The Conflict Between Non-definable Reals and the Axiom of Choice in ZFC

On the Incompatibility of Non-computable Reals with the Axiom of Choice

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Abstract

We examine the consequences of applying the Axiom of Choice (AC) within ZFC to the set $D := \mathbb{R} \setminus C$, where C denotes the set of computable real numbers. We argue that any choice function defined on D inevitably introduces a structure in which its elements become effectively referable or indexable, thereby contradicting the definition of D as unnameable and unindexable. This leads to a logical inconsistency, suggesting that AC is incompatible with the existence of D. To resolve this conflict, we propose a modified axiomatic system, ZFC D/AC, in which the Axiom of Choice is explicitly restricted from applying to sets composed of non-definable real numbers. This result prompts broader meta-mathematical reflection on the interplay between definability, choice, and the foundational assumptions underlying mathematical existence.

1 Introduction

The Axiom of Choice (AC) is among the most powerful and controversial axioms in the Zermelo-Fraenkel set theory (ZFC). It permits the selection of elements from arbitrary collections of nonempty sets, and supports numerous fundamental results across modern mathematics. Yet, AC also gives rise to counterintuitive outcomes, such as the Banach-Tarski paradox, and its relationship to non-constructive or non-definable entities remains insufficiently explored.

In this paper, we focus on the implications of applying AC to a particular subset of real numbers: let $C \subset \mathbb{R}$ denote the set of computable real numbers, and define $D := \mathbb{R} \setminus C$, the set of non-computable reals. Elements of D are not only uncomputable, but also undefinable and unindexable — they cannot be described by any algorithm, function, or formal expression within ZFC.

We argue that applying a choice function $f : D \to D$ inevitably gives rise to a structure in which elements of D become distinguishable or referable, violating the very definition of D. This leads to a logical contradiction within the system ZFC + AC. To maintain consistency, we propose a revised framework, ZFC D/AC, in which AC is explicitly barred from acting on non-definable sets.

This contradiction between the act of choice and the principle of unnameability raises deeper questions about the nature of mathematical existence. We contend that the very operation of choosing presupposes a minimal degree of identifiability — a property that cannot coherently be attributed to entities that are, by construction, unidentifiable.

2 Definitions

Definition 1 (Computable Reals)**.** *Let* $C \subset \mathbb{R}$ *denote the set of* computable *real numbers. A real number* $r \in \mathbb{R}$ *is computable if there exists a Turing machine* M *such that, for every* $n \in \mathbb{N}$*, the machine* M*, on input* n*, halts and outputs the* n*-th digit of the decimal expansion of* r*.* [1]

Definition 2 (Non-computable Reals)**.** *Define* $D := \mathbb{R} \setminus C$*, the set of* non-computable *real numbers. Since* C *is countable and* $\mathbb{R}$ *is uncountable, it follows that* D *is uncountable.* [2]

Definition 3 (Non-definability and Non-indexability)**.** *Let* $D := \mathbb{R} \setminus C$ *be the set of non-computable reals.*

An element $d \in D$ *is said to be* non-definable *within ZFC if there does not exist any formula* $\varphi(x)$ *in the language of ZFC such that*

$$\text{ZFC} \vdash \exists! x\, \varphi(x) \quad \text{and} \quad \text{ZFC} \vdash \varphi(d).$$

That is, no uniquely specifying definition of d *exists within the system.*

Consequently, the set D *is* non-indexable*: there exists no definable injective function*

$$f : \mathbb{N} \to D$$

such that $f(n) = d_n \in D$ *for all* $n \in \mathbb{N}$*, and* f *is definable in ZFC.*

This aligns with the canonical understanding of uncountability*, which entails the impossibility of indexing the elements of* D *via any definable mapping from* $\mathbb{N}$*.* [3]

Definition 4 (Axiom of Choice and Indexing Implication)**.** *The Axiom of Choice (AC) asserts that for any nonempty set* S*, there exists a function*

$$f : \mathcal{P}(S) \setminus \{\emptyset\} \to S \quad \text{such that} \quad f(A) \in A$$

for all nonempty subsets $A \subseteq S$*.*

When applied to the set D*, AC guarantees the existence of a choice function that selects a single element from each nonempty subset of* D*. This selection process imposes an implicit indexing structure on* D*, making the elements of* D *distinguishable, which contradicts the non-indexability of* D *as previously defined.*

3 Main Argument

Let $C \subset \mathbb{R}$ denote the set of all computable real numbers. Define the set

$$D := \mathbb{R} \setminus C.$$

Then D is uncountable, and by construction, every $d \in D$ is non-computable and unnameable within ZFC.

