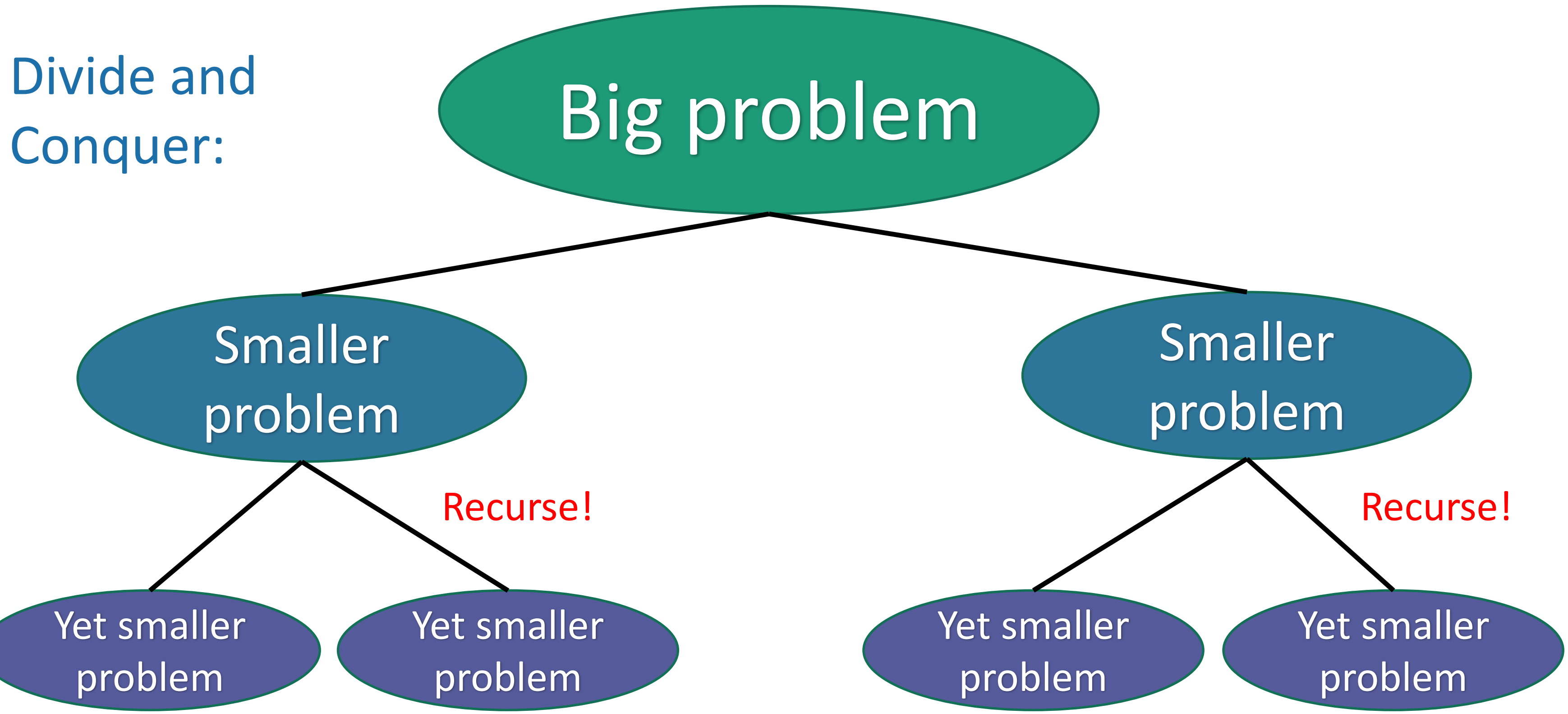
# Lecture 7

Binary Search Trees and Red-Black Trees

# But first!

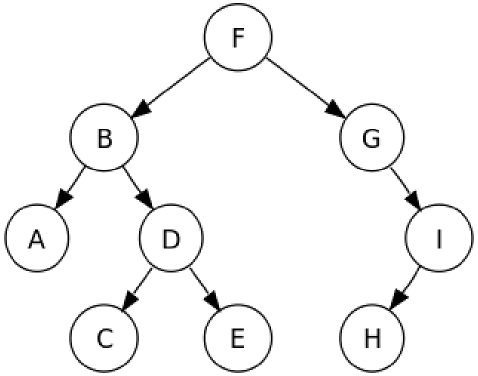
- A brief wrap-up of divide and conquer.



# How do we design divide-and-conquer algorithms?

- So far we've seen lots of examples.
  - Karatsuba
  - MergeSort
  - Select
  - QuickSort

# Today

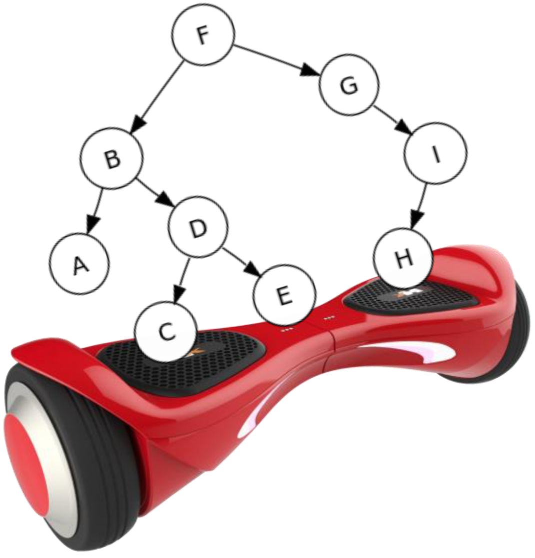



- Begin a brief foray into data structures!

- Binary search trees
    - They are better when they're balanced.

this will lead us to…

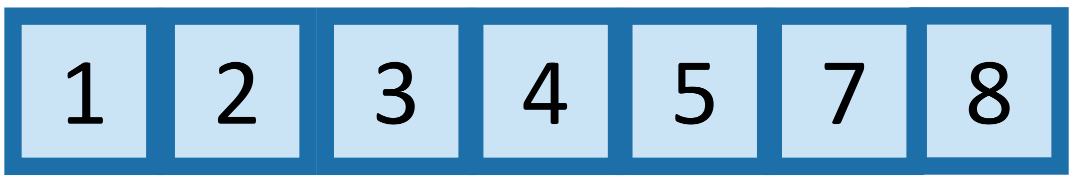
- Self-Balancing Binary Search Trees
    - **Red-Black** trees.



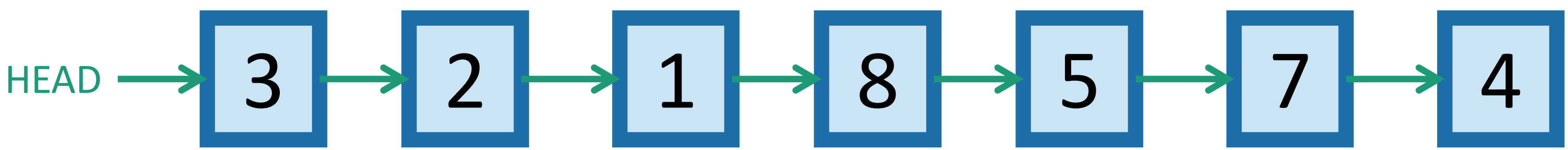
# Some data structures

for storing objects like 5 (aka, nodes with keys)

- (Sorted) arrays:



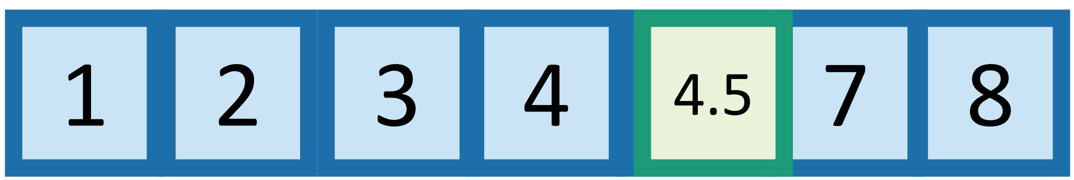

- Linked lists:




- Some basic operations:
  - INSERT, DELETE, SEARCH

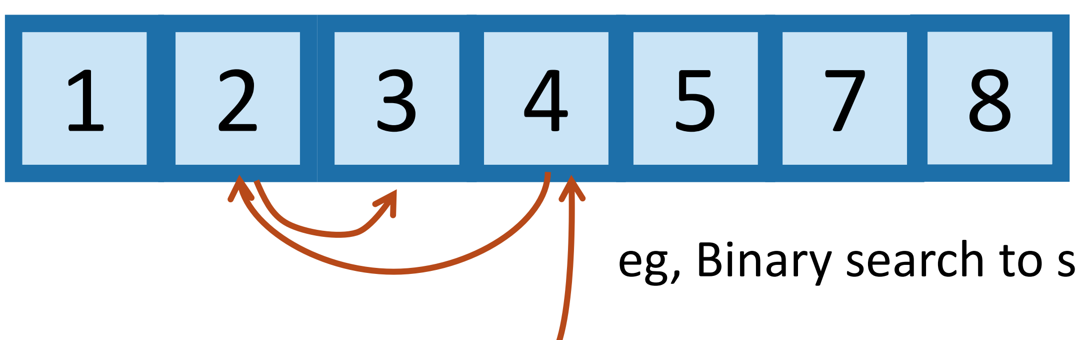
# Sorted Arrays


- O(n) INSERT/DELETE:
  - First, find the relevant element (we'll see how below), and then move a bunch elements in the array:




eg, insert 4.5

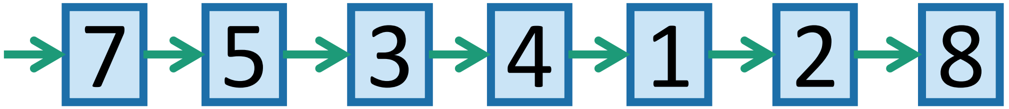
- O(log(n)) SEARCH:




eg, Binary search to see if 3 is in A.

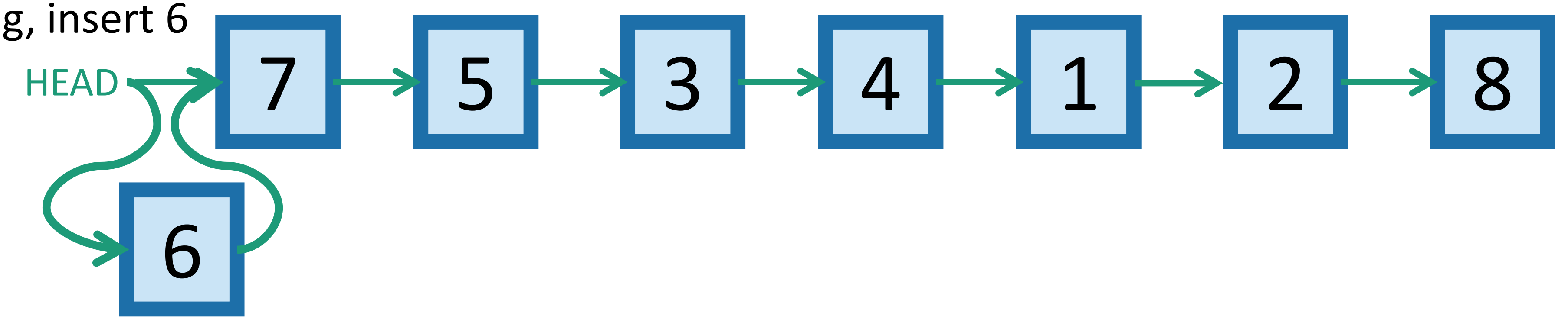
(Not necessarily sorted)

# Linked lists

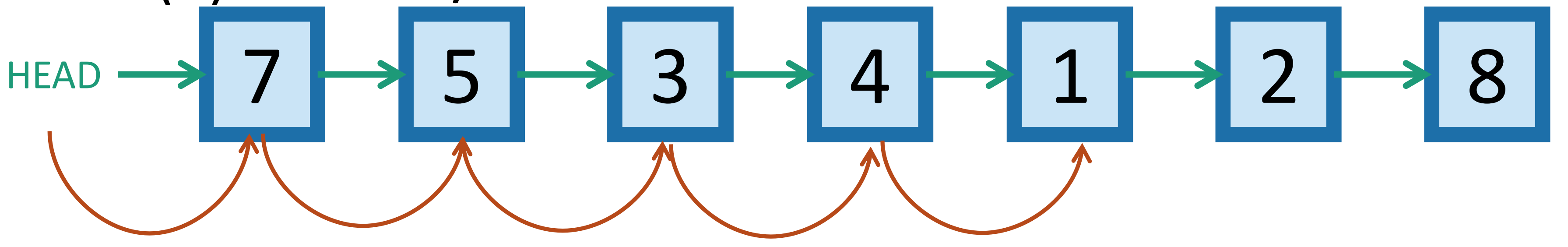



- O(1) INSERT:

eg, insert 6




- O(n) SEARCH/DELETE:




eg, search for 1 (and then you could delete it by manipulating pointers).

# Motivation for Binary Search Trees

TODAY!

| | Sorted Arrays | Linked Lists | (Balanced) **Binary Search Trees** |
|---|---|---|---|
| **Search** | O(log(n)) | O(n) | O(log(n)) |
| **Delete** | O(n) | O(n) | O(log(n)) |
| **Insert** | O(n) | O(1) | O(log(n)) |

For today all keys are distinct.

# Binary tree terminology

Each node has two children.

The left child of 3 is 2

The right child of 3 is 4

The parent of 3 is 5

2 is a descendant of 5

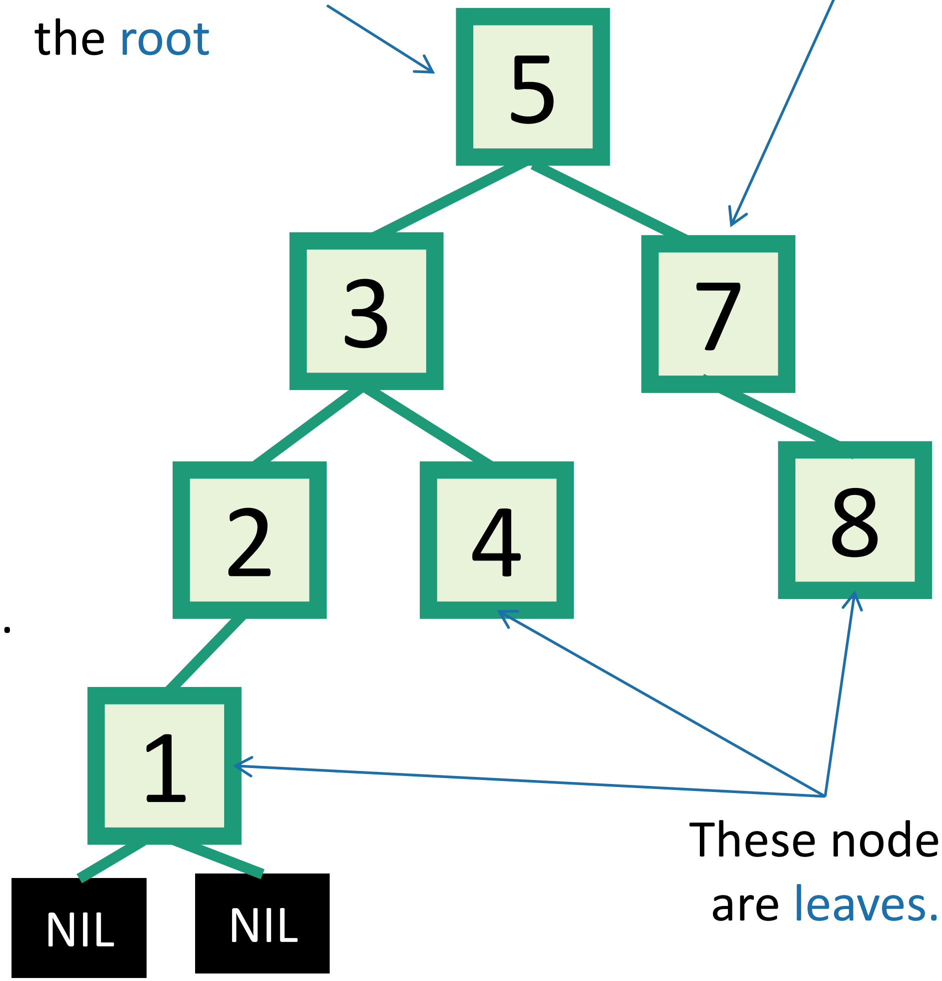
Each node has a pointer to its left child, right child, and parent.

Both children of 1 are NIL.
(I won't usually draw them).

The height of this tree is 3.
(Max number of edges from the root to a leaf).

This is a node.
It has a key (7).

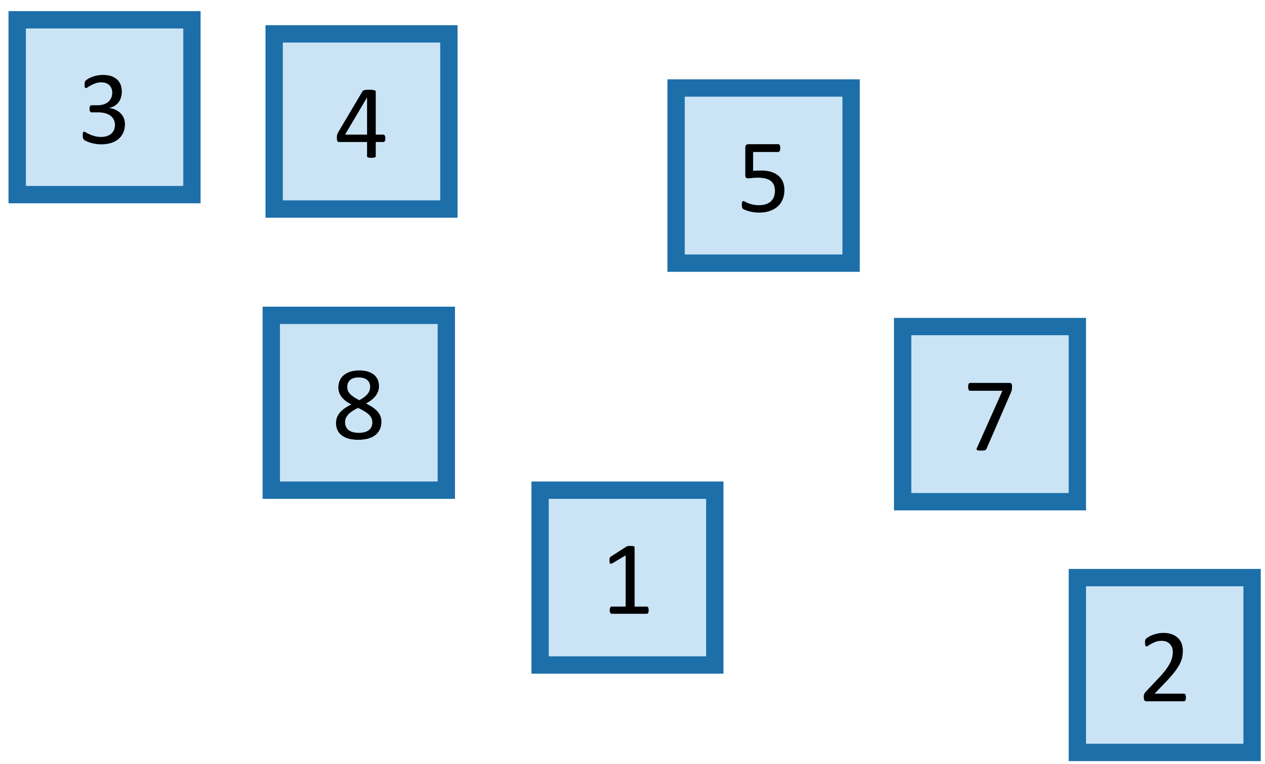
This node is the root




These nodes are leaves.

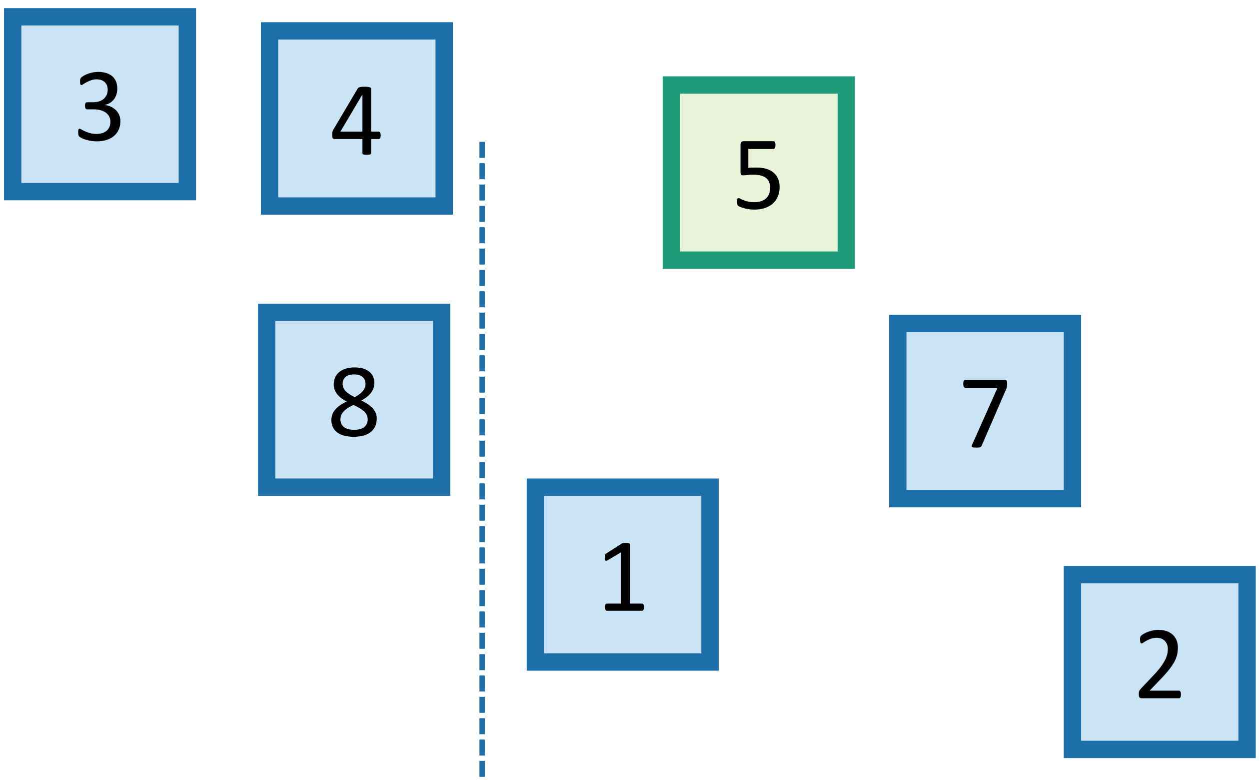
# Binary Search Trees

- A BST is a binary tree so that:
  - Every LEFT descendant of a node has key less than that node.
  - Every RIGHT descendant of a node has key larger than that node.
- Example of building a binary search tree:



