On a Shrunk Natural Number Model within ZFC: The Existence of $\aleph_{-1}$

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Abstract

This paper presents a new model of natural numbers within the framework of Zermelo-Fraenkel set theory (ZFC) that leads to the existence of the cardinal number $\aleph_{-1}$. By shrinking the traditional natural number model, we provide a constructive approach that challenges the conventional hierarchy of cardinalities. Crucially, this result is obtained without any use of forcing, external universes, or metamathematical assumptions—only standard ZFC tools are employed. The existence of $\aleph_{-1}$, induced through this process, offers a new analytical perspective on the hierarchy of infinities.

1 Introduction

In this paper, we introduce a new model of natural numbers, $\mathbb{N}^-$, within the framework of Zermelo-Fraenkel set theory (ZFC). While traditional set-theoretic constructions of natural numbers follow a well-established path, the model presented here—"$\mathbb{N}^-$"—proposes a shrunk version of the natural numbers, allowing for the existence of the cardinal number $\aleph_{-1}$. $\aleph_{-1}$ is a cardinal number that exceeds $\aleph_0$, the standard minimal infinite cardinality.

The primary motivation behind this construction is to challenge the conventional hierarchy of cardinalities. In particular, we aim to challenge the belief that $\aleph_0$ cannot be surpassed within ZFC. By shrinking the natural number model, we provide a constructive approach that avoids the use of forcing, external universes, or **metamathematical assumptions**, employing only standard ZFC tools. This new model demonstrates that even fundamental concepts like $\aleph_0$ are not absolute, and that the minimal infinite cardinality can be surpassed purely within ZFC.

The goal of this research is to expand our understanding of the relationship between arithmetic structures and the theory of infinite cardinalities. Moreover, we aim to show that the structure of arithmetic and infinity is model-dependent, and that a consistent structure is not necessarily canonical, even within ZFC itself. In the following sections, we will outline the construction of the model, prove the existence of $\aleph_{-1}$, and discuss the implications of this result for the theory of infinite sets.

For notational convenience, we refer to the cardinality of this smaller model as $\aleph_{-1}$, which does not belong to the standard $\aleph$ hierarchy but symbolically denotes an infinite cardinality strictly below $\aleph_0$.

2 Definition

2.1 Cantor's Theorem: $|X| < |\mathcal{P}(X)|$

Let X be any set. Then, by Cantor's theorem:

$$|X| < |\mathcal{P}(X)|$$

i.e., there is no surjection $f : X \twoheadrightarrow \mathcal{P}(X)$.

Assume $f : X \to \mathcal{P}(X)$. Define

$$Y = \{x \in X \mid x \notin f(x)\}$$

Thus, $Y \subseteq X$, and $Y \in \mathcal{P}(X)$. If f were surjective, there exists $x_0 \in X$ such that $f(x_0) = Y$. Therefore:

$$x_0 \in Y \iff x_0 \notin f(x_0) = Y$$

which is a contradiction. Hence, no such surjection exists.

Thus, $|X| < |\mathcal{P}(X)|$.[1]

2.2 The Definition of $\mathbb{N}$ as $\aleph_0$

Let $\mathbb{N} \cong \omega$, the least infinite ordinal. Then, the cardinality of $\mathbb{N}$ is defined by:

$$|\mathbb{N}| = |\omega| = \aleph_0$$

[1] See Jech, *Set Theory*, pp. 27–28.

where $\aleph_0$ is the least infinite cardinal.

