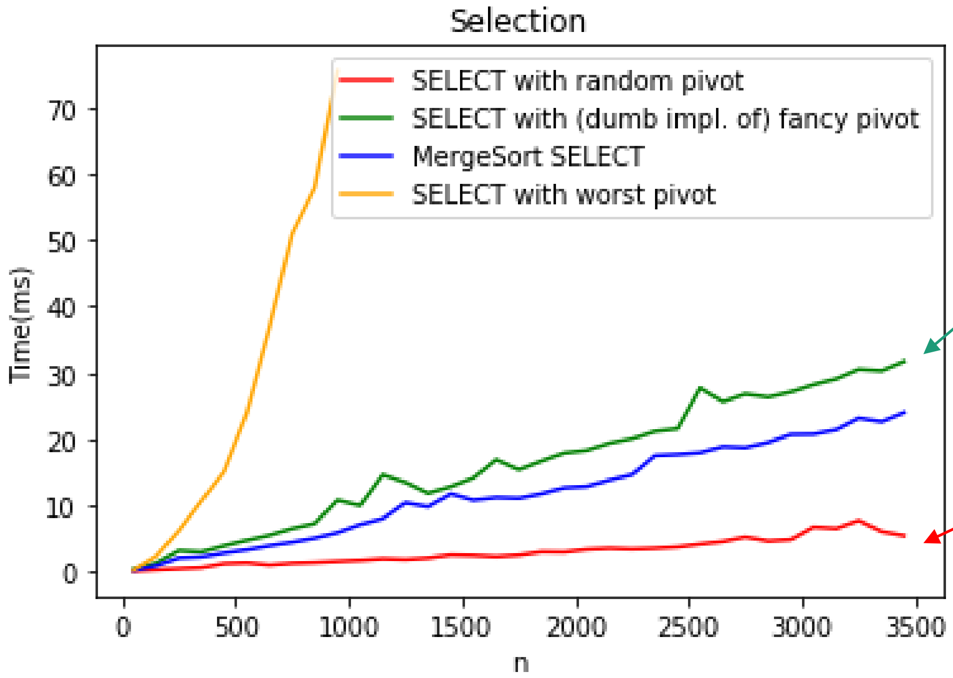
# Lecture 5

Randomized algorithms and QuickSort

# Last time

- We saw a divide-and-conquer algorithm to solve the **`Select`** problem in time O(n) in the worst-case.
- It all came down to picking the pivot…




We choose a pivot **cleverly**

We choose a pivot **randomly.**

# Randomized algorithms

- We make some random choices during the algorithm.
- We hope the algorithm works.
- We hope the algorithm is fast.

For today we will look at algorithms that always work and are probably fast.

e.g., **`Select`** with a random pivot is a randomized algorithm.

- Always works (aka, is correct).
- Probably fast.

# Today

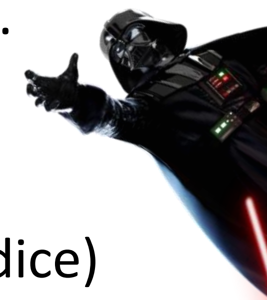
- How do we analyze randomized algorithms?
- A few randomized algorithms for sorting.
  - BogoSort
  - QuickSort

- BogoSort is a pedagogical tool.
- QuickSort is important to know. (in contrast with BogoSort…)

# How do we measure the runtime of a randomized algorithm?

### Scenario 1

1. You publish your algorithm.
2. Bad guy picks the input.
3. You run your randomized algorithm.



### Scenario 2

1. You publish your algorithm.
2. Bad guy picks the input.
3. Bad guy chooses the randomness (fixes the dice) and runs your algorithm.

- In **Scenario 1**, the running time is a **random variable.**
  - It makes sense to talk about **expected running time**.
- In **Scenario 2**, the running time is **not random**.
  - We call this the **worst-case running time** of the randomized algorithm.

# Today

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- A few randomized algorithms for sorting.
  - BogoSort
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- BogoSort is a pedagogical tool.
- QuickSort is important to know. (in contrast with BogoSort…)

Assume A has distinct entries

# From your pre-lecture exercise: BogoSort

Suppose that you can draw a random integer in {1,…,n} in time O(1). How would you randomly permute an array in-place in time O(n)?

Ollie the over-achieving ostrich

- **BogoSort**(A)
  - **While** true:
    - Randomly permute A.
    - Check if A is sorted.
    - **If** A is sorted, **return** A.

- Let $X_i = \begin{cases} 1 \text{ if A is sorted after iteration i} \\ 0 \text{ otherwise} \end{cases}$
- $E[X_i] = \frac{1}{n!}$
- $E[\text{number of iterations until A is sorted}] = n!$

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# Solutions to pre-lecture exercise 1

1. Let X be a random variable which is 1 with probability 1/100 and 0 with probability 99/100.
   a) E[X] = 1/100
   b) If $X_1, X_2, \dots X_n$ are iid copies of X, by linearity of expectation,
   $$E\left[\sum_{i=1}^{n} X\_i\right] = \sum_{i=1}^{n} E[X_i] = \frac{n}{100}$$
   c) Let N be the index of the first 1. Then E[N] = 100.