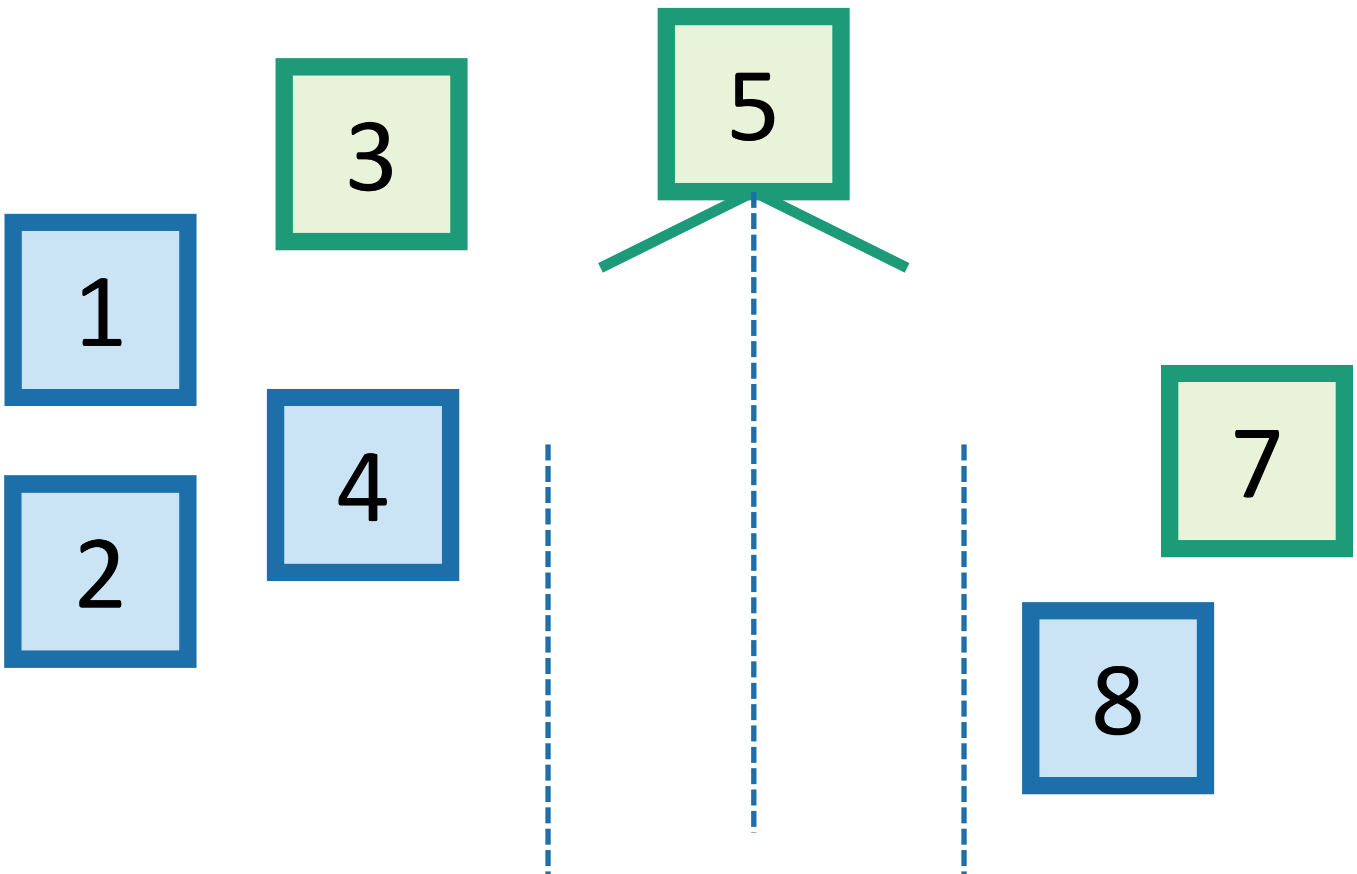
# Binary Search Trees

- A BST is a binary tree so that:
  - Every LEFT descendant of a node has key less than that node.
  - Every RIGHT descendant of a node has key larger than that node.
- Example of building a binary search tree:



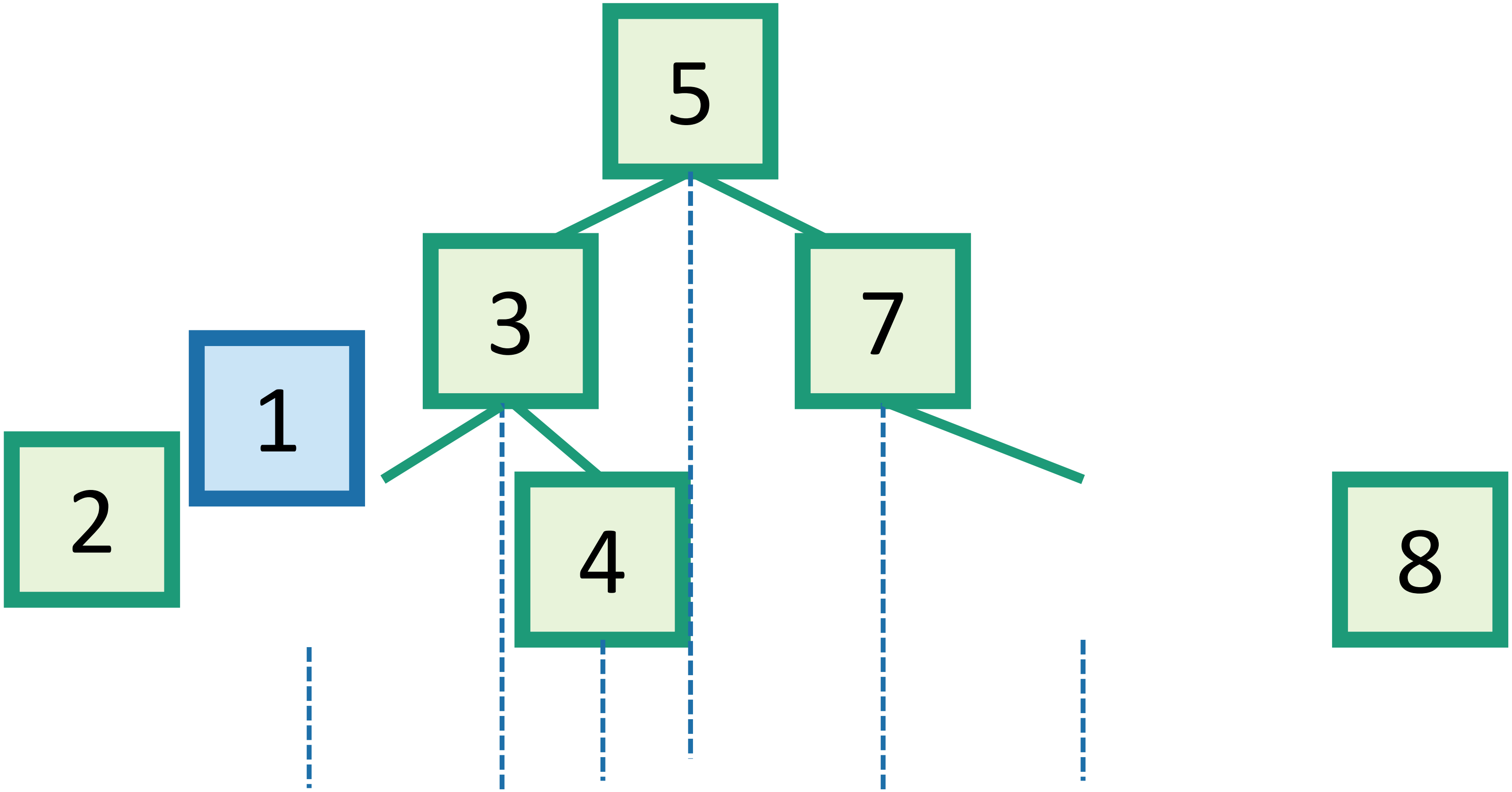
# Binary Search Trees

- A BST is a binary tree so that:
  - Every LEFT descendant of a node has key less than that node.
  - Every RIGHT descendant of a node has key larger than that node.
- Example of building a binary search tree:



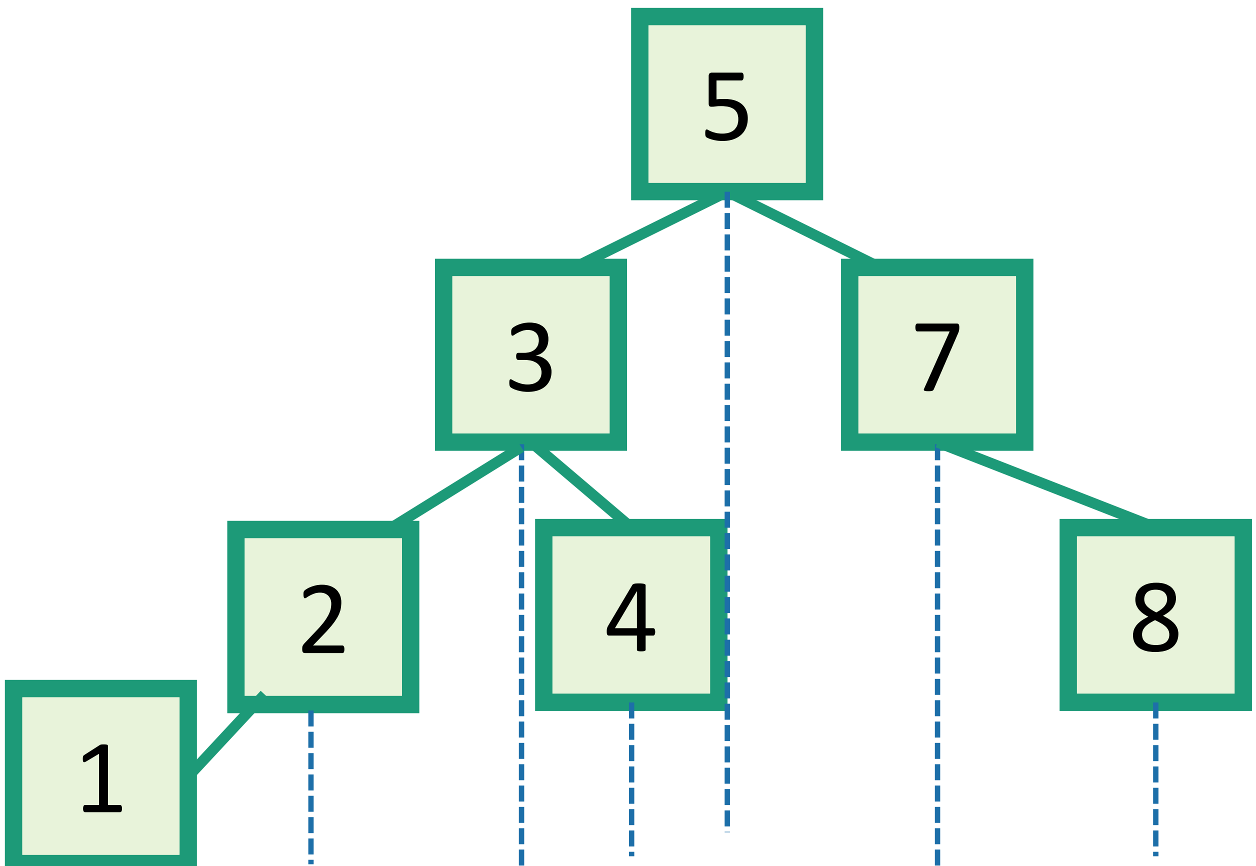
# Binary Search Trees

- A BST is a binary tree so that:
  - Every LEFT descendant of a node has key less than that node.
  - Every RIGHT descendant of a node has key larger than that node.
- Example of building a binary search tree:



# Binary Search Trees

- A BST is a binary tree so that:
  - Every LEFT descendant of a node has key less than that node.
  - Every RIGHT descendant of a node has key larger than that node.
- Example of building a binary search tree:

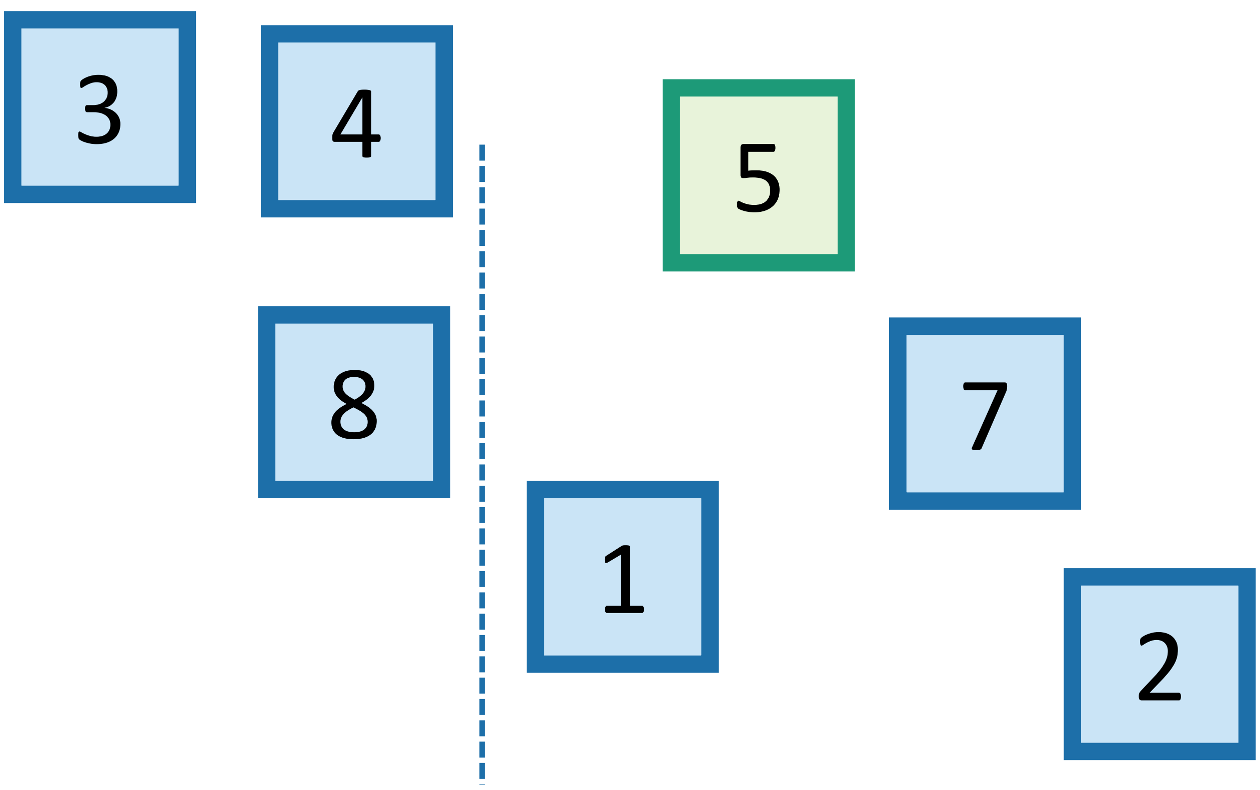



Q: Is this the only binary search tree I could possibly build with these values?

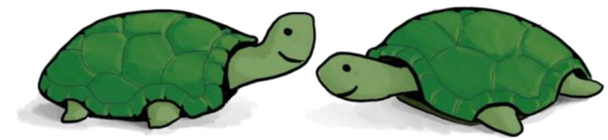
A: **No.** I made choices about which nodes to choose when. Any choices would have been fine.

# Aside: this should look familiar

## kinda like QuickSort

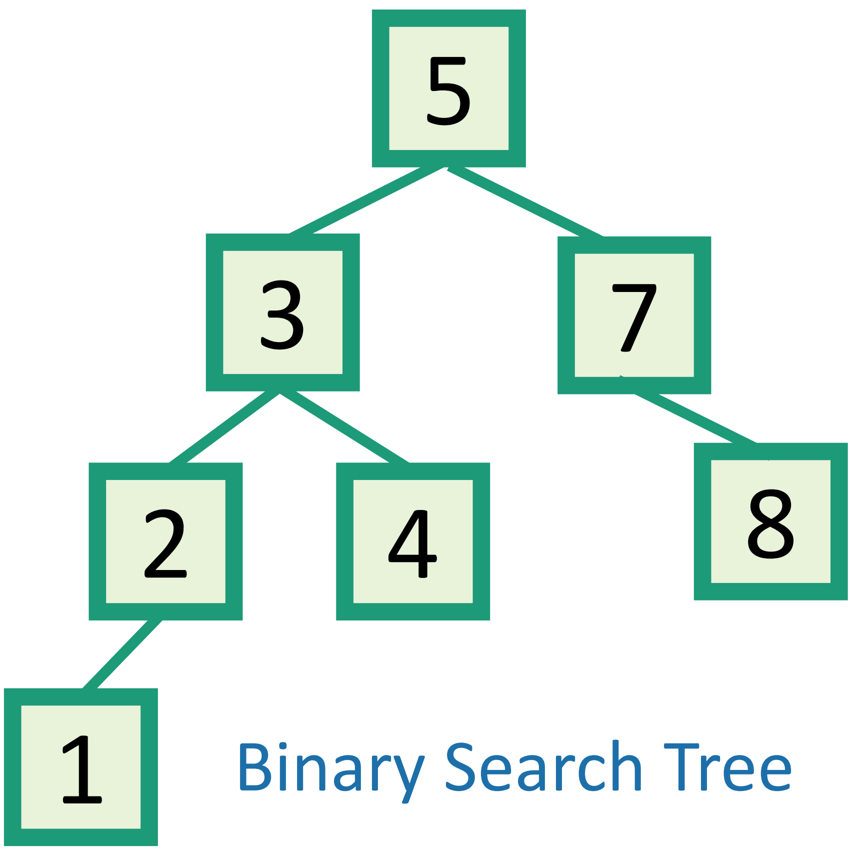


Which of these is a BST?
1 minute Think-Pair-Share

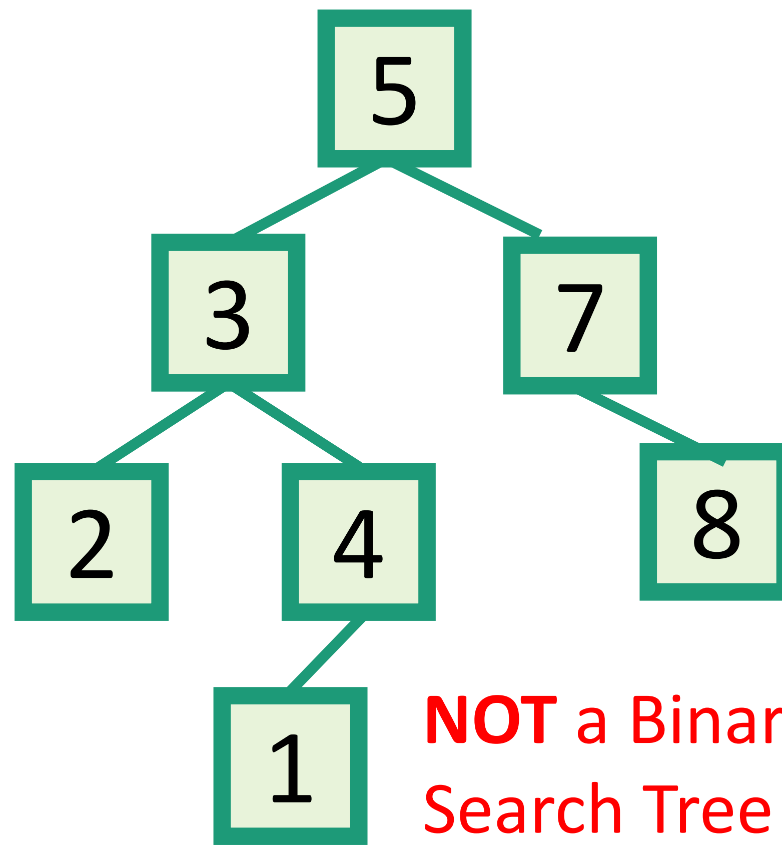


# Binary Search Trees

- A BST is a binary tree so that:
  - Every LEFT descendant of a node has key less than that node.
  - Every RIGHT descendant of a node has key larger than that node.




Binary Search Tree

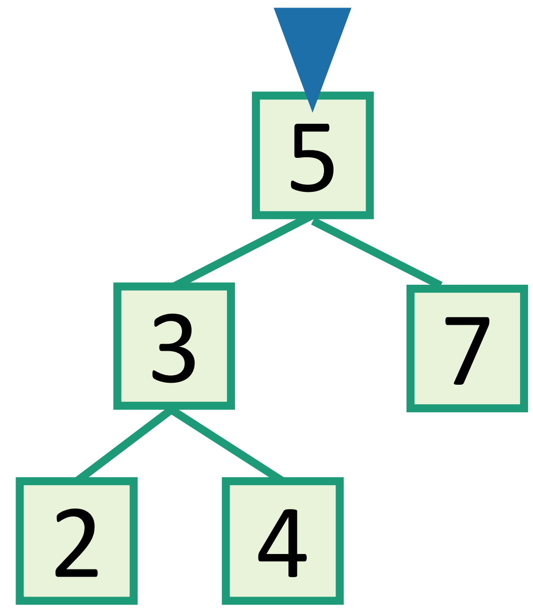



**NOT** a Binary Search Tree

# Aside: In-Order Traversal of BSTs

- Output all the elements in sorted order!

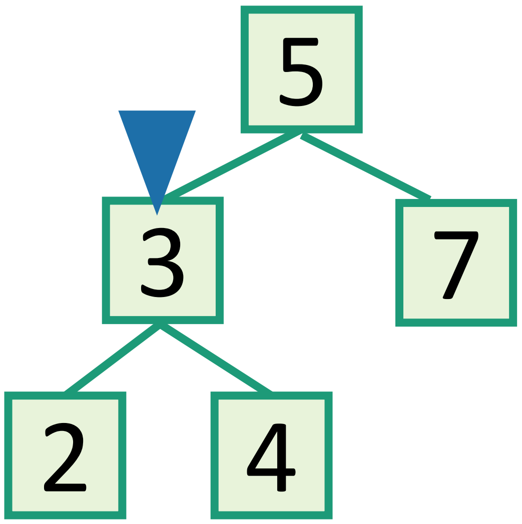
- inOrderTraversal(x):
  - if x!= NIL:
    - inOrderTraversal( x.left )
    - print( x.key )
    - inOrderTraversal( x.right )



# Aside: In-Order Traversal of BSTs

- Output all the elements in sorted order!

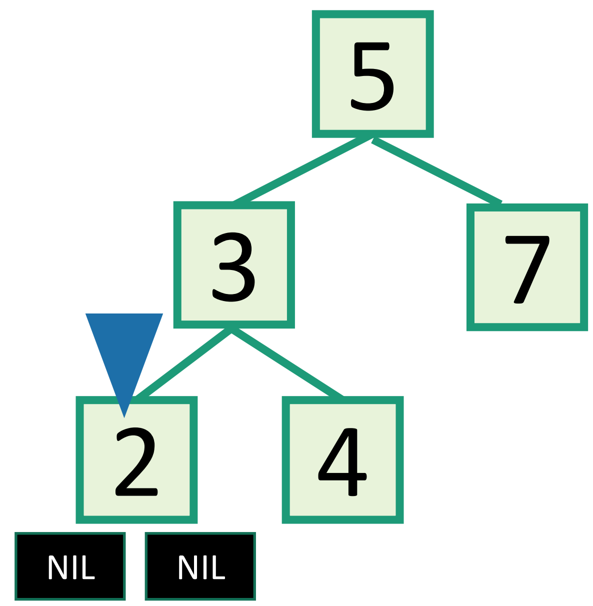
- inOrderTraversal(x):
  - if x!= NIL:
    - inOrderTraversal( x.left )
    - print( x.key )
    - inOrderTraversal( x.right )



# Aside: In-Order Traversal of BSTs

- Output all the elements in sorted order!

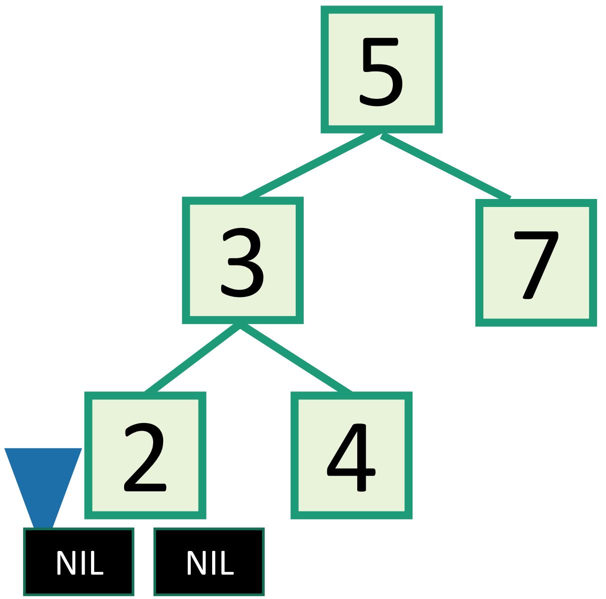
- inOrderTraversal(x):
  - if x!= NIL:
    - inOrderTraversal( x.left )
    - print( x.key )
    - inOrderTraversal( x.right )



# Aside: In-Order Traversal of BSTs

- Output all the elements in sorted order!

