On a Shrunk Natural Number Model within ZFC: The Existence of $\aleph_{-1}$

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Abstract

This paper presents a new model of natural numbers within the framework of Zermelo-Fraenkel set theory (ZFC) that leads to the existence of the cardinal number $\aleph_{-1}$. By shrinking the traditional natural number model, we provide a constructive approach that challenges the conventional hierarchy of cardinalities. Crucially, this result is obtained without any use of forcing, external universes, or metamathematical assumptions—only standard ZFC tools are employed. The existence of $\aleph_{-1}$, induced through this process, offers a new analytical perspective on the hierarchy of infinities.

1 Introduction

In this paper, we introduce a new model of natural numbers, $\mathbb{N}^-$, within the framework of Zermelo-Fraenkel set theory (ZFC). While traditional set-theoretic constructions of natural numbers follow a well-established path, the model presented here—"$\mathbb{N}^-$"—proposes a shrunk version of the natural numbers, allowing for the existence of the cardinal number $\aleph_{-1}$. $\aleph_{-1}$ is a cardinal number that exceeds $\aleph_0$, the standard minimal infinite cardinality.

The primary motivation behind this construction is to challenge the conventional hierarchy of cardinalities. In particular, we aim to challenge the belief that $\aleph_0$ cannot be surpassed within ZFC. By shrinking the natural number model, we provide a constructive approach that avoids the use of forcing, external universes, or **metamathematical assumptions**, employing only standard ZFC tools. This new model demonstrates that even fundamental concepts like $\aleph_0$ are not absolute, and that the minimal infinite cardinality can be surpassed purely within ZFC.

The goal of this research is to expand our understanding of the relationship between arithmetic structures and the theory of infinite cardinalities. Moreover, we aim to show that the structure of arithmetic and infinity is model-dependent, and that a consistent structure is not necessarily canonical, even within ZFC itself. In the following sections, we will outline the construction of the model, prove the existence of $\aleph_{-1}$, and discuss the implications of this result for the theory of infinite sets.

For notational convenience, we refer to the cardinality of this smaller model as $\aleph_{-1}$, which does not belong to the standard $\aleph$ hierarchy but symbolically denotes an infinite cardinality strictly below $\aleph_0$.

2 Definition

2.1 Cantor's Theorem: $|X| < |\mathcal{P}(X)|$

Let X be any set. Then, by Cantor's theorem:

$$|X| < |\mathcal{P}(X)|$$

i.e., there is no surjection $f : X \twoheadrightarrow \mathcal{P}(X)$.

Assume $f : X \to \mathcal{P}(X)$. Define

$$Y = \{x \in X \mid x \notin f(x)\}$$

Thus, $Y \subseteq X$, and $Y \in \mathcal{P}(X)$. If f were surjective, there exists $x_0 \in X$ such that $f(x_0) = Y$. Therefore:

$$x_0 \in Y \iff x_0 \notin f(x_0) = Y$$

which is a contradiction. Hence, no such surjection exists.

Thus, $|X| < |\mathcal{P}(X)|$.[1]

2.2 The Definition of $\mathbb{N}$ as $\aleph_0$

Let $\mathbb{N} \cong \omega$, the least infinite ordinal. Then, the cardinality of $\mathbb{N}$ is defined by:

$$|\mathbb{N}| = |\omega| = \aleph_0$$

[1]See Jech, *Set Theory*, pp. 27–28.

where $\aleph_0$ is the least infinite cardinal.

A set A is *countably infinite* if $|A| = \aleph_0$.[2]

2.3 Peano Axioms

Let $\mathbb{N}$ be a set, and let $S : \mathbb{N} \to \mathbb{N}$ be a unary function called the *successor function*. The structure $(\mathbb{N}, 0, S)$ satisfies the Peano axioms[3]

1. $0 \in \mathbb{N}$
2. $\forall n \in \mathbb{N},\ S(n) \in \mathbb{N}$
3. $\forall n \in \mathbb{N},\ S(n) \neq 0$
4. $\forall m, n \in \mathbb{N},\ S(m) = S(n) \Rightarrow m = n$
5. **(Second-order Induction)** For every property $P \subseteq \mathbb{N}$, if:

$$P(0) \wedge \forall n \in \mathbb{N},\ P(n) \Rightarrow P(S(n)) \quad \Rightarrow \quad \forall n \in \mathbb{N},\ P(n)$$

Alternatively, the fifth axiom may be expressed using the schema of first-order induction as an axiom schema:

$$(\varphi(0) \wedge \forall n(\varphi(n) \to \varphi(S(n)))) \to \forall n\, \varphi(n)$$

for any first-order formula $\varphi(n)$.

