Lecture 2

Karatsuba integer multiplication

InsertionSort, and MergeSort

Last time

Philosophy

- Algorithms are awesome!
- Our motivating questions:
  - Does it work?
  - Is it fast?
  - Can I do better?

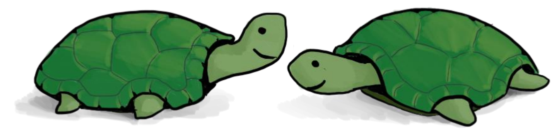
Technical content

- Karatsuba integer multiplication
- Example of “Divide and Conquer”
- Not-so-rigorous analysis

Cast

Plucky the pedantic penguin

Lucky the lackadaisical lemur



Think-Pair-Share Terrapins

Ollie the over-achieving ostrich

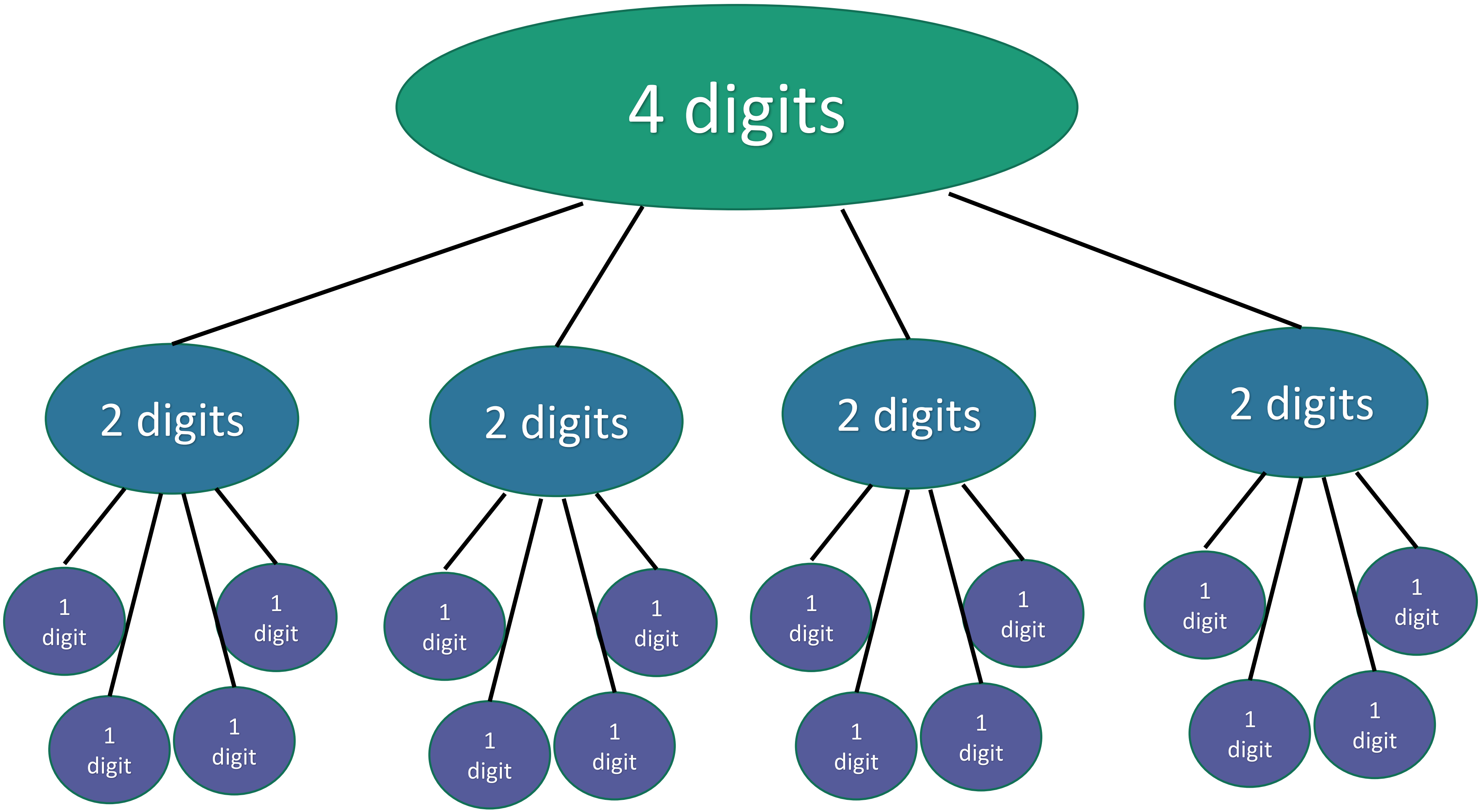
Siggi the studious stork

The Plan

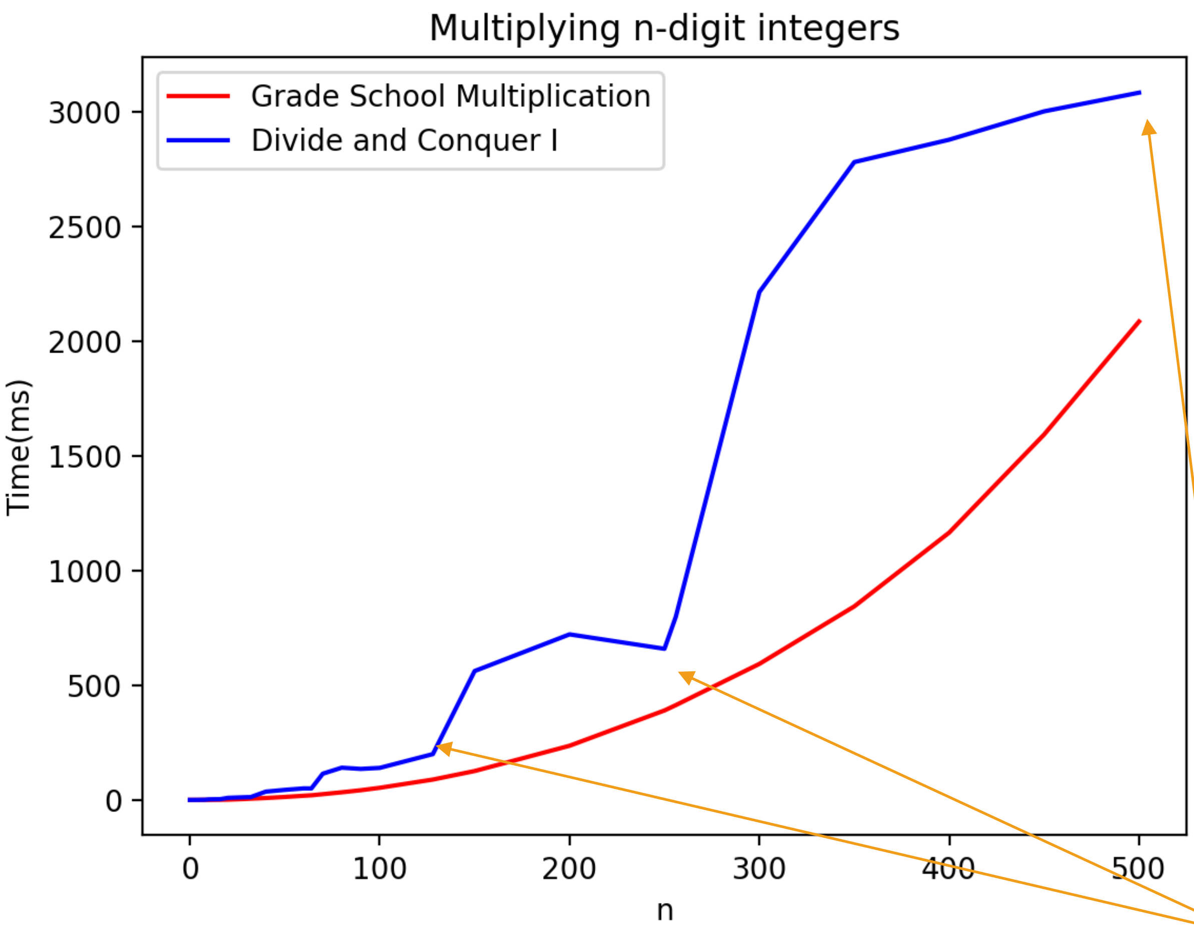
- Karatsuba integer multiplication
- InsertionSort
  - Does it work?
  - Is it fast?
- MergeSort
  - Does it work?
  - Is it fast?

Recursion Tree

16 one-digit multiplies!



1. Try it.



Conjectures about running time?

Doesn't look too good but hard to tell…

Maybe one implementation is slicker than the other?

Maybe if we were to run it to n=10000, things would look different.

Something funny is happening at powers of 2…

2. Try to understand the running time analytically

- ~~Proof by meta-reasoning:~~

~~It must be faster than the grade school algorithm, because we are learning it in an algorithms class.~~

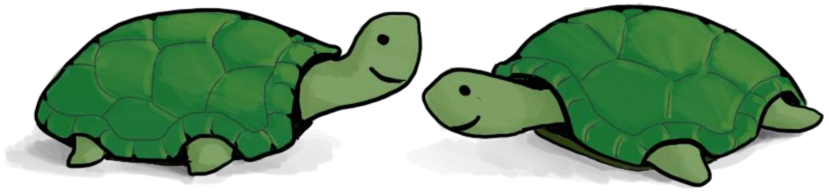
Not sound logic!

Plucky the Pedantic Penguin

2. Try to understand the running time analytically

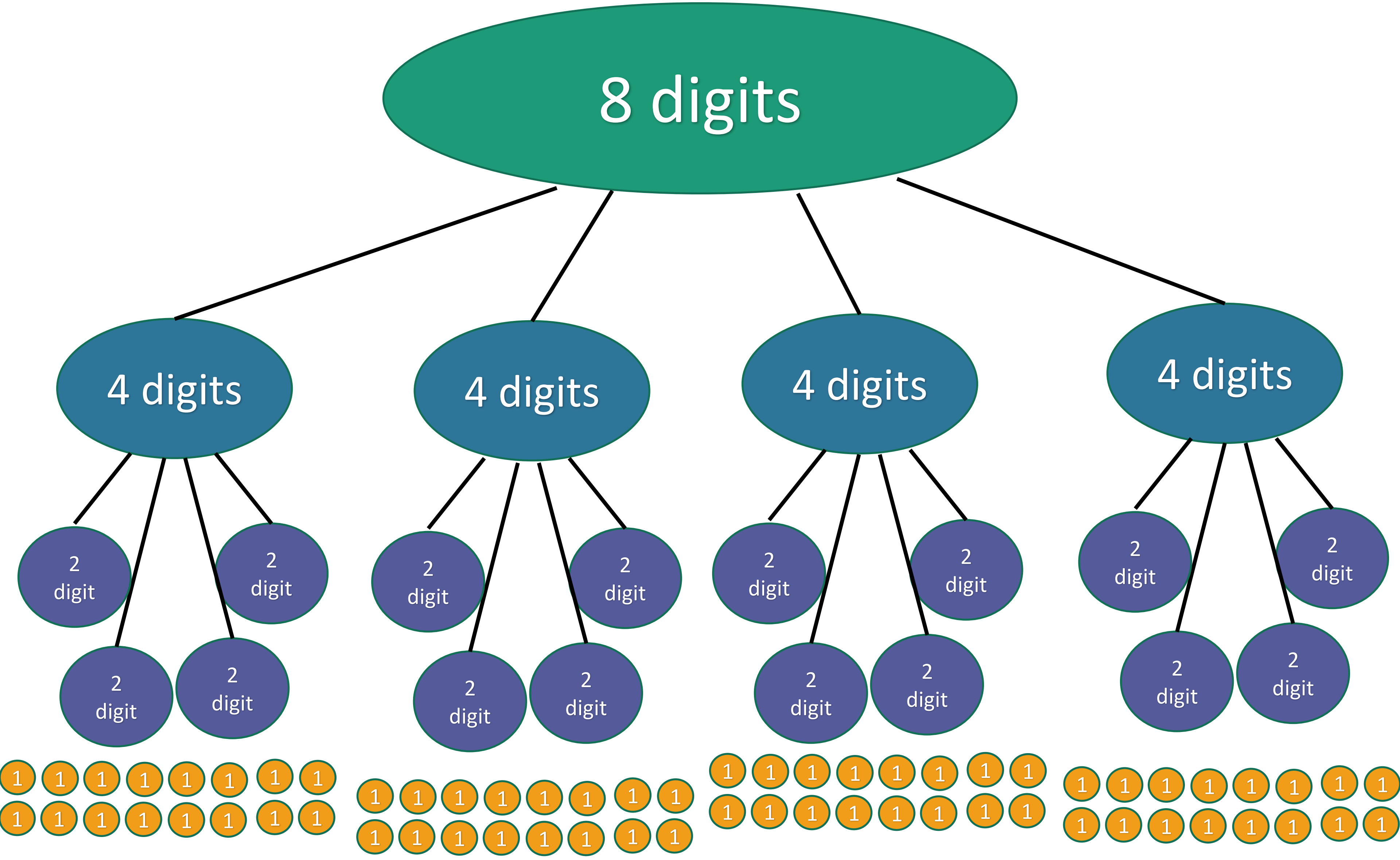
Think-Pair-Share:

- We saw that multiplying 4-digit numbers resulted in 16 one-digit multiplications.
- How about multiplying 8-digit numbers?
- What do you think about n-digit numbers?