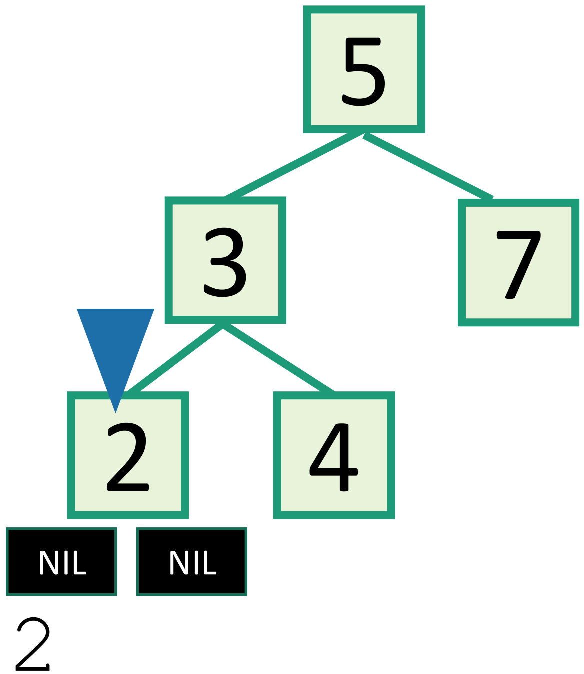
- inOrderTraversal(x):
  - if x!= NIL:
    - inOrderTraversal( x.left )
    - print( x.key )
    - inOrderTraversal( x.right )



# Aside: In-Order Traversal of BSTs

- Output all the elements in sorted order!

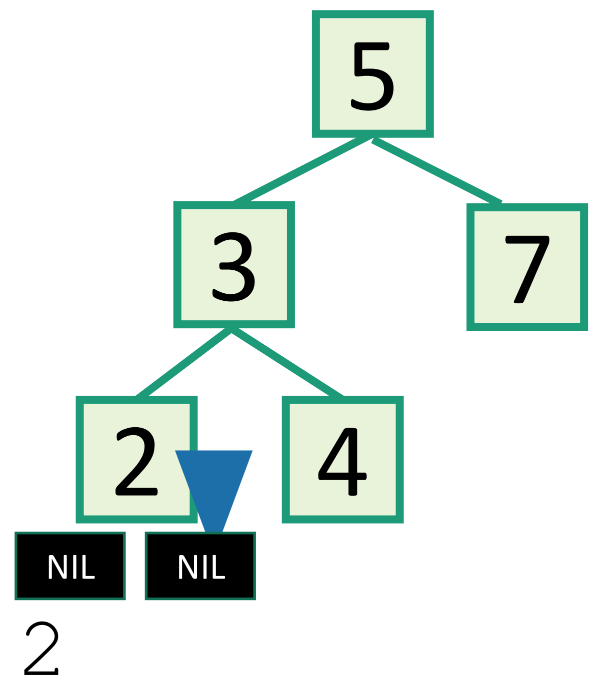
- inOrderTraversal(x):
  - if x!= NIL:
    - inOrderTraversal( x.left )
    - print( x.key )
    - inOrderTraversal( x.right )



# Aside: In-Order Traversal of BSTs

- Output all the elements in sorted order!

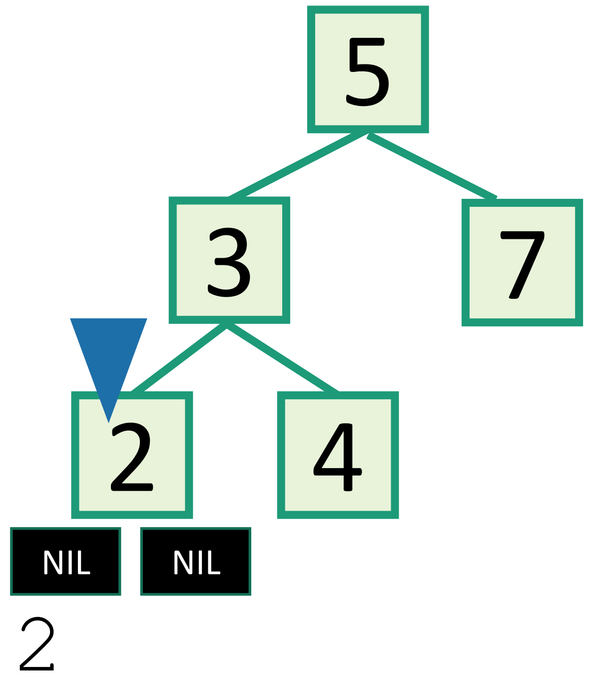
- inOrderTraversal(x):
  - if x!= NIL:
    - inOrderTraversal( x.left )
    - print( x.key )
    - inOrderTraversal( x.right )



# Aside: In-Order Traversal of BSTs

- Output all the elements in sorted order!

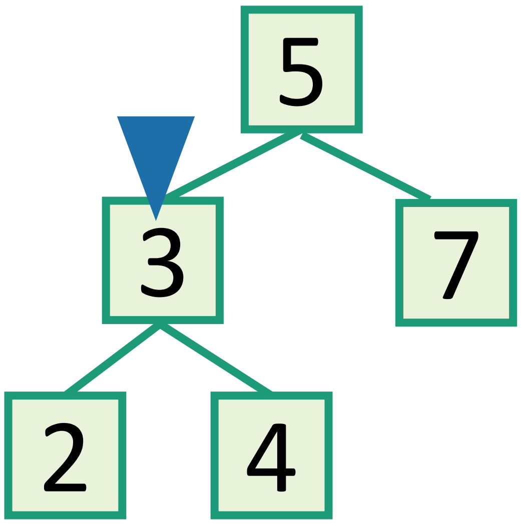
- inOrderTraversal(x):
  - if x!= NIL:
    - inOrderTraversal( x.left )
    - print( x.key )
    - inOrderTraversal( x.right )



# Aside: In-Order Traversal of BSTs

- Output all the elements in sorted order!

- inOrderTraversal(x):
  - if x!= NIL:
    - inOrderTraversal( x.left )
    - print( x.key )
    - inOrderTraversal( x.right )

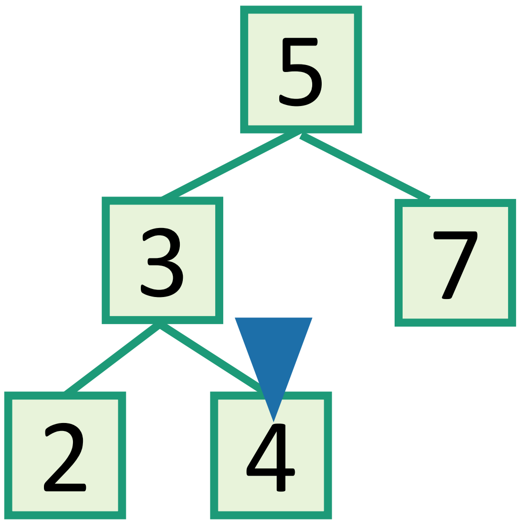



2 3

# Aside: In-Order Traversal of BSTs

- Output all the elements in sorted order!

- inOrderTraversal(x):
    - if x!= NIL:
        - inOrderTraversal( x.left )
        - print( x.key )
        - inOrderTraversal( x.right )

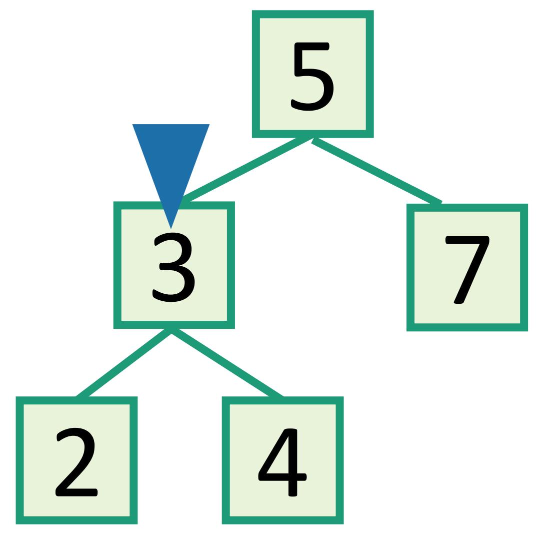



2 3 4

# Aside: In-Order Traversal of BSTs

- Output all the elements in sorted order!

- inOrderTraversal(x):
  - if x!= NIL:
    - inOrderTraversal( x.left )
    - print( x.key )
    - inOrderTraversal( x.right )

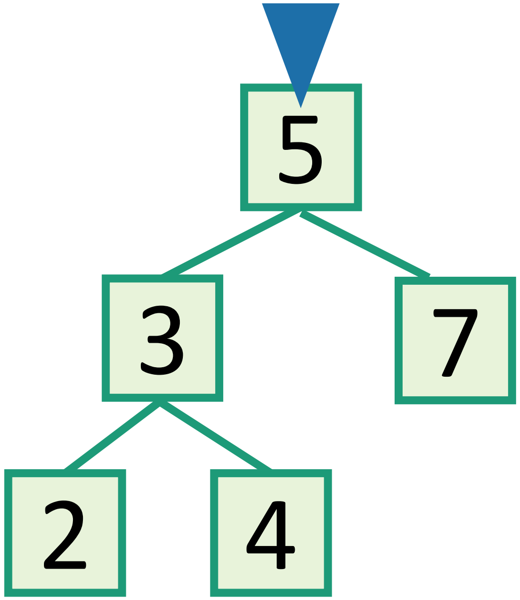



2 3 4

# Aside: In-Order Traversal of BSTs

- Output all the elements in sorted order!

- inOrderTraversal(x):
    - if x!= NIL:
        - inOrderTraversal( x.left )
        - print( x.key )
        - inOrderTraversal( x.right )

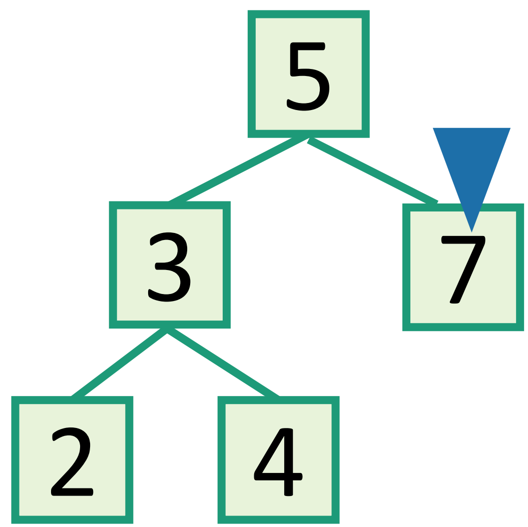



2 3 4 5

# Aside: In-Order Traversal of BSTs

- Output all the elements in sorted order!

- inOrderTraversal(x):
  - if x!= NIL:
    - inOrderTraversal( x.left )
    - print( x.key )
    - inOrderTraversal( x.right )

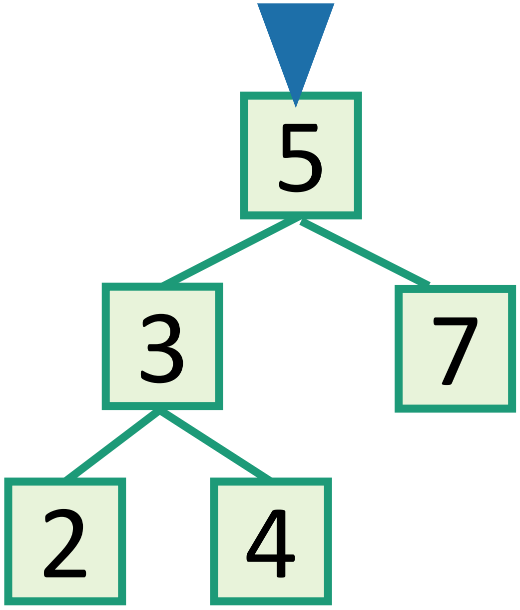



2 3 4 5 7

# Aside: In-Order Traversal of BSTs

- Output all the elements in sorted order!

- inOrderTraversal(x):
  - if x!= NIL:
    - inOrderTraversal( x.left )
    - print( x.key )
    - inOrderTraversal( x.right )




- Runs in time O(n).

2 3 4 5 7 Sorted!

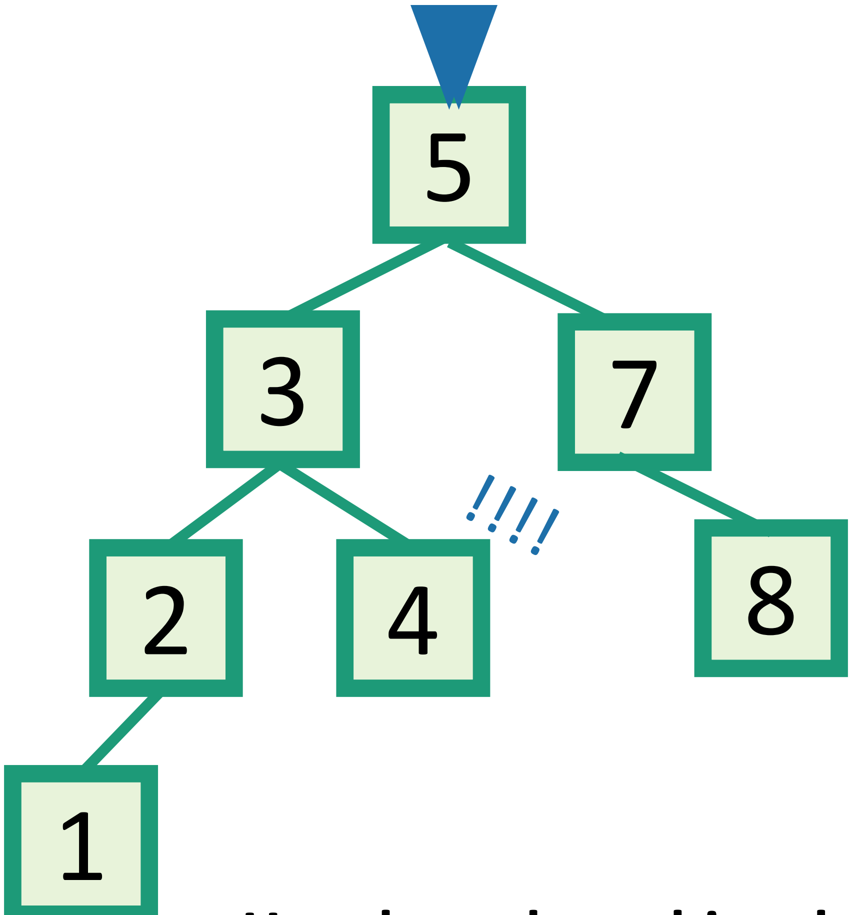
# Back to the goal

## Fast SEARCH/INSERT/DELETE

Can we do these?

# SEARCH in a Binary Search Tree

## definition by example




**EXAMPLE:** Search for 4.

**EXAMPLE:** Search for 4.5

- It turns out it will be convenient to **return 4** in this case
- (that is, **return** the last node before we went off the tree)

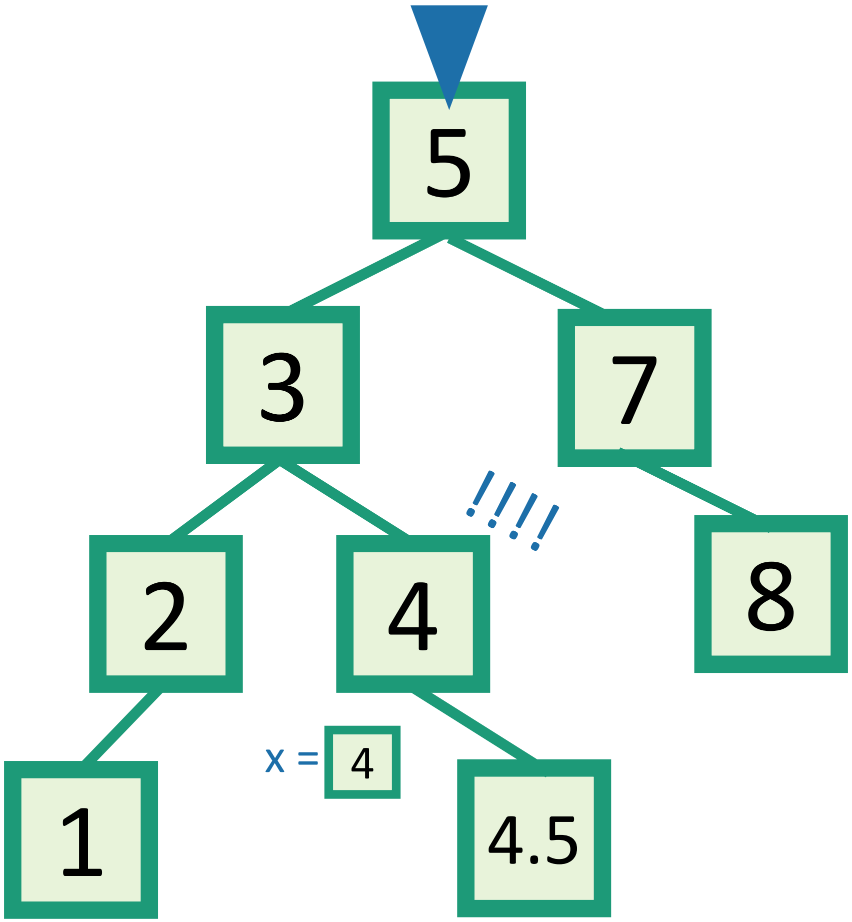
**How long does this take?**

O(length of longest path) = O(height)

Write pseudocode (or actual code) to implement this!

Ollie the over-achieving ostrich

# INSERT in a Binary Search Tree

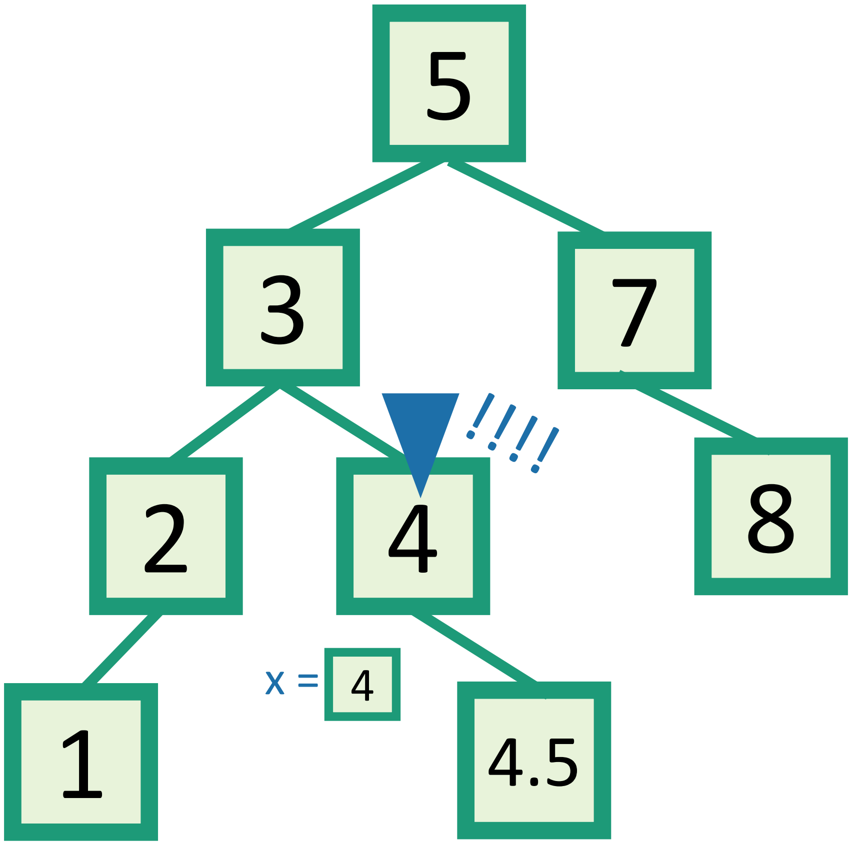



**EXAMPLE:** Insert 4.5

- INSERT(key):
  - x = SEARCH(key)
  - **Insert** a new node with desired key at x…

# INSERT in a Binary Search Tree

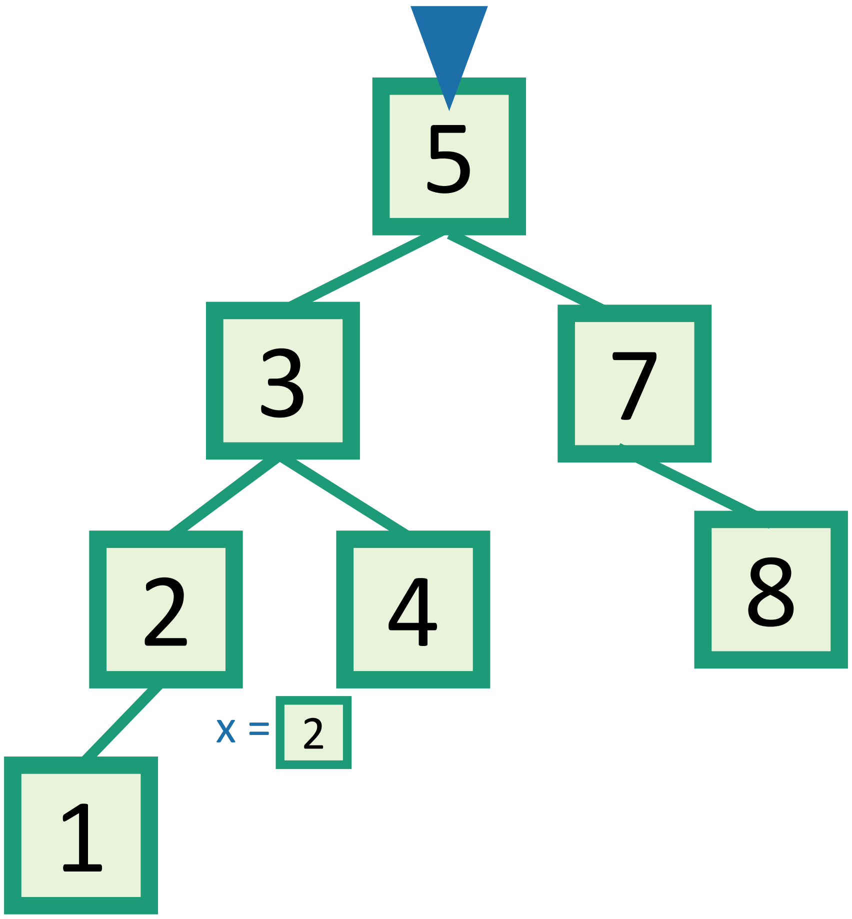
This slide skipped in class – here for reference




**EXAMPLE:** Insert 4.5

- INSERT(key):
  - x = SEARCH(key)
  - **if** key > x.key:
    - Make a new node with the correct key, and put it as the right child of x.
  - **if** key < x.key:
    - Make a new node with the correct key, and put it as the left child of x.
  - **if** x.key == key:
    - **return**

# DELETE in a Binary Search Tree




**EXAMPLE:** Delete 2

- DELETE(key):
  - x = SEARCH(key)
  - **if** x.key == key:
    - ....delete x....

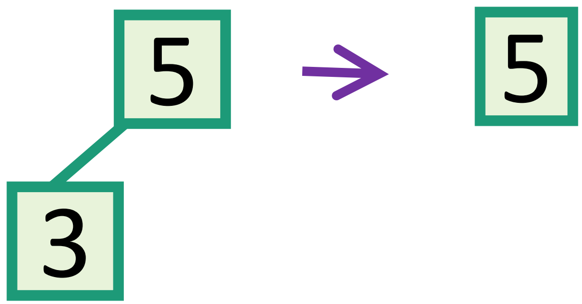
This is a bit more complicated...see the skipped slides for some pictures of the different cases.

# DELETE in a Binary Search Tree

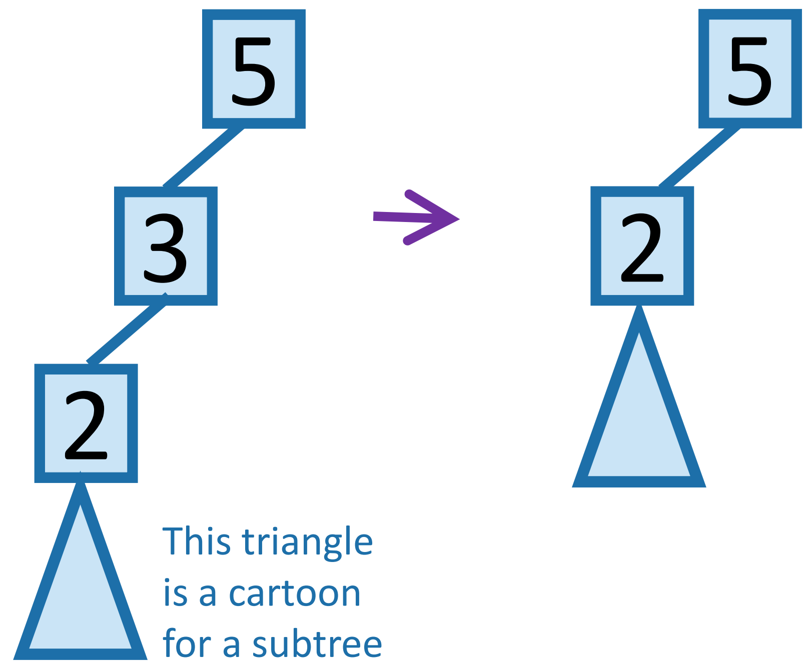
## several cases (by example)

## say we want to delete 3

This slide skipped in class – here for reference!