## To see part (c), either:

- You saw in CS109 that N is a geometric random variable, and you know a formula for that.
- Suppose you do the first trial. If it comes up 1 (with probability 1/100), then N=1. Otherwise, you start again except you've already used one trial. Thus:

$$E[N] = \frac{1}{100} \cdot 1 + \left(1 - \frac{1}{100}\right) \cdot (1 + E[N]) = 1 + \left(1 - \frac{1}{100}\right) E[N]$$

  Solving for E[N] we see E[N] = 100.

- (There are other derivations too).

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# Solutions to pre-lecture exercise 2

2. Let $X_i$ be 1 iff A is sorted on iteration i.
   a) Okay. (There wasn't actually a question for part (a)…)
   b) $E[X_i] = 1/n!$ since there are n! possible orderings of A and only one is sorted. (Suppose A has distinct entries).
   c) Let N be the index of the first 1. Then E[N] = n!.

## Part (c) is similar to part (c) in exercise 1:

- You saw in CS109 that N is a geometric random variable, and you know a formula for that. Or,
- Suppose you do the first trial. If it comes up 1 (with probability 1/n!), then N=1. Otherwise, you start again except you've already used one trial. Thus:

$$E[N] = \frac{1}{n!} \cdot 1 + \left(1 - \frac{1}{n!}\right) \cdot (1 + E[N]) = 1 + \left(1 - \frac{1}{n!}\right) E[N]$$

Solving for E[N] we see E[N] = n!

- (There are other derivations too).

Assume A has distinct entries

From your pre-lecture exercise:

# BogoSort

Suppose that you can draw a random integer in {1,…,n} in time O(1). How would you randomly permute an array in-place in time O(n)?

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- **BogoSort**(A)
  - **While** true:
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- Let $X_i = \begin{cases} 1 \text{ if A is sorted after iteration i} \\ 0 \text{ otherwise} \end{cases}$
- $E[X_i] = \frac{1}{n!}$
- $E[\text{number of iterations until A is sorted}] = n!$

# Expected Running time of BogoSort

This isn't random, so we can pull it out of the expectation.

E[ running time on a list of length n ]

= E[ (number of iterations) * (time per iteration) ]

= (time per iteration) * E[number of iterations]

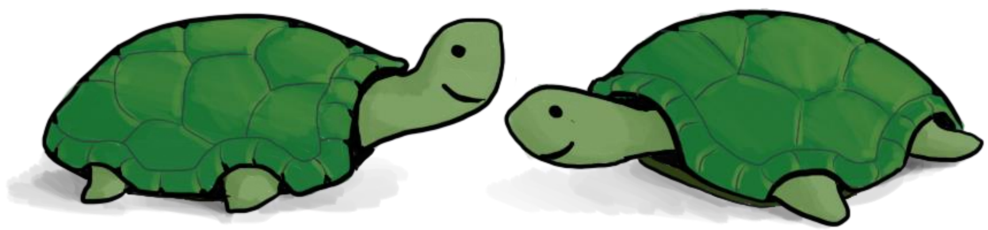
This is O(n) (to permute and then check if sorted)

We just computed this. It's n!.

= $O(n \cdot n!)$

= REALLY REALLY BIG.

# Worst-case running time of BogoSort?



Think-Pair-Share Terrapins!
1 minute: think
1 minute: pair and share

- **BogoSort**(A)
  - **While** true:
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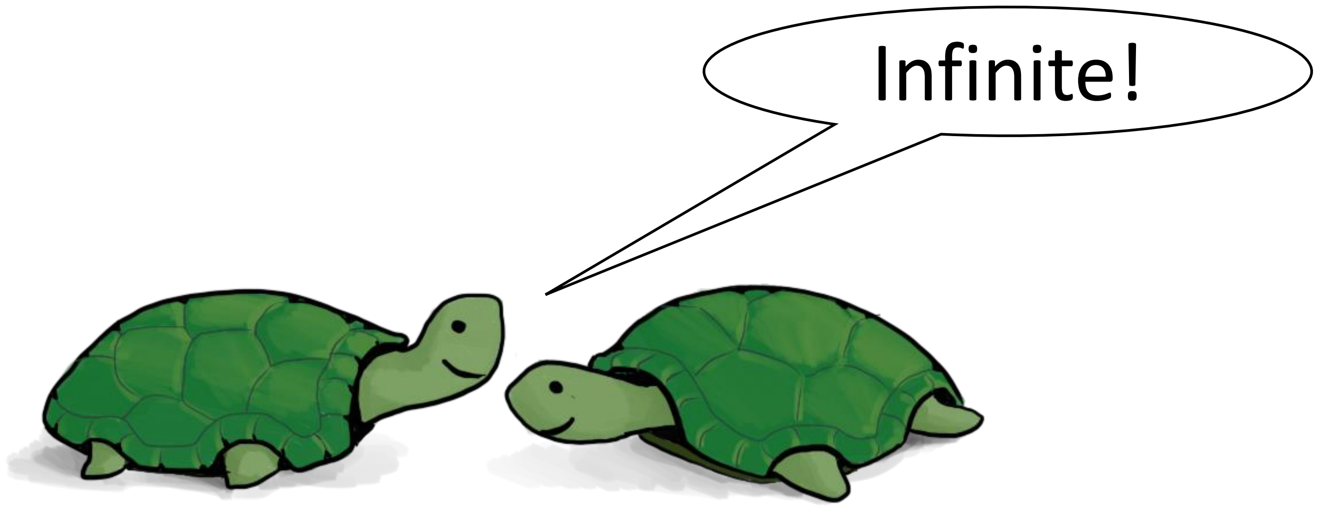
# Worst-case running time of BogoSort?