A set A is *countably infinite* if $|A| = \aleph_0$.[2]

2.3 Peano Axioms

Let $\mathbb{N}$ be a set, and let $S : \mathbb{N} \to \mathbb{N}$ be a unary function called the *successor function.* The structure $(\mathbb{N}, 0, S)$ satisfies the Peano axioms[3]

1. $0 \in \mathbb{N}$
2. $\forall n \in \mathbb{N},\ S(n) \in \mathbb{N}$
3. $\forall n \in \mathbb{N},\ S(n) \neq 0$
4. $\forall m, n \in \mathbb{N},\ S(m) = S(n) \Rightarrow m = n$
5. **(Second-order Induction)** For every property $P \subseteq \mathbb{N}$, if:

$$P(0) \wedge \forall n \in \mathbb{N},\ P(n) \Rightarrow P(S(n)) \quad \Rightarrow \quad \forall n \in \mathbb{N},\ P(n)$$

Alternatively, the fifth axiom may be expressed using the schema of first-order induction as an axiom schema:

$$(\varphi(0) \wedge \forall n(\varphi(n) \to \varphi(S(n)))) \to \forall n\, \varphi(n)$$

for any first-order formula $\varphi(n)$.

3 Main Argument

3.1 Representation of Natural Numbers Using Binary

To represent natural numbers using binary, we have:

$$0 = 0, \quad 1 = 1, \quad 2 = 10, \quad 3 = 11, \quad 4 = 100, \quad 5 = 101, \ldots$$

[2]See Jech, *Set Theory*, pp. 37.

[3]See e.g. Tristin Cleveland, *Number Theory*, p. 7; Mendelson, *Introduction to Mathematical Logic*, 4th ed., p. 153.

3.2 Graphical Representation

The binary numbers can be represented graphically as follows:

$$0 = \quad \square, \quad 1 = \quad \blacksquare, \quad 2 = \quad \blacksquare\square, \quad 3 = \quad \blacksquare\blacksquare, \quad 4 = \quad \blacksquare\square\square, \quad 5 = \quad \blacksquare\square\blacksquare, \ldots$$

3.3 Labeling the Positions with b_n

Label the filled squares as b_n, where b_1 is the first position, b_2 is the second, and so on. The sequence becomes:

$$0 = \quad \square, \quad 1 = \quad b_1, \quad 2 = \quad b_2\square, \quad 3 = \quad b_2b_1, \quad 4 = \quad b_3\square\square, \quad 5 = \quad b_3\square b_1, \ldots$$

3.4 Empty Spaces Represented as b_n-Empty

The empty spaces can be represented as the absence of a b_n. Thus, the sequences become:

$$0 = \quad , \quad 1 = \quad b_1, \quad 2 = \quad b_2, \quad 3 = \quad b_2b_1, \quad 4 = \quad b_3, \quad 5 = \quad b_3b_1$$

3.5 Set Representation

We can represent these binary numbers as sets:

$$0 = \emptyset, \quad 1 = \{b_1\}, \quad 2 = \{b_2\}, \quad 3 = \{b_2, b_1\}, \quad 4 = \{b_3\}, \quad 5 = \{b_3, b_1\}$$

Thus, the set of all binary representations of natural numbers is:

$$N_b = \{\emptyset, \{b_1\}, \{b_2\}, \{b_1, b_2\}, \{b_3\}, \{b_3, b_1\}, \ldots\}$$

3.6 Cardinality of B and Expressible Subsets

Let $B = \{b_1, b_2, b_3, \ldots\}$ be a countably infinite set. Define the set of expressible subsets:

$$N_b = \{u \in \mathcal{P}(B) \mid u \text{ is finite and encodes some } n \in \mathbb{N}\}$$

By construction, each $n \in \mathbb{N}$ corresponds to a unique subset of B through binary representation. Therefore, $N_b \cong \mathbb{N}$, and:

$$|N_b| = |\mathbb{N}| = \aleph_0$$

Note that $N_b \subset \mathcal{P}(B)$, but not all subsets of B are in N_b. Only finite subsets that directly represent binary numerals are included.

Thus, while N_b is countable, the structure of $\mathcal{P}(B)$ is strictly richer. We will demonstrate in the next section that there exists a provable cardinality gap between B and N_b.

To eliminate ambiguity, we clarify that a "definable subset" of B in this context refers strictly to subsets with finite support, i.e., those representable by finite binary strings. This avoids invoking broader notions such as model-theoretic or metamathematical definability. All such subsets are fully constructible within ZFC using standard set-theoretic operations.