**Case 1**: if 3 is a leaf, just delete it.




**Case 2:** if 3 has just one child, move that up.

Write pseudocode for all of these!

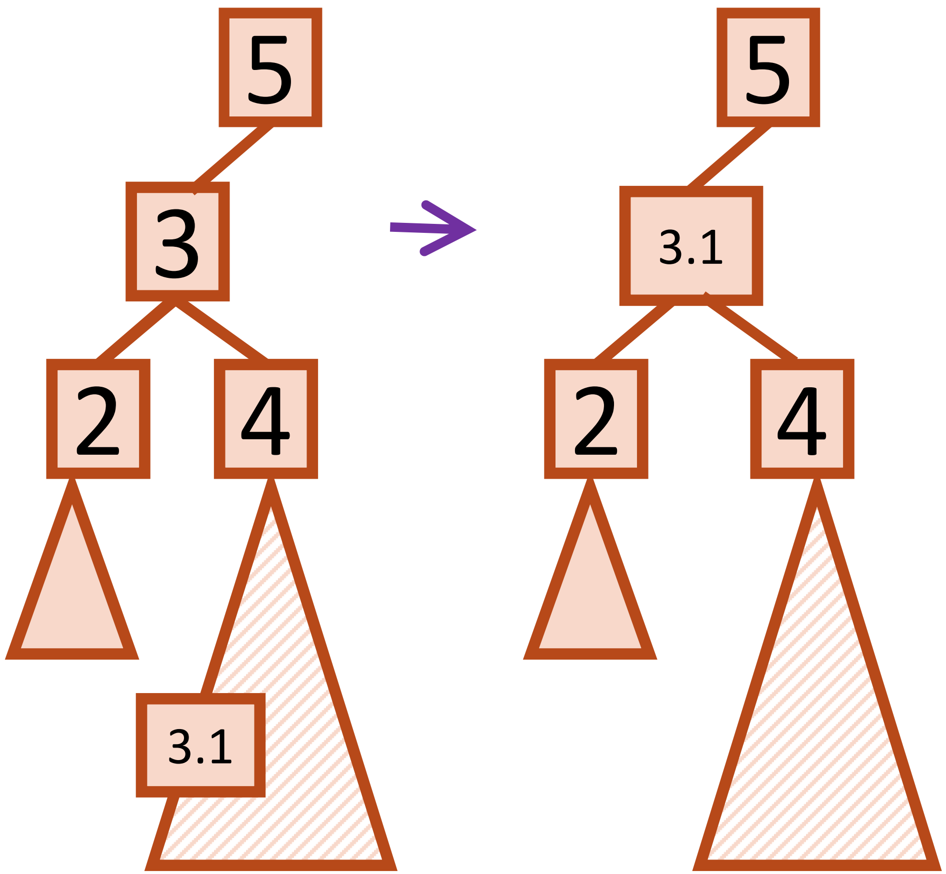
Siggi the Studious Stork

# DELETE in a Binary Search Tree

ctd.

This slide skipped in class – here for reference!

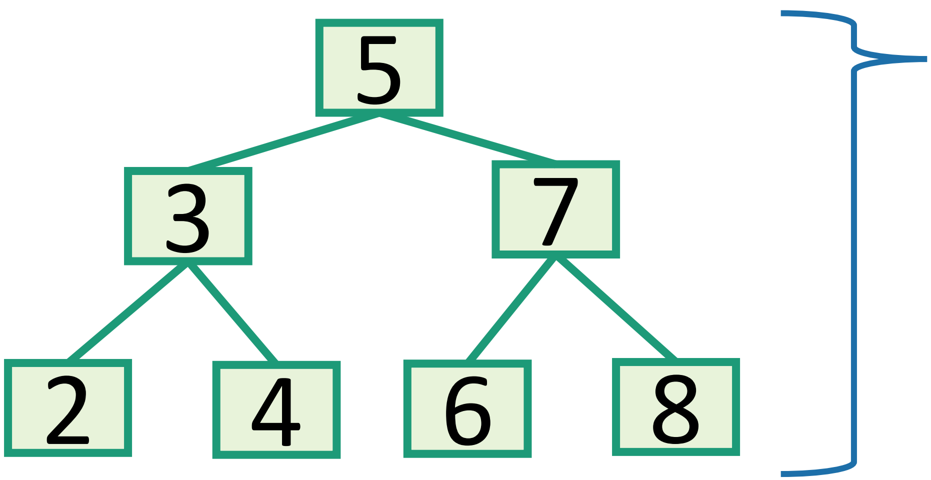
**Case 3**: if 3 has two children,
replace 3 with it's immediate successor.
(aka, next biggest thing after 3)




- Does this maintain the BST property?
  - Yes.
- How do we find the immediate successor?
  - SEARCH for 3 in the subtree under 3.right
- How do we remove it when we find it?
  - If [3.1] has 0 or 1 children, do one of the previous cases.
- What if [3.1] has two children?
  - It doesn't.

# How long do these operations take?

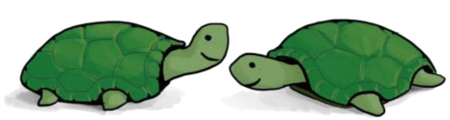
- SEARCH is the big one.
  - Everything else just calls SEARCH and then does some small O(1)-time operation.




Time = O(height of tree)

How long does search take?

1 minute think; 1 minute pair+share



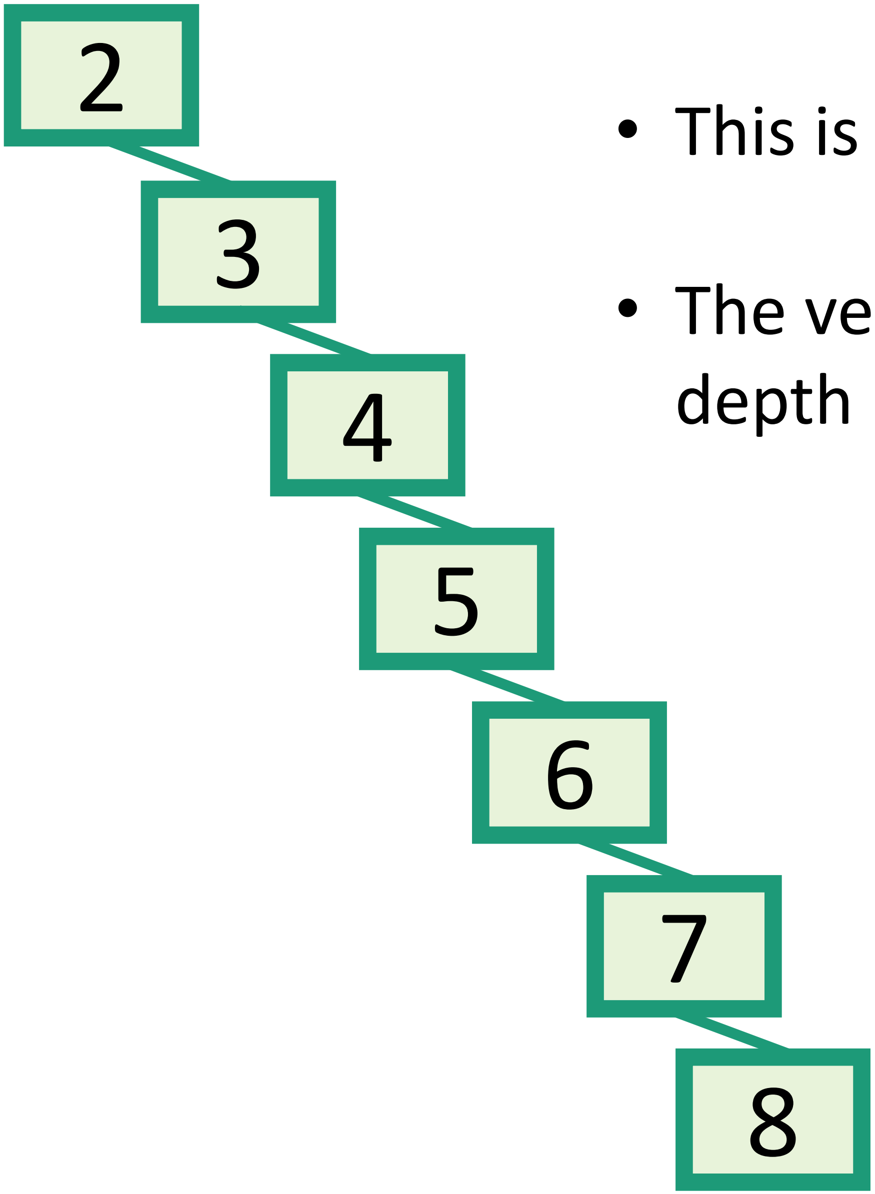
Trees have depth O(log(n)). **Done!**

Lucky the lackadaisical lemur.

Wait a second…

Plucky the Pedantic Penguin

# Search might take time O(n).




- This is a valid binary search tree.
- The version with n nodes has depth n, **not** O(log(n)).

How often is “every so often” in the worst case? It’s actually pretty often!

Ollie the over-achieving ostrich

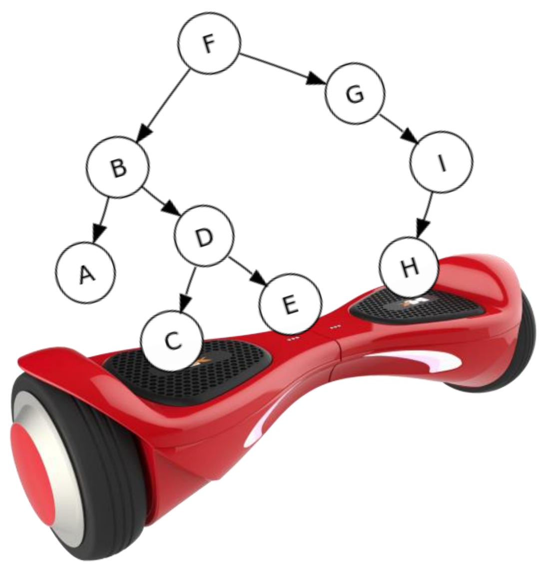
# What to do?

- Goal: Fast SEARCH/INSERT/DELETE
- All these things take time O(height)
- And the height might be big!!! ☹

- Idea 0:
  - Keep track of how deep the tree is getting.
  - If it gets too tall, re-do everything from scratch.
    - At least $\Omega(n)$ every so often....

- Turns out that’s not a great idea. Instead we turn to…

# Self-Balancing Binary Search Trees

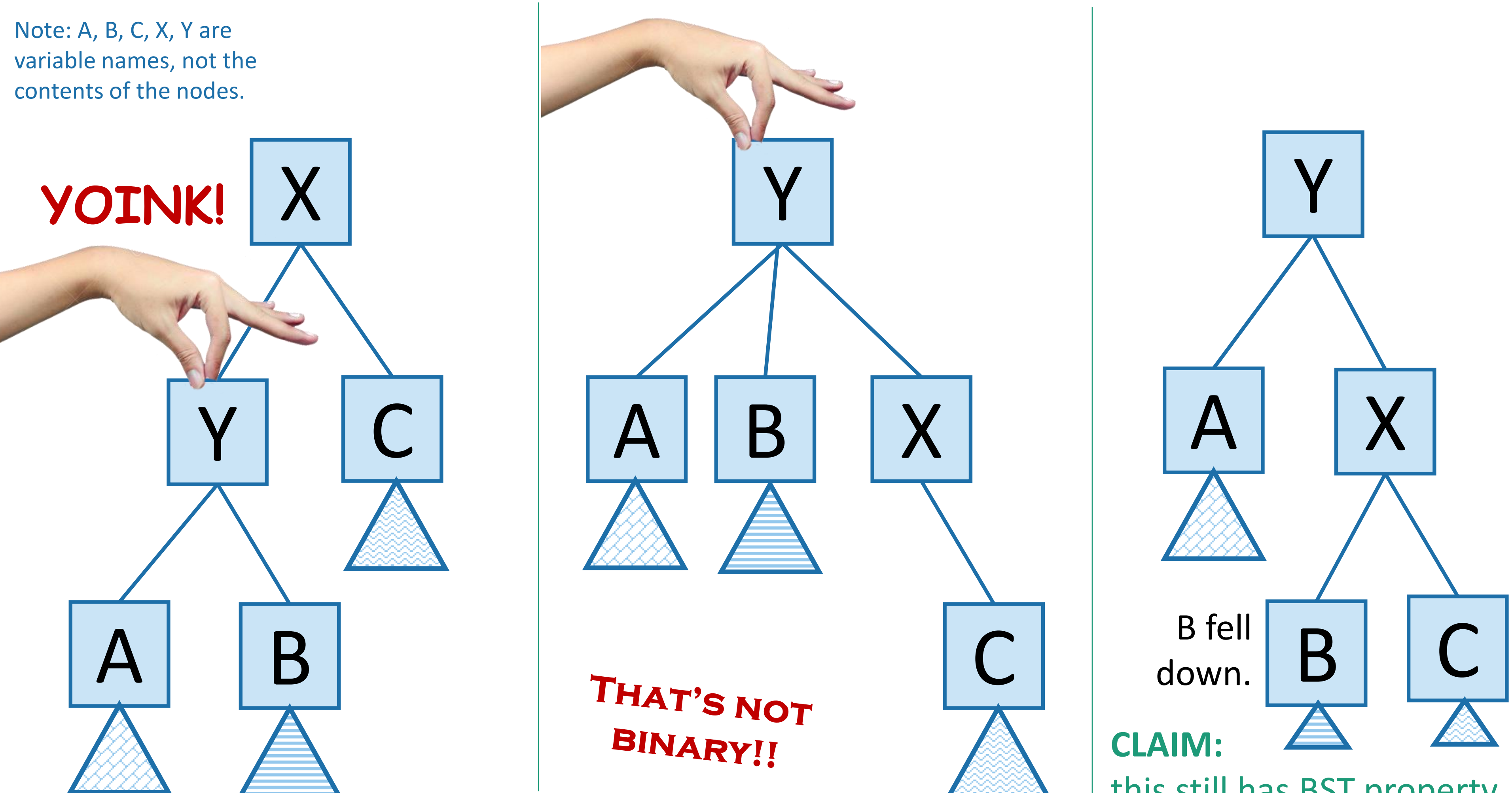


# Idea 1: Rotations

No matter what lives underneath A,B,C,
**this takes time O(1).** (Why?)

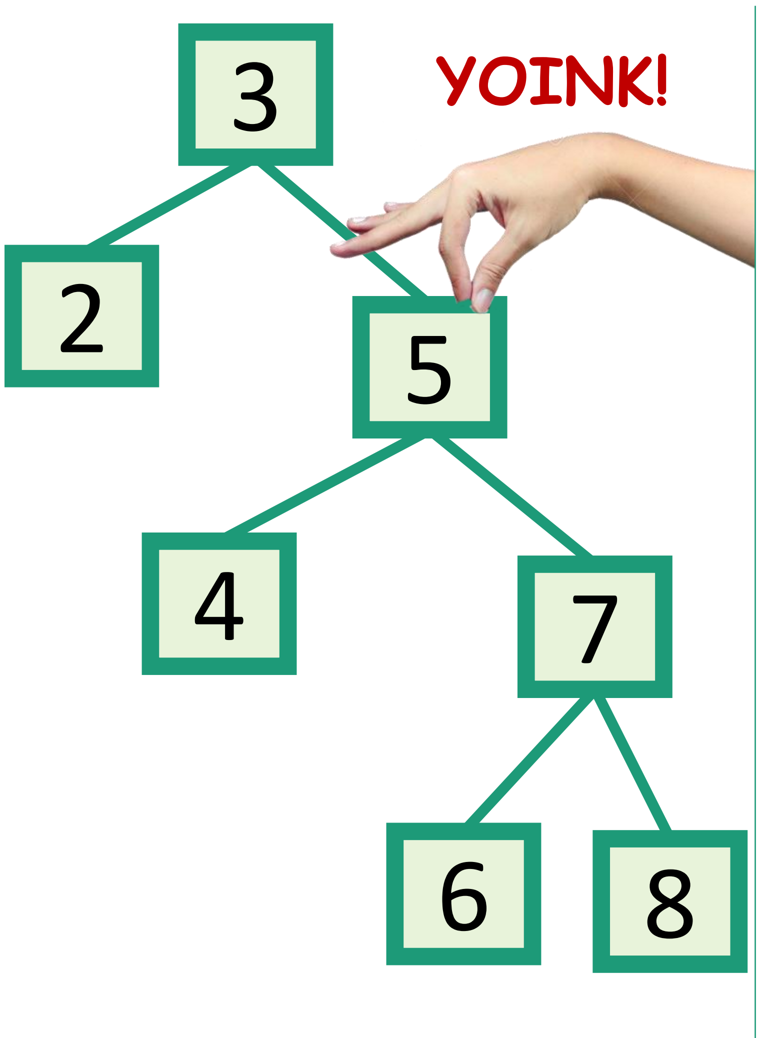
- Maintain Binary Search Tree (BST) property, while moving stuff around.

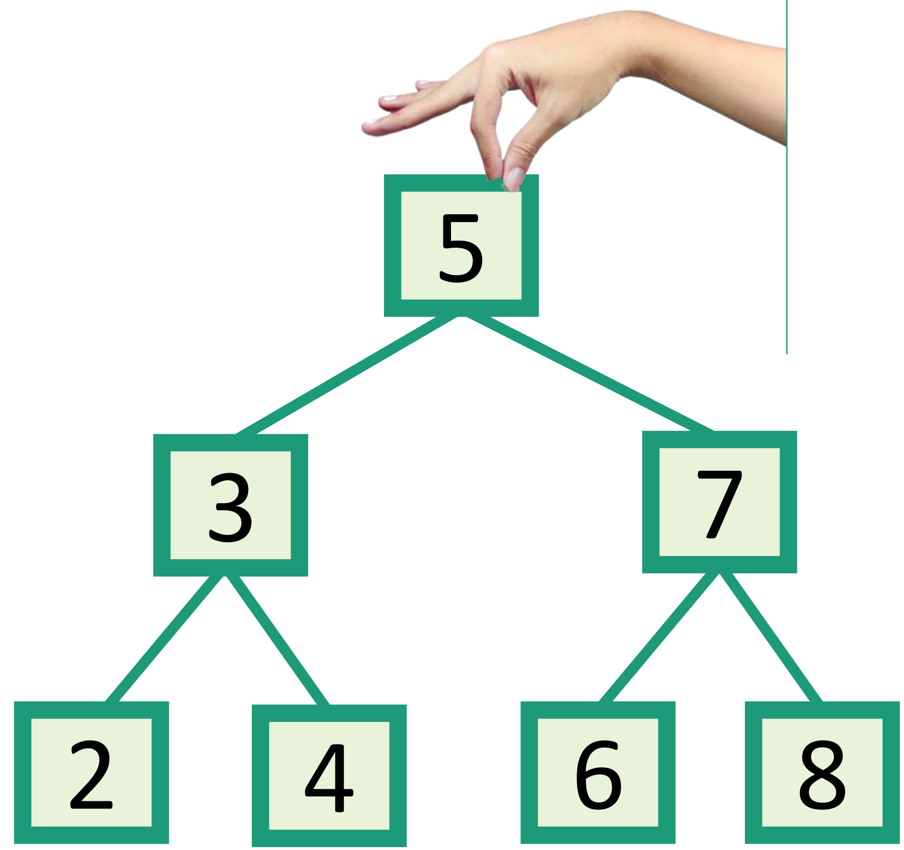
Note: A, B, C, X, Y are variable names, not the contents of the nodes.




**CLAIM:**
this still has BST property.

# This seems helpful



# Strategy?

- Whenever something seems unbalanced, do rotations until it's okay again.

Lucky the Lackadaisical Lemur

Even for Lucky this is pretty vague. What do we mean by "seems unbalanced"? What's "okay"?

# Idea 2: have some proxy for balance

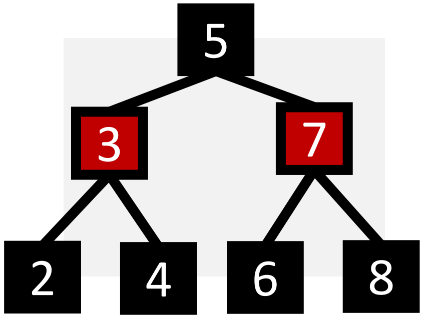
- Maintaining perfect balance is too hard.
- Instead, come up with some proxy for balance:
  - If the tree satisfies [SOME PROPERTY], then it's pretty balanced.
  - We can maintain [SOME PROPERTY] using rotations.

There are actually several ways to do this, but today we'll see…

# Red-Black Trees

- A Binary Search Tree that balances itself!
- No more time-consuming by-hand balancing!
- Be the envy of your friends and neighbors with the time-saving...

*Red-Black tree!*



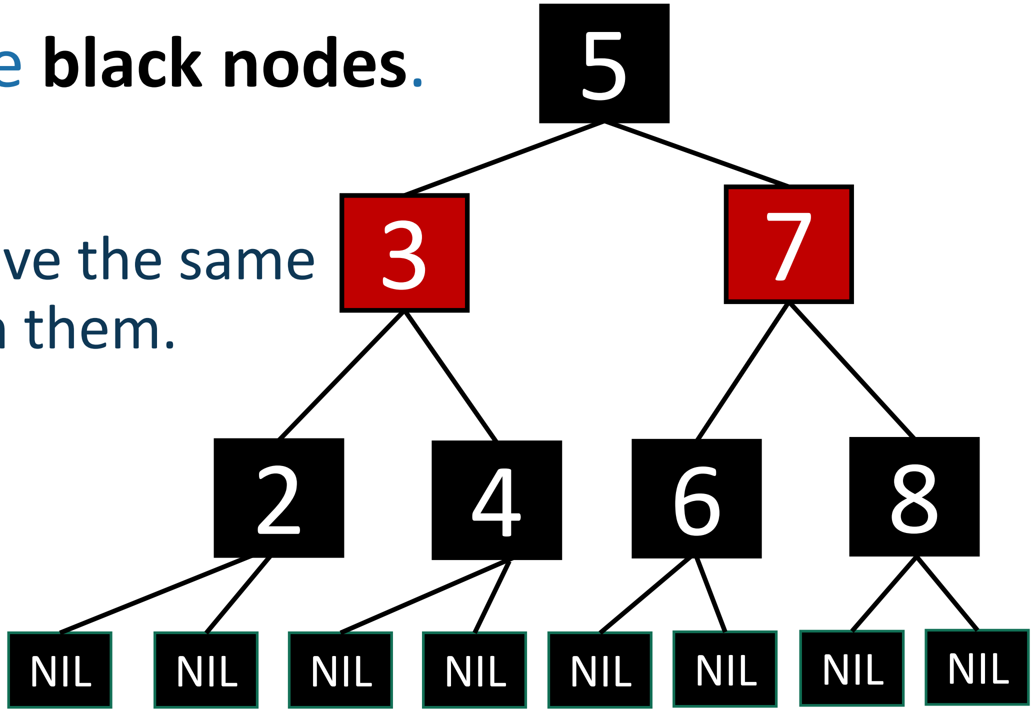

*Maintain balance* by stipulating that **black nodes** are balanced, and that there *aren't too many* **red nodes**.

*It's just good sense!*

# Red-Black Trees

obey the following rules (which are a proxy for balance)

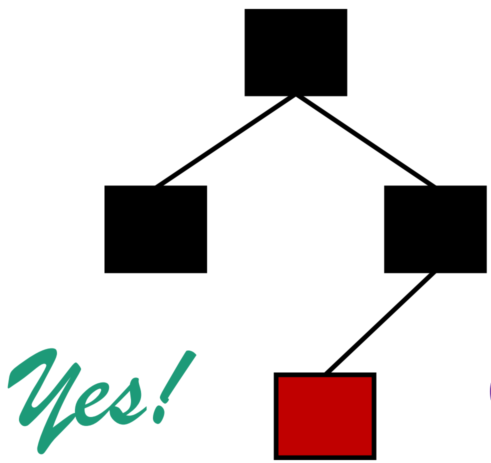
- Every node is colored **red** or **black.**
- The root node is a **black node**.
- NIL children count as **black nodes**.
- Children of a **red node** are **black nodes**.
- For all nodes x:
  - all paths from x to NIL's have the same number of **black nodes** on them.