Recursion Tree

64 one-digit multiplies!



2. Try to understand the running time analytically

Claim:

The running time of this algorithm is AT LEAST n^2 operations.

There are n^2 1-digit problems

1 problem
of size n

4 problems
of size n/2

…

4^t problems
of size $n/2^t$

Note: this is just a cartoon – I'm not going to draw all 4^t circles!

…

$\underline{n^2}$ problems
of size 1

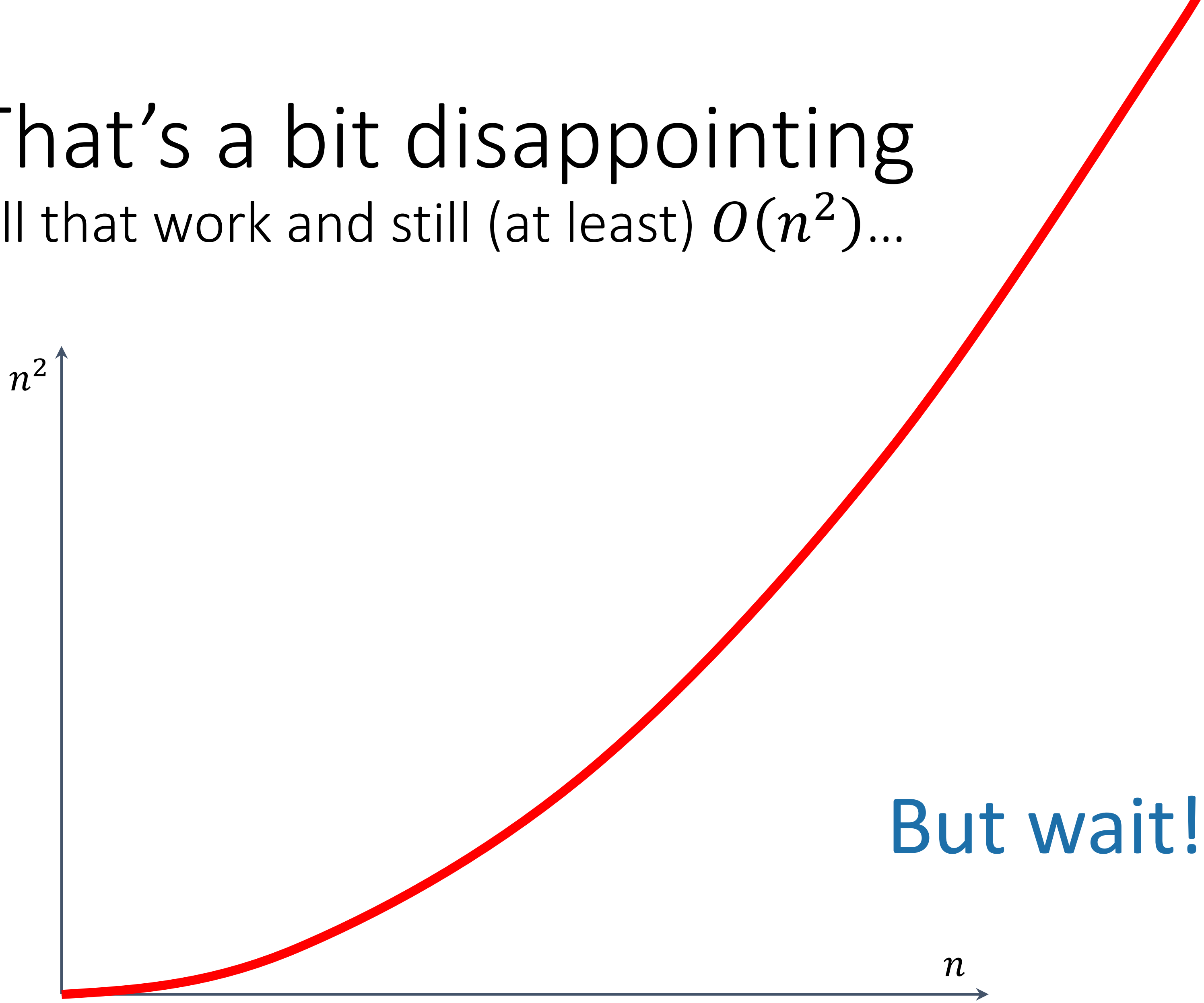
- If you cut n in half $\log_2(n)$ times, you get down to 1.
- So at level
  $t = \log_2(n)$
  we get…

$$4^{\log_2 n} = n^{\log_2 4} = n^2$$

problems of size 1.

That’s a bit disappointing

All that work and still (at least) $O(n^2)$…



But wait!!

Divide and conquer can actually make progress

- Karatsuba figured out how to do this better!

$$xy = (a \cdot 10^{n/2} + b)(c \cdot 10^{n/2} + d)$$
$$= ac \cdot 10^{n} + (ad + bc)10^{n/2} + bd$$

Need these three things

- If only we could recurse on three things instead of four…

Karatsuba integer multiplication

- Recursively compute these THREE things:
  - ac
  - bd
  - (a+b)(c+d)

Subtract these off

get this

(a+b)(c+d) = ac + bd + bc + ad

- Assemble the product:

$$xy = (a \cdot 10^{n/2} + b)(c \cdot 10^{n/2} + d)$$
$$= ac \cdot 10^{n} + (ad + bc)10^{n/2} + bd$$

How would this work?

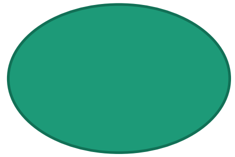
x,y are n-digit numbers

Multiply(x, y):

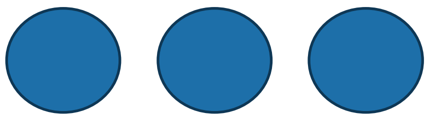
- **If** n=1:
  - **Return** xy
- Write $x = a\,10^{\frac{n}{2}} + b$ and $y = c\,10^{\frac{n}{2}} + d$
- ac = **Multiply**(a, c)
- bd = **Multiply**(b, d)
- z = **Multiply**(a+b, c+d)
- xy = ac 10^n + (z – ac - bd) $10^{n/2}$ + bd
- Return xy

a, b, c, d are n/2-digit numbers

What’s the running time?

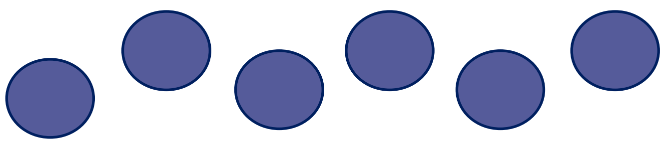


1 problem
of size n



3 problems
of size n/2

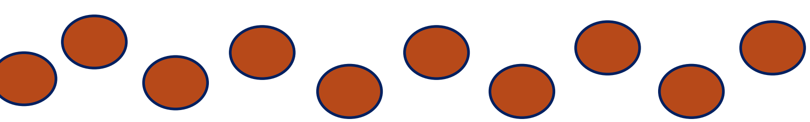
…



3^t problems
of size $n/2^t$

Note: this is just a cartoon – I’m not going to draw all 3^t circles!

…



$\underline{n^{1.6}}$ problems
of size 1

- If you cut n in half $\log_2(n)$ times, you get down to 1.
- So at level
  $t = \log_2(n)$
  we get…

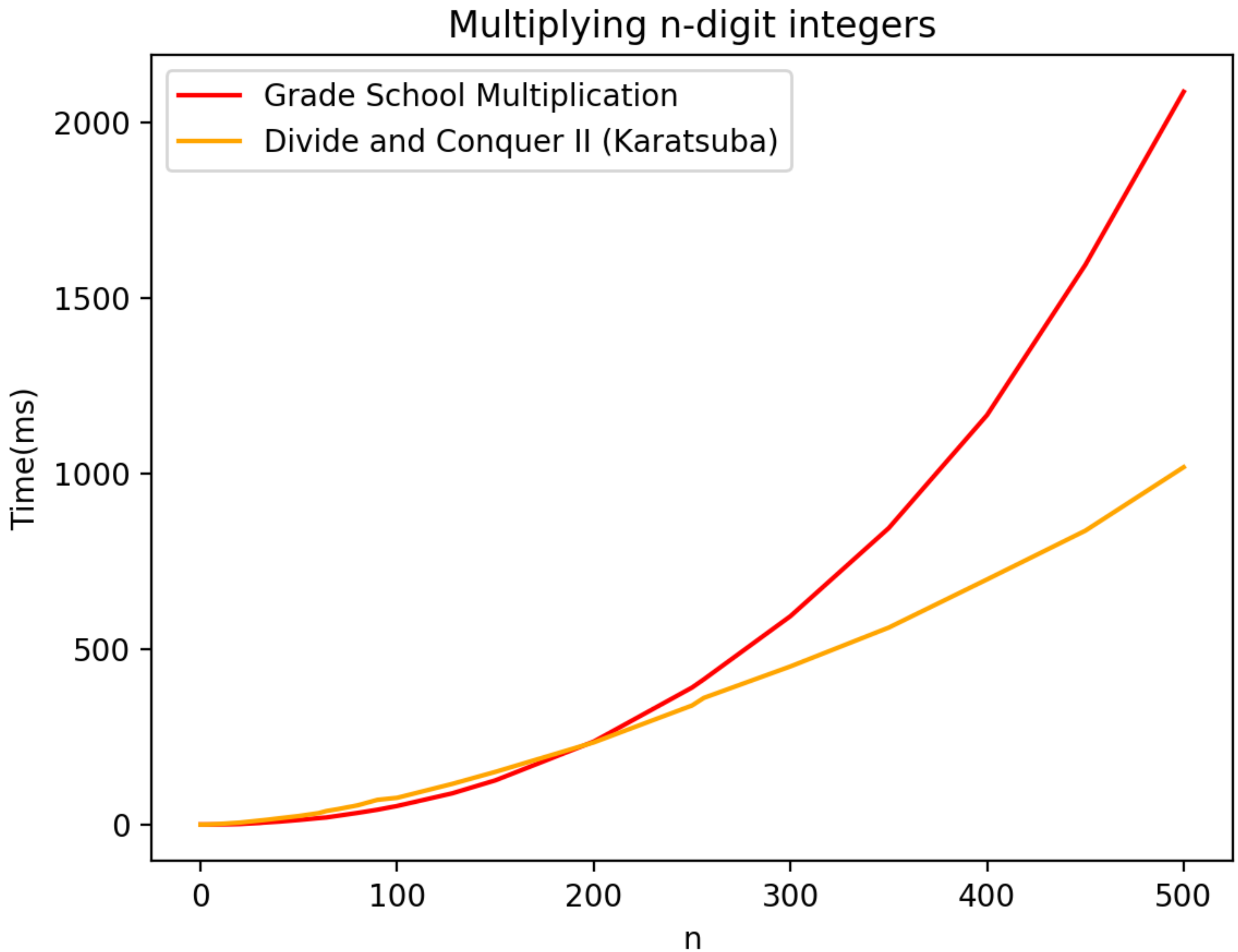
$$3^{\log_2 n} = n^{\log_2 3} \approx n^{1.6}$$

problems of size 1.

We aren’t accounting for the work at the higher levels! But we’ll see later that this turns out to be okay.

This is much better!

We can even see it in real life!



Can we do better?

- **Toom-Cook** (1963): instead of breaking into three n/2-sized problems, break into five n/3-sized problems.
  - Runs in time $O(n^{1.465})$

Try to figure out how to break up an n-sized problem into five n/3-sized problems! **(Hint: start with nine n/3-sized problems).**

Given that you can break an n-sized problem into five n/3-sized problems, where does the 1.465 come from?