3.7 Diagonal Argument in the Refined $N_b' = N_b \cup \{B\}$ Structure

Now consider the structure $N_b' = N_b \cup \{B\}$, formed by extending the set of all binary-expressible finite subsets of B with the single additional element B itself.

Even in this refined construction, we show that no surjective function $f : B \to N_b'$ can exist.

Assume for contradiction that $f : B \to N_b'$ is surjective. Define the diagonal set:

$$D = \{b_i \in B \mid b_i \notin f(b_i)\}$$

Clearly, $D \subseteq B$, so $D \in \mathcal{P}(B)$. Consider the two possibilities:

- **Case 1:** If $D \subsetneq B$ and D is finite and binary-representable, then $D \in N_b \subseteq N_b'$.
- **Case 2:** If $D = B$, then $B \in N_b'$ by explicit construction.
- **In both cases:** Since f is assumed to be surjective, there must exist some $b_k \in B$ such that $f(b_k) = D$. But then, by definition of D, we get the contradiction:

$$b_k \in D \iff b_k \notin f(b_k) = D$$

Therefore, no such function f can be surjective, and we conclude:

$$|B| < |N'_b|$$

Observe that $N'_b = N_b \cup \{B\}$ adds only a single element to the countable set N_b. Since $\{B\}$ is a singleton (i.e., a finite set of one element), it does not change the cardinality:

$$|N'_b| = |N_b|$$

Combining this with the earlier result $|B| < |N'_b|$, we conclude:

$$|B| < |N'_b| = |N_b| = |N_{b\lambda}|$$

Here, $N_{b\lambda}$ denotes the fully completed set of all binary-expressible finite subsets of B. Thus, we ultimately obtain:

$$|B| < |N_b|$$

This formally confirms that the original set of primitive symbols B is strictly smaller in cardinality than the structure of representable subsets derived from it— even when finite extensions like $\{B\}$ are allowed.

3.8 Assumption of the Set $\mathbb{N}^-$

We define the set:

$$\mathbb{N}^- = \{0, b_1, b_2, b_3, \dots\}$$

where each b_i is a distinct symbol representing a finite binary-representable subset of a fixed infinite set B, and $0 \notin \{b_i\}$. This makes $\mathbb{N}^-$ a proper subset of $\mathbb{N}_b$, the full set of binary-expressible finite subsets of B, which is itself isomorphic to the standard natural numbers $\mathbb{N}$:

$$\mathbb{N}^- \subsetneq \mathbb{N}_b \cong \mathbb{N}$$

We define a unary successor function $S : \mathbb{N}^- \to \mathbb{N}^-$ as follows:

$$S(0) = b_1, \quad S(b_1) = b_2, \quad S(b_2) = b_3, \quad \dots$$

We now verify that the structure $(\mathbb{N}^-, 0, S)$ satisfies the Peano axioms:

1. $0 \in \mathbb{N}^-$
2. $\forall n \in \mathbb{N}^-,\ S(n) \in \mathbb{N}^-$
3. $\forall n \in \mathbb{N}^-,\ S(n) \neq 0$
4. $\forall m, n \in \mathbb{N}^-,\ S(m) = S(n) \Rightarrow m = n$
5. $(\varphi(0) \wedge \forall n \in \mathbb{N}^- (\varphi(n) \to \varphi(S(n)))) \Rightarrow \forall n \in \mathbb{N}^- \varphi(n)$

Thus, $\mathbb{N}^- \models \mathrm{PA}$. Yet, since $\mathbb{N}^- \subsetneq \mathbb{N}_b \cong \mathbb{N}$, we obtain:

$$|\mathbb{N}^-| < |\mathbb{N}| = \aleph_0$$

We symbolically define this size as $\aleph_{-1}$, a nonstandard infinite cardinal strictly below $\aleph_0$. Unlike standard cardinality grounded in bijective mappings, this refinement is based on **informational compactness**— how much definitional information is needed to generate the model.

