The Diversity of Natural Number Models and the Model-Dependence of Infinite Interpretation

Using a Peano-Compliant Extension to Collapse the $\aleph_0 < \aleph_1$ Hierarchy

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Abstract

This paper presents a ZFC-internal construction of an extended natural number model, $\mathbb{N}^+$, which fully satisfies the Peano axioms while admitting a surjection onto the powerset of standard naturals $\mathcal{P}(\mathbb{N})$. Crucially, this result is obtained without any use of forcing, external universes, or meta-mathematical assumptions—only standard ZFC tools and the Axiom of Choice (AC) are employed. The construction demonstrates that even the foundational notion of $\aleph_0$ is not absolute within ZFC: the minimal infinite cardinality can be exceeded by a Peano-compliant structure constructed purely internally. This challenges the conventional belief that the Peano axioms uniquely determine $\mathbb{N}$, and shows that the structure of arithmetic and infinity is model-relative, not canonical, even inside ZFC itself.

1 Introduction

The set of natural numbers $\mathbb{N}$ is widely regarded as the unique and foundational base of the infinite hierarchy of cardinalities, with size $\aleph_0$. It serves as the starting point for most constructions in mathematics, from ordinals to real numbers. This belief in its uniqueness is reinforced by the Peano axioms, which define the properties of natural numbers in a first-order logical system.

However, the Peano axioms do not guarantee uniqueness — they merely specify a class of models. Within ZFC, and particularly under the Axiom of Choice (AC), it is possible to construct *nonstandard* models of $\mathbb{N}$ that still satisfy all Peano axioms. These models may contain "foreign" elements not present in the standard conception of $\mathbb{N}$, and can even exhibit cardinality greater than $\aleph_0$.

This paper introduces a method for explicitly constructing such a model, denoted $\mathbb{N}^+$, which satisfies all Peano axioms but has strictly larger cardinality than $\aleph_0$. By embedding unindexable elements into the successor function and defining a consistent structure over them, we demonstrate that the infinite hierarchy itself — assumed to be absolute — can be internally violated.

The implications are foundational: if two distinct models of natural numbers exist within ZFC and are both Peano-valid, yet differ in cardinality, then the notion of $\aleph_0$ as a fixed point in the infinite landscape becomes unstable. In such a framework, even the base of the transfinite tower is no longer absolute — and ZFC, as currently conceived, cannot detect this divergence.

2 Definitions

2.1 Peano Axioms

Let $\mathbb{N}$ be a set, and let $S : \mathbb{N} \to \mathbb{N}$ be a unary function called the *successor function.* The structure $(\mathbb{N}, 0, S)$ satisfies the Peano axioms[1]

1. $0 \in \mathbb{N}$
2. $\forall n \in \mathbb{N},\ S(n) \in \mathbb{N}$
3. $\forall n \in \mathbb{N},\ S(n) \neq 0$
4. $\forall m, n \in \mathbb{N},\ S(m) = S(n) \Rightarrow m = n$

[1] See e.g. Tristin Cleveland, *Number Theory*, p. 7; Mendelson, *Introduction to Mathematical Logic*, 4th ed., p. 153.

5. **(Second-order Induction)** For every property $P \subseteq \mathbb{N}$, if:

$$P(0) \land \forall n \in \mathbb{N},\ P(n) \Rightarrow P(S(n)) \quad \Rightarrow \quad \forall n \in \mathbb{N},\ P(n)$$

Alternatively, the fifth axiom may be expressed using the schema of first-order induction as an axiom schema:

$$(\varphi(0) \land \forall n(\varphi(n) \to \varphi(S(n)))) \to \forall n\, \varphi(n)$$

for any first-order formula $\varphi(n)$.

2.2 Countable Infinity

A set A is said to be **countably infinite** if there exists a bijection $f : A \to \mathbb{N}$. This implies that the cardinality of A is equal to $\aleph_0$, the smallest infinite cardinal[2].

2.3 Powerset Cardinality

For any set A, the cardinality of its powerset satisfies:

$$|\mathcal{P}(A)| = 2^{|A|}$$

That is, if $|A| = \kappa$, then $|\mathcal{P}(A)| = 2^{\kappa}$[3].

This identity defines the exponential operation in cardinal arithmetic. It is used to compare the size of a set and its powerset, as in:

$$|\mathbb{N}| = \aleph_0 \quad \Rightarrow \quad |\mathcal{P}(\mathbb{N})| = 2^{\aleph_0}$$

Combined with Cantor's Theorem, this yields the classical inequality:

$$\aleph_0 < 2^{\aleph_0}$$

which is a foundational step in defining the infinite cardinal hierarchy.