- **Schönhage–Strassen** (1971):
  - Runs in time $O(n \log(n) \log\log(n))$
- **Furer** (2007)
  - Runs in time $n \log(n) \cdot 2^{O(\log^*(n))}$
- **Harvey and van der Hoeven** (2019)
  - Runs in time $O(n \log(n))$

[This is just for fun, you don't need to know these algorithms!]

Today

- We are going to ask:
  - Does it work?
  - Is it fast?

- We'll start to see how to answer these by looking at some examples of sorting algorithms.
  - InsertionSort
  - MergeSort

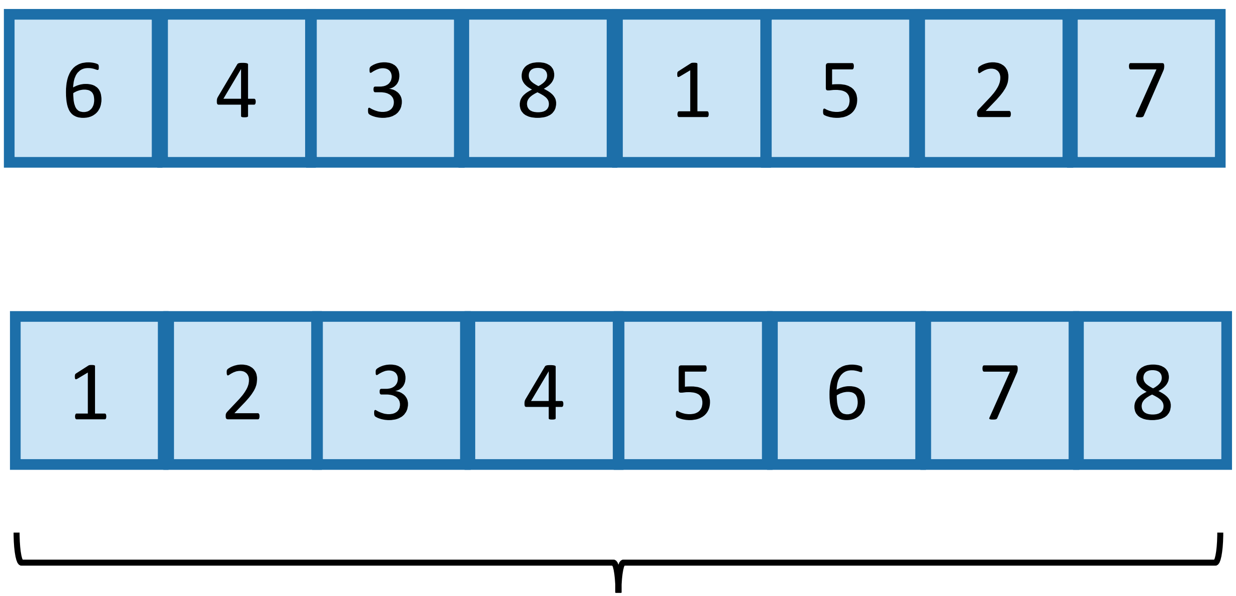
SortingHatSort not discussed

The Plan

- Karatsuba integer multiplication
- InsertionSort
  - Does it work?
  - Is it fast?
- MergeSort
  - Does it work?
  - Is it fast?

Sorting

- Important primitive
- For today, we'll pretend all elements are distinct.



Length of the list is n

InsertionSort

example

6	4	3	8	5

Start by moving A[1] toward the beginning of the list until you find something smaller (or can't go any further):

6	4	3	8	5
4	6	3	8	5

Then move A[2]:

4	6	3	8	5
3	4	6	8	5

Then move A[3]:

3	4	6	8	5
3	4	6	8	5

Then move A[4]:

3	4	6	8	5
3	4	5	6	8

Then we are done!

Insertion Sort

1. Does it work?
2. Is it fast?

What does that mean???

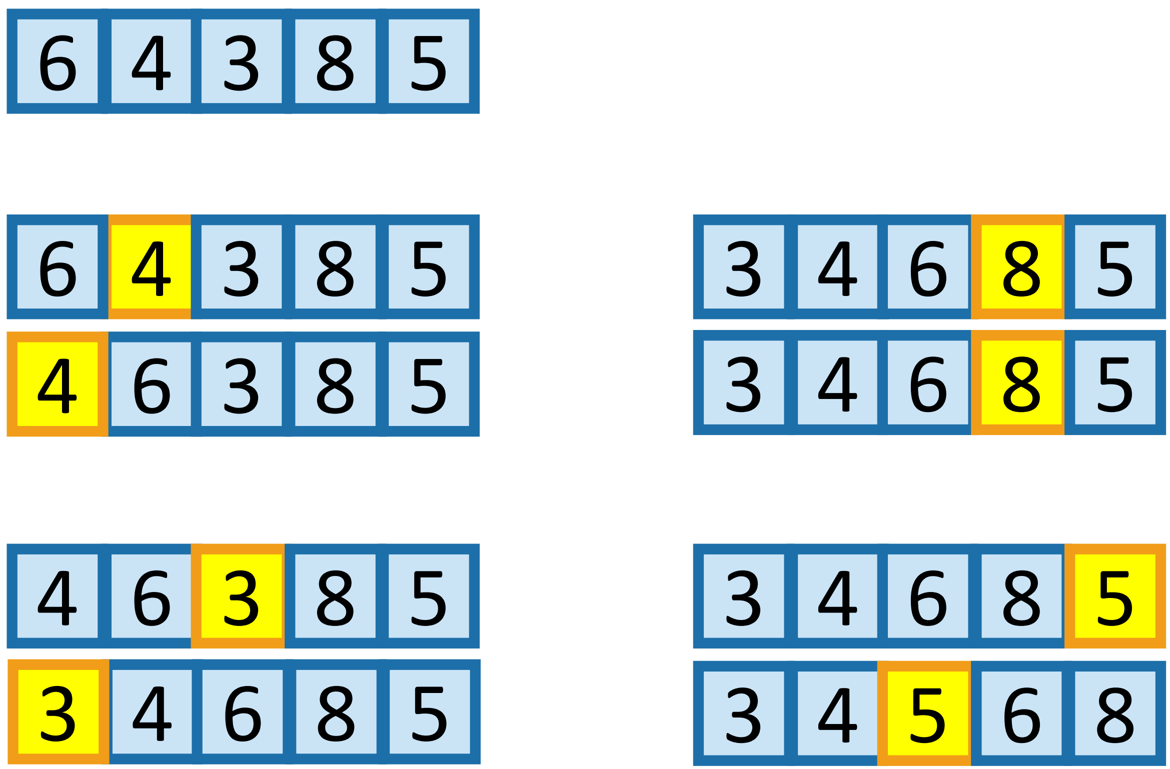
Plucky the
Pedantic Penguin

The Plan

- Karatsuba integer multiplication
- InsertionSort
  - Does it work?
  - Is it fast?
- MergeSort
  - Does it work?
  - Is it fast?

Claim: InsertionSort "works"

- "Proof:" It just worked in this example:



Sorted!

Claim: InsertionSort "works"

```
A = [1,2,3,4,5,6,7,8,9,10]
for trial in range(100):
    shuffle(A)
    InsertionSort(A)
    if is_sorted(A):
        print('YES IT IS SORTED!')
```

```
YES IT IS SORTED!
YES IT IS SORTED!
YES IT IS SORTED!
YES IT IS SORTED!
YES IT IS SORTED!
YES IT IS SORTED!
YES IT IS SORTED!
YES IT IS SORTED!
YES IT IS SORTED!
YES IT IS SORTED!
YES IT IS SORTED!
YES IT IS SORTED!
YES IT IS SORTED!
YES IT IS SORTED!
YES IT IS SORTED!
YES IT IS SORTED!
YES IT IS SORTED!
YES IT IS SORTED!
YES IT IS SORTED!
YES IT IS SORTED!
YES IT IS SORTED!
YES IT IS SORTED!
YES IT IS SORTED!
YES IT IS SORTED!
YES IT IS SORTED!
YES IT IS SORTED!
YES IT IS SORTED!
YES IT IS SORTED!
YES IT IS SORTED!
YES IT IS SORTED!
YES IT IS SORTED!
YES IT IS SORTED!
YES IT IS SORTED!
YES IT IS SORTED!
YES IT IS SORTED!
YES IT IS SORTED!
YES IT IS SORTED!
YES IT IS SORTED!
YES IT IS SORTED!
YES IT IS SORTED!
YES IT IS SORTED!
YES IT IS SORTED!
YES IT IS SORTED!
YES IT IS SORTED!
YES IT IS SORTED!
YES IT IS SORTED!
YES IT IS SORTED!
YES IT IS SORTED!
YES IT IS SORTED!
YES IT IS SORTED!
YES IT IS SORTED!
YES IT IS SORTED!
YES IT IS SORTED!
YES IT IS SORTED!
YES IT IS SORTED!
YES IT IS SORTED!
YES IT IS SORTED!
YES IT IS SORTED!
YES IT IS SORTED!
YES IT IS SORTED!
YES IT IS SORTED!
YES IT IS SORTED!
YES IT IS SORTED!
YES IT IS SORTED!
YES IT IS SORTED!
YES IT IS SORTED!
YES IT IS SORTED!
YES IT IS SORTED!
YES IT IS SORTED!
YES IT IS SORTED!
YES IT IS SORTED!
YES IT IS SORTED!
YES IT IS SORTED!
YES IT IS SORTED!
YES IT IS SORTED!
YES IT IS SORTED!
YES IT IS SORTED!
YES IT IS SORTED!
YES IT IS SORTED!
YES IT IS SORTED!
YES IT IS SORTED!
YES IT IS SORTED!
YES IT IS SORTED!
YES IT IS SORTED!
YES IT IS SORTED!
YES IT IS SORTED!
YES IT IS SORTED!
YES IT IS SORTED!
YES IT IS SORTED!
YES IT IS SORTED!
YES IT IS SORTED!
YES IT IS SORTED!
YES IT IS SORTED!
YES IT IS SORTED!
YES IT IS SORTED!
YES IT IS SORTED!
YES IT IS SORTED!
YES IT IS SORTED!
YES IT IS SORTED!
YES IT IS SORTED!
YES IT IS SORTED!
YES IT IS SORTED!
YES IT IS SORTED!
YES IT IS SORTED!
YES IT IS SORTED!
YES IT IS SORTED!
YES IT IS SORTED!
YES IT IS SORTED!
YES IT IS SORTED!
YES IT IS SORTED!
YES IT IS SORTED!
YES IT IS SORTED!
YES IT IS SORTED!
YES IT IS SORTED!
YES IT IS SORTED!
YES IT IS SORTED!
YES IT IS SORTED!
```

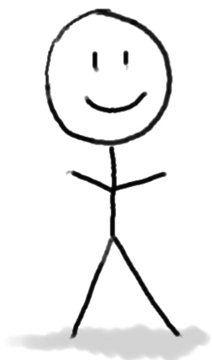
What does it mean to “work”?

- Is it enough to be correct on only one input?
- Is it enough to be correct on most inputs?

- In this class, we will use **worst-case analysis**:
  - An algorithm must be correct on **all possible** inputs.
  - The running time of an algorithm is the worst possible running time over all inputs.