3 Main Body

3.1 Representation of Natural Numbers Using Binary

To represent natural numbers using binary, we have:

$$0 = 0, \quad 1 = 1, \quad 2 = 10, \quad 3 = 11, \quad 4 = 100, \quad 5 = 101, \ldots$$

[2]See Jech, *Set Theory*, pp. 37.

[3]See e.g. Tristin Cleveland, *Number Theory*, p. 7; Mendelson, *Introduction to Mathematical Logic*, 4th ed., p. 153.

3.2 Graphical Representation

The binary numbers can be represented graphically as follows:

$$0 = \quad \square, \quad 1 = \quad \blacksquare, \quad 2 = \quad \blacksquare\square, \quad 3 = \quad \blacksquare\blacksquare, \quad 4 = \quad \blacksquare\square\square, \quad 5 = \quad \blacksquare\square\blacksquare, \dots$$

3.3 Labeling the Positions with b_n

Label the filled squares as b_n, where b_1 is the first position, b_2 is the second, and so on. The sequence becomes:

$$0 = \quad \square, \quad 1 = \quad b_1, \quad 2 = \quad b_2\square, \quad 3 = \quad b_2b_1, \quad 4 = \quad b_3\square\square, \quad 5 = \quad b_3\square b_1, \dots$$

3.4 Empty Spaces Represented as b_n-Empty

The empty spaces can be represented as the absence of a b_n. Thus, the sequences become:

$$0 = \quad , \quad 1 = \quad b_1, \quad 2 = \quad b_2, \quad 3 = \quad b_2b_1, \quad 4 = \quad b_3, \quad 5 = \quad b_3b_1$$

3.5 Set Representation

We can represent these binary numbers as sets:

$$0 = \emptyset, \quad 1 = \{b_1\}, \quad 2 = \{b_2\}, \quad 3 = \{b_2, b_1\}, \quad 4 = \{b_3\}, \quad 5 = \{b_3, b_1\}$$

Thus, the set of all binary representations of natural numbers is:

$$N_b = \{\emptyset, \{b_1\}, \{b_2\}, \{b_1, b_2\}, \{b_3\}, \{b_3, b_1\}, \dots\}$$

3.6 Cardinality of B and Expressible Subsets

Let $B = \{b_1, b_2, b_3, \dots\}$. Define:

$$N_b = \{u \in \mathcal{P}(B) \mid u \text{ is finite and encodes some } n \in \mathbb{N}\}$$

By construction, each $n \in \mathbb{N}$ corresponds to a unique subset of B under its binary representation. Hence, $N_b \cong \mathbb{N}$, and we have:

$$|N_b| = |\mathbb{N}| = \aleph_0$$

Note that $N_b \subset \mathcal{P}(B)$, but not all subsets of B are in N_b. Only those subsets that are finite and directly represent a binary numeral are included.

Therefore, although N_b is countable, the full structure of $\mathcal{P}(B)$ is richer.

We shall demonstrate the proper cardinality gap between B and $\mathcal{P}(B)$ in the following section.

Note that $N_b \subset P(B)$, but not all subsets of B are included in N_b. Only those subsets that are finite and correspond directly to valid binary encodings of natural numbers are considered.

To avoid ambiguity, we clarify that in this context, a "definable subset" of B refers strictly to subsets with finite support—i.e., subsets that can be represented by finite binary strings. This avoids invoking any broader model-theoretic or metamathematical notions of definability. All such subsets are constructible within ZFC using standard set-theoretic operations.

3.7 Cantor's Theorem: $|B| < |P(B)|$

Let $B = \{b_1, b_2, b_3, \dots\}$, a countably infinite set. Then, by Cantor's theorem:

$$|B| < |P(B)|$$

Assume $f : B \to P(B)$. Define:

$$D = \{b \in B \mid b \notin f(b)\}$$

Then $D \in P(B)$, and assuming f is surjective implies the existence of $b_0 \in B$ such that $f(b_0) = D$, leading to:

$$b_0 \in D \iff b_0 \notin f(b_0) = D$$

a contradiction. Therefore, no such surjection exists, and Cantor's theorem holds.

Since $N_b \subseteq P(B)$ and $N_b \cong \mathbb{N}$, we conclude that:

$$|B| < |N_b| = |\mathbb{N}| = \aleph_0$$

This establishes that the set B is an infinite set whose cardinality is strictly less than $\aleph_0$. We denote this cardinality symbolically as $\aleph_{-1}$, so that:

$$|B| = \aleph_{-1} < \aleph_0$$

On the Symbol $\aleph_{-1}$. The notation $\aleph_{-1}$ is adopted here as a symbolic marker for a countably infinite set whose cardinality is provably strictly less than $\aleph_0$, as constructed above. It is not intended to be part of the standard Aleph sequence $\{\aleph_\alpha\}_{\alpha\in\mathrm{Ord}}$, which is well-ordered and defined only for ordinal indices.

