The Conflict Between Non-definable Reals and the Axiom of Choice in ZFC

On the Incompatibility of Non-computable Reals with the Axiom of Choice

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June 7, 2025

Abstract

We examine the consequences of applying the Axiom of Choice (AC) within ZFC to the set $D := \mathbb{R} \setminus C$, where C denotes the set of computable real numbers. We argue that any choice function defined on D inevitably introduces a structure in which its elements become effectively referable or indexable, thereby contradicting the definition of D as unnameable and unindexable. This leads to a logical inconsistency, suggesting that AC is incompatible with the existence of D. To resolve this conflict, we propose a modified axiomatic system, ZFC D/AC, in which the Axiom of Choice is explicitly restricted from applying to sets composed of non-definable real numbers. This result prompts broader meta-mathematical reflection on the interplay between definability, choice, and the foundational assumptions underlying mathematical existence.

1 Introduction

The Axiom of Choice (AC) is among the most powerful and controversial axioms in the Zermelo-Fraenkel set theory (ZFC). It permits the selection of elements from arbitrary collections of nonempty sets, and supports numerous fundamental results across modern mathematics. Yet, AC also gives rise to counterintuitive outcomes, such as the Banach-Tarski paradox, and its relationship to non-constructive or non-definable entities remains insufficiently explored.

In this paper, we focus on the implications of applying AC to a particular subset of real numbers: let $C \subset \mathbb{R}$ denote the set of computable real numbers, and define $D := \mathbb{R} \setminus C$, the set of non-computable reals. Elements of D are not only uncomputable, but also undefinable and unindexable — they cannot be described by any algorithm, function, or formal expression within ZFC.

We argue that applying a choice function $f : D \to D$ inevitably gives rise to a structure in which elements of D become distinguishable or referable, violating the very definition of D. This leads to a logical contradiction within the system ZFC + AC. To maintain consistency, we propose a revised framework, ZFC D/AC, in which AC is explicitly barred from acting on non-definable sets.

This contradiction between the act of choice and the principle of unnameability raises deeper questions about the nature of mathematical existence. We contend that the very operation of choosing presupposes a minimal degree of identifiability — a property that cannot coherently be attributed to entities that are, by construction, unidentifiable.

2 Definitions

Definition 1 (Computable Reals). Let $C \subset \mathbb{R}$ denote the set of computable real numbers. A real number $r \in \mathbb{R}$ is computable if there exists a Turing machine M such that, for every natural number $n \in \mathbb{N}$, the machine outputs the n-th digit of r.

Definition 2 (Non-computable Reals). We define $D := \mathbb{R} \setminus C$, the set of non-computable real numbers. That is, D contains all real numbers for which no Turing machine can produce their decimal expansion digit by digit. This set is uncountable.

Definition 3 (Non-definability and Indexing). Since the output of a Turing machine corresponds to a finite symbolic expression in a formal language, no element of D is finitely definable within ZFC. As a result, D is not indexable: there exists no function $f : \mathbb{N} \to D$ that can map each natural number to a unique element of D in a definable way.

Definition 4 (Axiom of Choice). The Axiom of Choice (AC) states that for any nonempty set S, there exists a choice function $f : S \to S$ that selects a unique element from each nonempty subset of S. In the case of D, AC implies that a choice function $f : D \to D$ must exist.

3 Main Argument

We now examine the logical consequences of applying the Axiom of Choice (AC) to the set $D := \mathbb{R} \setminus C$, where C is the set of all computable real numbers.

1. **Application of Choice.** By AC, there exists a choice function $f : D \to D$ that selects exactly one representative $d_i \in D$ from each nonempty subset of D.

2. **Selection of a Single Element.** Consequently, for every subset $A \subseteq D$, including the entire set D itself, the function f yields a specific element $d_i = f(A) \in A$. In particular, singleton subsets $\{d_i\} \subseteq D$ are uniquely identifiable via f.

3. **Formation of Singleton Subsets.** Each selected element $d_i \in D$ determines a singleton set $D_i := \{d_i\}$, which is trivially a subset of D. Since $f(D)$ produces individual elements d_i, we obtain a collection of definable singleton sets $\{D_i\} \subset \mathcal{P}(D)$, each corresponding to a distinct element of D.

4. **Implicit Indexing.** The function f thereby induces a structure in which each element of D becomes individually identifiable. Although this indexing may not be algorithmically or constructively defined, its logical presence contradicts the foundational characterization of D as unindexable and undefinable.

5. **Contradiction.** By definition, no element of D is definable or indexable within ZFC. The existence of a function $f : D \to D$ that distinguishes individual elements implies otherwise. Hence, assuming the Axiom of Choice for D leads to a contradiction. It follows that AC is incompatible with the existence of D as defined.