3 Main Argument

3.1 Coexisting Successor Chains in N^+

We define $N^+ := \mathbb{N} \cup Q$, where

- $\mathbb{N} = \{0, 1, 2, 3, \dots\}$
- $Q = \{n_1, n_2, n_3, \dots\}$, a disjoint set starting from n_1

We define the successor function $S : N^+ \to N^+$ by

$$S(x) = \begin{cases} x + 1 & \text{if } x \in \mathbb{N} \\ n_{k+1} & \text{if } x = n_k \in Q \end{cases}$$

This function is injective and total over N^+, and satisfies:

1. $0 \in N^+$
2. $\forall x \in N^+,\ S(x) \in N^+$
3. $\forall x \in N^+,\ S(x) \neq 0$
4. $\forall x, y \in N^+,\ S(x) = S(y) \Rightarrow x = y$
5. If $P(0) \land \forall x \in N^+,\ P(x) \Rightarrow P(S(x)),$ then $\forall x \in N^+,\ P(x)$

Hence, N^+ is a valid Peano model with coexisting successor chains.

[2]See Jech, *Set Theory* (2002), p. 28.
[3]See Jech, *Set Theory* (2002), p. 34.

3.2 Classical Indexing over $\mathbb{N}$

The standard natural number set $\mathbb{N}$ admits bijections with other countable sets:

$$\begin{array}{l} \mathbb{N} \leftrightarrow \mathbb{Z} \\ \mathbb{N} \leftrightarrow \mathbb{Q} \\ \mathbb{N} \leftrightarrow \mathbb{N}^k \quad (k \in \mathbb{N}) \end{array}$$

These bijections demonstrate that all such sets share the cardinality $\aleph_0$.

However, **Cantor's Theorem** asserts:

$$\nexists f : \mathbb{N} \to \mathcal{P}(\mathbb{N}) \text{ such that } f \text{ is surjective}$$

Consequently,

$$|\mathbb{N}| = \aleph_0 < |\mathcal{P}(\mathbb{N})| = 2^{\aleph_0}$$

This inequality[4] implies that any function $f : \mathbb{N} \to \mathcal{P}(\mathbb{N})$ satisfies:

$$\mathrm{Im}(f) \subsetneq \mathcal{P}(\mathbb{N})$$

Therefore, classical indexing mechanisms are inherently restricted to enumerating countably many subsets:

$$\text{Indexable Subsets} := \mathrm{Im}(f) \subset \mathcal{P}(\mathbb{N})$$

This subsection formally sets the boundary of countable enumeration, which we aim to transcend in the next step using the Axiom of Choice.

3.3 The Unindexable Remainder Set Q_r

Let $f : \mathbb{N} \to \mathcal{P}(\mathbb{N})$ be any function attempting to enumerate subsets of naturals. Since:

$$|\mathrm{Im}(f)| = \aleph_0 \quad \text{and} \quad |\mathcal{P}(\mathbb{N})| = 2^{\aleph_0},$$

the image of f necessarily omits a proper class of subsets.

We define the
emphremainder set as:

$$Q_r := \mathcal{P}(\mathbb{N}) \setminus \mathrm{Im}(f)$$

By Cantor's Theorem, $Q_r \neq \emptyset$ and:

$$|Q_r| = 2^{\aleph_0}$$

The elements of Q_r represent subsets of $\mathbb{N}$ that are
textbfunreachable by any countable indexing function. That is:

$$\forall g : \mathbb{N} \to \mathcal{P}(\mathbb{N}), \quad Q_r \cap \mathrm{Im}(g) = \emptyset$$

Without the Axiom of Choice (AC), no function $s : Q_r \to \mathcal{P}(\mathbb{N})$ can guarantee the selection of a specific representative $s(S) \in S$ for each $S \in Q_r$.

Hence, Q_r forms an unstructured and nonconstructive surplus beyond the reach of classical enumeration.

The next subsection applies AC to extract definable representatives from this uncountable set.

3.4 Representative Selection via the Axiom of Choice

The Axiom of Choice (AC) asserts that for any collection of non-empty sets, there exists a function selecting exactly one element from each set. Applying this to the unindexable remainder set Q_r, we define a choice function:

$$f_{\mathrm{AC}} : Q_r \to \bigcup Q_r \quad \text{such that} \quad \forall S \in Q_r,\ f_{\mathrm{AC}}(S) \in S$$

This yields the representative set:

$$Q_{rr} := \{ f_{\mathrm{AC}}(S) \mid S \in Q_r \}$$

Then:

[4] See Jech, *Set Theory* (2002), Section 3.3.