Worst-case analysis

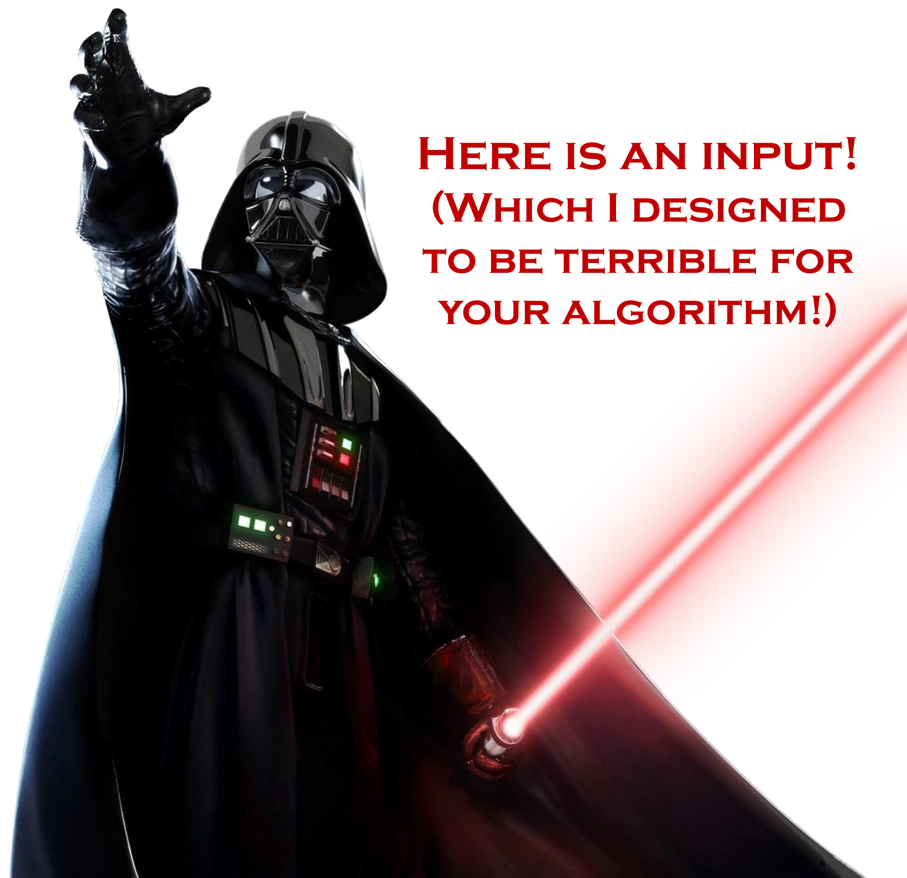
Think of it like a game:



Here is my algorithm!

```
Algorithm:
  Do the thing
  Do the stuff
  Return the answer
```

Algorithm designer



HERE IS AN INPUT!
(WHICH I DESIGNED
TO BE TERRIBLE FOR
YOUR ALGORITHM!)

Insertion Sort

1. Does it work?
2. Is it fast?

- Okay, so it's pretty obvious that it works.

- HOWEVER! In the future it won't be so obvious, so let's take some time now to see how we would prove this rigorously.

Why does this work?

- Say you have a sorted list, $\boxed{3}\boxed{4}\boxed{6}\boxed{8}$, and another element $\boxed{5}$.
- Insert $\boxed{5}$ right after the largest thing that's still smaller than $\boxed{5}$. (Aka, right after $\boxed{4}$).
- Then you get a sorted list: $\boxed{3}\boxed{4}\boxed{5}\boxed{6}\boxed{8}$

So just use this logic at every step.

6	4	3	8	5

The first element, [6], makes up a sorted list.

4	6	3	8	5

So correctly inserting 4 into the list [6] means that [4,6] becomes a sorted list.

4	6	3	8	5

The first two elements, [4,6], make up a sorted list.

3	4	6	8	5

So correctly inserting 3 into the list [4,6] means that [3,4,6] becomes a sorted list.

3	4	6	8	5

The first three elements, [3,4,6], make up a sorted list.

3	4	6	8	5

So correctly inserting 8 into the list [3,4,6] means that [3,4,6,8] becomes a sorted list.

3	4	6	8	5

The first four elements, [3,4,6,8], make up a sorted list.

3	4	5	6	8

So correctly inserting 5 into the list [3,4,6,8] means that [3,4,5,6,8] becomes a sorted list.

YAY WE ARE DONE!

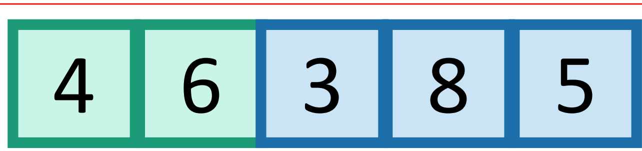
This sounds like a job for…

Proof By Induction!

Outline of a proof by induction

Let A be a list of length n

- Inductive Hypothesis:
  - A[:i+1] is sorted at the end of the i^{th} iteration (of the outer loop).
- Base case (i=0):
  - `A[:1]` is sorted at the end of the 0'th iteration. ✓
- Inductive step:
  - For any 0 < k < n, if the inductive hypothesis holds for i=k-1, then it holds for i=k.
  - Aka, if A[:k] is sorted at step k-1, then A[:k+1] is sorted at step k
- Conclusion:
  - The inductive hypothesis holds for i = 0, 1, …, n-1.
  - In particular, it holds for i=n-1.
  - At the end of the n-1'st iteration (aka, at the end of the algorithm), `A[:n] = A` is sorted.
  - That's what we wanted! ✓



The first two elements, [4,6], make up a sorted list.

So correctly inserting 3 into the list [4,6] means that [3,4,6] becomes a sorted list.

This was iteration i=2.

What have we learned?

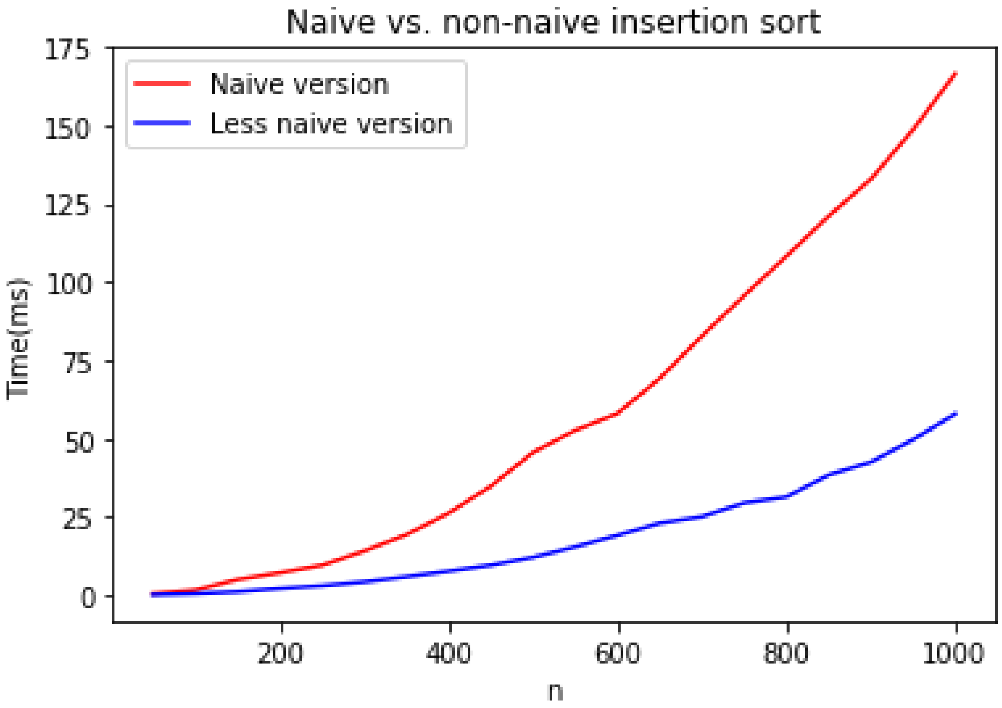
- In this class we will use worst-case analysis:
  - We assume that a “bad guy” comes up with a worst-case input for our algorithm, and we measure performance on that worst-case input.
- With this definition, InsertionSort “works”
  - Proof by induction!

The Plan

- Karatsuba integer multiplication
- InsertionSort
  - Does it work?
  - Is it fast?
- MergeSort
  - Does it work?
  - Is it fast?

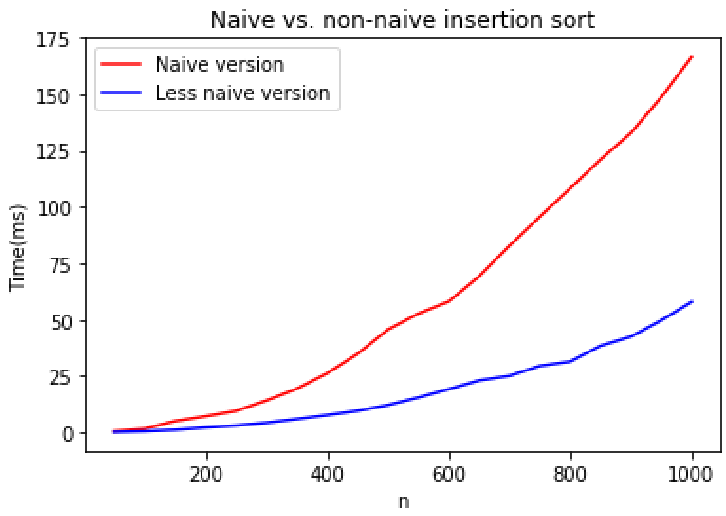
How fast is InsertionSort?

- This fast:



Issues with this answer?

- The “same” algorithm can be slower or faster depending on the implementations.
- It can also be slower or faster depending on the hardware that we run it on.



With this answer, “running time” isn’t even well-defined!

How fast is InsertionSort?

- Let’s count the number of operations!

```
def InsertionSort(A):
    for i in range(1,len(A)):
        current = A[i]
        j = i-1
        while j >= 0 and A[j] > current:
            A[j+1] = A[j]
            j -= 1
        A[j+1] = current
```

By my count*…

- $2n^2 - n - 1$ variable assignments
- $2n^2 - n - 1$ increments/decrements
- $2n^2 - 4n + 1$ comparisons
- …

*Do not pay attention to these formulas, they do not matter.
Also not valid for bug bounty points.

Issues with this answer?

- It's very tedious!
- In order to use this to understand running time, I need to know how long each operation takes, plus a whole bunch of other stuff…

```
def InsertionSort(A):
    for i in range(1,len(A)):
        current = A[i]
        j = i-1
        while j >= 0 and A[j] > current:
            A[j+1] = A[j]
            j -= 1
        A[j+1] = current
```

Counting individual operations is a lot of work and doesn't seem very helpful!

Lucky the lackadaisical lemur

In this class we will use…

- **Big-Oh notation!**
- Gives us a meaningful way to talk about the running time of an algorithm, independent of programming language, computing platform, etc., without having to count all the operations.

Main idea:

Focus on how the runtime **scales** with n (the input size).

Some examples…

(Only pay attention to the largest function of n that appears.)

Number of operations	Asymptotic Running Time
$\frac{1}{10} \cdot n^2 + 100$	$O(n^2)$
$0.063 \cdot n^2 - .5\, n + 12.7$	$O(n^2)$
$100 \cdot n^{1.5} - 10^{10000}\sqrt{n}$	$O(n^{1.5})$
$11 \cdot n\log(n) + 1$	$O(n\log(n))$

We say this algorithm is "asymptotically faster" than the others.

Why is this a good idea?

- Suppose the running time of an algorithm is:

$$T(n) = 10n^2 + 3n + 7 \quad \text{ms}$$

This constant factor of 10 depends a lot on my computing platform…

These lower-order terms don't really matter as n gets large.

We're just left with the n^2 term! That's what's meaningful.

Pros and Cons of Asymptotic Analysis

Pros:

- Abstracts away from hardware- and language-specific issues.
- Makes algorithm analysis much more tractable.
- Allows us to meaningfully compare how algorithms will perform on large inputs.

Cons:

- Only makes sense if n is large (compared to the constant factors).

1000000000 n
is "better" than n^2 ?!?!

pronounced "big-oh of …" or sometimes "oh of …"

Informal definition for O(…)

- Let $T(n)$, $g(n)$ be functions of positive integers.
  - Think of $T(n)$ as a runtime: positive and increasing in n.