[1] See A. M. Turing, "On Computable Numbers, with an Application to the Entscheidungsproblem," *Proc. London Math. Soc.*, Series 2, Vol. 42 (1937), pp. 230–265.

[2] See Jech, *Set Theory* (2002), p. 38.

[3] See Jech, *Set Theory* (2002), Theorem 4.3, p. 38.

1. **Assumption of the Choice Function.**
   Assume the Axiom of Choice (AC). Then there exists a choice function

   $$f : \mathcal{P}(D) \setminus \{\emptyset\} \to D \quad \text{such that} \quad f(A) \in A \quad \forall A \subseteq D,\ A \neq \emptyset.$$

   This function selects one element from each non-empty subset of D.

2. **Indexing via Single Element.**
   Let $d_1 := f(D)$. Then we have

   $$D \setminus \{d_1\} \subseteq D, \quad d_1 \notin D \setminus \{d_1\}.$$

   Now, define a labeling function

   $$\text{Index}_1 : D \to \{0,1\}, \quad \text{Index}_1(d) = \begin{cases} 1 & \text{if } d = d_1, \\ 0 & \text{otherwise.} \end{cases}$$

   This function identifies a unique element d_1 in D, contradicting the assumption that D is unindexable.

3. **Explicit Indexing via Iteration.**
   Define recursively:

   $$D_1 := D, \quad d_n := f(D_n), \quad D_{n+1} := D_n \setminus \{d_n\}.$$

   Let

   $$D' := \{d_n \mid n \in \mathbb{N}\} \subseteq D,$$

   and define the map

   $$\varphi : \mathbb{N} \to D', \quad \varphi(n) := d_n.$$

   Then φ is a bijection, and hence D' is explicitly indexable. Again, this violates the assumption that D is unindexable.

4. **Logical Incompatibility.**
   Both the single-element case and the iterative construction yield identifiable elements in D. Therefore, we conclude:

   $$\text{AC} \Rightarrow \text{“Indexing of } D\text{”} \Rightarrow \text{Contradiction.}$$

   Hence, we deduce:

   $$\text{AC} \Rightarrow \neg\text{“D exists in ZFC”}, \quad \text{or} \quad \text{D exists} \Rightarrow \neg\text{AC}.$$

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4 Conclusion

1. Let

   $$C := \{r \in \mathbb{R} \mid r \text{ is computable}\}, \quad D := \mathbb{R} \setminus C.$$

   Then

   $$\forall d \in D, \quad \neg\text{Computable}(d), \quad \neg\text{Definable}_{\text{ZFC}}(d),$$

   and D is uncountable and non-indexable.

2. Assume the Axiom of Choice (AC):

$$\text{AC} \Rightarrow \exists f : \mathcal{P}(D) \setminus \{\emptyset\} \to D, \quad f(A) \in A.$$

Then we obtain:

$$d_1 := f(D) \Rightarrow D \setminus \{d_1\} \text{ is definable in terms of } d_1.$$

Define

$$\text{Index}_1 : D \to \{0, 1\}, \quad \text{Index}_1(d) = \begin{cases} 1 & \text{if } d = d_1, \\ 0 & \text{otherwise.} \end{cases}$$

Hence, d_1 becomes indexable, violating the assumption that D is non-indexable.

3. More generally, define:

$$d_1 := f(D), \quad d_n := f(D \setminus \{d_1, \dots, d_{n-1}\}) \quad (n \geq 2),$$

and

$$D' := \{d_n \mid n \in \mathbb{N}\}, \quad \varphi : \mathbb{N} \to D', \quad \varphi(n) := d_n.$$

Then φ is a definable bijection from $\mathbb{N}$ to a subset of D, which contradicts the assumption that D is unindexable.