- $|Q_{rr}| = |Q_r| = 2^{\aleph_0}$
- $Q_{rr} \subseteq \mathcal{P}(\mathbb{N})$, but not enumerable by any $f : \mathbb{N} \to \mathcal{P}(\mathbb{N})$
- Each $q_i \in Q_{rr}$ originates from a unique $S_i \in Q_r$

Thus, Q_{rr} forms a definable subcollection of representatives selected from an otherwise unstructured surplus. While non-constructive, this extraction is legitimized within ZFC by AC. Symbolically:

$$\exists f_{\mathrm{AC}},\ \forall S \in Q_r,\ f_{\mathrm{AC}}(S) \in S \quad \Rightarrow \quad Q_{rr} \text{ is well-defined}$$

This step enables the construction of new successor chains extending $\mathbb{N}$, as shown in the next section.

3.5 Extending the Natural Numbers

Given the representative set Q_{rr}, we define the extended natural number model as:

$$\mathbb{N}^+ := \mathbb{N} \cup Q_{rr}$$

Let $Q_{rr} = \{q_1, q_2, q_3, \dots\}$ be well-ordered by the Axiom of Choice.

We now define a unified successor function $S : \mathbb{N}^+ \to \mathbb{N}^+$ by:

$$S(x) = \begin{cases} x+1 & \text{if } x \in \mathbb{N} \\ q_{k+1} & \text{if } x = q_k \in Q_{rr} \end{cases}$$

This function is total, injective, and satisfies the Peano axioms over $\mathbb{N}^+$:

1. $0 \in \mathbb{N}^+$
2. $\forall x \in \mathbb{N}^+,\ S(x) \in \mathbb{N}^+$
3. $\forall x \in \mathbb{N}^+,\ S(x) \neq 0$
4. $\forall x, y \in \mathbb{N}^+,\ S(x) = S(y) \Rightarrow x = y$
5. For any property P, if $P(0) \wedge \forall x \in \mathbb{N}^+,\ P(x) \Rightarrow P(S(x))$, then $\forall x \in \mathbb{N}^+,\ P(x)$

Hence, $(\mathbb{N}^+, 0, S)$ is a valid model of Peano Arithmetic.

Furthermore, since $|Q_{rr}| = 2^{\aleph_0}$ and $|\mathbb{N}| = \aleph_0$, we have:

$$|\mathbb{N}^+| = |\mathbb{N}| + |Q_{rr}| = 2^{\aleph_0}$$

This construction bridges the gap between countable and uncountable domains, demonstrating that Peano-compliant arithmetic need not be limited to countable models. In particular, it reveals that the separation $\aleph_0 < 2^{\aleph_0}$ is not absolute, but model-dependent within ZFC.

3.6 Completion: Surjection onto $\mathcal{P}(\mathbb{N})$

By construction, we have two partial mappings:

$$\mathbb{N} \to \mathrm{Im}(f) \subset \mathcal{P}(\mathbb{N})$$
$$Q_{rr} \to Q_r = \mathcal{P}(\mathbb{N}) \setminus \mathrm{Im}(f)$$

We now define a total function:

$$F : \mathbb{N}^+ = \mathbb{N} \cup Q_{rr} \longrightarrow \mathcal{P}(\mathbb{N})$$

by:

- $F(n) := f(n) \in \mathrm{Im}(f)$ for all $n \in \mathbb{N}$
- $F(q_i) := S_i \in Q_r$ where $q_i = f_{\mathrm{AC}}(S_i) \in Q_{rr}$

Hence:

$$\mathrm{Im}(F) = \mathrm{Im}(f) \cup Q_r = \mathcal{P}(\mathbb{N})$$

This establishes:

$$F : \mathbb{N}^+ \twoheadrightarrow \mathcal{P}(\mathbb{N})$$

Therefore, $\mathbb{N}^+$ is sufficient to index the entire powerset of $\mathbb{N}$, something $\mathbb{N}$ alone cannot achieve. The cardinality of $\mathbb{N}^+$ matches that of $\mathcal{P}(\mathbb{N})$:

$$|\mathbb{N}^+| = 2^{\aleph_0} = |\mathcal{P}(\mathbb{N})|$$

This final step completes the construction of a Peano-compliant model that surjects onto its own power set—collapsing the conventional $\aleph_0 < 2^{\aleph_0}$ hierarchy and demonstrating that the foundation of infinity is model-dependent within ZFC.

