb{N}, 0, n \mapsto n + 1)$$

as interpreted within a set-theoretic meta-theory such as ZFC, where $\mathbb{N}$ denotes the set of finite von Neumann ordinals. Although commonly accepted as the canonical model of PA, this model is not definable within PA itself, nor can it be uniquely characterized by any first-order axiomatization.

3 Main Argument

3.1 Trivial Truth of the Statement g in the Standard Model

We consider the following formula:

$$g := \forall x \, (x = 0 \lor \exists y \, (S(y) = x))$$

We define the standard model $\mathbb{N}$ as follows:

$$\mathrm{Dom}(\mathbb{N}) = \{0, S(0), S(S(0)), \dots\}$$

Here, S is interpreted as the successor function, i.e., $S(n) = n + 1$.

According to this definition, every element of $\mathbb{N}$ falls into one of the following two cases:

- $x = 0$, or
- There exists some $y \in \mathbb{N}$ such that $x = S(y)$.

Therefore, all elements of $\mathbb{N}$ satisfy the condition of g, and no counterexample exists. Hence:

$$\mathbb{N} \models g$$

In conclusion, the formula g holds in the standard model by construction. This is not a theorem derived from the axioms, but rather a direct consequence of the model-theoretic definition.

We now introduce a new nonstandard model $\mathbb{N} \cup O$, where O is defined as follows:

$$O := \{o_1, o_2, o_3, o_4, o_5\}$$

The successor function S is extended over O such that:

$$S(o_1) = o_2, \quad S(o_2) = o_3, \quad S(o_3) = o_4, \quad S(o_4) = o_5, \quad S(o_5) = o_1$$

In this model, the domain is:

$$\mathrm{Dom} = \mathbb{N} \cup O$$

Each element in this domain satisfies the formula g. For all $x \in \mathbb{N}$, the truth of g has already been shown. For each $x \in O$, there exists a $y \in O$ such that $S(y) = x$ (by construction). Therefore:

$$\mathbb{N} \cup O \models g$$

This demonstrates that g is satisfied not only in the standard model, but also in a strictly larger nonstandard model.

3.2 A Model Where g is False

We now present a model in which the statement g is false, while all Peano axioms remain satisfied. This construction demonstrates that g is independent of PA.

For reference, the formula g is defined as:

$$g := \forall x \, (x = 0 \lor \exists y \, (S(y) = x))$$

In words: every number is either 0 or the successor of some number.

We define a new model $N^+ := \mathbb{N} \cup Q$, where

- $\mathbb{N} = \{0, 1, 2, 3, \dots\}$
- $Q = \{n_1, n_2, n_3, \dots\}$, a disjoint set starting from n_1

The successor function $S : N^+ \to N^+$ is defined by

$$S(x) = \begin{cases} x + 1 & \text{if } x \in \mathbb{N} \\ n_{k+1} & \text{if } x = n_k \in Q \end{cases}$$

Verification of Peano Axioms in N^+

We now verify that this structure satisfies all five Peano axioms.

1. **0 is a natural number.**

$$0 \in \mathbb{N} \subseteq N^+$$

2. **The successor of any number is a number.** For all $x \in \mathbb{N}$, $S(x) = x + 1 \in \mathbb{N} \subseteq N^+$. For all $x = n_k \in Q$, $S(x) = n_{k+1} \in Q \subseteq N^+$.

3. **0 is not the successor of any number.** $S(x) \geq 1$ for all $x \in \mathbb{N}$, and $S(x) \in Q$ for all $x \in Q$, so $S(x) \neq 0$ in both cases.

4. **The successor function is injective.** On $\mathbb{N}$, $x+1 = y+1 \Rightarrow x = y$. On Q, $S(n_k) = n_{k+1}$ is injective by definition. $\mathbb{N} \cap Q = \emptyset$, so the images of both chains are disjoint.

5. **The fifth axiom.**

   Let $P(x)$ be any unary predicate over N^+. The axiom states:

$$[P(0) \land \forall x \, (P(x) \to P(S(x)))] \to \forall x \, P(x)$$

   In this model:

   - If $P(0)$ is true, and
   - $P(x) \to P(S(x))$ holds for all $x \in N^+$,

   then $P(x)$ holds for all $x \in N^+$, since both $\mathbb{N}$ and Q are closed under S.

   However, the formula g is not an instance of this axiom. Let:

$$P(x) := x = 0 \lor \exists y \, (S(y) = x)$$

   Then:

   - $P(0)$ is true
   - $P(x) \to P(S(x))$ holds for all $x \in N^+ \setminus \{n_0\}$
   - But $P(n_1)$ is false, since:

$$n_1 \neq 0 \quad \text{and} \quad \forall y \in N^+, \; S(y) \neq n_1$$

   So g is false in this model, but the fifth axiom is not violated—since it does not assert that arbitrary P is true, only that the implication schema above holds.