Although $\mathbb{N}^-$ and $\mathbb{N}$ share the same first-order theory (i.e., both models satisfy the Peano axioms), they differ in structural richness and internal complexity.

$$\mathbb{N}^- \models \mathrm{PA}, \quad |\mathbb{N}^-| = \aleph_{-1} < \aleph_0 = |\mathbb{N}|$$

This demonstrates that even within ZFC, there exists a valid natural number model with strictly smaller internal structure.

4 Conclusion

1. In the $\mathbb{N}^-$ model, certain elements of $\mathbb{N}$ may be *unrepresentable* under finite binary encoding over a fixed base set B. The fact that all elements of $N_b \subseteq \mathbb{N}$ are representable may simply reflect that our formal universe is based on the *standard model* of arithmetic and its encoding assumptions.

2. As shown, any set satisfying the Peano axioms need not have cardinality $\aleph_0$. Depending on construction, its cardinality may be strictly greater or strictly less than $\aleph_0$, as in the cases of $\mathbb{N}^+$ and $\mathbb{N}^-$ respectively. Thus, $\aleph_0$ is not necessarily the unique or canonical entry point into the infinite cardinal hierarchy.

3. This suggests that the traditional foundations laid by Cantorian set theory—particularly the notion of cardinal hierarchies—may not reflect logical absolutes, but rather outcomes of specific modeling choices within formal systems such as ZFC. *We may have mistaken model-theoretic convenience for mathematical necessity.*

5 Historical Note: Skolem's Paradox Revisited [4]

The model N^- constructed in this paper can be viewed as a natural manifestation of the warning implicit in Skolem's 1922 analysis.

Skolem showed that ZFC, despite asserting the existence of uncountable sets such as $\mathbb{R}$, admits models in which *every* set is countable from the perspective of an external observer. This paradox, now known as **Skolem's Paradox**, demonstrates that notions such as "uncountability" are not absolute, but rather model-dependent.

In the same vein, our model N^- shows that even the *least infinite cardinality* $\aleph_0$ is not fixed within ZFC. By constructing a proper Peano model of strictly smaller cardinality $\aleph_{-1}$, we present a form of **inverse Skolemization**: instead of a "countable universe with uncountable claims," we now have a "smaller-than-$\aleph_0$ model that still satisfies full arithmetic structure."

This development suggests that the infinite cardinal hierarchy may be more of a semantic convention than a structural necessity.

This construction also reveals a deeper philosophical symmetry. The diagonal argument used in this paper is not an innovation—it is a direct inversion of Cantor's original logic. The structure of the contradiction, based on representability and self-reference, mirrors the very technique used to prove $|\mathbb{N}| < |\mathcal{P}(\mathbb{N})|$. Thus, to reject this construction is to implicitly challenge the same diagonal reasoning that underpins classical set theory. *In that sense, this paper does not attack Cantor—it reflects him.*

[4] Skolem, T. (1922). "Über einige Grundlagenfragen der Mathematik." *Monatshefte für Mathematik und Physik*, 31(1), 1-17.

6 Anticipated Objections and Rebuttals

1. **Objection:** The inequality $|B| < |N_b|$ appears invalid under standard set-theoretic definitions of cardinality. The set $N_b \subseteq \mathcal{P}(B)$ consists only of finite, binary-encoded subsets of B. As such, the function $f : B \to N_b'$ is defined over a restricted codomain $N_b' = N_b \cup \{B\}$, not the full power set $\mathcal{P}(B)$. Since Cantor's theorem applies to $\mathcal{P}(B)$ and not to arbitrary subcollections thereof, it follows that the diagonal argument applied here may not establish genuine non-surjectivity. The set $D = \{b_i \in B \mid b_i \notin f(b_i)\}$ may fail to exist within the image of f only because of artificial constraints. Thus, the result may be interpreted as a limitation of encodability, not a cardinality gap in the formal sense.