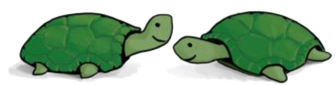

I'm not going to draw the NIL children in the future, but they are treated as black nodes.

# Examples(?)

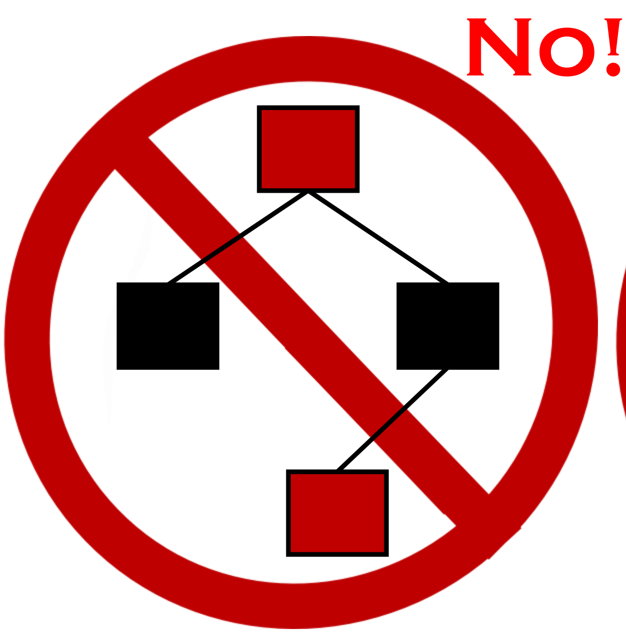
- Every node is colored **red** or **black.**
- The root node is a **black node**.
- NIL children count as **black nodes**.
- Children of a **red node** are **black nodes**.
- For all nodes x:
  - all paths from x to NIL's have the same number of **black nodes** on them.

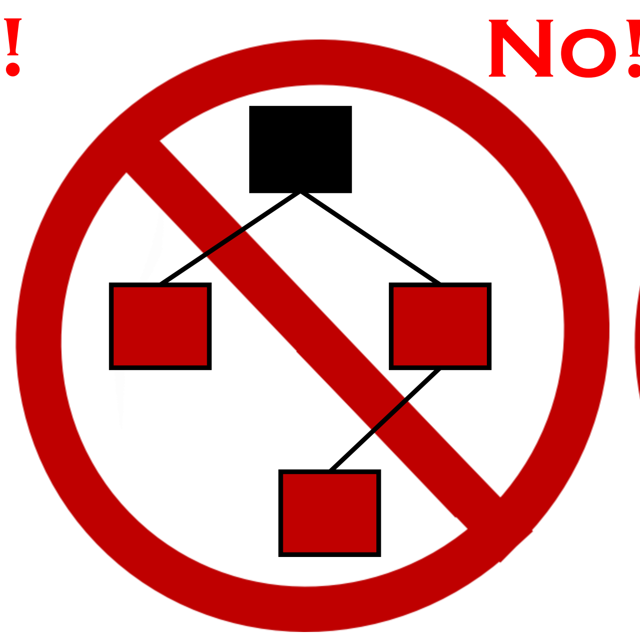


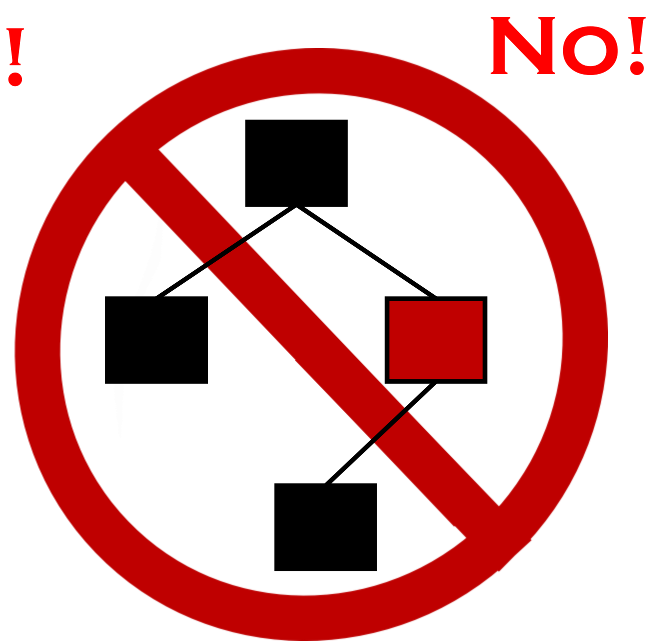

Which of these are red-black trees? (NIL nodes not drawn)



1 minute think
1 minute pair+share

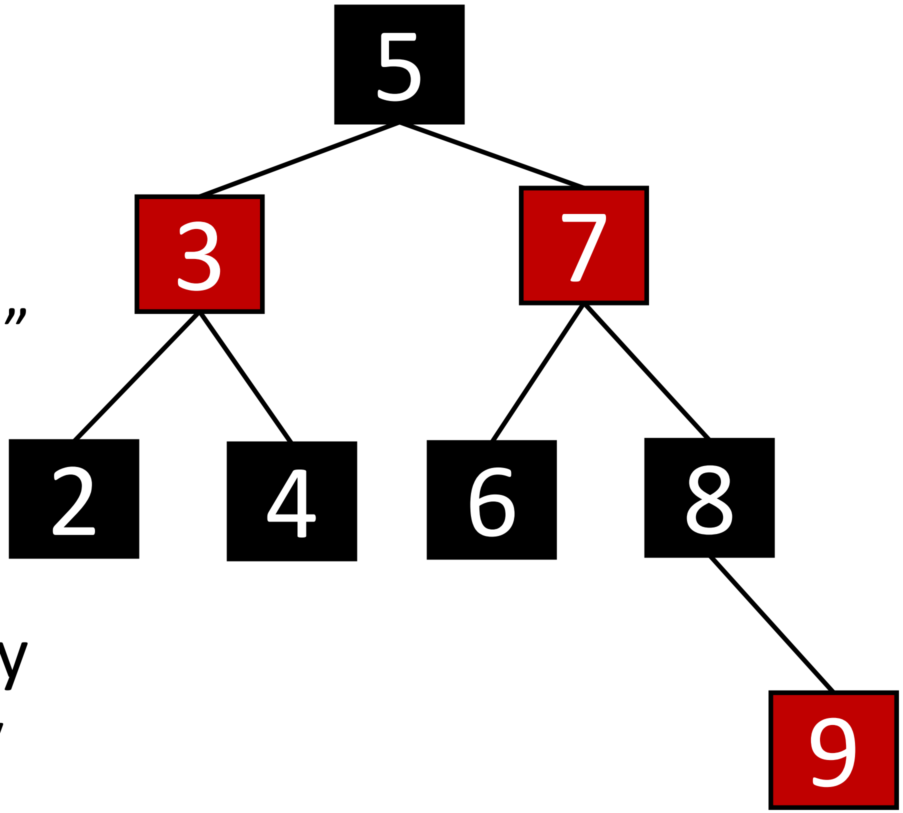


# Why **these** rules???????

- This is pretty balanced.
  - The **black nodes** are balanced
  - The **red nodes** are "spread out" so they don't mess things up too much.




- We can maintain this property as we insert/delete nodes, by using rotations.

This is the really clever idea!

This **Red-Black** structure is a proxy for balance.

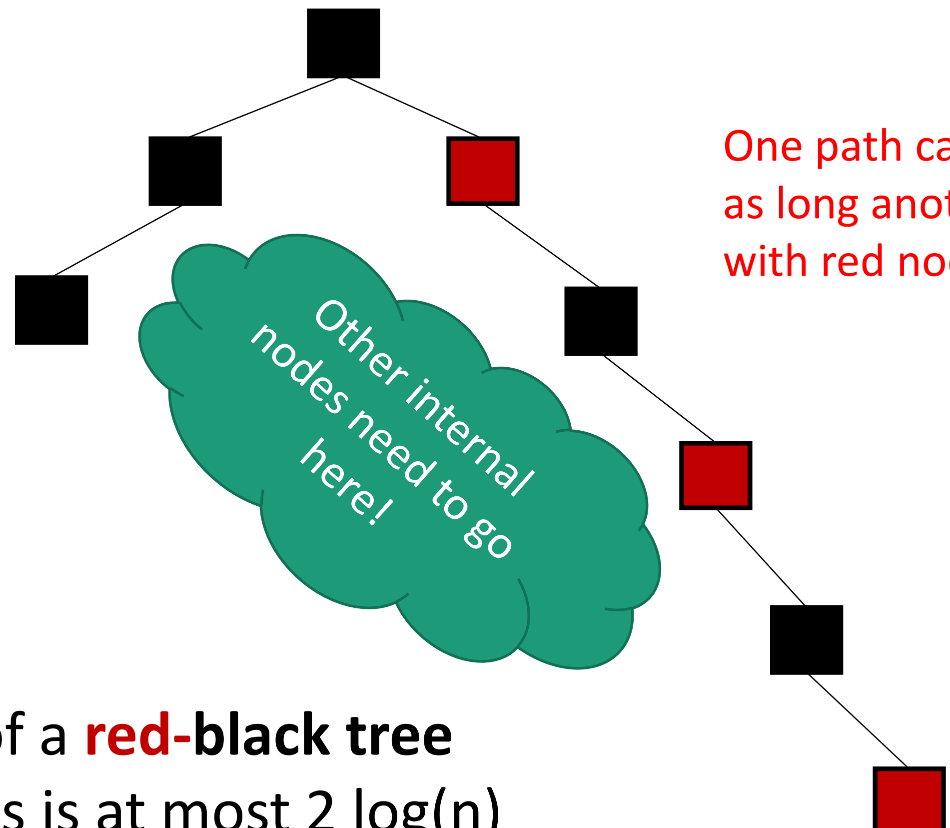
It's just a smidge weaker than perfect balance, but we can actually maintain it!

Let's build some intuition!

Lucky the lackadaisical lemur

# This is "pretty balanced"

- To see why, intuitively, let's try to build a Red-Black Tree that's unbalanced.

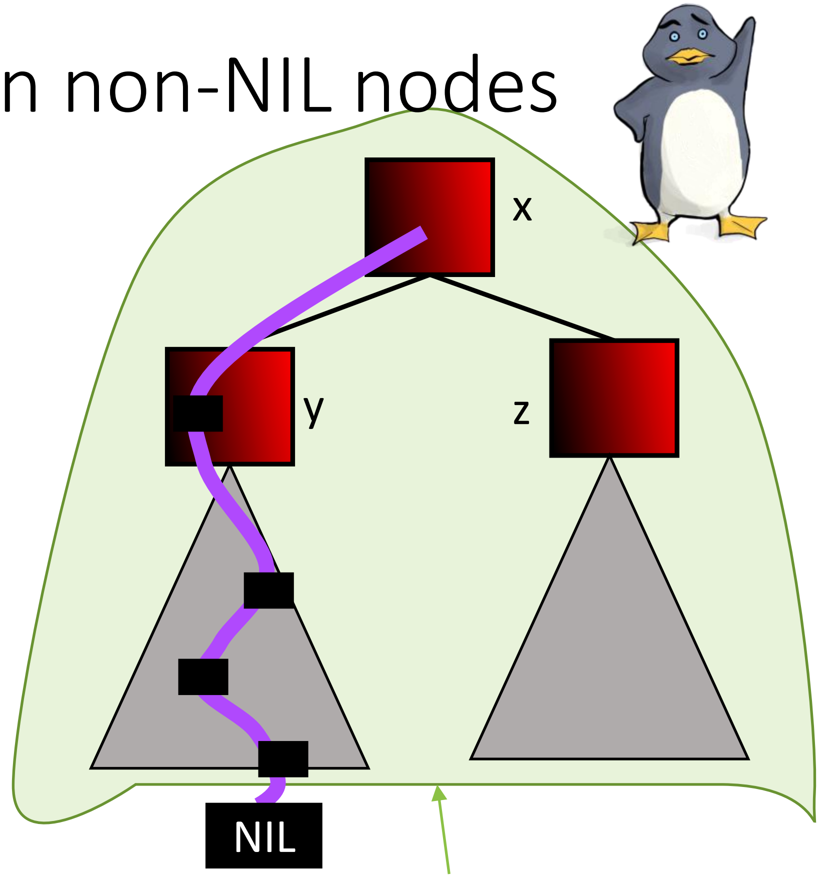



One path can be at most twice as long another if we pad it with red nodes.

**Conjecture**:
the height of a **red-black tree** with n nodes is at most 2 log(n)

# The height of a RB-tree with n non-NIL nodes is at most $2\log(n+1)$

- Define b(x) to be the number of black nodes in any path from x to NIL.
  - (excluding x, including NIL).
- Claim:
  - There are at least $2^{b(x)} - 1$ non-NIL nodes in the subtree underneath x. (Including x).
- [Proof by induction – on board if time]




Then:

$$n \geq 2^{b(root)} - 1$$ using the Claim

$$\geq 2^{height/2} - 1$$ b(root) >= height/2 because of RBTree rules.

Rearranging:

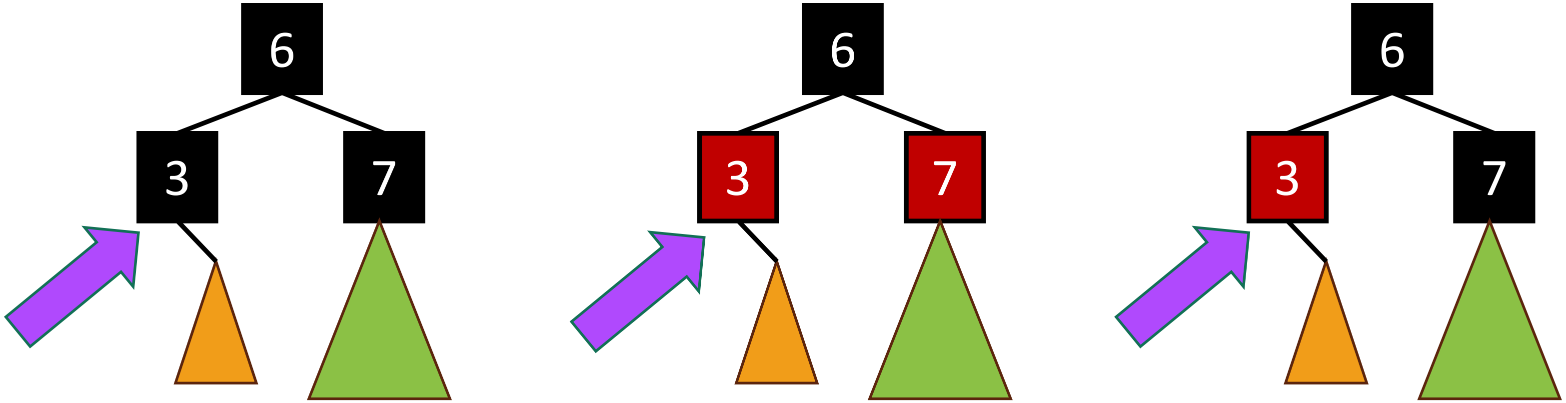
$$n + 1 \geq 2^{height/2} \Rightarrow height \leq 2\log(n+1)$$

# This is great!

- SEARCH in an RBTree is immediately O(log(n)), since the depth of an RBTree is O(log(n)).

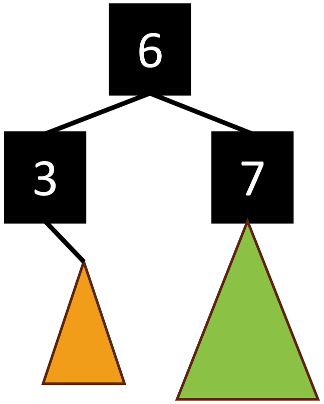
- What about INSERT/DELETE?
  - Turns out, you can INSERT and DELETE items from an RBTree in time O(log(n)), while *maintaining* the RBTree property.
  - That's why this is a good property!

# INSERT: Many cases



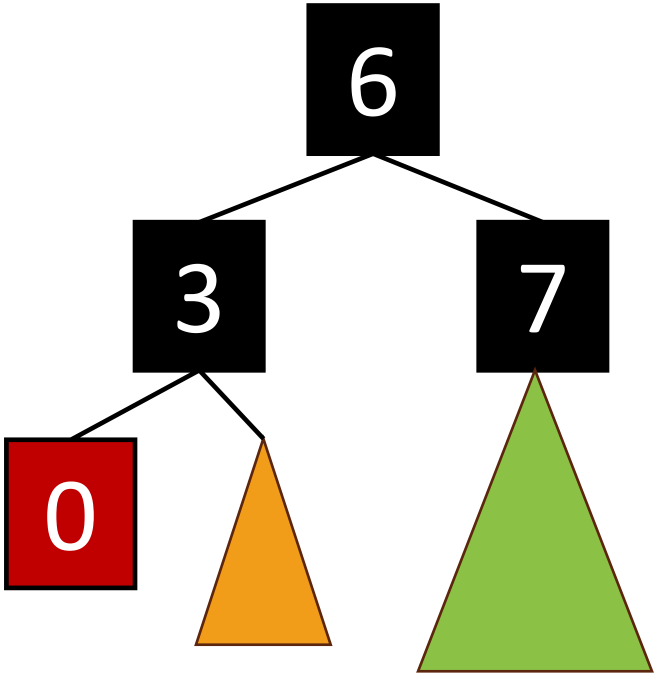

- Suppose we want to insert 0 **here**.
- There are 3 “important” cases for different colorings of the existing tree.

# INSERT: Case 1



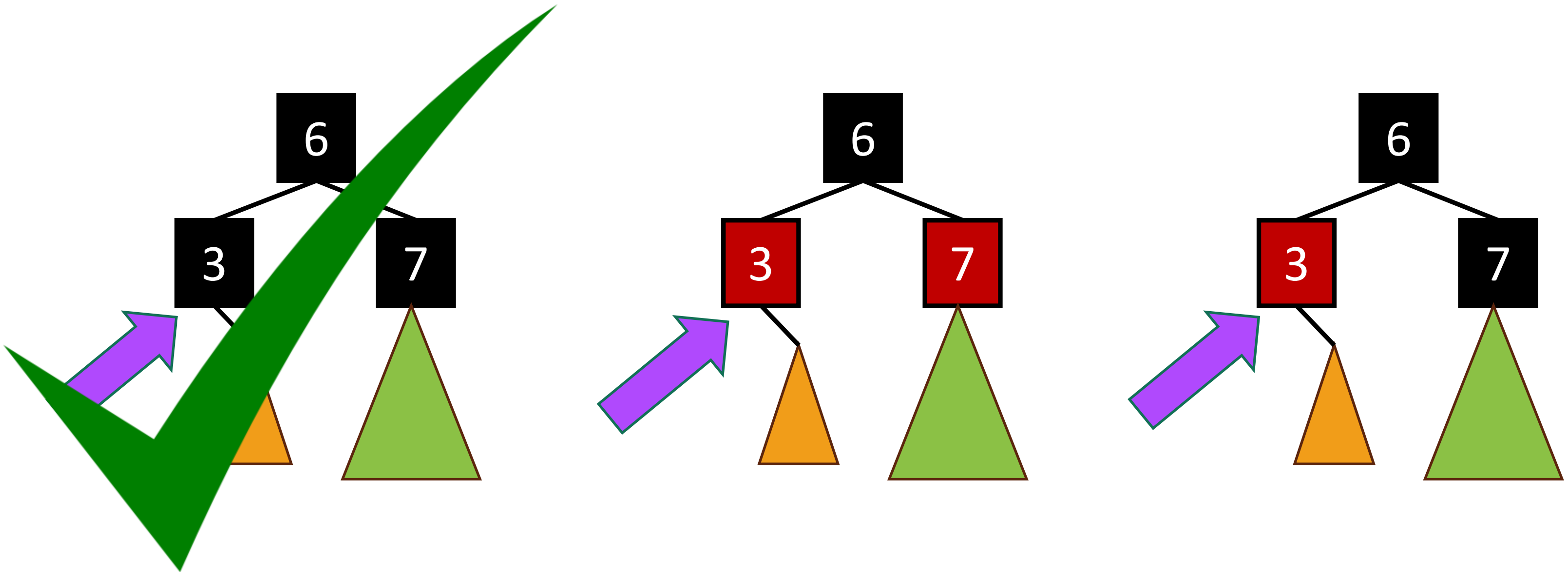

What if it looks like this?

- Make a new **red node**.
- Insert it as you would normally.




Example: insert 0

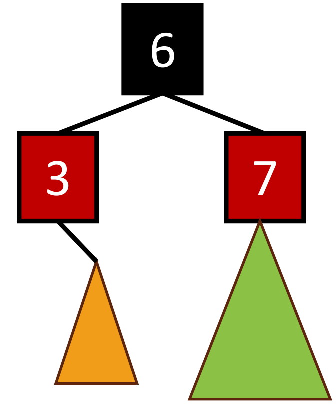
# INSERT: Many cases




- Suppose we want to insert 0 **here**.
- There are 3 "important" cases for different colorings of the existing tree.

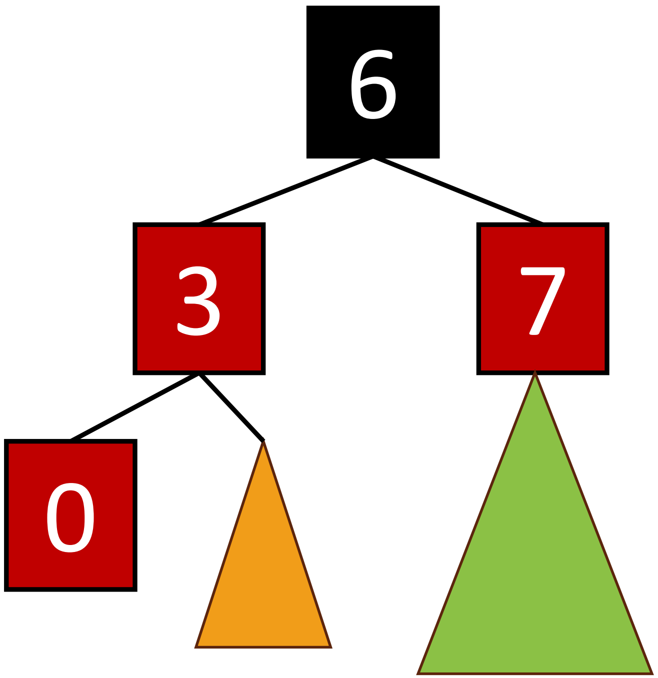
# INSERT: Case 2

- Make a new **red node.**
- Insert it as you would normally.
- Fix things up if needed.



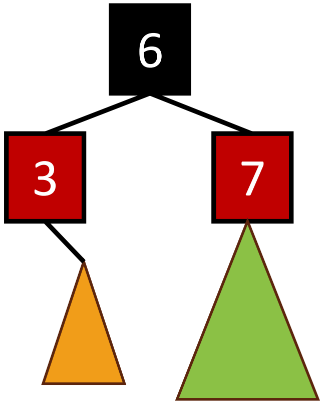

What if it looks like this?

Example: insert 0



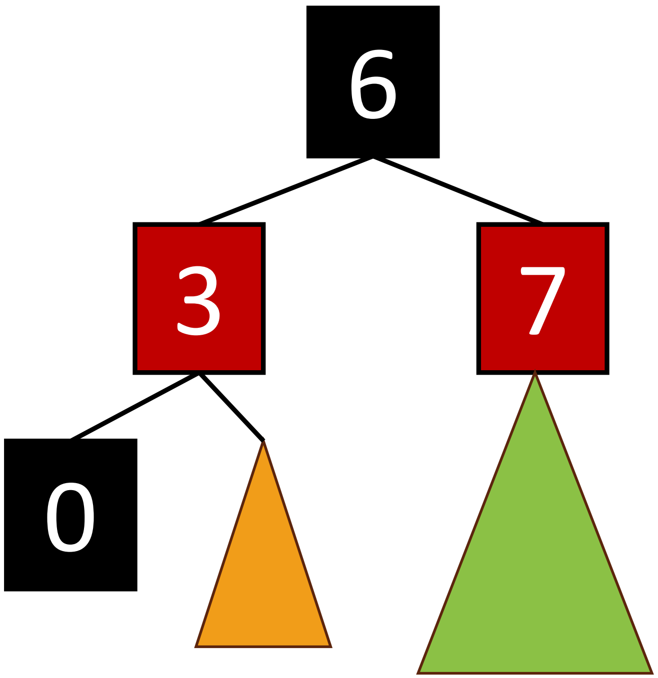

# INSERT: Case 2

- Make a new **red node.**
- Insert it as you would normally.
- Fix things up if needed.