In particular, the negation of g:

$$\exists x\,(x \neq 0 \land \forall y\, S(y) \neq x)$$

is satisfied by $x = n_1$. Therefore, some $P(x)$ does satisfy the fifth axiom, and the axiom itself remains valid.

Hence, $(N^+, S, 0)$ is a valid model of PA in which the formula g is false.

Combined with the earlier standard model in which g is true, this construction shows that g is true in some models of PA and false in others. Therefore, the formula

$$g := \forall x\,(x = 0 \lor \exists y\,(S(y) = x))$$

is independent of PA: it is true in the standard model of $\mathbb{N}$, but false in a nonstandard model that satisfies all Peano axioms.[2]

Further Justification via Herbrand's Theorem

We further validate the independence of the formula g using a proof-theoretic approach via Herbrand's theorem.

The negation of g is:

$$\exists x\,(x \neq 0 \land \forall y\, S(y) \neq x)$$

To prove this in PA, we would need to derive a Herbrand disjunction of the form:

$$(t_1 \neq 0 \land \forall y\, S(y) \neq t_1) \lor \cdots \lor (t_k \neq 0 \land \forall y\, S(y) \neq t_k)$$

where each t_i is a ground term (e.g., $0, S(0), S(S(0)), \ldots$) from the Herbrand universe.

However, since $y \in N^+ = \mathbb{N} \cup Q$, and $Q = \{n_1, n_2, \ldots\}$ is infinite, verifying $\forall y\, S(y) \neq t_i$ requires checking infinitely many cases for each t_i. Therefore, no finite Herbrand disjunction can suffice.

By Herbrand's theorem, this means the negation of g cannot be proven in PA. Hence, g is not provable in PA either—establishing its independence from PA from both model-theoretic and proof-theoretic perspectives.

[2] See Gödel's 1931 paper: *Über formal unentscheidbare Sätze der Principia Mathematica und verwandter Systeme I*, which established the existence of undecidable propositions within formal systems such as PA.

3.3 A Model Where All Numbers Are Less Than or Equal to Zero

We conclude by presenting a minimal nonstandard model $\mathbb{N}^-$, defined as:

$$\mathbb{N}^- := \{0, -1, -2, -3, -4, \dots\}$$

The successor function is defined by:

$$S(x) := x - 1$$

This model satisfies the schema:

$$\forall x \in \mathbb{N}^-,\ x \leq 0$$

It clearly satisfies all five axioms of PA.

4 Conclusion

The formula $g := \forall x\, (x = 0 \lor \exists y\, (S(y) = x))$ is merely one example among infinitely many first-order statements that are independent of PA. For each such independent statement, there exists a corresponding nonstandard model in which the statement is either true or false, without violating any of the five axioms of PA.

This demonstrates that PA does not uniquely characterize the structure of the natural numbers, but instead admits a vast space of models, each with its own internal arithmetic.

Independence is not an anomaly—it is the norm. The belief that "every natural number is either zero or a successor" is not a theorem of PA, but a condition satisfied only in certain models.

In conclusion, PA is not a definition of the natural numbers. It is a framework within which multiple, often incompatible, interpretations of number coexist.

Anticipated Objections and Rebuttals

This section outlines several possible objections that may arise in response to the models presented in this paper, and provides brief clarifications addressing each one.

1. **"Your model is not a model of the natural numbers."**
   This criticism conflates philosophical intuition with formal semantics. In first-order logic, a model of PA is defined solely by satisfying its five axioms. All models presented in this paper do exactly that.

2. **"This is just a trivial model trick; it's not mathematically meaningful."**
   While the existence of nonstandard models is known, this paper explicitly constructs such models around a concrete and intuitive formula g, making the independence phenomenon transparent and pedagogically effective.

3. **"Your model violates the fifth axiom."**
   This is incorrect. The fifth axiom, stated as

   $$[P(0) \wedge \forall x\,(P(x) \to P(S(x)))] \to \forall x\,P(x),$$

   is satisfied in all models presented. That g is false in some models merely reflects that g is not an instance of this axiom.

4. **"Of course PA can't prove that—it's only first-order. That's expected."**
   This is precisely the point. The fact that such an intuitive property is undecidable within PA exposes the deep limitations of first-order formalization. The goal is not to prove this is unknown, but to demonstrate it vividly.

5. **"Is g even expressible in PA?"**
   Yes. The formula $g := \forall x\,(x = 0 \vee \exists y\,(S(y) = x))$ is a valid first-order sentence using only the symbols allowed in the PA language.

5 Contact

For questions, feedback, or collaboration inquiries regarding this paper, please contact:

Kim Altair
Independent Researcher
Email: `dbnjk515@gmail.com`
ORCID: `https://orcid.org/0009-0002-1479-2769`
GitHub: `https://github.com/KimAltair`

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