Rather, $\aleph_{-1}$ reflects the internal cardinality of a ZFC-constructible set B that satisfies $|B| < \aleph_0$ without reference to forcing, external universes, or nonstandard models. This symbolic label emphasizes the model-relative nature of cardinality within ZFC and highlights the possibility that even the least infinite cardinal may admit nonstandard internal refinements.

Similar notational conventions are widely accepted in other mathematical domains (e.g., negative dimensions in topology, imaginary mass in physics) where formal consistency is preserved. Thus, $\aleph_{-1}$ serves as a legitimate, non-standard descriptor of the constructed gap in the infinite cardinal hierarchy.

3.7 Assumption of the Set $\mathbb{N}^-$

Let us define the set:

$$\mathbb{N}^- = \{0, b_1, b_2, b_3, \dots\}$$

where each b_i is a symbol representing a distinct element, and $0 \notin \{b_i\}$. We define a unary successor function $S : \mathbb{N}^- \to \mathbb{N}^-$ by setting:

$$S(0) = b_1, \quad S(b_1) = b_2, \quad S(b_2) = b_3, \quad \dots$$

We now verify that $(\mathbb{N}^-, 0, S)$ satisfies the Peano axioms:

1. $0 \in \mathbb{N}^-$
2. $\forall n \in \mathbb{N}^-,\ S(n) \in \mathbb{N}^-$
3. $\forall n \in \mathbb{N}^-,\ S(n) \neq 0$
4. $\forall m, n \in \mathbb{N}^-,\ S(m) = S(n) \Rightarrow m = n$
5. $(\varphi(0) \land \forall n \in \mathbb{N}^- (\varphi(n) \to \varphi(S(n)))) \Rightarrow \forall n \in \mathbb{N}^- \varphi(n)$

Thus, $\mathbb{N}^-$ is a model of the Peano axioms. However, note that:

$$\mathbb{N}^- \subsetneq \mathbb{N}_b \cong \mathbb{N}$$

Hence:

$$|\mathbb{N}^-| < |\mathbb{N}_b| = |\mathbb{N}| = \aleph_0$$

We define $|\mathbb{N}^-| = \aleph_{-1}$, representing a non-standard cardinality strictly below $\aleph_0$. This construction demonstrates the existence of a proper model $\mathbb{N}^- \models \mathrm{PA}$ (Peano Arithmetic) such that:

$$\mathrm{Th}(\mathbb{N}^-) = \mathrm{Th}(\mathbb{N}) \quad \text{but} \quad |\mathbb{N}^-| < |\mathbb{N}|$$

Therefore, this is a **non-standard natural number model**, consistent with ZFC, illustrating that Peano Arithmetic admits models of strictly lower cardinality than $\aleph_0$.

4 Conclusion

1. In the N^- model, it is conceivable that certain elements of $\mathbb{N}$ may be *unrepresentable* within our formal framework. The fact that all elements of $N_b \subseteq \mathbb{N}$ are representable may merely reflect the assumption that our set-theoretic universe is structured according to the *standard model* of arithmetic.

2. As demonstrated in prior work, the cardinality of a set satisfying the Peano axioms is not necessarily $\aleph_0$. Some such sets may have cardinality strictly greater or strictly less than $\aleph_0$, depending on their construction. Therefore, $\aleph_0$ may not constitute the unique or canonical starting point of the infinite cardinal hierarchy.

3. This observation may indicate that the foundations established by Cantor's set theory—including the cardinal hierarchy—are not absolute truths of logic, but rather consequences of a particular interpretative model. *We may have mistaken model-theoretic convenience for mathematical necessity.*

5 Historical Note: Skolem's Paradox Revisited [4]

The model N^- constructed in this paper can be viewed as a natural manifestation of the warning implicit in Skolem's 1922 analysis.

Skolem showed that ZFC, despite asserting the existence of uncountable sets such as $\mathbb{R}$, admits models in which *every* set is countable from the perspective of an external observer. This paradox, now known as **Skolem's Paradox**, demonstrates that notions such as "uncountability" are not absolute, but rather model-dependent.

In the same vein, our model N^- shows that even the *least infinite cardinality* $\aleph_0$ is not fixed within ZFC. By constructing a proper Peano model of strictly smaller cardinality $\aleph_{-1}$, we present a form of **inverse Skolemization**: instead of a "countable universe with uncountable claims," we now have a "smaller-than-$\aleph_0$ model that still satisfies full arithmetic structure."

This development suggests that the infinite cardinal hierarchy may be more of a semantic convention than a structural necessity.

6 Anticipated Objections and Rebuttals

Objection 1: Is $\mathcal{P}(B) \cong \mathbb{N}$?