- We say "$T(n)$ is $O\big(g(n)\big)$" if:

for large enough n,

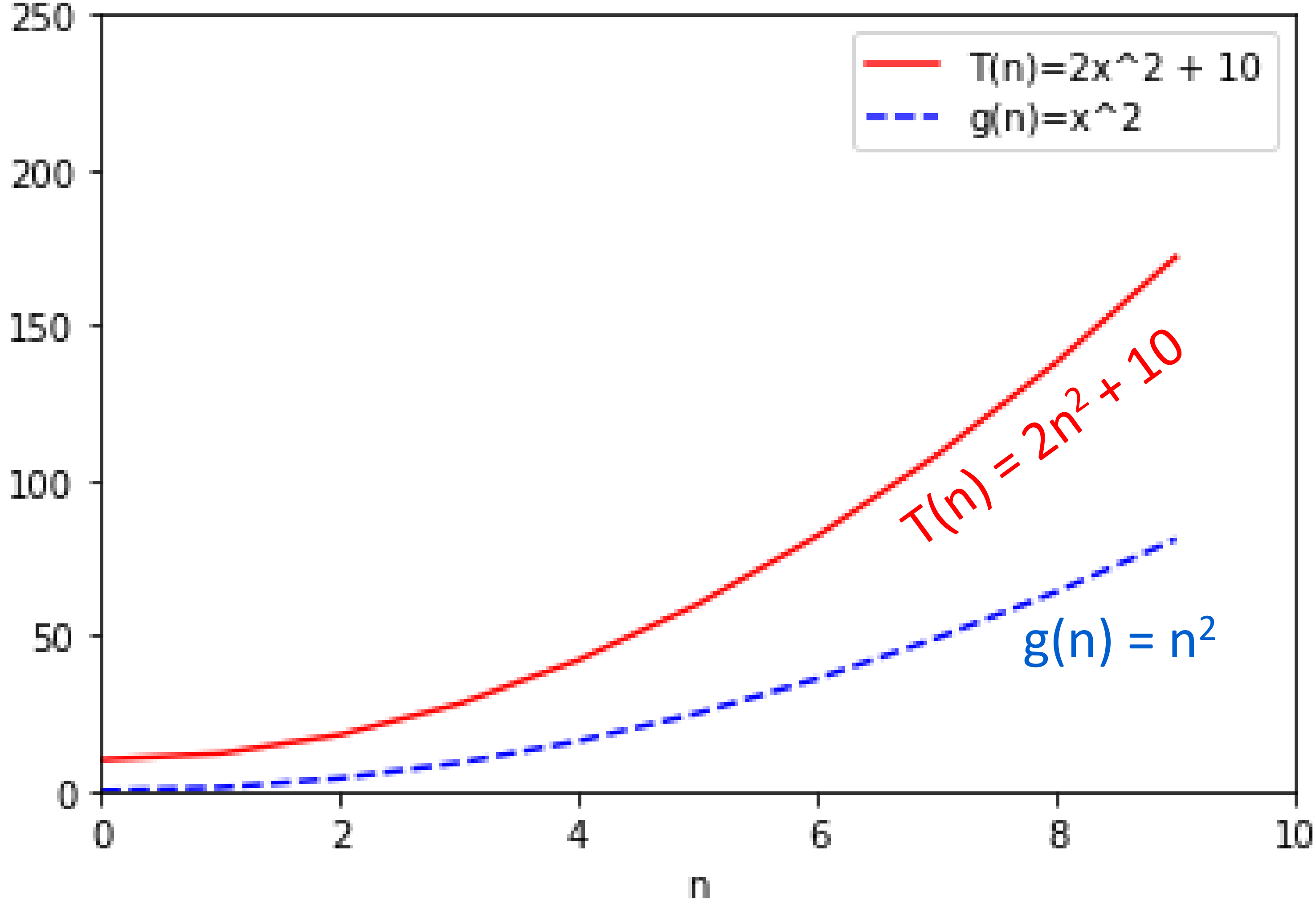
$T(n)$ is at most some constant multiple of $g(n)$.

Here, "constant" means "some number that doesn't depend on n."

Example

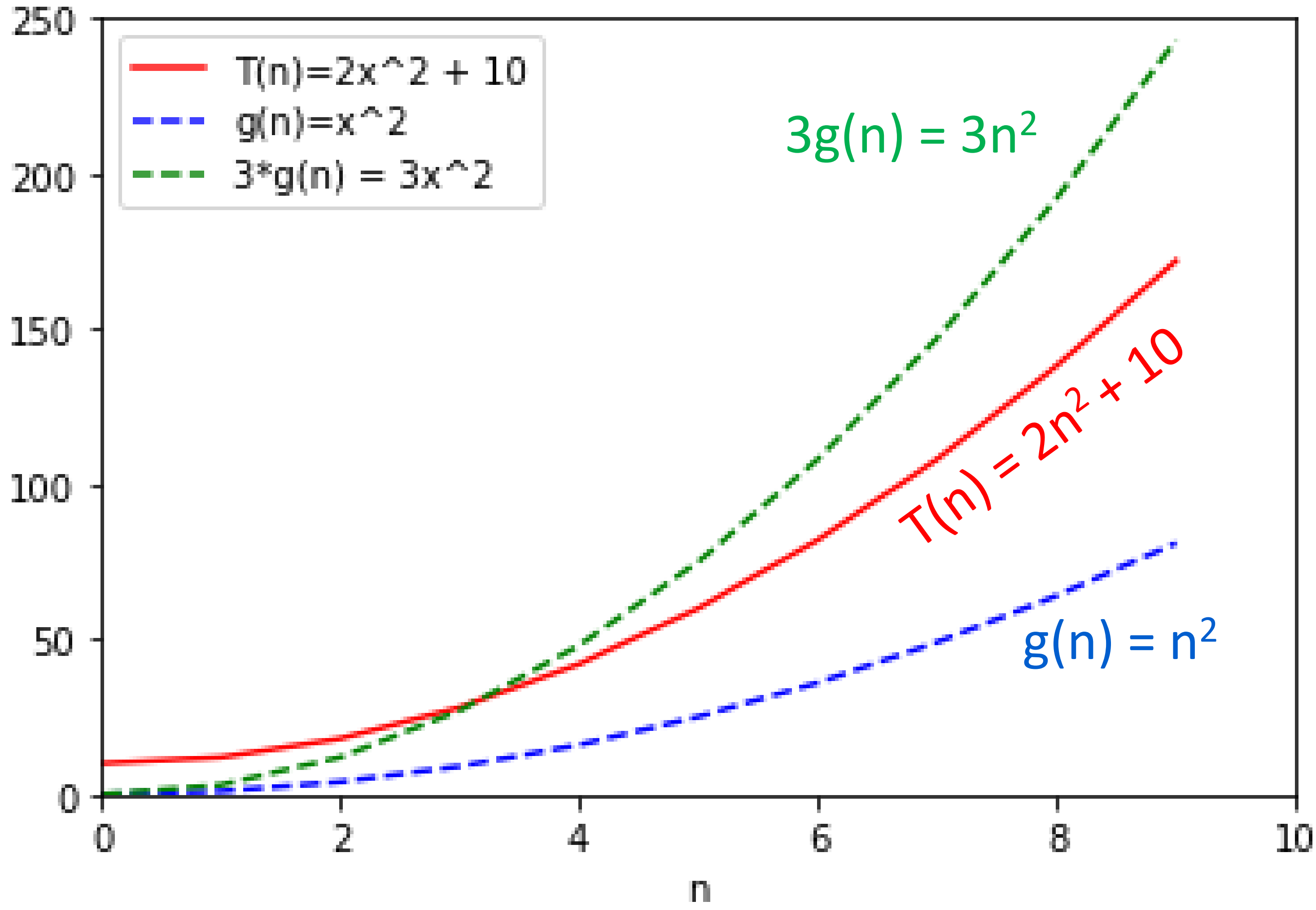
$2n^2 + 10 = O(n^2)$

for large enough n,
$T(n)$ is at most some constant
multiple of $g(n)$.



Example
$2n^2 + 10 = O(n^2)$

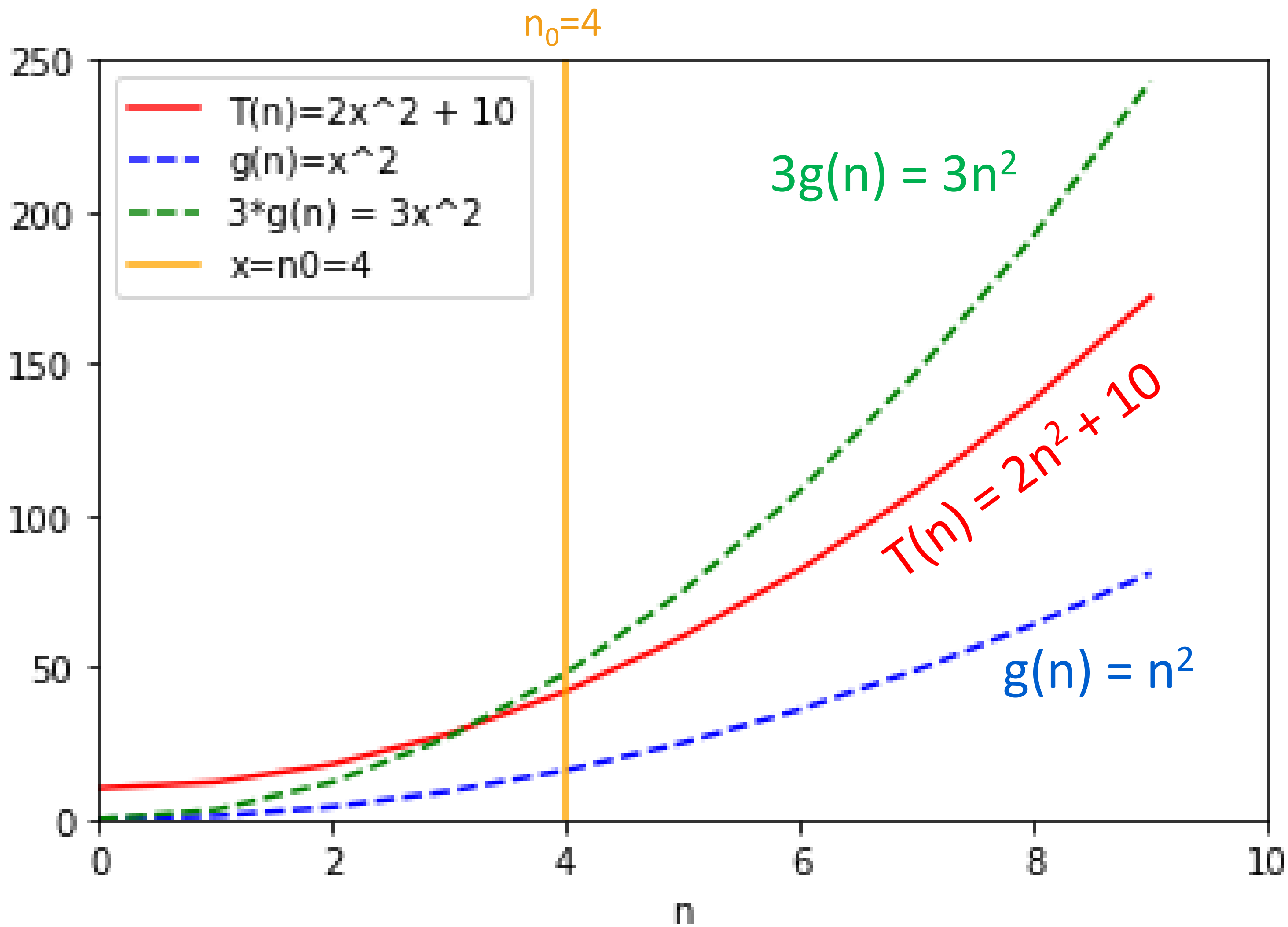
for large enough n,
$T(n)$ is at most some constant
multiple of $g(n)$.



Example

$2n^2 + 10 = O(n^2)$

for large enough n,
$T(n)$ is at most some constant multiple of $g(n)$.



Formal definition of O(...)

- Let $T(n)$, $g(n)$ be functions of positive integers.
  - Think of $T(n)$ as a runtime: positive and increasing in n.