Think-Pair-Share Terrapins!

- **BogoSort**(A)
  - **While** true:
    - Randomly permute A.
    - Check if A is sorted.
    - **If** A is sorted, **return** A.

# What have we learned?

- Don’t use bogoSort.

# Today

- How do we analyze randomized algorithms?
- A few randomized algorithms for sorting.
  - BogoSort
  - QuickSort

- BogoSort is a pedagogical tool.
- QuickSort is important to know. (in contrast with BogoSort…)

# a better randomized algorithm: QuickSort

- Expected runtime $O(n\log(n))$.
- Worst-case runtime $O(n^2)$.
- In practice works great!

# Quicksort

We want to sort this array.

For the rest of the lecture, assume all elements of A are distinct.

First, pick a "pivot."
**Do it at random.**

| | | | | | | |
|---|---|---|---|---|---|---|
| 7 | 6 | 3 | 5 | 1 | 2 | 4 |

random pivot!

This PARTITION step takes time O(n). (Notice that we don't sort each half). [same as in SELECT]

Next, partition the array into "bigger than 5" or "less than 5"

Arrange them like so:

L = array with things smaller than A[pivot]

R = array with things larger than A[pivot]

Recurse on L and R:

| | | | | | | |
|---|---|---|---|---|---|---|
| 1 | 2 | 3 | 4 | 5 | 6 | 7 |

# PseudoPseudoCode for what we just saw

- QuickSort(A):
  - **If** len(A) <= 1:
    - **return**
  - Pick some x = A[i] at random. Call this the **pivot**.
  - PARTITION the rest of A into:
    - L (less than x) and
    - R (greater than x)
  - Replace A with [L, x, R] (that is, rearrange A in this order)
  - QuickSort(L)
  - QuickSort(R)

Assume that all elements of A are distinct. How would you change this if that's not the case?

# Running time?

- $T(n) = T(|L|) + T(|R|) + O(n)$

- In an ideal world…
  - if the pivot splits the array exactly in half…

$$T(n) = 2 \cdot T\left(\frac{n}{2}\right) + O(n)$$


- We've seen that a bunch:

$$T(n) = O(n\log(n)).$$

# The expected running time of QuickSort is O(nlog(n)).

**Proof:** *

- $E[|L|] = E[|R|] = \frac{n-1}{2}$.
  - The expected number of items on each side of the pivot is half of the things.

Remember, we are assuming all elements of A are distinct

# Aside

## why is $E[|L|] = \frac{n-1}{2}$ ?

- $E[|L|] = E[|R|]$
  - by symmetry
- $E[|L| + |R|] = n - 1$
  - because L and R make up everything except the pivot.
- $E[|L|] + E[|R|] = n - 1$
  - By linearity of expectation
- $2E[|L|] = n - 1$
  - Plugging in the first bullet point.
- $E[|L|] = \frac{n-1}{2}$
  - Solving for $E[|L|]$.

# The expected running time of QuickSort is O(nlog(n)).

**Proof:***

- $E[|L|] = E[|R|] = \frac{n-1}{2}$.
  - The expected number of items on each side of the pivot is half of the things.
- If that occurs, the running time is $T(n) = O(n\log(n))$.
  - Since the relevant recurrence relation is $T(n) = 2T\left(\frac{n-1}{2}\right) + O(n)$
- Therefore, the expected running time is $O(n\log(n))$.

***Disclaimer: this proof is wrong.**

# Red flag

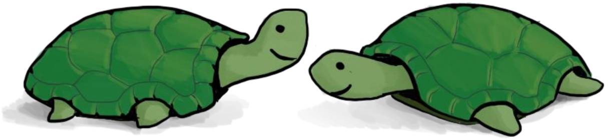
We can use the same argument to prove something false.