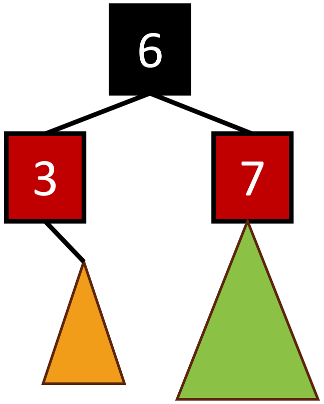

What if it looks like this?

Example: insert 0



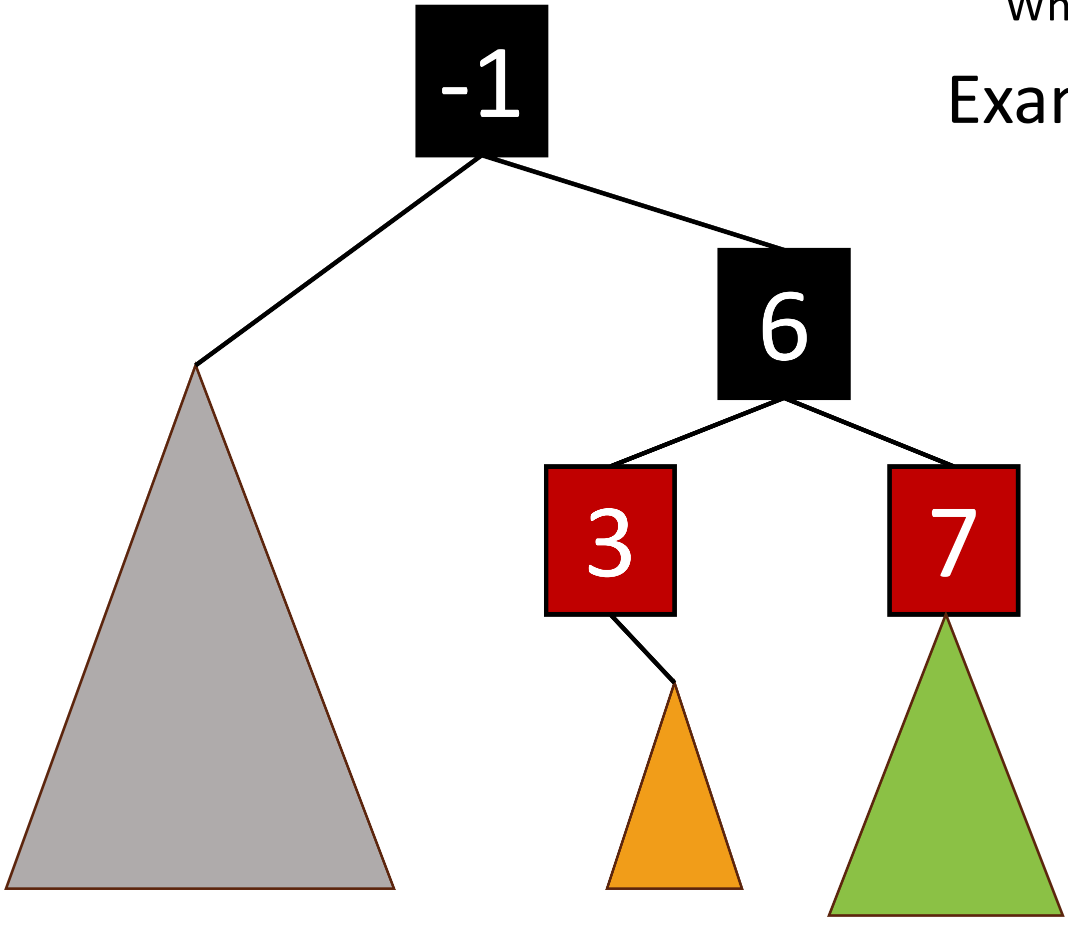

Can’t we just insert 0 as a **black node?**

# We need a bit more context



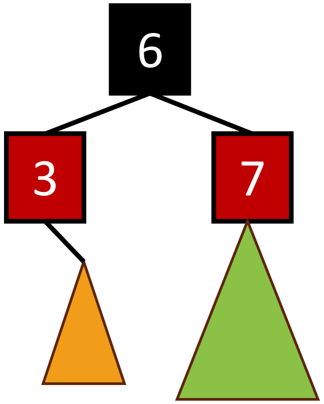

What if it looks like this?

Example: insert 0



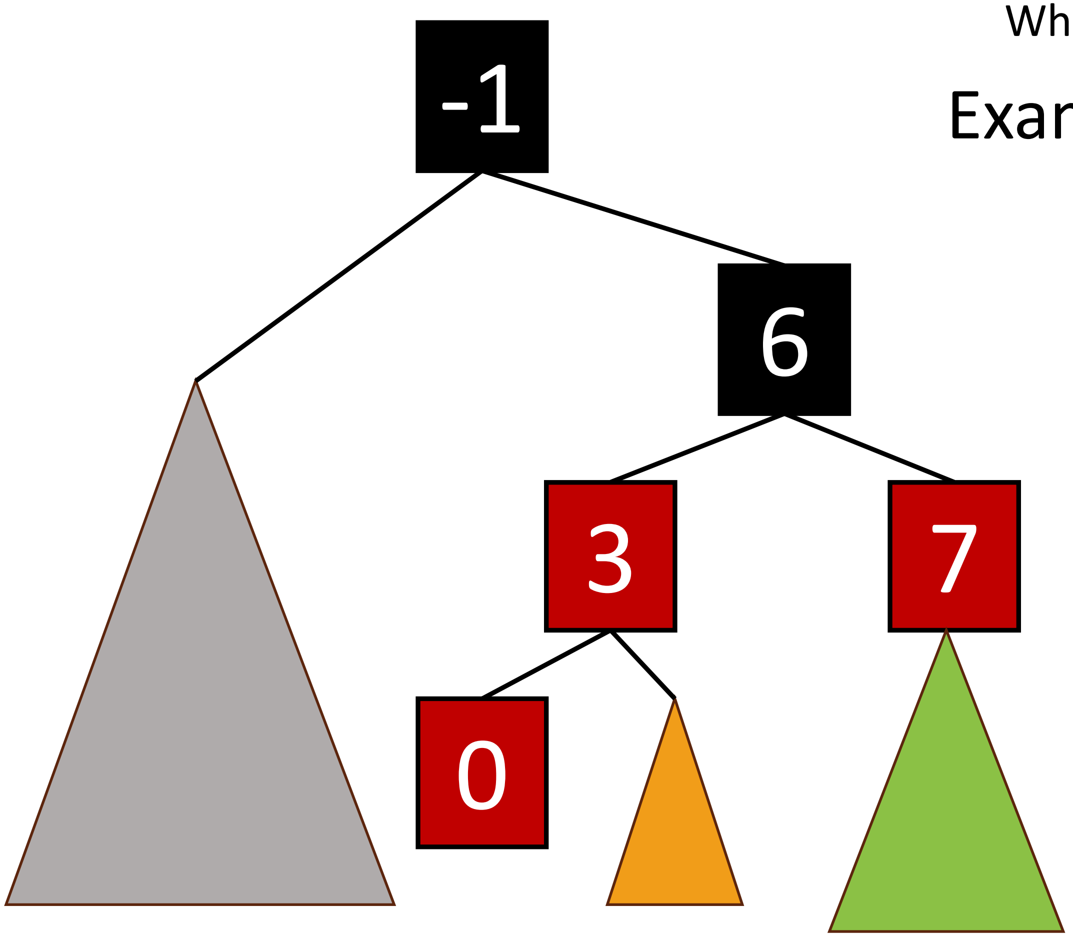
# We need a bit more context

- Add 0 as a red node.



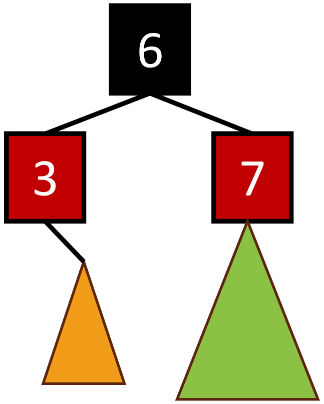

What if it looks like this?

Example: insert 0



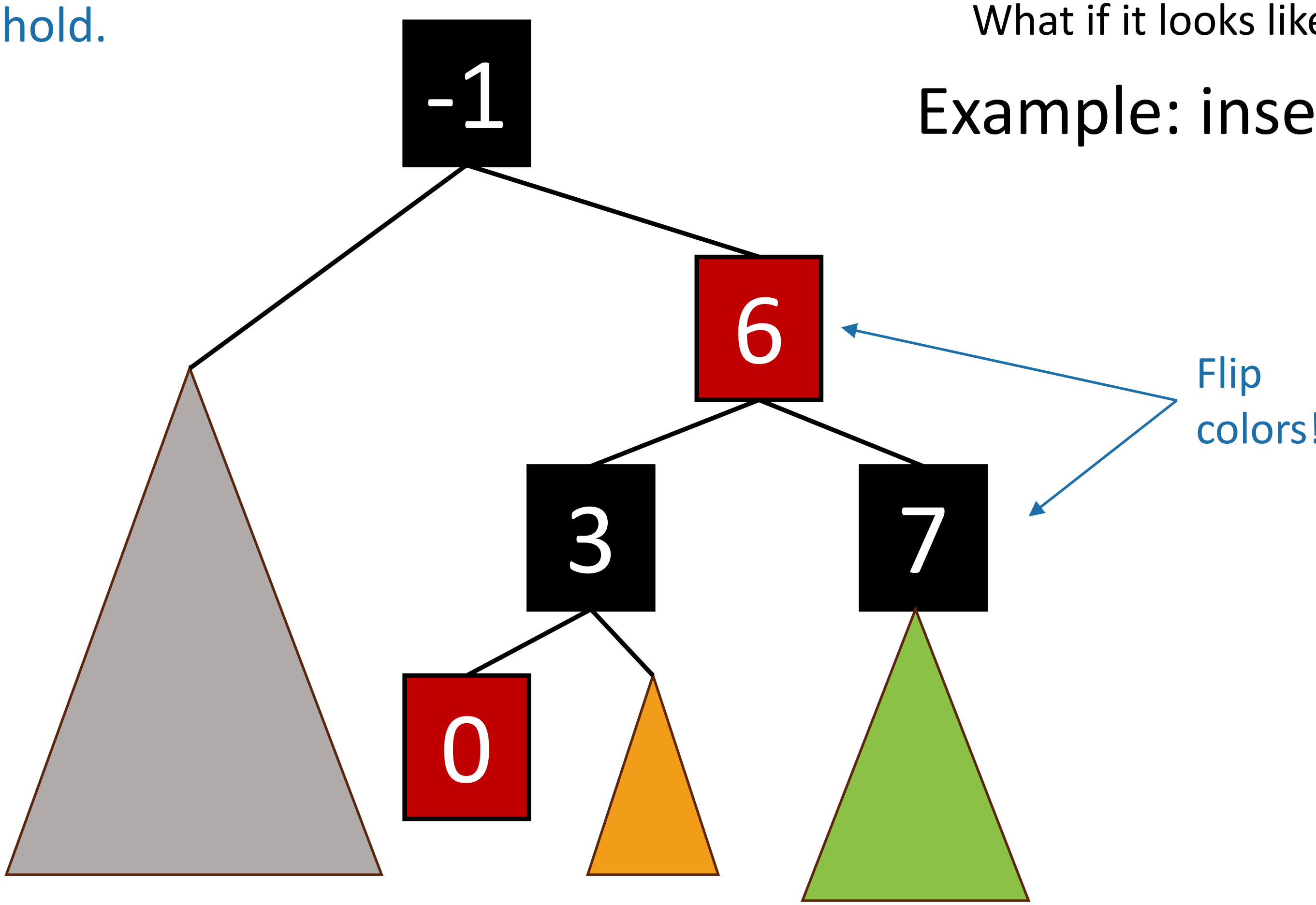
# We need a bit more context

- Add 0 as a red node.
- **Claim:** RB-Tree properties still hold.

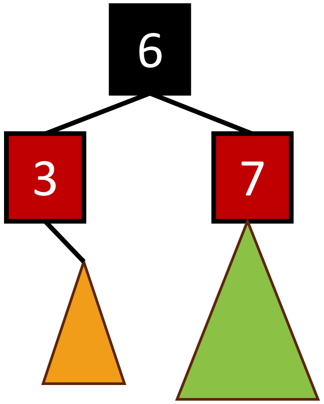



What if it looks like this?

Example: insert 0

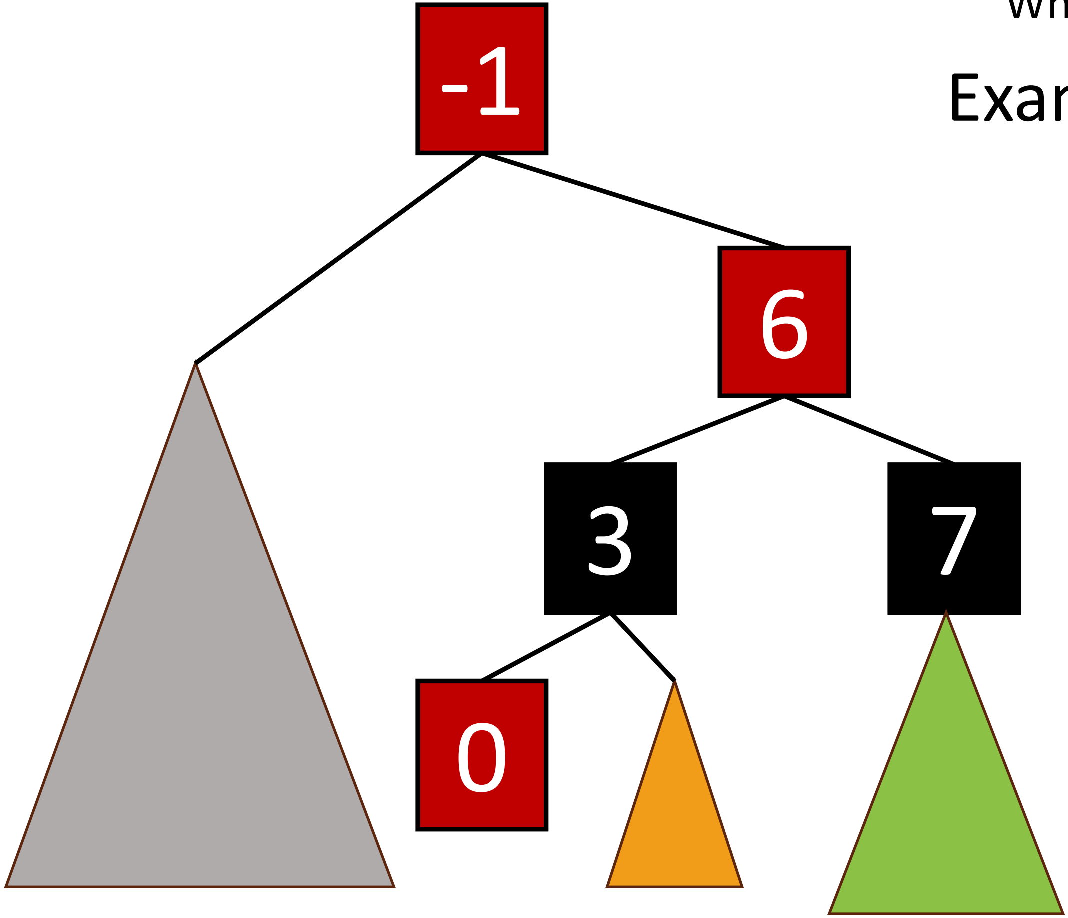


# But what if that was red?

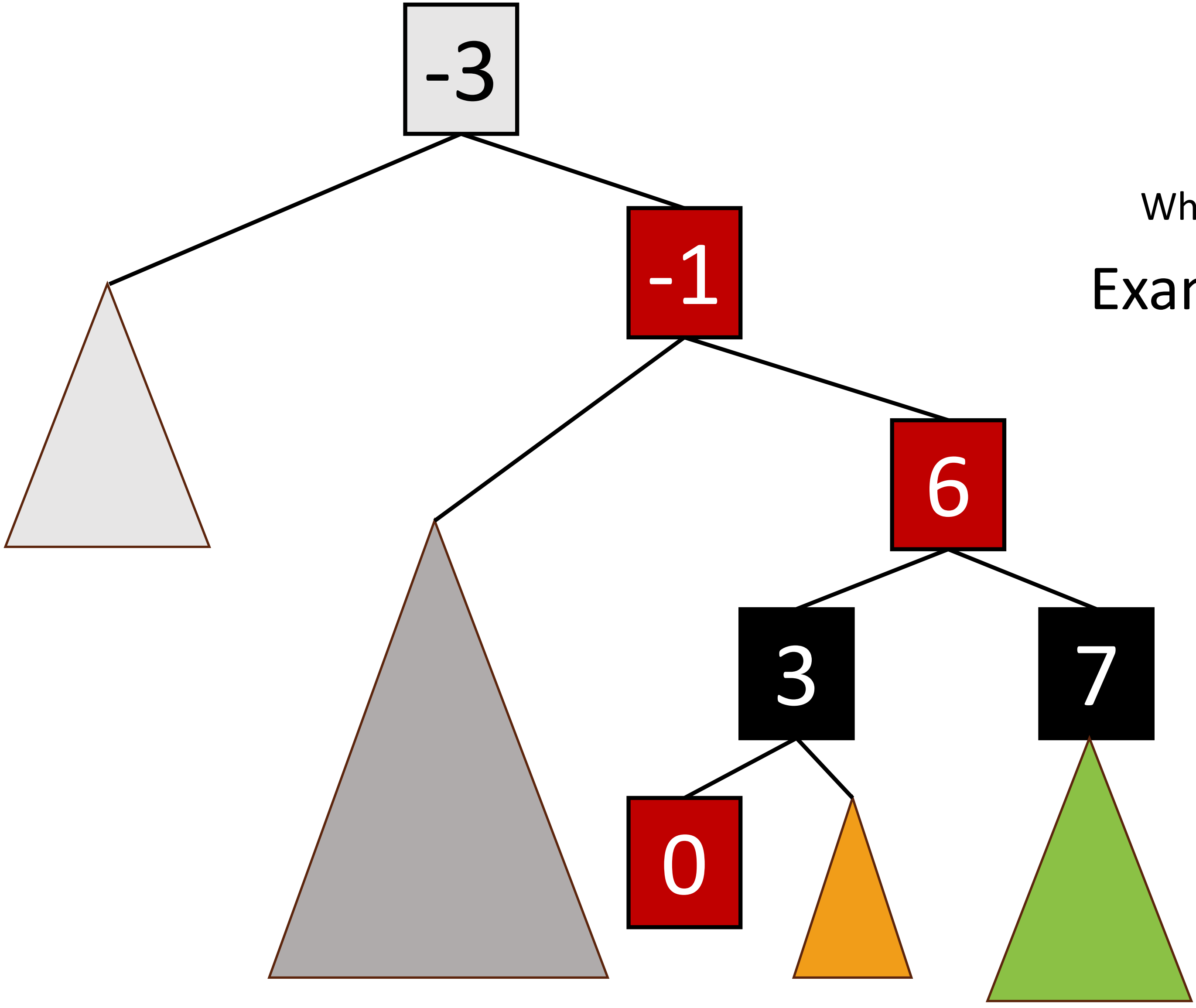


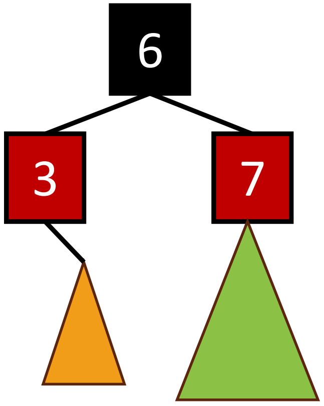

What if it looks like this?

Example: insert 0



# More context…

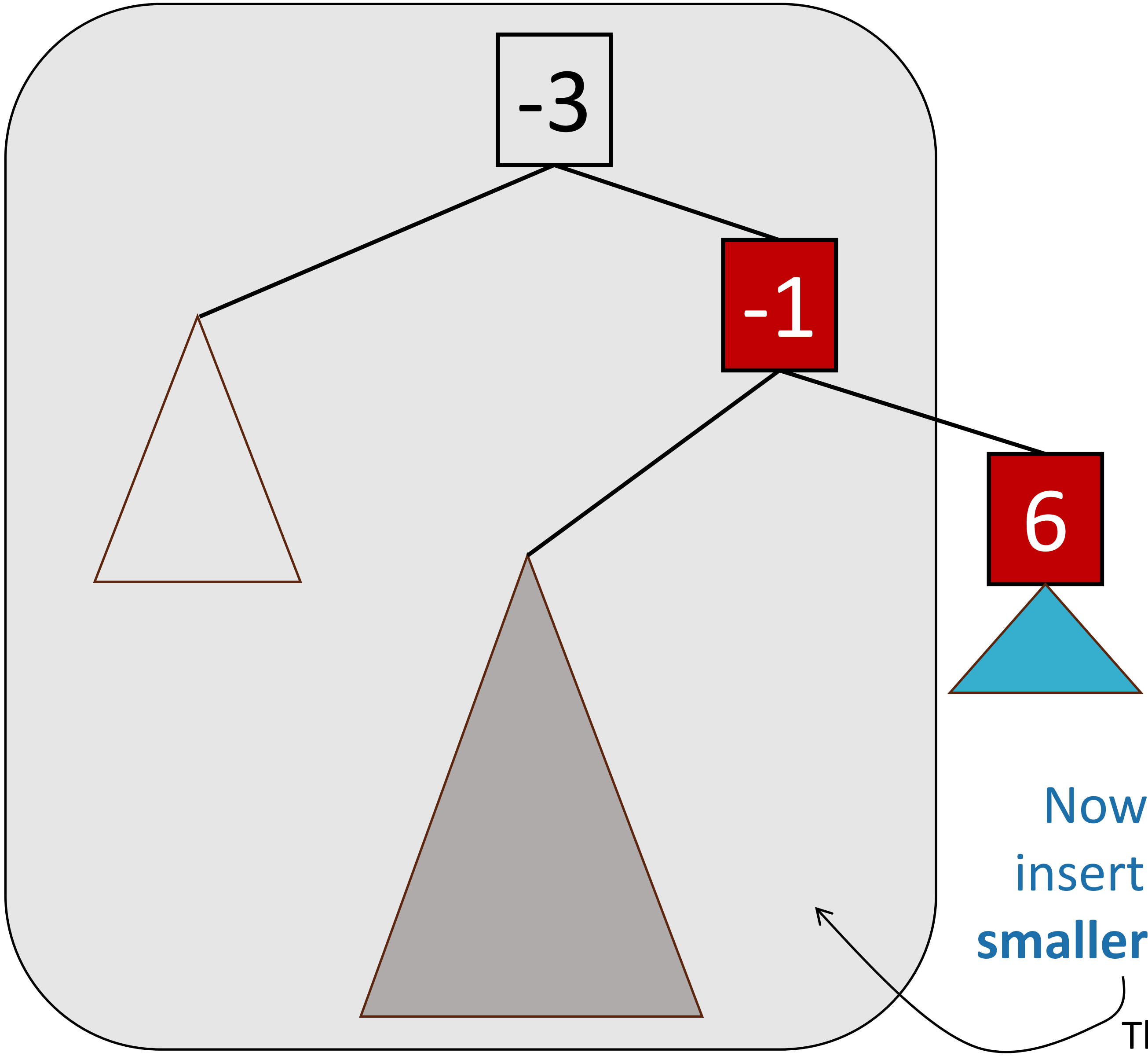


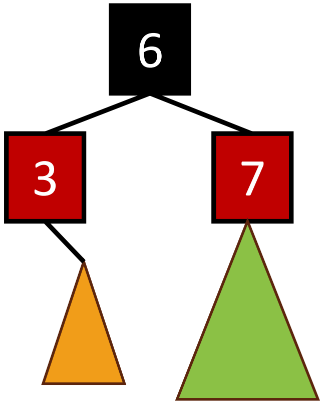




What if it looks like this?