- Formally,

$$T(n) = O\big(g(n)\big)$$

“If and only if”

$$\iff$$

“For all”

$$\exists c, n_0 > 0 \;\; s.t. \;\; \forall n \geq n_0,$$

$$T(n) \leq c \cdot g(n)$$

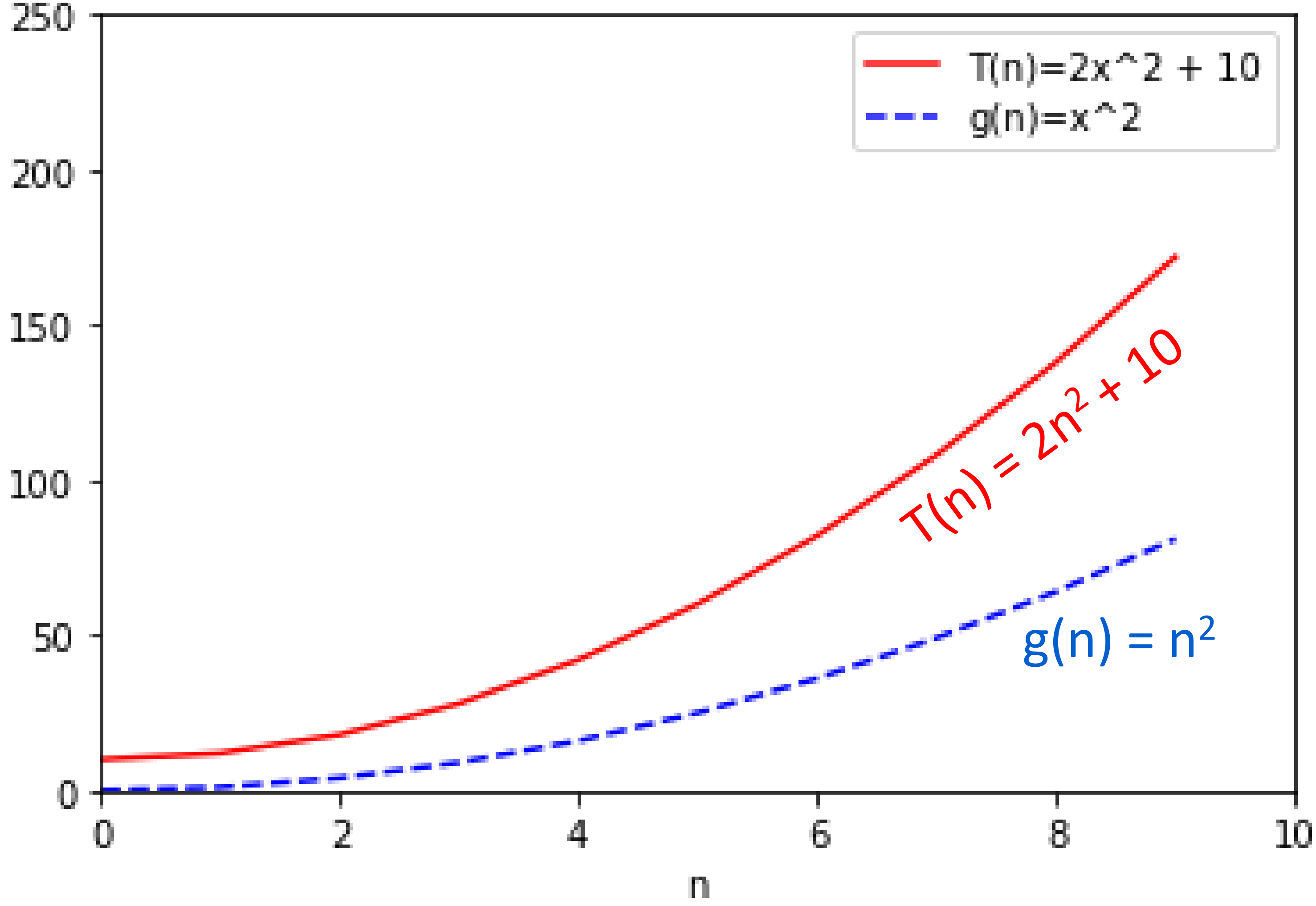
“There exists”

“such that”

Example

$2n^2 + 10 = O(n^2)$

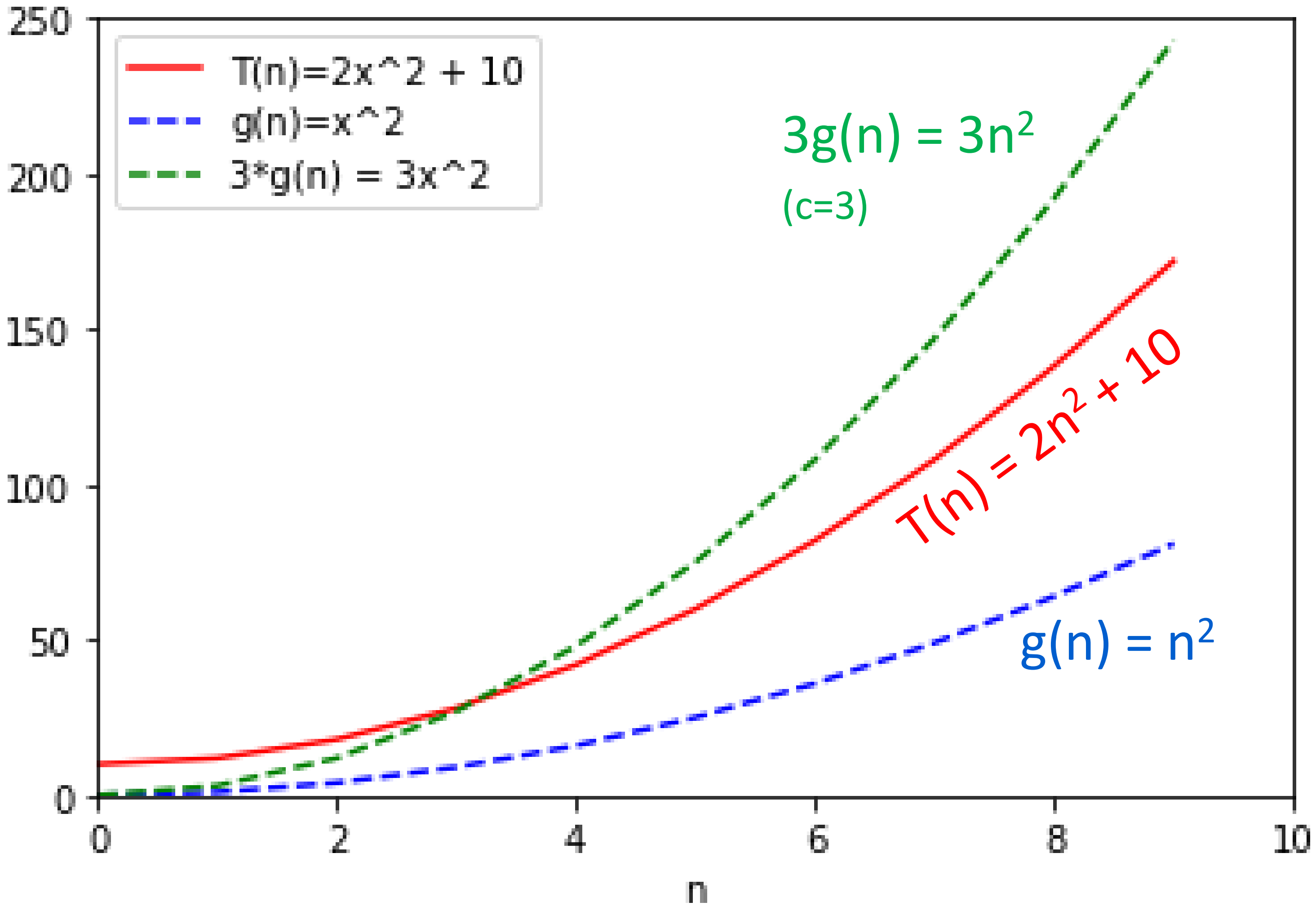
$$T(n) = O\big(g(n)\big)$$
$$\Leftrightarrow$$
$$\exists c, n_0 > 0 \;\; s.t. \;\; \forall n \geq n_0,$$
$$T(n) \leq c \cdot g(n)$$



Example

$2n^2 + 10 = O(n^2)$

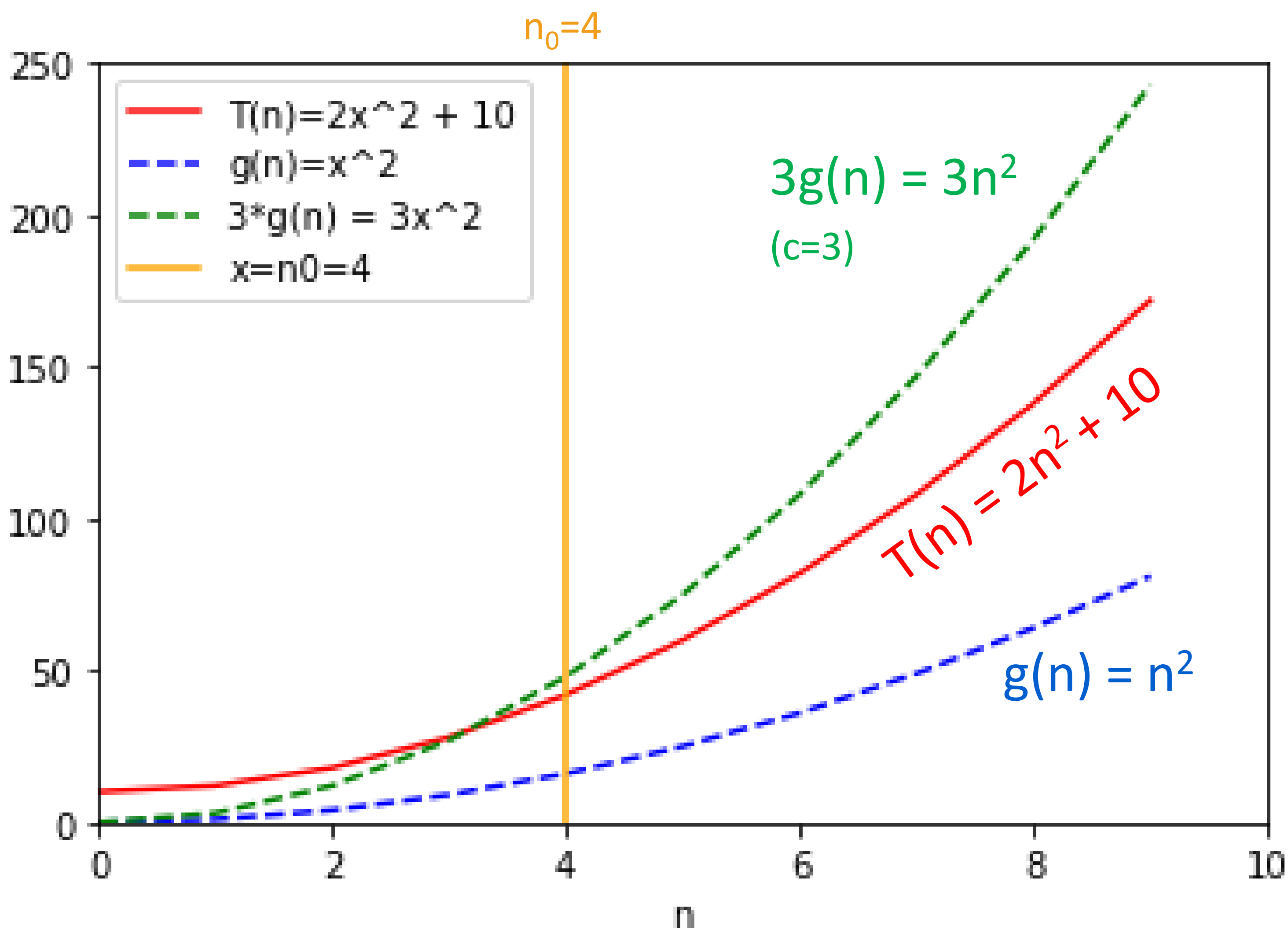
$$T(n) = O(g(n))$$
$$\Leftrightarrow$$
$$\exists c, n_0 > 0 \;\; s.t. \;\; \forall n \geq n_0,$$
$$T(n) \leq c \cdot g(n)$$