- **Slow** Sort(A):
  - **If** len(A) <= 1:
    - **return**
  - **Pick the pivot x to be either max(A) or min(A), randomly**
    - \\ We can find the max and min in O(n) time
  - PARTITION the rest of A into:
    - L (less than x) and
    - R (greater than x)
  - Replace A with [L, x, R] (that is, rearrange A in this order)
  - **Slow** Sort(L)
  - **Slow** Sort(R)

- Same recurrence relation:
  $$T(n) = T(|L|) + T(|R|) + O(n)$$
- We still have $E[|L|] = E[|R|] = \frac{n-1}{2}$
- But now, one of |L| or |R| is always n-1.
- You check: Running time is $\Theta(n^2)$, with probability 1.

# The expected running time of SlowSort is O(nlog(n)).

What's wrong???
2 minutes: think
1 minute: pair and share



**Proof:***

- $E[|L|] = E[|R|] = \frac{n-1}{2}$.
  - The expected number of items on each side of the pivot is half of the things.
- If that occurs, the running time is $T(n) = O(n\log(n))$.
  - Since the relevant recurrence relation is $T(n) = 2T\left(\frac{n-1}{2}\right) + O(n)$
- Therefore, the expected running time is $O(n\log(n))$.

***Disclaimer: this proof is wrong.**

# What's wrong?

- $E[|L|] = E[|R|] = \frac{n-1}{2}$.
  - The expected number of items on each side of the pivot is half of the things.
- If that occurs, the running time is $T(n) = O(n\log(n))$.
  - Since the relevant recurrence relation is $T(n) = 2T\left(\frac{n-1}{2}\right) + O(n)$
- Therefore, the expected running time is $O(n\log(n))$.

***That's not how expectations work!***

Plucky the Pedantic Penguin

- The running time in the "expected" situation is not the same as the expected running time.
- Sort of like how $E[X^2]$ is not the same as $(E[X])^2$