3.7 On the Possibility of an Injection $\mathbb{N} \to \mathbb{N}^+$

We now examine whether the standard natural numbers $\mathbb{N}$ admit an injective mapping into the extended set $\mathbb{N}^+$.

Case 1: $\mathbb{N} \hookrightarrow \mathbb{N}^+$ is possible. If such an injection exists, then the entirety of $\mathbb{N}$ can be embedded into $\mathbb{N}^+$ without loss of structure. Since we have previously shown a surjection $\mathbb{N}^+ \twoheadrightarrow \mathcal{P}(\mathbb{N})$, it would follow that:

$$\mathbb{N} \to \mathbb{N}^+ \to \mathcal{P}(\mathbb{N}) \quad \Rightarrow \quad \exists f : \mathbb{N} \to \mathcal{P}(\mathbb{N}) \text{ (surjection)}$$

This contradicts Cantor's Theorem. Therefore, such a chain would collapse the classical cardinal hierarchy $\aleph_0 < 2^{\aleph_0}$, and implies that $\mathbb{N} \not\hookrightarrow \mathbb{N}^+$ if $\mathbb{N}^+$ surjects onto $\mathcal{P}(\mathbb{N})$.

Case 2: $\mathbb{N} \not\hookrightarrow \mathbb{N}^+$. If no injective function exists from $\mathbb{N}$ into $\mathbb{N}^+$, then within a model where $\mathbb{N}^+$ is treated as the standard natural numbers, the original set $\mathbb{N}$ becomes a **strictly smaller** countable subset. Symbolically:

$$|\mathbb{N}| < |\mathbb{N}^+| = 2^{\aleph_0}$$

This implies more than just the demotion of $\mathbb{N}$; it reveals that even the most basic layer of the infinite hierarchy—namely $\aleph_0$—is not absolute within ZFC.

That is, depending on the model, a fully Peano-compliant natural number set smaller than the traditional $\mathbb{N}$ can exist. This challenges the classical assumption that $\aleph_0$ represents the minimal infinite cardinality.

$$|\mathbb{N}| < |\mathbb{N}^+| = 2^{\aleph_0}$$

This would imply the existence of a model of ZFC in which the so-called "standard" naturals are no longer the base layer of arithmetic. The classical notion of $\mathbb{N}$ becomes a countable substructure within a richer and uncountable Peano-compliant system.

Conclusion. Whether or not an injection $\mathbb{N} \to \mathbb{N}^+$ exists has profound implications:

- If yes, then the traditional infinite hierarchy collapses. - If not, then $\mathbb{N}$ itself is demoted to a non-universal status—merely one countable model among many possible foundations.

This ambiguity reinforces the model-relativity of arithmetic and infinity within ZFC.

4 Conclusion

Through the constructions and arguments presented in this paper, we have demonstrated that the set traditionally denoted by $\mathbb{N}$ can be extended to a larger Peano-compliant model $\mathbb{N}^+$ with cardinality $2^{\aleph_0}$. Within a model where $\mathbb{N}^+$ is treated as the standard natural numbers, the original set $\mathbb{N}$ may be reinterpreted as a smaller infinite set—denoted informally as $\aleph_{-1}$. In such a setting, familiar subsets such as even and odd numbers would also belong to this reduced hierarchy.

This result goes beyond mere model-theoretic variation. It raises deep ontological questions about the foundations of mathematics:

Is the $\mathbb{N}$ we trust truly fundamental?

Or is it just one of many possible interpretations?

In this light, the notion of "infinity" ceases to be a fixed magnitude. Instead, it emerges as a model-relative value—an interpretive artifact rather than a universal constant. This realization strikes at the metaphysical core of mathematics and opens the door to a broader view: that the truth or falsity of certain mathematical statements may not be absolute, but contingent upon the underlying model.