Example

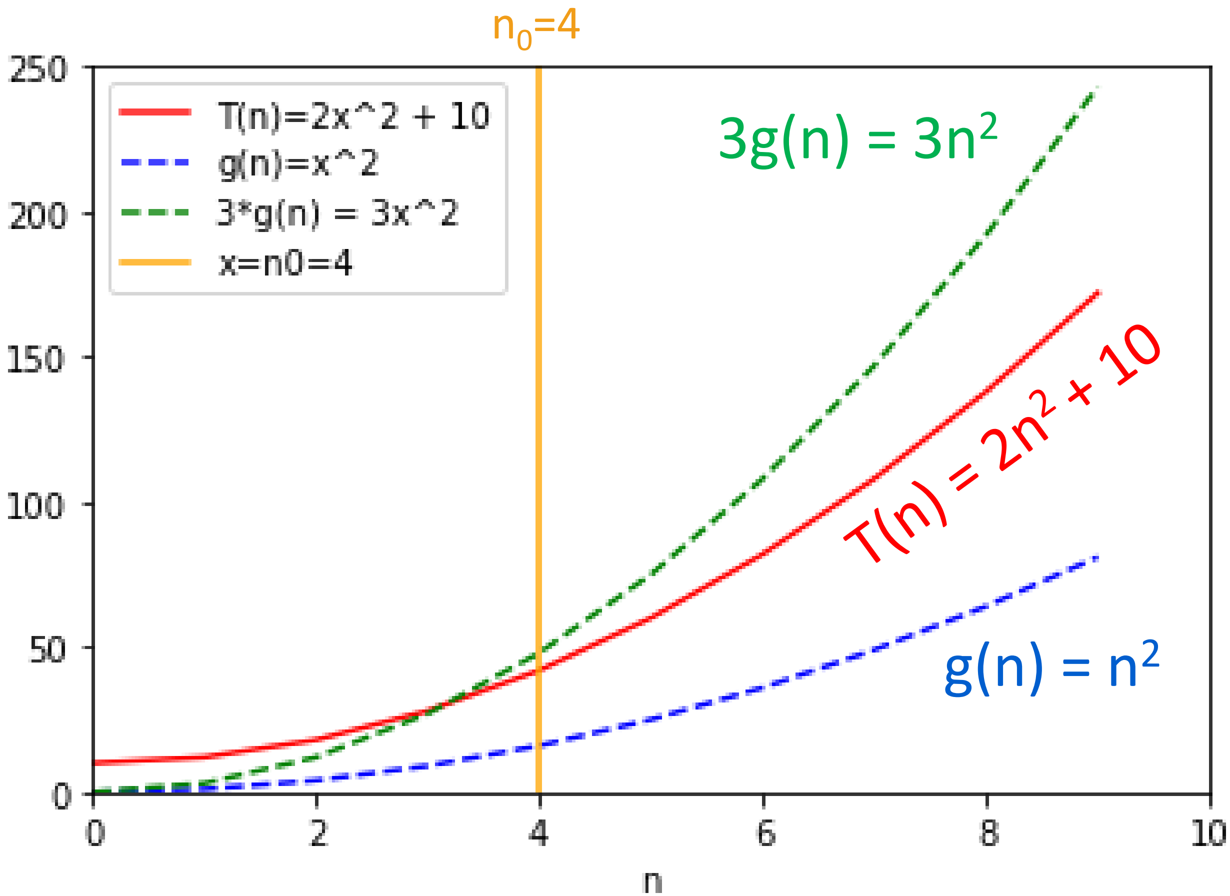
$2n^2 + 10 = O(n^2)$

$$T(n) = O(g(n))$$
$$\Leftrightarrow$$
$$\exists c, n_0 > 0 \;\; s.t. \;\; \forall n \geq n_0,$$
$$T(n) \leq c \cdot g(n)$$



Example
$2n^2 + 10 = O(n^2)$

$$T(n) = O\big(g(n)\big)$$
$$\Leftrightarrow$$
$$\exists c, n_0 > 0 \;\; s.t. \;\; \forall n \geq n_0,$$
$$T(n) \leq c \cdot g(n)$$



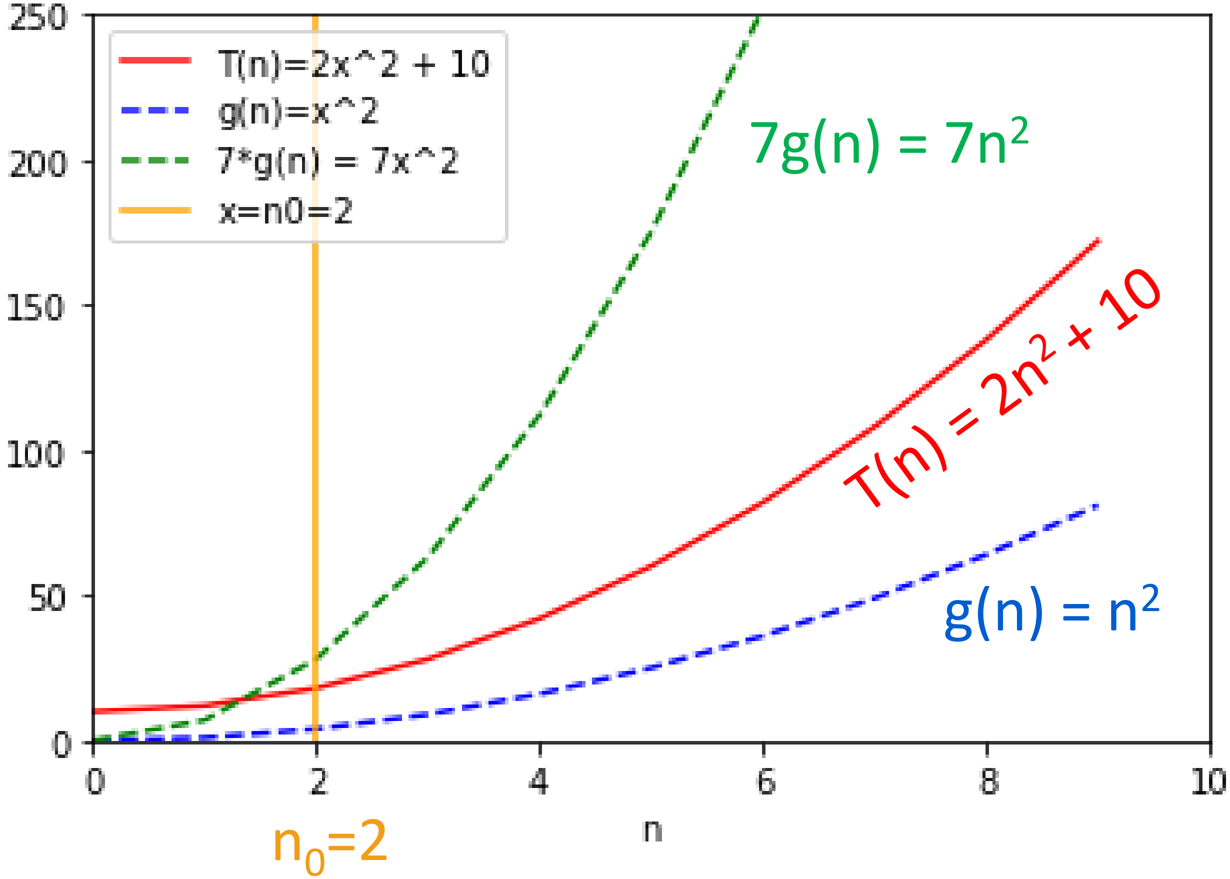
Formally:

- Choose c = 3
- Choose n_0 = 4
- Then:

$$\forall n \geq 4,$$
$$2n^2 + 10 \leq 3 \cdot n^2$$

Same example
$2n^2 + 10 = O(n^2)$

$$T(n) = O\big(g(n)\big) \iff \exists c, n_0 > 0 \ \ s.t. \ \ \forall n \geq n_0, \ T(n) \leq c \cdot g(n)$$



Formally:

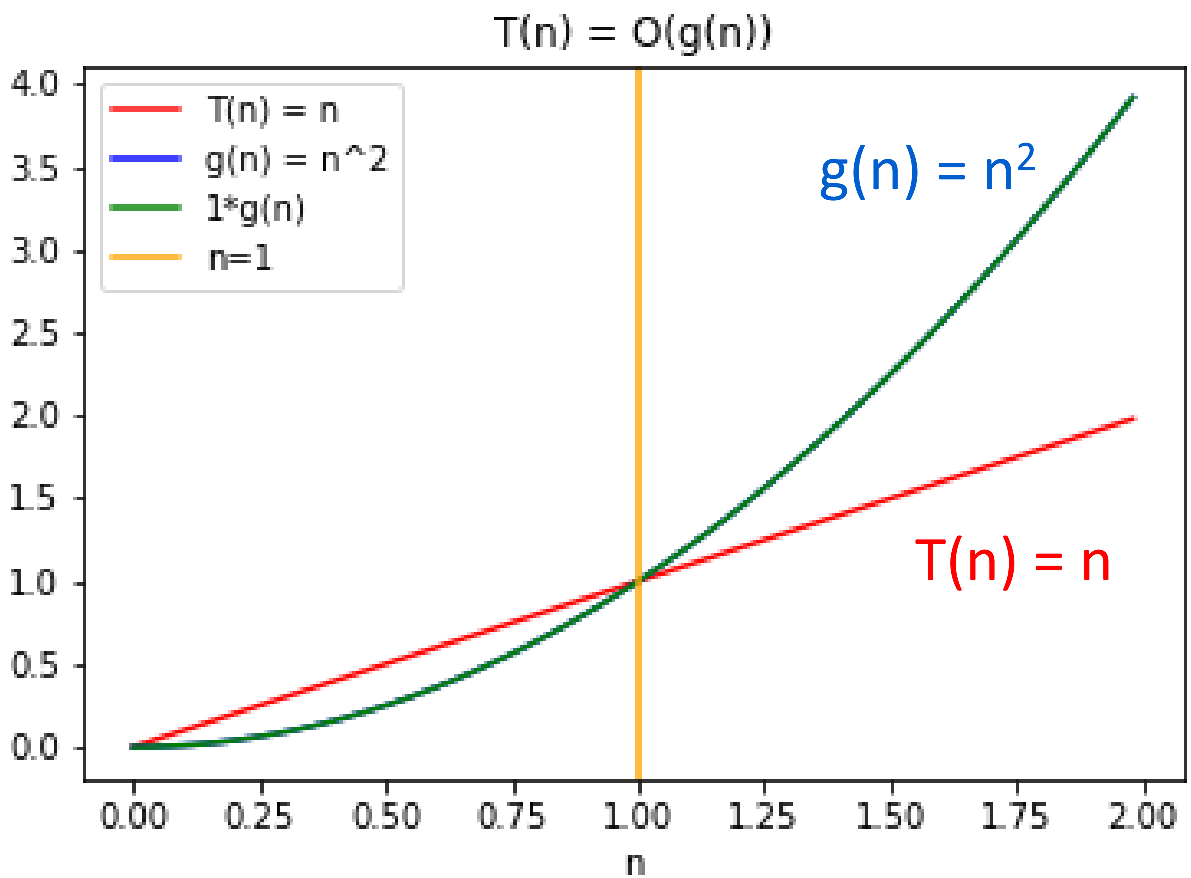
- Choose c = 7
- Choose n_0 = 2
- Then:

$$\forall n \geq 2,$$
$$2n^2 + 10 \leq 7 \cdot n^2$$

There is not a "correct" choice of c and n_0

O(...) is an upper bound:
$n = O(n^2)$

$$T(n) = O(g(n))$$
$$\Longleftrightarrow$$
$$\exists c, n_0 > 0 \;\; s.t. \;\; \forall n \geq n_0,$$
$$T(n) \leq c \cdot g(n)$$



- Choose c = 1
- Choose $n_0 = 1$
- Then

$$\forall n \geq 1,$$
$$n \leq n^2$$

Ω(…) means a lower bound

- We say "$T(n)$ is $\Omega\big(g(n)\big)$" if, for large enough n, $T(n)$ is at least as big as a constant multiple of $g(n)$.

- Formally,

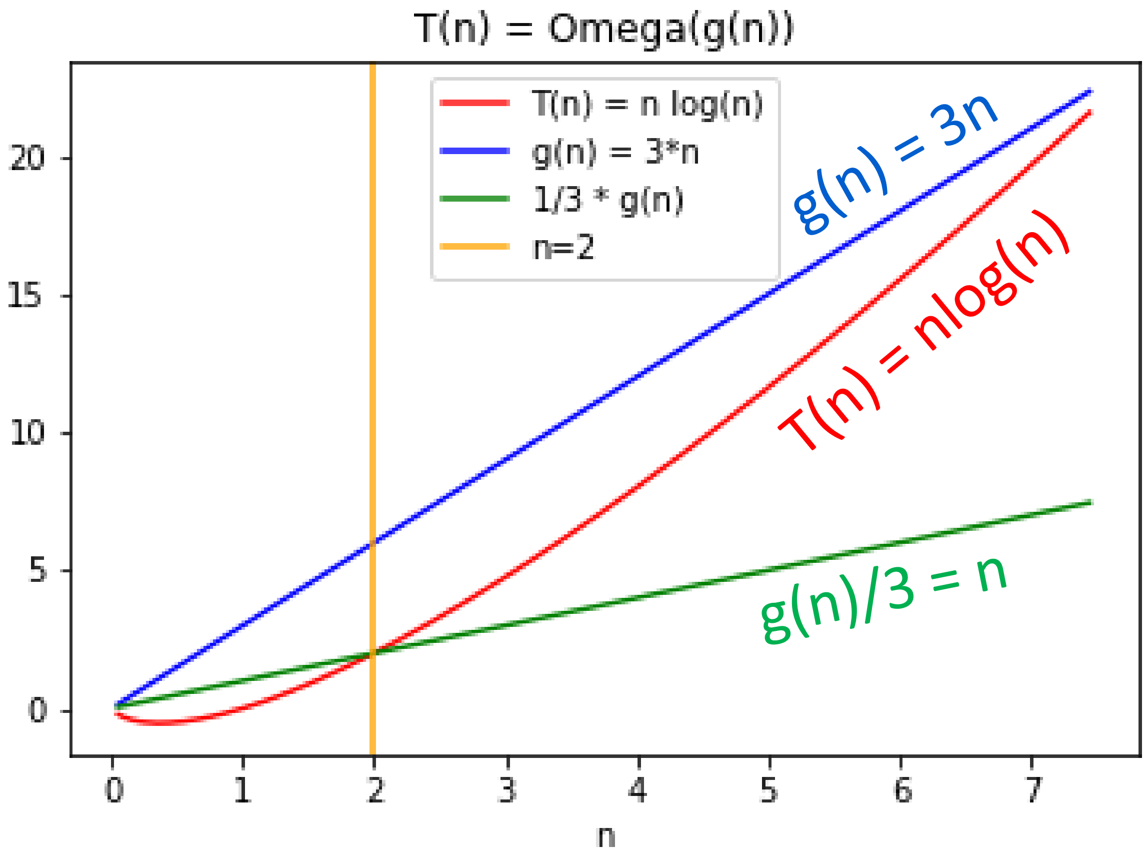
$$T(n) = \Omega\big(g(n)\big)$$
$$\Leftrightarrow$$
$$\exists c, n_0 > 0 \;\; s.t. \;\; \forall n \geq n_0,$$
$$c \cdot g(n) \leq T(n)$$

Switched these!!