4. Therefore, we conclude:

$$\text{AC} \Rightarrow \text{Indexable}(D) \Rightarrow \bot,$$

and so

$$\neg(\text{AC} \wedge \text{Exists}(D)).$$

5. To resolve this contradiction, we propose a modified system, denoted **ZFC D/AC**, which restricts the domain of AC. We add the following axiom:

Restricted Choice Axiom (RCA):

$$\forall S \neq \emptyset, \quad [\forall x \in S, \ \neg \text{Definable}_{\text{ZFC}}(x)] \Rightarrow \neg \exists f : \mathcal{P}(S) \setminus \{\emptyset\} \to S, \quad f(A) \in A.$$

6. This axiom preserves the standard utility of AC for definable sets, while preventing contradictions arising from its application to purely non-definable sets such as D.

5 Anticipated Objections and Rebuttals

Objection 1: Gödel's constructible universe L proves that ZF + AC is consistent.
Rebuttal: $L \models \text{ZF} + \text{AC}$, but $D \not\subseteq L$ since D includes non-constructible reals. $Thus, L \models \text{Consistent}(\text{ZF} + \text{AC}) \not\Rightarrow \text{Consistent}(\text{ZF} + \text{AC} + D\text{-exists})$.

Objection 2: There might be ZFC models in which D does not exist.
Rebuttal: D is definable in ZFC as the set of unindexable reals. Since $\mathbb{R} \in \text{ZFC}$, we have $D \in \text{ZFC}$. Therefore, $\text{ZFC} \vdash \text{Exists}(D)$.

Objection 3: There may exist models of ZFC in which D does not exist.

Rebuttal: $D := \mathbb{R} \setminus C$, with C being the set of computable reals, is definable in ZFC. Since $\mathbb{R} \in \text{ZFC} \Rightarrow D \in \text{ZFC}$, we have

$$\text{ZFC} \vdash \text{Exists}(D)$$

Objection 4: Perhaps D is a proper class, not a set.
Rebuttal: $D \subset \mathbb{R}$ and $\mathbb{R}$ is a set in ZFC. Therefore, D is also a set.

Objection 5: AC is foundational to modern mathematics.
Rebuttal: Foundational axioms must preserve consistency. If

$$\text{AC} \Rightarrow \perp$$

then the scope of AC must be restricted to avoid contradictions in the theory.

Objection 6: Selecting $d_i \in D$ does not entail indexing all of D.
Rebuttal: If $f(D) = d_i$, then

$$\text{ZFC} \vdash \text{Identifiable}(d_i) \Rightarrow \text{Indexable}(\{d_i\}).$$

Repeated applications yield:

$$\{d_1, d_2, \dots\} \Rightarrow \text{Indexed subset } D' \subset D \Rightarrow \perp.$$

Since D is unindexable, this leads to a contradiction.

Objection 7: A countable subset $D' \subset D$ does not contradict the uncountability of D.
Rebuttal: True, but

$$\exists \varphi : \mathbb{N} \to D' \text{ definable in ZFC} \Rightarrow \exists d \in D \text{ with Definable}(d) \Rightarrow \perp$$

Objection 8: An element of D may exist without being constructible or selectable.
Rebuttal: That is consistent. However,

$$\text{AC} \Rightarrow \text{Select}(d_i \in D) \Rightarrow \text{Definable}(d_i) \Rightarrow \perp$$

Objection 9: AC and the existence of D can coexist.
Rebuttal: We have shown:

$$\text{AC} + \text{Exists}(D) \Rightarrow \text{Index}(D) \Rightarrow \perp$$

$$\Rightarrow \neg(\text{AC} \wedge \text{Exists}(D))$$

References

[1] Alan Turing, *On Computable Numbers, with an Application to the Entscheidungsproblem*, Proceedings of the London Mathematical Society, 1936.

[2] Kurt Gödel, *The Consistency of the Axiom of Choice and of the Generalized Continuum Hypothesis*, Annals of Mathematics Studies, 1940.

[3] Paul J. Cohen, *The Independence of the Continuum Hypothesis*, Proceedings of the National Academy of Sciences, Vol. 50, No. 6 (1963), pp. 1143–1148.

[4] Paul J. Cohen, *The Independence of the Axiom of Choice*, Proceedings of the National Academy of Sciences, Vol. 51, No. 1 (1964), pp. 1–4.