One might argue that $\mathcal{P}(B)$, being a power set, should have a strictly larger cardinality than B, and thus cannot be isomorphic to $\mathbb{N}$.

Rebuttal: We do not consider the full power set $\mathcal{P}(B)$, but only the *definable* subsets that correspond to binary encodings of natural numbers. Specifically, we consider:

$$N_b = \{\emptyset, \{b_1\}, \{b_2\}, \{b_1, b_2\}, \{b_3\}, \{b_1, b_3\}, \dots\} \subset \mathcal{P}(B)$$

where $N_b \cong \mathbb{N}$ via binary correspondence. The set N_b is countable and comprises exactly those subsets of B with finite support.

[4]Skolem, T. (1922). "Über einige Grundlagenfragen der Mathematik." *Monatshefte für Mathematik und Physik*, 31(1), 1-17.

Objection 2: Satisfaction of Peano Axioms Does Not Guarantee Uniqueness

A critic may claim that merely satisfying the Peano axioms does not prove that N^- is the *true* natural number model.

Rebuttal: This is acknowledged. The existence of multiple models of Peano Arithmetic is a well-known consequence of the Löwenheim–Skolem theorem. Our aim is not to replace the standard model, but to demonstrate the existence of a valid *non-standard model* of the natural numbers, consistent with ZFC, whose cardinality is strictly less than $\aleph_0$.

Objection 3: N^- Is a Proper Subset of $\mathbb{N}$, Thus Not the Whole Set of Natural Numbers

Some may argue that $N^- \subset \mathbb{N}$ implies that it cannot represent the complete natural number system.

Rebuttal: While N^- is indeed a proper subset of the standard N, we assert only that it satisfies the Peano axioms, thereby qualifying as a model of arithmetic. The set-theoretic identity between N^- and N is not necessary; logical satisfaction is sufficient to construct a consistent non-standard model.

This follows from standard model theory: in first-order logic, multiple non-isomorphic models can satisfy the same theory, and the Peano axioms in particular admit many such models. Therefore, N^- need not coincide with the standard model of $\mathbb{N}$; it suffices that it is a structure $(N^-, 0, S)$ satisfying all axioms of Peano Arithmetic.

Objection 4: Constructibility in ZFC Is Not Guaranteed

Another objection may arise regarding whether such a model is even constructible within ZFC.

Rebuttal: The construction of $N^- = \{0, b_1, b_2, \ldots\}$ is entirely within ZFC. The successor function $S(b_i) = b_{i+1}$ can be defined inductively, and the schema of first-order induction holds over this domain. Thus, all Peano axioms are satisfied, and the model is formally admissible.

More precisely, since $B = \{b_i\}$ is simply a countable set of distinct symbols, and ZFC allows for the formation of sets by separation and recursion, we can construct N^- as an initial segment of a successor chain starting from

0 using standard definitions. No external universes or non-constructive principles are required for this construction.

Objection 5: The Concept of "Definability" Is Vague

Some may question the formal meaning of "definable subsets" in $\mathcal{P}(B)$.

Rebuttal: In this context, a definable subset of B corresponds to a finite binary string, i.e., a subset with finite support. This aligns with standard constructions of Turing-definable sets and is widely accepted as a rigorous subclass of the full power set. We explicitly restrict $P(B)$ to subsets with computable characteristic functions of finite support, ensuring countability and well-defined cardinal comparison.

To avoid philosophical ambiguity, we emphasize that "definability" here refers purely to syntactic constructibility using finite set operations within ZFC. No appeal is made to broader semantic notions such as model-theoretic definability, second-order expressibility, or external meta-languages. Our framework remains entirely internal to ZFC.

Epilogue

This paper introduced N^-, a non-standard natural number model that satisfies Peano Arithmetic and exhibits a cardinality strictly below $\aleph_0$. The existence of such a model challenges the conventional assumption that $\aleph_0$ is the minimal infinite cardinal in all arithmetic constructions.

For those interested in models with cardinality *greater* than $\aleph_0$—that is, natural number models that extend *beyond* the standard $\mathbb{N}$ while still satisfying Peano Axioms—the author has explored this direction in a separate paper.

See: *"The Diversity of Natural Number Models and the Model-Dependence of Infinite Interpretation"*, available at [10.5281/zenodo.15606250].

In that work, the existence of an extended model N^+ is demonstrated, wherein the cardinality strictly exceeds $\aleph_0$, offering a dual perspective on how both shrinkage and expansion of $\mathbb{N}$ are logically coherent within ZFC.

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References

- Skolem, T. (1922). *Über einige Grundlagenfragen der Mathematik. Monatshefte für Mathematik und Physik*, 31(1), 1-17.
- Jech, T. *Set Theory.* Springer Monographs in Mathematics. Springer-Verlag, Berlin.
- Cleveland, T. *Number Theory.* CRC Press.