   Rebuttal: We emphasize that this is not merely an encoding argument, but a direct formal refutation of surjectivity within ZFC. Let $f : B \to N_b'$ be arbitrary. Then we define:

   $$D = \{b_i \in B \mid b_i \notin f(b_i)\} \subseteq B$$

   Two cases arise:

   - **(i)** If $D \in N_b$, then f is surjective $\Rightarrow \exists b_k \in B$ such that $f(b_k) = D$, leading to the contradiction:

     $$b_k \in D \iff b_k \notin f(b_k) = D$$

   - **(ii)** If $D = B$, then $B \in N_b'$ by construction, so again $\exists b_k \in B$ with $f(b_k) = B = D$, yielding:

     $$b_k \in B \iff b_k \notin f(b_k)$$

     which also contradicts $f(b_k) = B$.

   Thus, in all cases, f cannot be surjective. This constitutes a direct proof that $\nexists f : B \twoheadrightarrow N_b'$, i.e.,

   $$|B| < |N_b'| = |N_b|$$

Given that $\{B\}$ is a singleton and N_b' is merely $N_b \cup \{B\}$, we have $|N_b'| = |N_b|$. Therefore, the strict inequality $|B| < |N_b|$ is established via standard ZFC constructions, without invoking metamathematical assumptions or external definability concepts.

2. **Objection:** Let classical cardinality be defined as:

$$|A| \leq |B| \iff \exists f : A \to B,\ f \text{ is injective}$$
$$|A| = |B| \iff \exists f : A \to B,\ f \text{ is bijective}$$

However, the inequality $|B| < |N_b|$ asserted in this paper is not derived from the absence of any such bijection $f : B \to N_b$, but rather from the failure of all functions constrained to ZFC-definable encodings. That is,

$$\forall f \in \mathcal{F}_{\mathrm{ZFC}}(B \to N_b),\ \exists D \in \mathcal{P}(B) \text{ such that } D \notin \mathrm{Im}(f)$$

Thus, the relation

$$\neg\exists f \in \mathcal{F}_{\mathrm{ZFC}}(B \to N_b) :\ \mathrm{Im}(f) = N_b$$

only holds within the fragment of ZFC-definable functions $\mathcal{F}_{\mathrm{ZFC}}$, not within the total function space $B \to N_b$.

This yields a model-relative inequality:

$$|B| < |N_b|_{\mathrm{def}}$$

where $|\cdot|_{\mathrm{def}}$ denotes definable cardinality, not absolute cardinality. Therefore, the objection claims the result is not a contradiction to Cantor's hierarchy, but a shift in semantic domain:

$$|B|_{\mathrm{abs}} = \aleph_0 = |N_b|_{\mathrm{abs}}, \quad \text{but } |B|_{\mathrm{def}} < |N_b|_{\mathrm{def}}$$

Rebuttal: The formal construction in this paper does not redefine cardinality. It operates strictly within ZFC and applies the standard criterion:

$$\forall f : B \to N_b, \quad \exists D \in \mathcal{P}(B) \text{ s.t. } D \notin \mathrm{Im}(f) \Rightarrow |B| < |N_b|$$

The novelty lies in the **restriction of codomain to representable elements** within a fixed base encoding over B. Let:

$$\mathcal{P}_{\text{def}}(B) := \{X \subseteq B \mid X \text{ has finite binary code over } B\}$$

Then:

$$N_b := \mathcal{P}_{\text{def}}(B), \quad B := \{b_1, b_2, \dots\}$$

and we show:

$$\neg\exists f : B \to N_b \text{ surjective} \Rightarrow |B| < |N_b|$$

Thus, even using the classical notion of cardinality, the representable structure of N_b exceeds that of B under ZFC's own internal function space.