Example: insert 0

# More context…



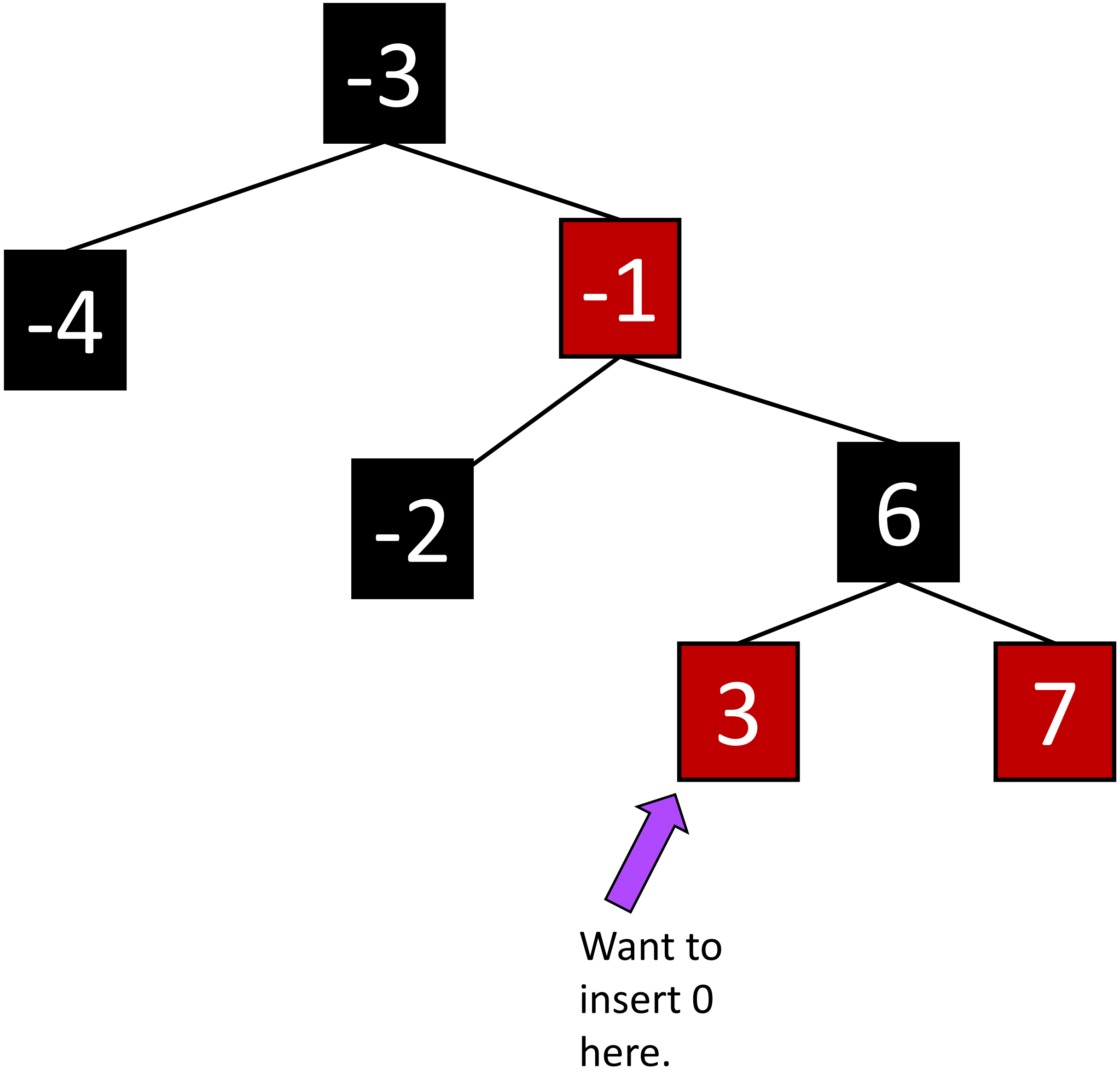

What if it looks like this?

Example: insert 0

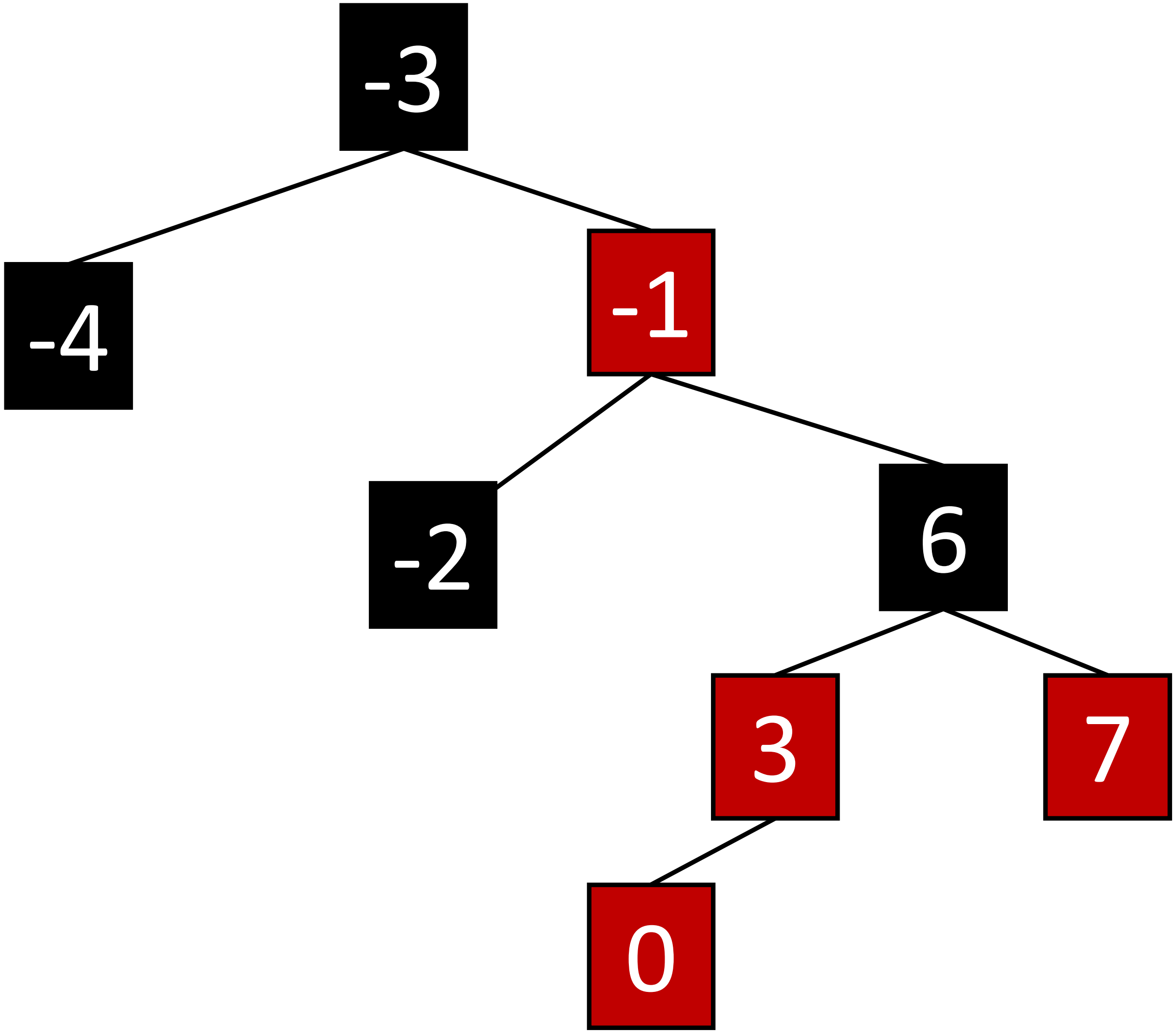
Now we're basically inserting 6 into some **smaller tree**. Recurse!

This one!

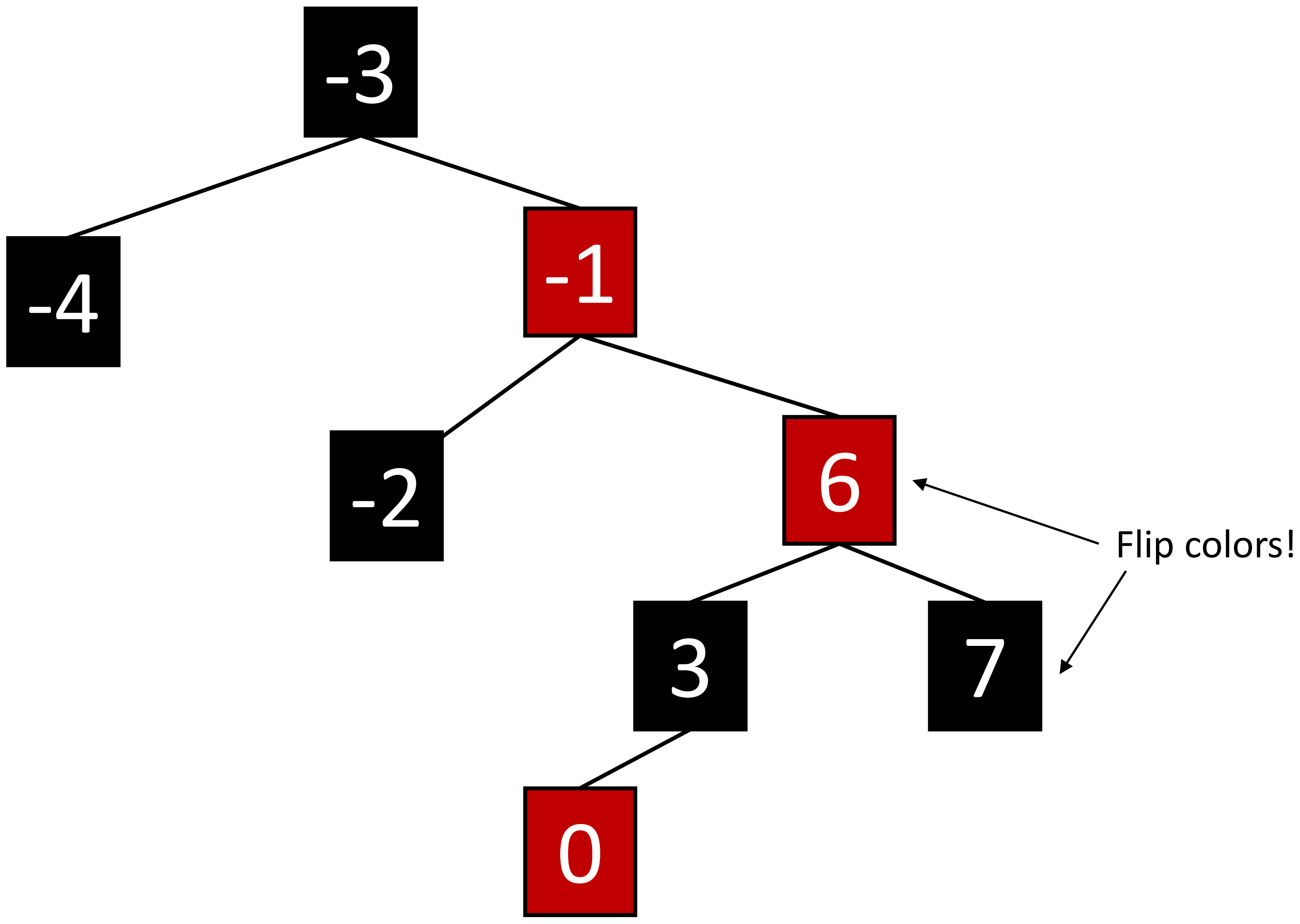
# Example, part I



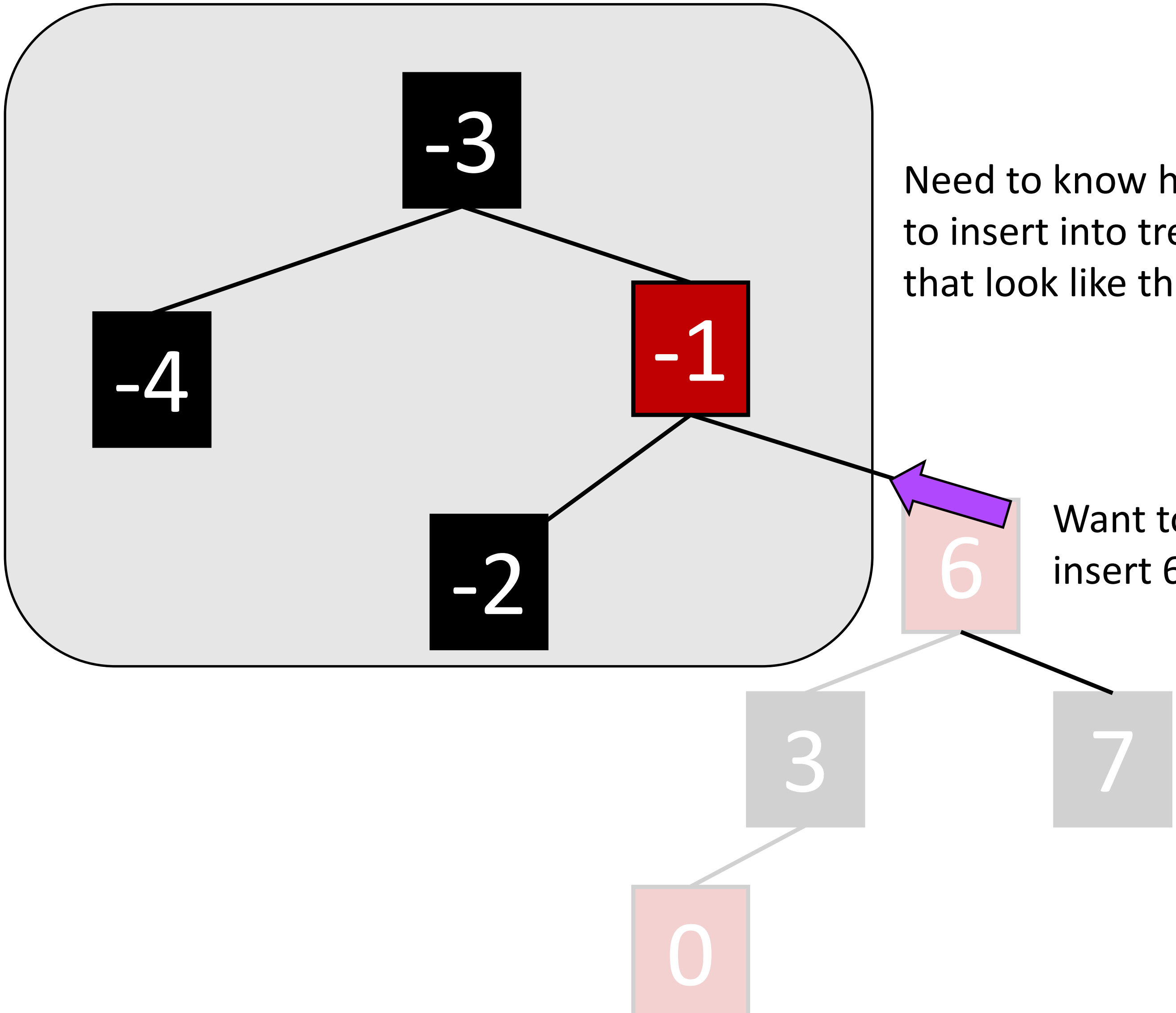
# Example, part I



# Example, part I



# Example, part I

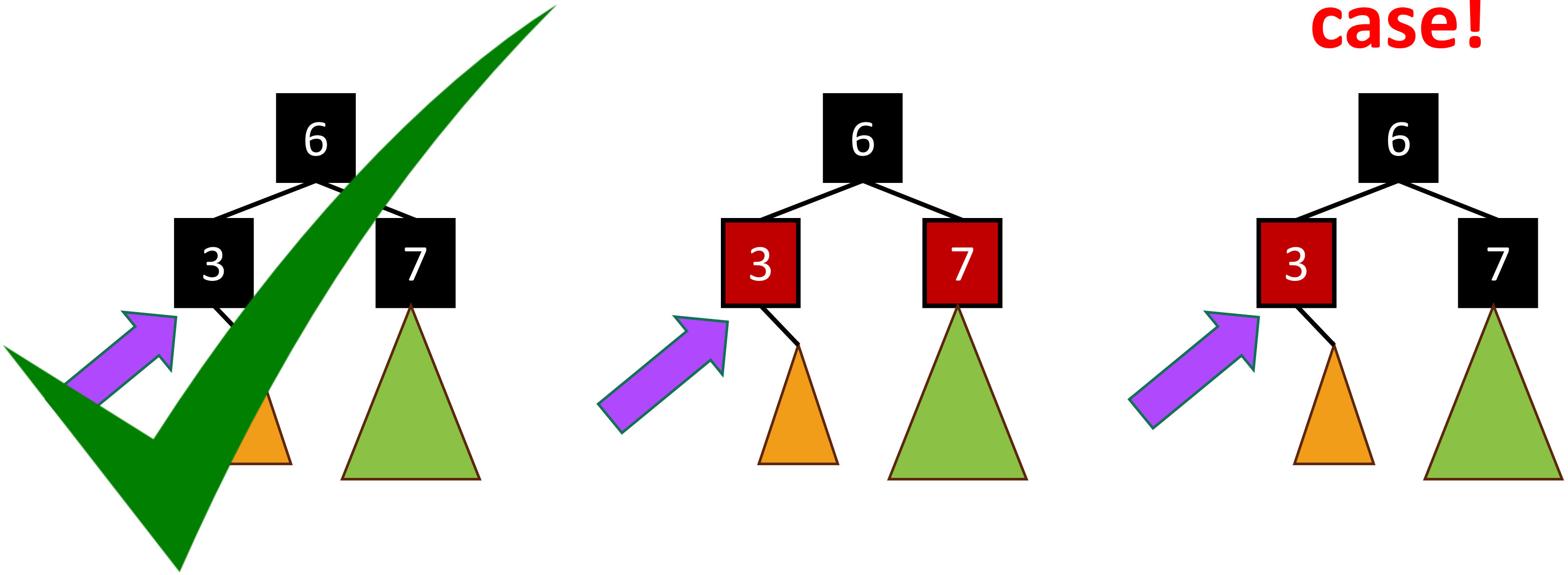



Need to know how to insert into trees that look like this…

Want to insert 6 here.

# INSERT: Many cases

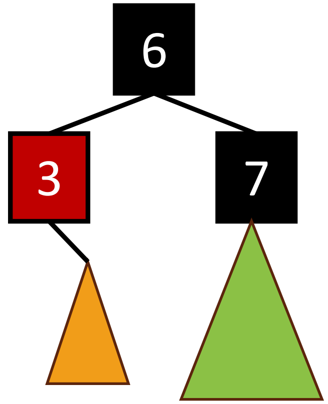
**That's this case!**



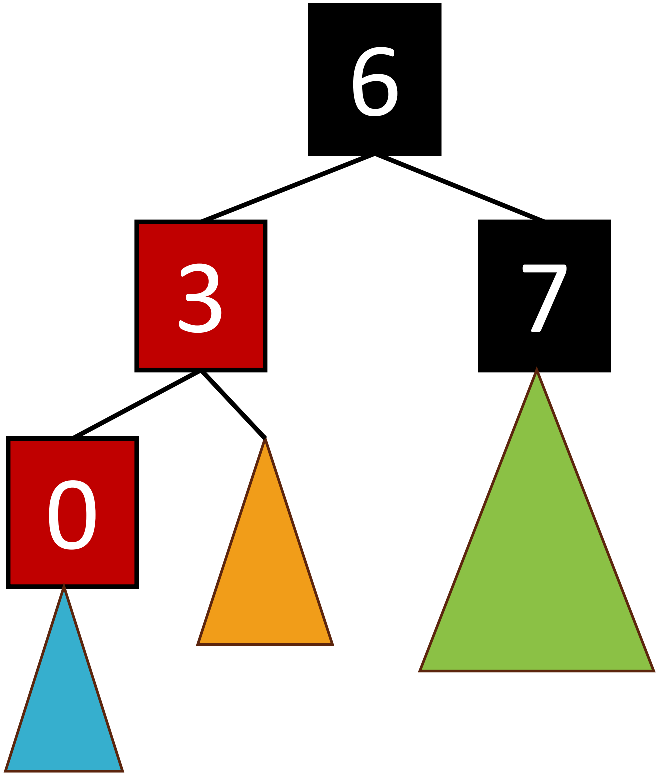

- Suppose we want to insert 0 **here**.
- There are 3 "important" cases for different colorings of the existing tree.

# INSERT: Case 3

- Make a new **red node.**
- Insert it as you would normally.
- Fix things up if needed.




What if it looks like this?

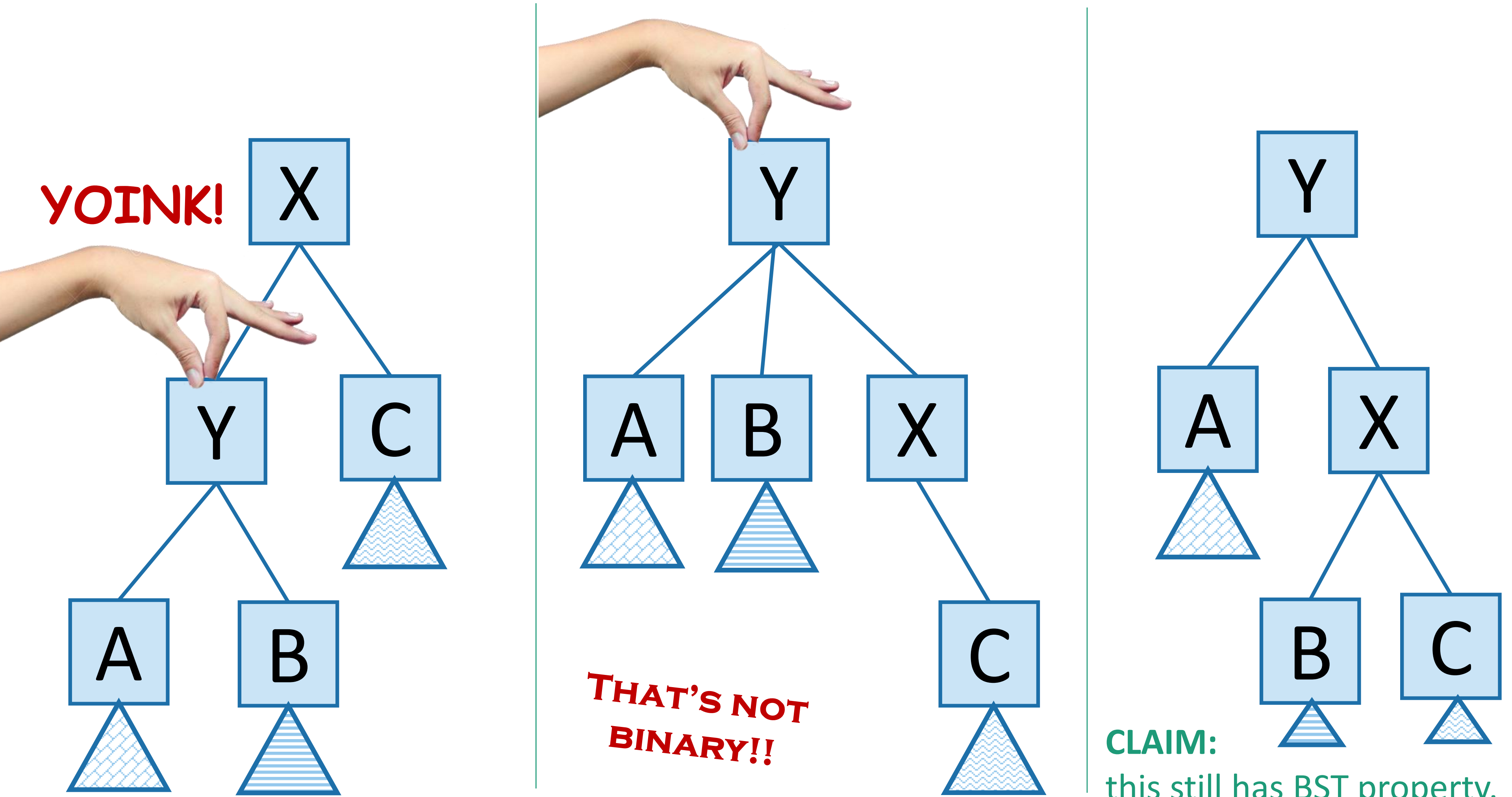



Example: Insert 0.

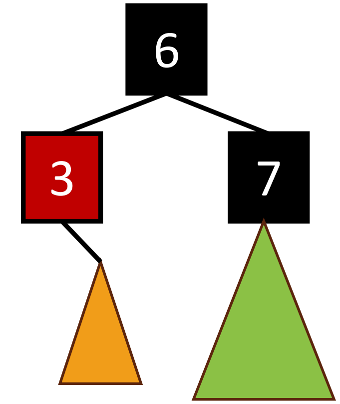
- Maybe with a subtree below it.