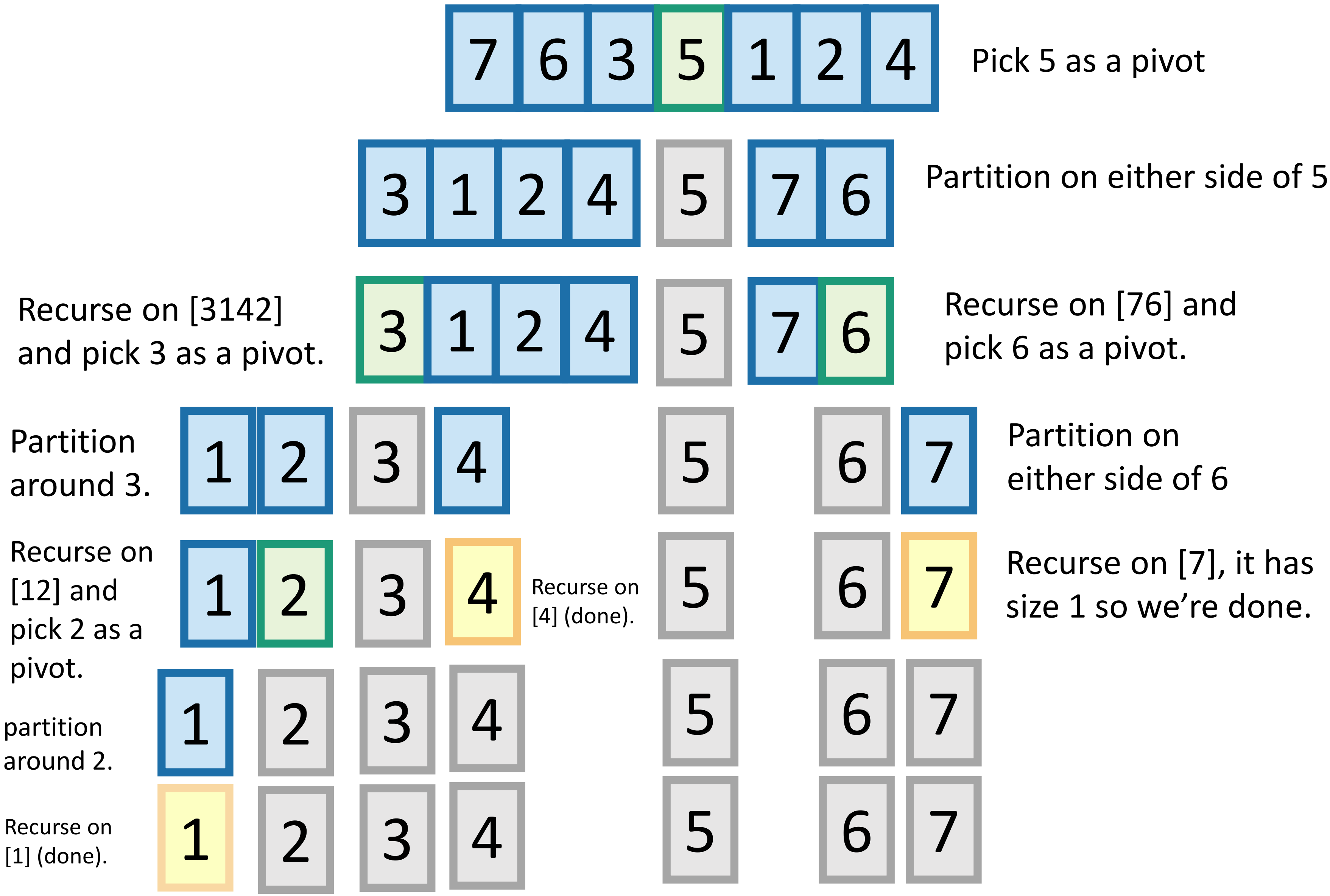
# Instead

- We’ll have to think a little harder about how the algorithm works.

# Next goal:

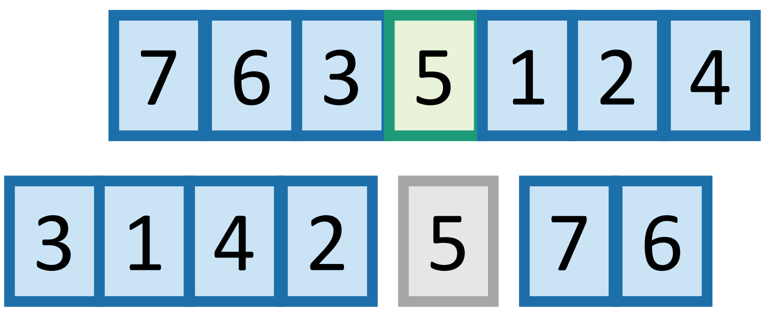
- Get the same conclusion, correctly!

# Example of recursive calls

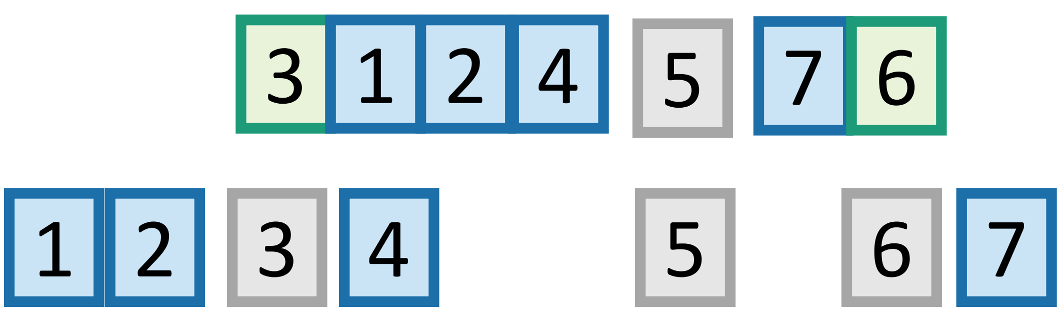


# How long does this take to run?

- We will count the number of comparisons that the algorithm does.
  - This turns out to give us a good idea of the runtime. (Not obvious).
- How many times are any two items compared?




In the example before, everything was compared to 5 once in the first step....and never again.




But not everything was compared to 3.
5 was, and so were 1,2 and 4.
But not 6 or 7.

# Each pair of items is compared either 0 or 1 times. Which is it?

| | | | | | | |
|---|---|---|---|---|---|---|
| 7 | 6 | 3 | 5 | 1 | 2 | 4 |

Let's assume that the numbers in the array are actually the numbers 1,…,n

Of course this doesn't have to be the case! It's a good exercise to convince yourself that the analysis will still go through without this assumption. (Or see the book)

- **Whether or not a,b are compared** is a random variable, that depends on the choice of pivots. Let's say

$$X_{a,b} = \begin{cases} 1 & \text{if } a \text{ and } b \text{ are ever compared} \\ 0 & \text{if } a \text{ and } b \text{ are never compared} \end{cases}$$

- In the previous example $X_{1,5} = 1$, because item 1 and item 5 were compared.
- But $X_{3,6} = 0$, because item 3 and item 6 were NOT compared.

# Counting comparisons

- The number of comparisons total during the algorithm is

$$\sum_{a=1}^{n-1} \sum_{b=a+1}^{n} X_{a,b}$$

- The expected number of comparisons is

$$E\left[\sum_{a=1}^{n-1} \sum_{b=a+1}^{n} X_{a,b}\right] = \sum_{a=1}^{n-1} \sum_{b=a+1}^{n} E[\, X_{a,b}]$$

using linearity of expectations.