5 Anticipated Objections and Rebuttals

1. **Objection: The extended model $\mathbb{N}^+$ is not truly Peano-compliant because it incorporates non-constructive elements.**
   Rebuttal: Let $\mathbb{N}^+ := \mathbb{N} \cup Q_{rr}$, where $Q_{rr} \subset \mathcal{P}(\mathbb{N})$ is a representative set selected via a choice function $f_{\mathrm{AC}} : Q_r \to \bigcup Q_r$, and ordered accordingly. Define the unified successor function $S : \mathbb{N}^+ \to \mathbb{N}^+$ by:

   $$S(x) = \begin{cases} x+1 & \text{if } x \in \mathbb{N} \\ q_{i+1} & \text{if } x = q_i \in Q_{rr} \end{cases}$$

   Then S is total, injective, and satisfies the Peano axioms:

   $$\begin{aligned} &0 \in \mathbb{N}^+, \quad \forall x \in \mathbb{N}^+, S(x) \in \mathbb{N}^+, \\ &\forall x \in \mathbb{N}^+, S(x) \neq 0, \quad S(x) = S(y) \Rightarrow x = y, \\ &P(0) \wedge \forall x \in \mathbb{N}^+, P(x) \Rightarrow P(S(x)) \Rightarrow \forall x \in \mathbb{N}^+, P(x) \end{aligned}$$

   The construction is formally valid in ZFC. Non-constructivity is permitted, as the Axiom of Choice allows existence without effective enumeration.

2. **Objection: The function $F : \mathbb{N}^+ \to \mathcal{P}(\mathbb{N})$ lacks explicit construction, so its surjectivity is illusory.**
   Rebuttal: Define F casewise as:

   $$F(x) := \begin{cases} f(x) & \text{if } x \in \mathbb{N}, \quad (\text{where } f : \mathbb{N} \to \mathrm{Im}(f) \subset \mathcal{P}(\mathbb{N})) \\ S_i & \text{if } x = f_{\mathrm{AC}}(S_i) \in Q_{rr} \end{cases}$$

   Then:

   $$\mathrm{Im}(F) = \mathrm{Im}(f) \cup Q_r = \mathcal{P}(\mathbb{N})$$

   So F is surjective: $F : \mathbb{N}^+ \twoheadrightarrow \mathcal{P}(\mathbb{N})$. This suffices for:

   $$|\mathbb{N}^+| \geq |\mathcal{P}(\mathbb{N})| = 2^{\aleph_0}$$

   Constructibility is not required in ZFC to prove surjectivity. Existence guaranteed by AC is sufficient.

3. **Objection: Comparing $\mathbb{N}$ and $\mathbb{N}^+$ violates model-theoretic independence.**
   Rebuttal: Both $\mathbb{N}$ and $\mathbb{N}^+$ are definable sets within the same ZFC model $\mathcal{M}$. Since:

   $$\mathbb{N} \subset \mathbb{N}^+ \subset V_\kappa \text{ for some } \kappa,$$

   we may analyze the existence (or failure) of injections $\mathbb{N} \to \mathbb{N}^+$ internally to $\mathcal{M}$. This does not compare across models but within one, so the hierarchy:

   $$|\mathbb{N}| < |\mathbb{N}^+|$$

   is meaningful and consistent in ZFC.

4. **Objection: The ontological implications of this model are philosophical, not mathematical.**
Rebuttal: The collapse $\aleph_0 = 2^{\aleph_0}$ within a ZFC-consistent Peano model is a formal result. Let $\kappa = |\mathbb{N}|$, and we obtain a Peano structure $\mathbb{N}^+ \models \mathrm{PA}$ such that:

$$|\mathbb{N}^+| = 2^\kappa$$

with a surjection $\mathbb{N}^+ \twoheadrightarrow \mathcal{P}(\mathbb{N})$. This demonstrates that the minimality of $\aleph_0$ is **not absolute**, but **model-relative** within ZFC. While the philosophical consequences are profound, the construction itself is rigorously mathematical.

Postscript: On the Root of the Collapse

If the results presented in this paper feel unsettling, even unacceptable, one might ask:

> Was the collapse of the infinite hierarchy truly a structural phenomenon?
>
> Or is it a symptom of something deeper—something hiding within the foundations themselves?

The answer lies in the Axiom of Choice.

The extended natural number model $\mathbb{N}^+$ constructed here owes its power entirely to the ability to choose representatives from an unindexable remainder set. But what if this ability—the Axiom of Choice—contains the contradiction?

A full exploration of this question is presented in the companion paper:

The Conflict Between Non-definable Reals and the Axiom of Choice in ZFC

There, we demonstrate that attempting to apply AC to the set of non-definable reals leads to a self-contradictory structure within ZFC, thus motivating the D/AC framework: ZFC with a restricted form of choice.

`"If this paper kept you awake, the next one will burn your bed."`

6 References

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