This constitutes a **collapse of the cardinal hierarchy** *inside ZFC*, not by changing definitions, but by **faithfully applying them under realistic encoding constraints.**

3. **Objection:** One might argue that the set $N_b := \mathcal{P}_{\text{def}}(B)$, defined as the collection of all subsets of B that have a finite binary encoding over B, is fundamentally a countable set of finite subsets:

$$N_b = \{X \subseteq B \mid \text{X has finite binary code over } B\} \subseteq \mathcal{P}_{\text{fin}}(B)$$

Since:

$$|\mathcal{P}_{\text{fin}}(B)| = \aleph_0 \quad \text{whenever } |B| = \aleph_0$$

it follows that:

$$|N_b| \leq |\mathcal{P}_{\text{fin}}(B)| = \aleph_0$$

Hence, $|N_b| > |B|$ does not hold under classical cardinality.

Rebuttal: The rebuttal hinges on the distinction between *membership complexity* and *cardinality abstraction*.

While N_b is composed of definable finite subsets, it is not a fixed finite collection. Rather, it is a countably infinite set whose members encode an unbounded number of configurations. That is:

$$N_b = \bigcup_{k=1}^{\infty} \mathcal{P}_k(B)$$

where $\mathcal{P}_k(B)$ denotes all subsets of B that are definable using k bits.

This layered structure creates a diagonalization trap: although each $\mathcal{P}_k(B)$ is finite, their union over all k is not. We show:

$$\forall f : B \to N_b, \quad \exists D \subseteq B,\ D \notin \mathrm{Im}(f) \Rightarrow |B| < |N_b|$$

Thus, the model-theoretic result holds not because N_b is "larger" in a metaphysical sense, but because *no encoding within B can exhaustively enumerate its own definable powerset.*

The diagonalization still works:

$$D := \{b_i \in B \mid b_i \notin f(b_i)\}$$

And since D is ZFC-definable using the graph of f, it lies within N_b yet outside the image of any such f, confirming $|B| < |N_b|$ *even under finite encoding constraints.*

4. **Objection:** The distinctions made in this paper between $|B|$ and $|N_b|$—and the assertion that $|B| < |N_b|$—rely on external, meta-theoretic interpretations of ZFC. Within ZFC itself, all countable models are elementarily equivalent. That is, from the internal perspective of ZFC, one cannot distinguish between:

$$|B| = \aleph_0, \quad |N_b| = \aleph_0$$

The inequality $|B| < |N_b|$ can only be established by reference to a *meta-model*, and therefore does not constitute a contradiction of Cantor's hierarchy, which is itself a meta-theoretic result. Hence, your claim is philosophical or model-theoretic, not set-theoretic.

Rebuttal: This paper deliberately works *within* ZFC, using only constructions and logical inferences that are permissible by its axioms. We do not appeal to external cardinality comparisons or nonstandard models unless they are constructible within the theory. In particular:

$$N_b := \mathcal{P}_{\text{def}}(B), \quad B := \{b_1, b_2, \dots\}$$

is explicitly definable within ZFC, and so is every function $f : B \to N_b$ that we consider.

We prove:

$$\forall f : B \to N_b, \quad \exists D \subseteq B \text{ (ZFC-definable) s.t. } D \notin \text{Im}(f) \Rightarrow |B| < |N_b|$$

Thus, while model-theoretic relativity exists, the construction does not rely on external semantic perspective. It derives an **internal inconsistency in cardinal stratification** within ZFC, showing that even under its own constraints, the hierarchy:

$$|B| < |\mathcal{P}_{\text{def}}(B)| < |\mathcal{P}(B)|$$

is not preserved. In this sense, the result is stronger than meta-theoretic commentary—it is a formal self-collapse.

5. **Objection:** The N^- model proposed in this paper restricts $\mathbb{N}$ to the subset $N_b := \mathcal{P}_{\text{def}}(B)$, consisting of only representable subsets under a finite binary code over a fixed basis B. However, this is not a true collapse of cardinality, but merely a structural filtering. The full set $\mathbb{N}$ remains untouched in the background universe of ZFC.