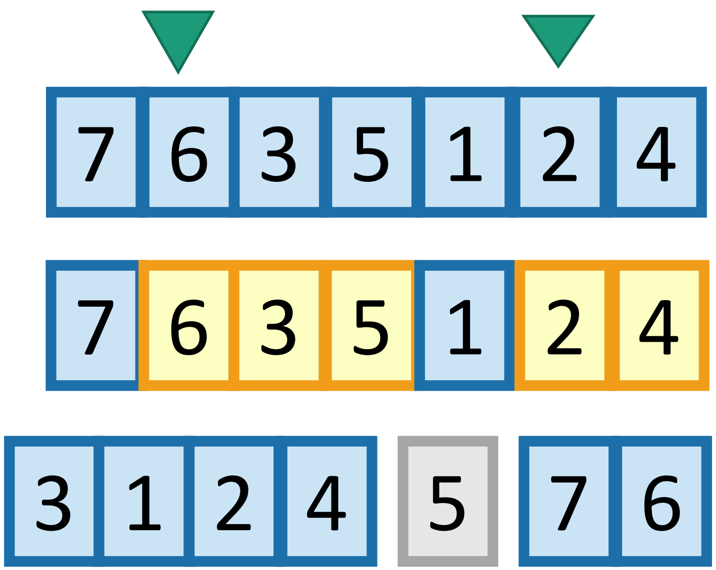
# Counting comparisons

expected number of comparisons:

$$\sum_{a=1}^{n-1}\sum_{b=a+1}^{n} E[\,X_{a,b}]$$

- So we just need to figure out $E[\,X_{a,b}\,]$
- $E\left[X_{a,b}\right] = P(X_{a,b} = 1)\cdot 1 + P(X_{a,b} = 0)\cdot 0 = P(X_{a,b} = 1)$
  - (using definition of expectation)
- So we need to figure out:

$P(X_{a,b} = 1)$ = the probability that a and b are ever compared.




Say that a = 2 and b = 6. What is the probability that 2 and 6 are ever compared?

This is exactly the probability that either 2 or 6 is first picked to be a pivot out of the highlighted entries.

If, say, 5 were picked first, then 2 and 6 would be separated and never see each other again.

# Counting comparisons

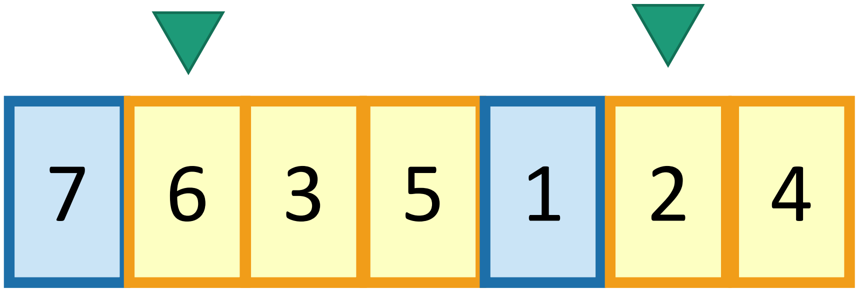
$P(\ X_{a,b} = 1\ )$

= probability a,b are ever compared

= probability that one of a,b are picked first out of all of the b – a +1 numbers between them.

2 choices out of b-a+1…

$$= \frac{2}{b - a + 1}$$



All together now…

# Expected number of comparisons

- $E\left[\sum_{a=1}^{n-1} \sum_{b=a+1}^{n} X_{a,b}\right]$ This is the expected number of comparisons throughout the algorithm
- $= \sum_{a=1}^{n-1} \sum_{b=a+1}^{n} E[\, X_{a,b}]$ linearity of expectation
- $= \sum_{a=1}^{n-1} \sum_{b=a+1}^{n} P(\, X_{a,b} = 1)$ definition of expectation
- $= \sum_{a=1}^{n-1} \sum_{b=a+1}^{n} \frac{2}{b-a+1}$ the reasoning we just did

- This is a big nasty sum, but we can do it.
- We get that this is less than 2n ln(n).

Do this sum!

Ollie the over-achieving ostrich

# Almost done

- We saw that E[ number of comparisons ] = O(n log(n))
- Is that the same as E[ running time ]?

- In this case, **yes**.
- We need to argue that the running time is dominated by the time to do comparisons.
- See Lemma 5.2 in Algs. Illuminated.

- QuickSort(A):
  - **If** len(A) <= 1:
    - **return**
  - Pick some x = A[i] at random. Call this the pivot.
  - PARTITION the rest of A into:
    - L (less than x) and
    - R (greater than x)
  - Replace A with [L, x, R] (that is, rearrange A in this order)
  - QuickSort(L)
  - QuickSort(R)

# What have we learned?

- The expected running time of QuickSort is $O(n\log(n))$

# Worst-case running time

- Suppose that an adversary is choosing the "random" pivots for you.
- Then the running time might be $O(n^2)$
  - Eg, they'd choose to implement SlowSort
  - In practice, this doesn't usually happen.