6. **Alternative Contradiction via Subset Construction.** Alternatively, consider the image set $f(D) = \{d_1, d_2, d_3, \ldots\} \subset D$, where each d_i is selected by the choice function. For each d_i, define the singleton set $D_i := \{d_i\}$, and let $D' := \bigcup_{i \in \mathbb{N}} D_i$. Then D' is a countably infinite subset of D, explicitly constructed via choice.

   The set D' is clearly indexable, as it is in explicit bijection with $\mathbb{N}$ via the map $i \mapsto d_i$. This entails the existence of a nonempty, countably infinite, explicitly indexed subset of D, contradicting the assumption that no such indexing is possible. Therefore, this construction independently reinforces the conclusion that AC is logically incompatible with the set D.

4 Conclusion

5 Conclusion

1. The Axiom of Choice in ZFC cannot be coherently applied to the set $D := \mathbb{R} \setminus C$, where C is the set of computable real numbers. Any such application results in contradiction, as it implicitly introduces indexing and identifiability for elements that are, by definition, unnameable and unindexable.

2. To preserve the coexistence of the Axiom of Choice with the existence of D, we propose the introduction of a supplementary axiom that restricts the domain of choice. Specifically, the Axiom of Choice must not apply to sets whose elements are not formally definable within the system. We designate this modified framework as **ZFC D/AC**, in which the Axiom of Choice is explicitly inapplicable to non-definable subsets of the reals.

   Formally, we propose the following additional axiom to be appended to the standard ZFC framework:

   > **Restricted Choice Axiom:** For any nonempty set S, if no element $x \in S$ is definable within ZFC (i.e., $\forall x \in S, \neg\text{Definable}(x)$), then there does not exist a choice function $f : S \to S$ such that $\forall x \in S, f(x) \in S$.

   This axiom preserves the utility of AC on all well-defined sets, while preventing contradictions that arise from its unrestricted application to definitional voids such as D.

3. The existence of D invites a deeper meta-mathematical inquiry. It challenges us to clarify whether a set composed entirely of uncomputable and undefinable elements can be meaningfully treated within a formal system. In particular, it raises the question of whether such a set can be regarded as mathematically manipulable or referable, even in principle.

6 Anticipated Objections and Rebuttals

- **Objection 1:** Gödel's constructible universe L proves that ZF + AC is consistent.

  Rebuttal: The model L includes only constructible sets and excludes non-constructible reals such as those comprising D. Since $D \subseteq \mathbb{R}$ is assumed to exist in any standard model of ZFC due to the uncountability and non-constructibility of reals, L is an incomplete submodel. Thus, the consistency of AC within L does not imply its consistency within the full ZFC framework in which D exists.

- **Objection 2:** A choice function f over D may exist even if we cannot explicitly describe $f(D)$.

  Rebuttal: The assumption that $f(D) \in D$ exists entails the ability to distinguish at least one specific element of D, thereby assigning it a reference. This contradicts the definition of D as a collection of unnameable and unindexable elements.

- **Objection 3:** There may exist models of ZFC in which the set D does not exist.

  Rebuttal: The set $D := \mathbb{R} \setminus C$, where C denotes the computable reals, is definable via standard constructions within ZFC. Its existence is guaranteed in any standard ZFC model that contains $\mathbb{R}$.

- **Objection 4:** Perhaps D is a proper class, not a set, and thus AC is inapplicable.

  Rebuttal: Since $\mathbb{R}$ is a set in ZFC and $D \subset \mathbb{R}$, it follows directly that D is also a set.

- **Objection 5:** The Axiom of Choice is foundational to modern mathematics; restricting it could be too disruptive.

  Rebuttal: Foundational principles must yield to consistency. If AC yields contradiction when applied to certain sets, then its scope must be appropriately delimited to maintain coherence.

- **Objection 6:** Selecting a single element $d_i \in D$ does not entail indexing the entire set.

  Rebuttal: Any selection operation presupposes the ability to distinguish the chosen element from all others, establishing minimal identifiability. Repeated applications of such selections amount to a form of implicit indexing, violating the intended non-definability of D.

- **Objection 7:** Constructing a countable subset of D does not contradict its uncountability.

  Rebuttal: While uncountability precludes enumeration of the full set, it does not guarantee unindexability. The existence of a countably infinite, explicitly indexed subset of D contradicts the principle that no element of D is definable or referable.

- **Objection 8:** An element of D may exist without being constructible or explicitly selectable.

  Rebuttal: This is consistent with the definition of D. However, the Axiom of Choice postulates the ability to select such elements regardless of definability, and this assumption collapses in the context of D.

- **Objection 9:** The Axiom of Choice and the existence of D can coexist without contradiction.

  Rebuttal: We have shown that applying AC to D entails implicit indexing, which contradicts the definitional constraints of D. Therefore, AC and D, as defined, cannot coexist within a consistent formal system.

References

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