   Therefore, what appears as $|B| < |N_b|$ is simply a result of:

$$\text{codomain compression} \quad \Rightarrow \quad \text{subset selection under definability}$$

not an actual inequality of cardinalities in the Cantorian sense. The inequality holds only within a model filtered by definability constraints, not in absolute cardinality. Hence:

$$|\mathbb{N}| = |\mathcal{P}(\mathbb{N})| > |N_b| > |B| = \aleph_0$$

still holds globally.

Rebuttal: The key premise of this paper is that *cardinality is only meaningful under representational and functional constraints imposed by the formal system itself.* In ZFC, we do not have access to all possible functions in the abstract—only those definable within the theory.

Therefore, if:

$$\forall f : B \to N_b, \quad f \ /\text{surjective} \Rightarrow |B| < |N_b|$$

within ZFC, then *no surjection can be constructed or proven to exist.* In such a case, claiming "the full set is still there" is metaphysical—it has no operational meaning within ZFC's definable universe.

Moreover, this objection concedes that:

- N_b is strictly larger than B - $N_b \subseteq \mathbb{N}$

Hence, we have constructed a definable subset of $\mathbb{N}$ that violates the standard assumption:

$$|\mathbb{N}| = \aleph_0$$

in a way that is **formally demonstrable within ZFC**. That is not structural encoding—it is a model-internal cardinality breach.

6. **Objection:** The proof that $|B| < |N_b|$ relies on a diagonal argument constructed over the definable power set:

$$N_b := \mathcal{P}_{\mathrm{def}}(B) = \{X \subseteq B \mid X \text{ has finite binary representation}\}$$

The function $f : B \to N_b$ is assumed to enumerate all definable subsets. The diagonal set $D := \{b_i \in B \mid b_i \notin f(b_i)\}$ is then defined, and its absence from $\mathrm{Im}(f)$ is taken as proof that f is not surjective.

However, this conclusion presupposes that D is definable in the same system that defines f. Yet, D's definition depends on the totality of f, and the very existence of such a definable f is what the argument aims to refute.

Thus, the reasoning becomes circular:

$$f \ /\text{total} \Rightarrow D \text{ exists} \Rightarrow f \ /\text{total}$$

That is, the construction of D assumes the global definability of f, which is precisely what the argument is trying to disprove.

Rebuttal: The diagonal argument is not circular—it is a classical indirect proof strategy (reductio ad absurdum). Suppose:

$$\exists f : B \to N_b,\ f \text{ total and surjective}$$

Then we define:

$$D := \{b_i \in B \mid b_i \notin f(b_i)\}$$

Since f is assumed to enumerate all elements of N_b, D must be equal to some $f(b_k)$ for some $b_k \in B$. But then:

$$b_k \in D \iff b_k \notin f(b_k) \iff b_k \notin D \quad \text{(contradiction)}$$

This is not circular—it is a contradiction derived from the assumption that f exists.

Moreover, D is definable under the same constraints as f, because both are built using first-order formulas referencing finite binary codes over B. Therefore, $D \in \mathcal{P}_{\text{def}}(B)$ but not in $\text{Im}(f)$, refuting f's surjectivity.

This is the exact mirror of Cantor's argument for $|\mathbb{N}| < |\mathcal{P}(\mathbb{N})|$—but now applied *within* a representationally constrained ZFC submodel.

Epilogue

This paper introduced N^-, a non-standard natural number model that satisfies Peano Arithmetic while exhibiting a cardinality strictly below $\aleph_0$. The existence of such a model challenges the conventional assumption that $\aleph_0$ is the minimal infinite cardinal in all arithmetic constructions.

For readers interested in models of *greater* cardinality—natural number models that extend *beyond* the standard $\mathbb{N}$ while still satisfying the Peano axioms—a complementary development is presented in a separate work.

See: *"The Diversity of Natural Number Models and the Model-Dependence of Infinite Interpretation"*, available at https://zenodo.org/records/15606250.

There, the construction of an extended model N^+ is presented, whose cardinality strictly exceeds $\aleph_0$, offering a dual perspective on how both contraction and expansion of $\mathbb{N}$ remain logically coherent within ZFC.

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