# Recall Rotations

- Maintain Binary Search Tree (BST) property, while moving stuff around.

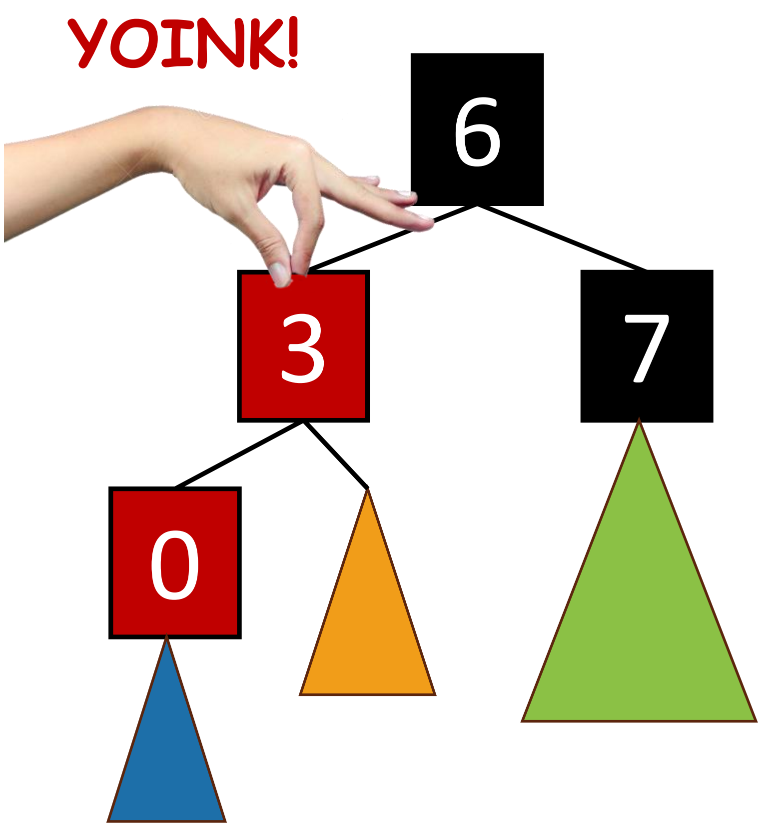


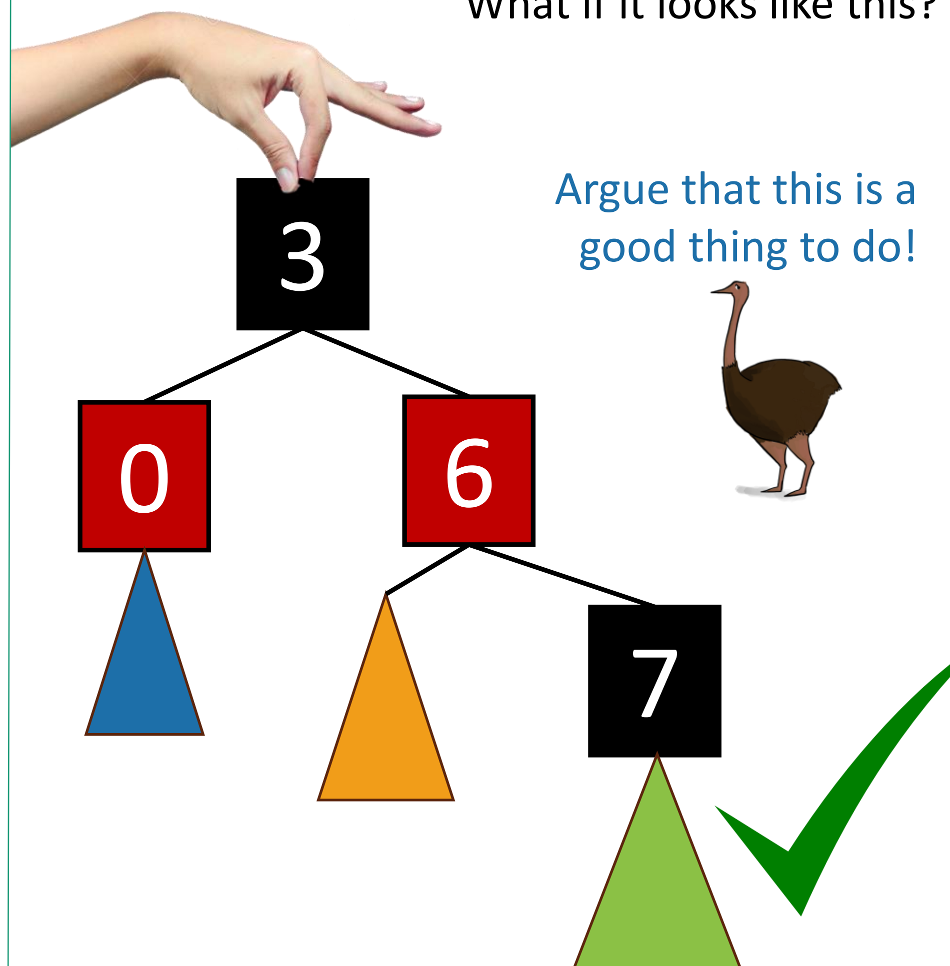
# Inserting into a Red-Black Tree



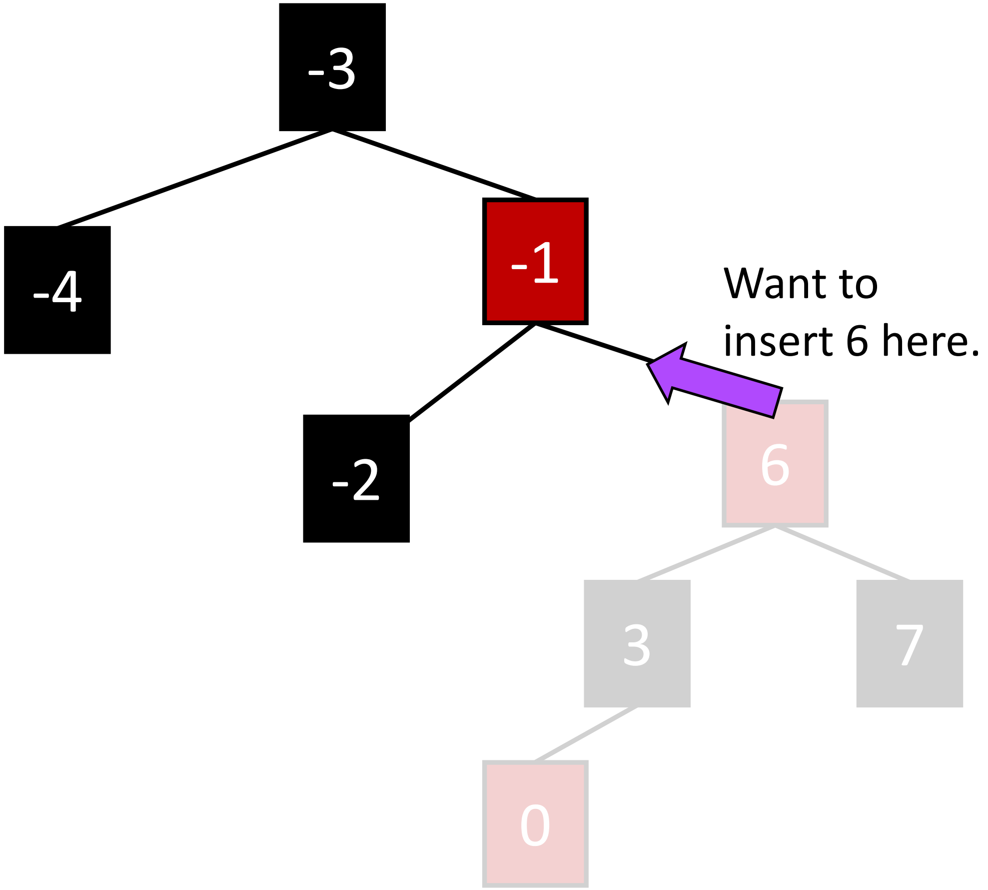

What if it looks like this?

- Make a new **red node**.
- Insert it as you would normally.
- Fix things up if needed.

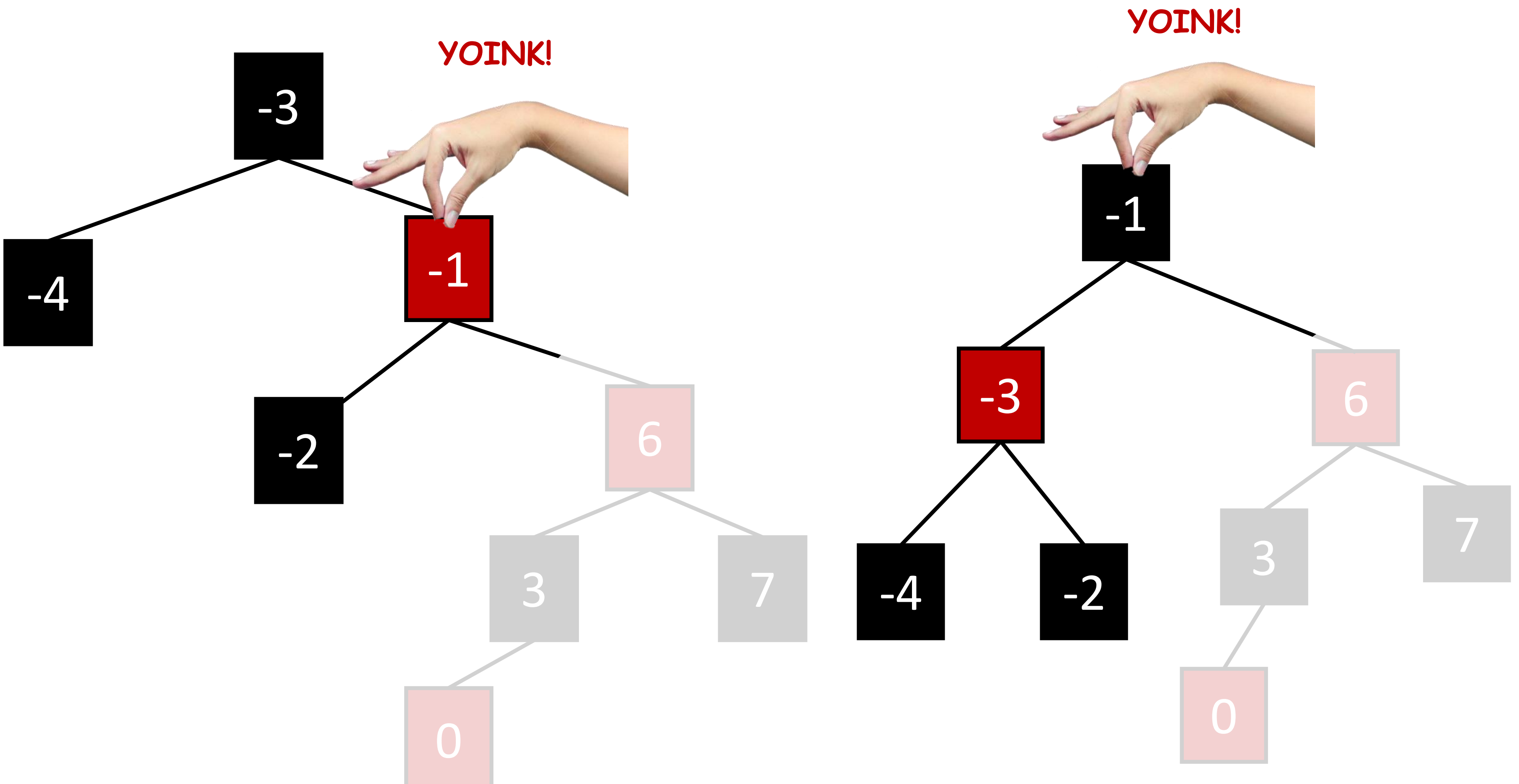


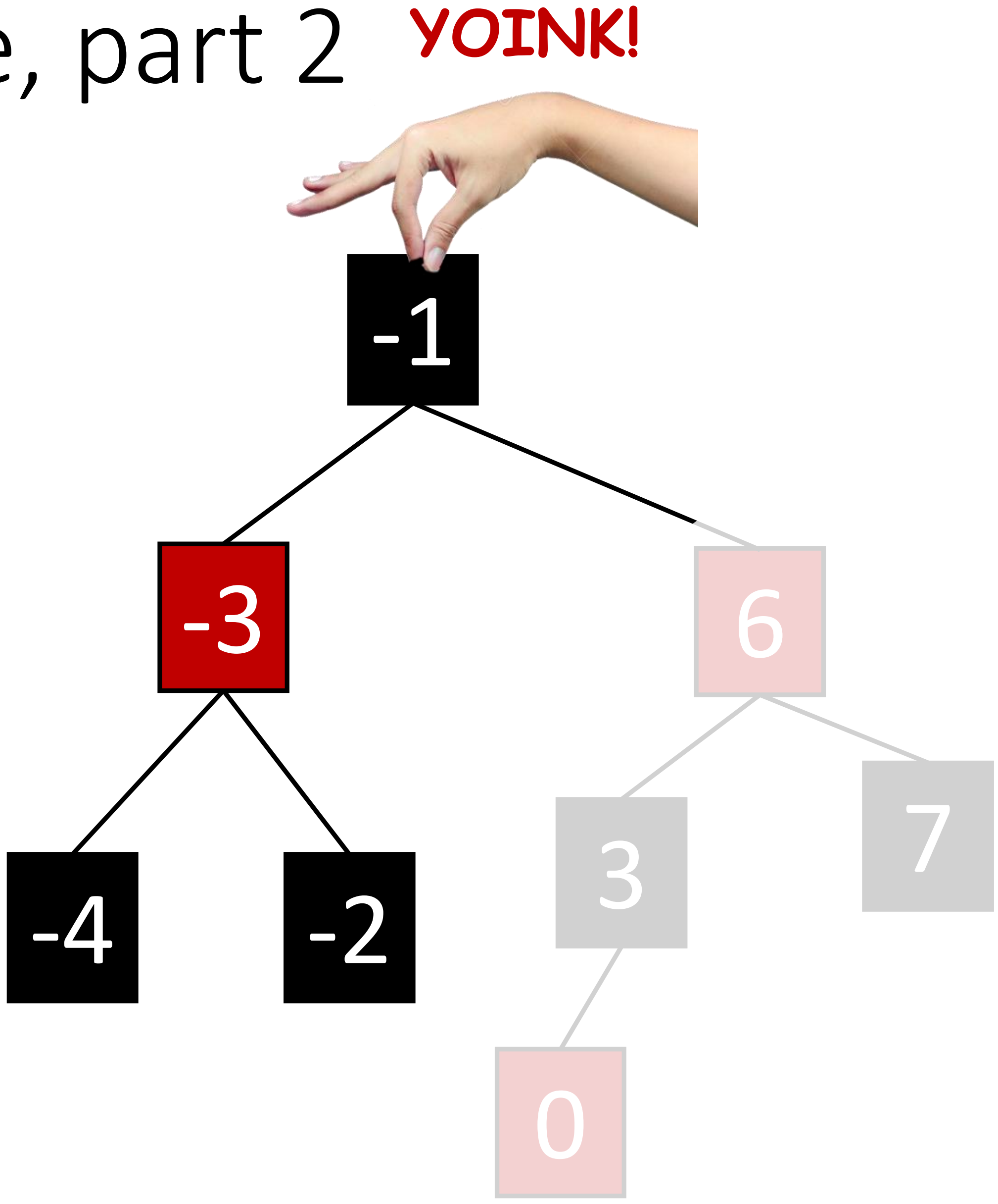
# Example, part 2



# Example, part 2

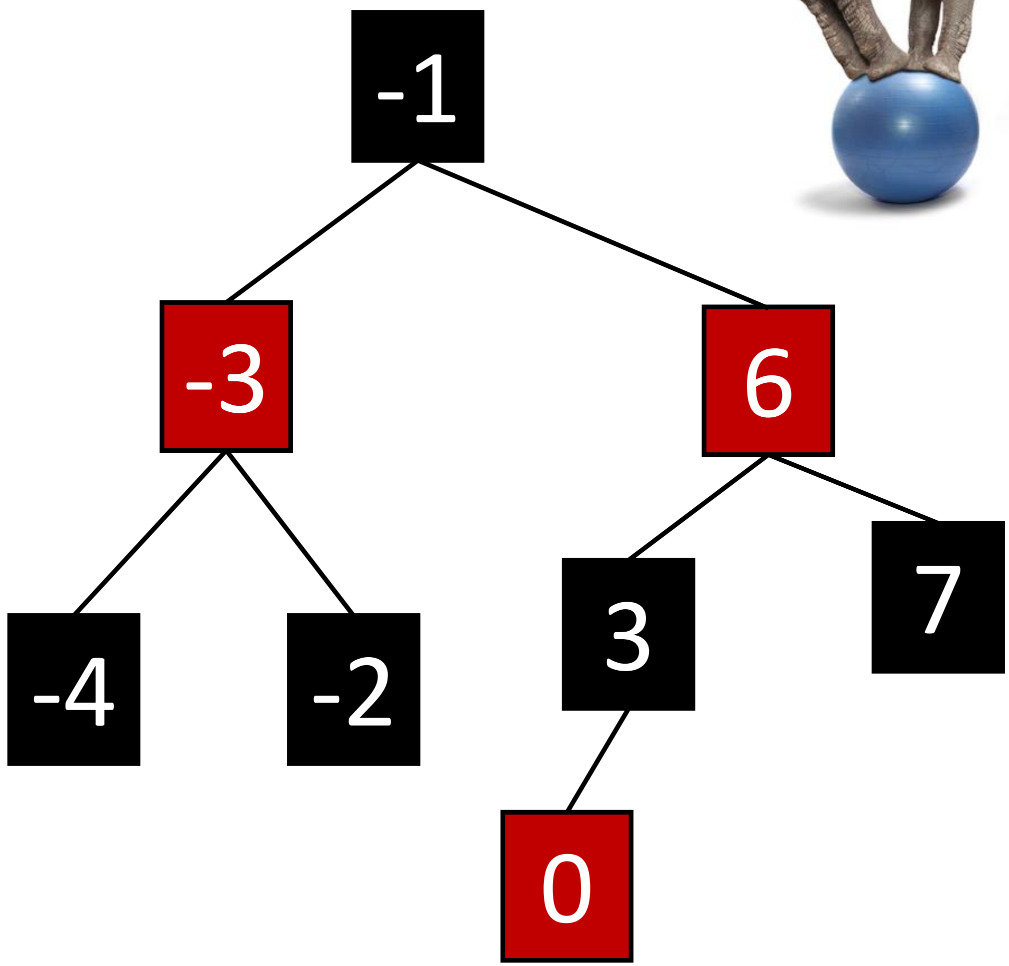


# Example, part 2

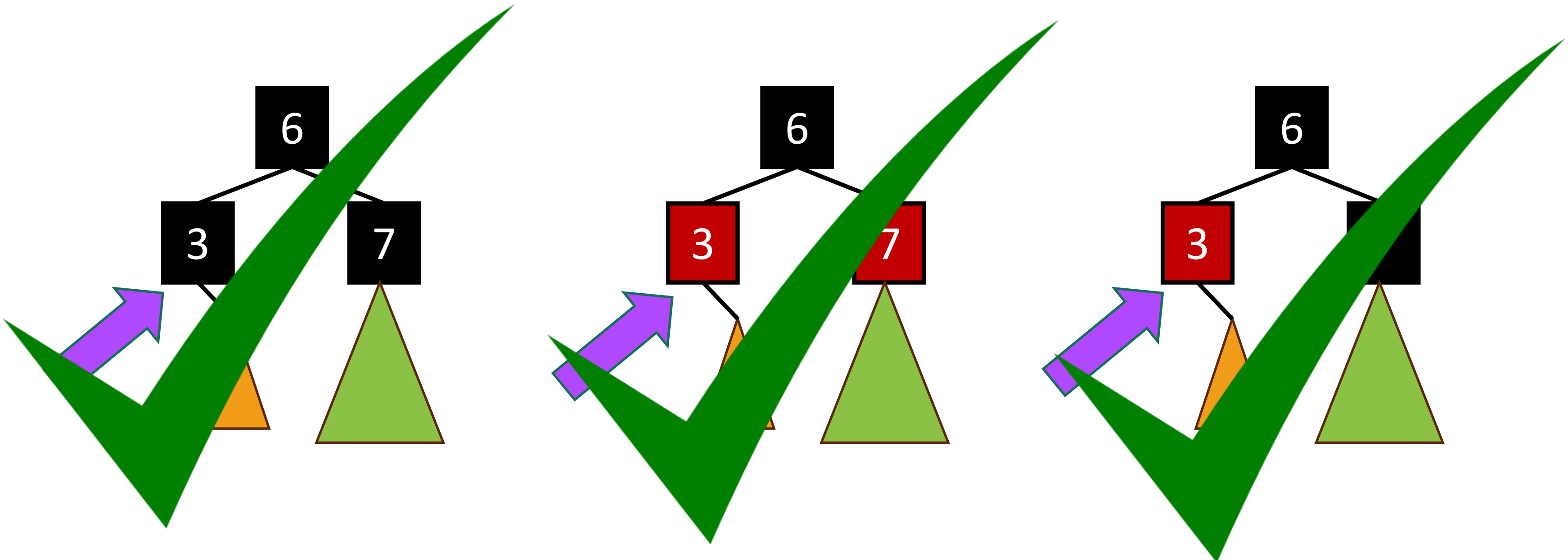


# Example, part 2

**_TA-DA!_**



# Many cases




- Suppose we want to insert 0 **here**.
- There are 3 “important” cases for different colorings of the existing tree.

# That’s a lot of cases!

- You are **not responsible** for the details of Red-Black Trees. (For this class)
- You should know:
  - What are the properties of an RB tree?
  - And (more important) why does that guarantee that they are balanced?

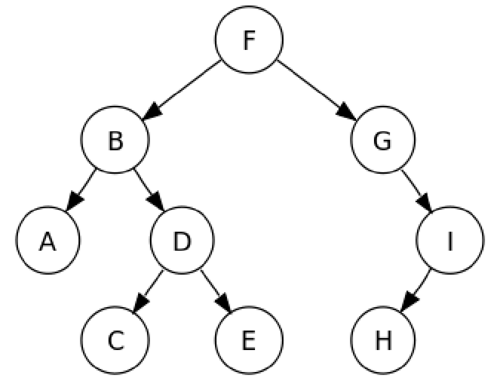
# What have we learned?

- Red-Black Trees always have height at most 2log(n+1).
- As with general Binary Search Trees, all operations are O(height)
- So all operations with RBTrees are O(log(n)).

# Conclusion: The best of both worlds

| | Sorted Arrays | Linked Lists | Binary Search Trees* |
|---|---|---|---|
| **Search** | O(log(n)) | O(n) | O(log(n)) |
| **Delete** | O(n) | O(n) | O(log(n)) |
| **Insert** | O(n) | O(1) | O(log(n)) |

# Today

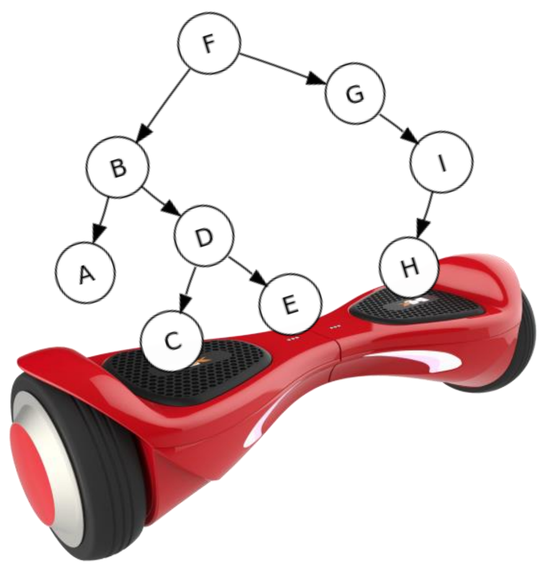



- Begin a brief foray into data structures!
  - See CS 166 for more!
- Binary search trees
  - You may remember these from CS 106B
  - They are better when they’re balanced.

this will lead us to…

- Self-Balancing Binary Search Trees
  - **Red-Black** trees.

Recap



# Recap

- Balanced binary trees are the best of both worlds!
- But we need to keep them balanced.
- **Red-Black** **Trees** do that for us.
  - We get O(log(n))-time INSERT/DELETE/SEARCH
  - Clever idea: have a proxy for balance

# Next time

- **Graph!**