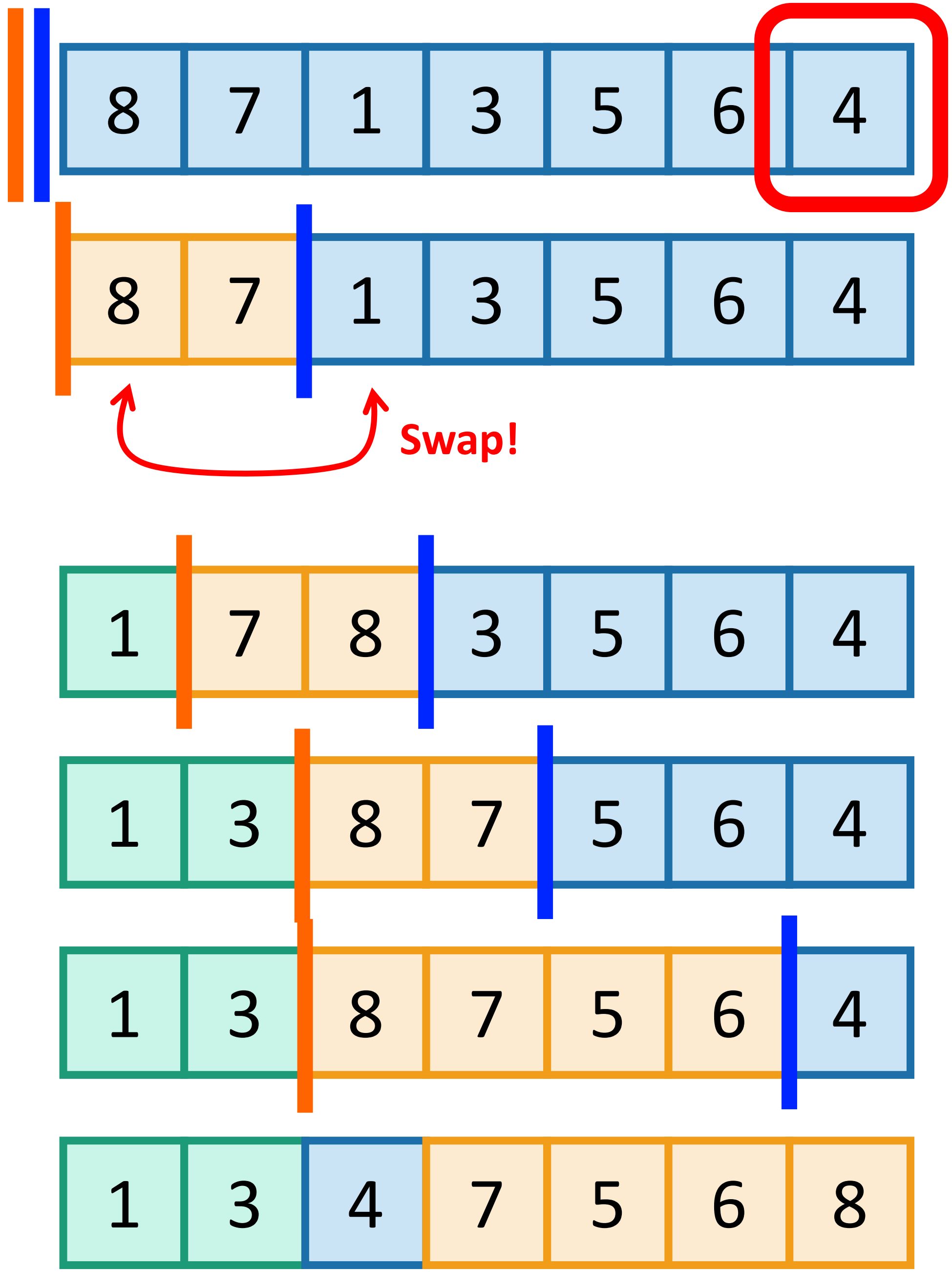
# How should we implement this?

- Our pseudocode is easy to understand and analyze, but is not a good way to implement this algorithm.

- QuickSort(A):
    - **If** len(A) <= 1:
        - **return**
    - Pick some x = A[i] at random. Call this the pivot.
    - PARTITION the rest of A into:
        - L (less than x) and
        - R (greater than x)
    - Replace A with [L, x, R] (that is, rearrange A in this order)
    - QuickSort(L)
    - QuickSort(R)

- Instead, implement it **in-place** (without separate L and R)
    - Here are some Hungarian Folk Dancers showing you how it's done: https://www.youtube.com/watch?v=ywWBy6J5gz8

# A better way to do Partition




**Pivot**

Choose it randomly, then swap it with the last one, so it's at the end.

Initialize and

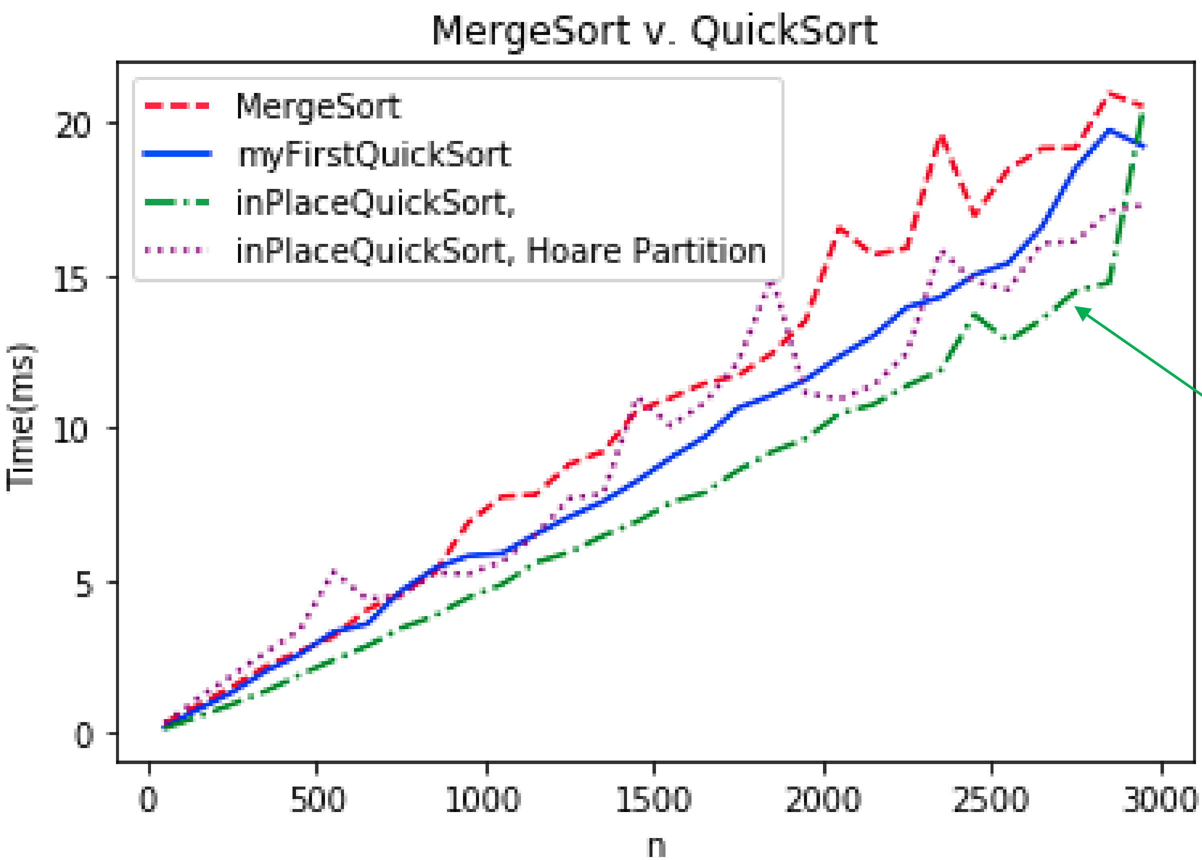
Step forward.

When sees something smaller than the pivot, **swap** the things ahead of the bars and increment both bars.

Repeat till the end, then put the pivot in the right place.

# QuickSort vs. smarter QuickSort vs. Mergesort?


- All seem pretty comparable…




Hoare Partition is a different way of doing it (c.f. CLRS Problem 7-1), which you might have seen elsewhere. You are not responsible for knowing it for this class.

In-place partition function uses less space, and also is a smidge faster on my system.

# QuickSort vs MergeSort

*What if you want O(nlog(n)) worst-case runtime and stability? Check out "Block Sort" on Wikipedia!

| | QuickSort (random pivot) | MergeSort (deterministic) |
|---|---|---|
| **Running time** | • Worst-case: $O(n^2)$<br>• Expected: O(n log(n)) | Worst-case: O(n log(n)) |
| **Used by** | • Java for primitive types<br>• C qsort<br>• Unix<br>• g++ | • Java for objects<br>• Perl |
| **In-Place?** (With O(log(n)) extra memory) | Yes, pretty easily | Not easily* if you want to maintain both stability and runtime. (But pretty easily if you can sacrifice runtime). |
| **Stable?** | No | Yes |
| **Other Pros** | Good cache locality if implemented for arrays | Merge step is really efficient with linked lists |

Understand this

These are just for fun. (Not on exam).

# Today

- How do we analyze randomized algorithms?
- A few randomized algorithms for sorting.
  - BogoSort
  - QuickSort

- BogoSort is a pedagogical tool.
- QuickSort is important to know. (in contrast with BogoSort…)

Recap

# Recap

- How do we measure the runtime of a randomized algorithm?
  - Expected runtime
  - Worst-case runtime

- QuickSort (with a random pivot) is a randomized sorting algorithm.
  - In many situations, QuickSort is nicer than MergeSort.
  - In many situations, MergeSort is nicer than QuickSort.

Code up QuickSort and MergeSort in a few different languages, with a few different implementations of lists A (array vs linked list, etc). What's faster?

Ollie the over-achieving ostrich

# Next time

- Can we sort faster than $\Theta(n\log(n))$??