Example

$n\log_2(n) = \Omega(3n)$

$$T(n) = \Omega(g(n)) \Leftrightarrow \exists c, n_0 > 0 \;\; s.t. \;\; \forall n \geq n_0, \; c \cdot g(n) \leq T(n)$$



- Choose c = 1/3
- Choose n_0 = 2
- Then

$$\forall n \geq 2,$$

$$\frac{3n}{3} \leq n\log_2(n)$$

$\Theta(\ldots)$ means both!

- We say "$T(n)$ is $\Theta(g(n))$" iff both:

$$T(n) = O\big(g(n)\big)$$

and

$$T(n) = \Omega\big(g(n)\big)$$

Non-Example: n^2 is not $O(n)$

$$T(n) = O(g(n))$$
$$\Longleftrightarrow$$
$$\exists c, n_0 > 0 \;\; s.t. \;\; \forall n \geq n_0,$$
$$T(n) \leq c \cdot g(n)$$

- Proof by contradiction:
- Suppose that $n^2 = O(n)$.
- Then there is some positive c and n_0 so that:

$$\forall n \geq n_0, \qquad n^2 \leq c \cdot n$$

- Divide both sides by n:

$$\forall n \geq n_0, \qquad n \; \leq c$$

- That's not true!!! What about, say, $n_0 + \mathrm{c} + 1$?
  - Then $n \geq n_0$, but , $n \; > c$
- Contradiction!

Take-away from examples

- To prove $T(n) = O(g(n))$, you have to come up with c and n_0 so that the definition is satisfied.

- To prove $T(n)$ is NOT $O(g(n))$, one way is **proof by contradiction**:
  - Suppose (to get a contradiction) that someone gives you a c and an n_0 so that the definition ***is*** satisfied.
  - Show that this someone must by lying to you by deriving a contradiction.

Another example: polynomials

- Say $p(n) = a_k n^k + a_{k-1} n^{k-1} + \cdots + a_1 n + a_0$ is a polynomial of degree $k \geq 1$.

- Then:
  1. $p(n) = O\left(n^k\right)$
  2. $p(n)$ is **not** $O\left(n^{k-1}\right)$

- See the book for a proof.

More examples

- $n^3 + 3n = O(n^3 - n^2)$
- $n^3 + 3n = \Omega(n^3 - n^2)$
- $n^3 + 3n = \Theta(n^3 - n^2)$

- 3^n is **NOT** $O(2^n)$
- $\log_2(n) = \Omega(\ln(n))$
- $\log_2(n) = \Theta(2^{\log\log(n)})$

Recap: Asymptotic Notation

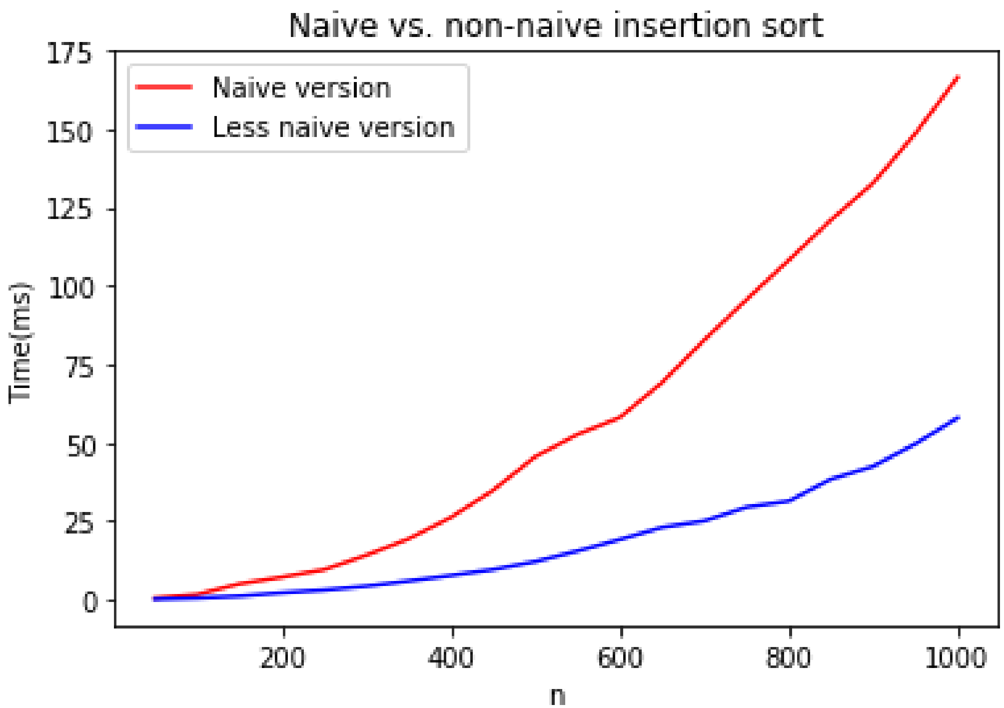
- This makes both Plucky and Lucky happy.
  - **Plucky the Pedantic Penguin** is happy because there is a precise definition.
  - **Lucky the Lackadaisical Lemur** is happy because we don't have to pay close attention to all those pesky constant factors.

- But we should always be careful not to abuse it.

- In the course, (almost) every algorithm we see will be actually practical, without needing to take $n \geq n_0 = 2^{10000000}$.

Back Insertion Sort

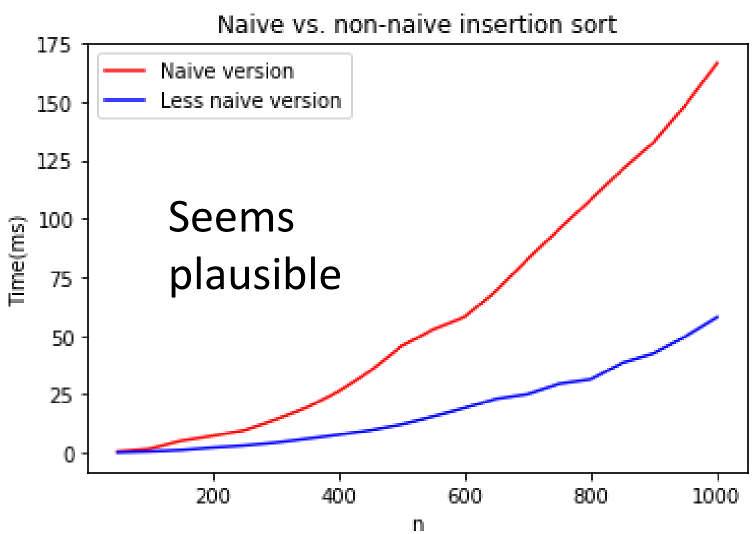
1. Does it work?
2. Is it fast?



Insertion Sort: running time

- Operation count was:
    - $2n^2 - n - 1$ variable assignments
    - $2n^2 - n - 1$ increments/decrements
    - $2n^2 - 4n + 1$ comparisons
    - …

- The running time is $O(n^2)$



Insertion Sort: running time

As you get more used to this, you won't have to count up operations anymore. For example, just looking at the pseudocode below, you might think…

```
def InsertionSort(A):
    for i in range(1,len(A)):
        current = A[i]
        j = i-1
        while j >= 0 and A[j] > current:
            A[j+1] = A[j]
            j -= 1
        A[j+1] = current
```

n-1 iterations of the outer loop

In the worst case, about n iterations of this inner loop

"There's O(1) stuff going on inside the inner loop, so each time the inner loop runs, that's O(n) work. Then the inner loop is executed O(n) times by the outer loop, so that's O(n^2)."

What have we learned?

InsertionSort is an algorithm that correctly sorts an arbitrary n-element array in time $O(n^2)$.

Can we do better?

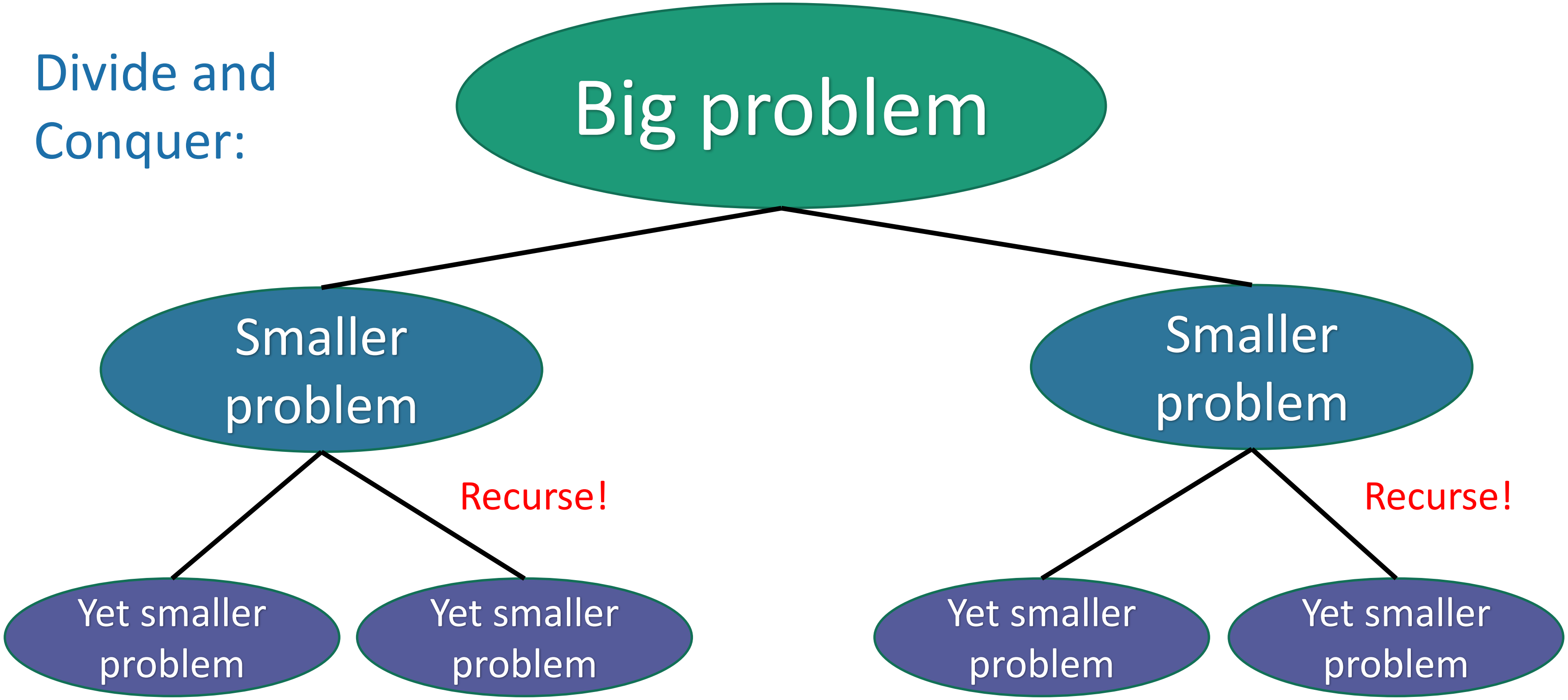
The Plan

- Karatsuba integer multiplication
- InsertionSort
  - Does it work?
  - Is it fast?
- MergeSort
  - Does it work?
  - Is it fast?

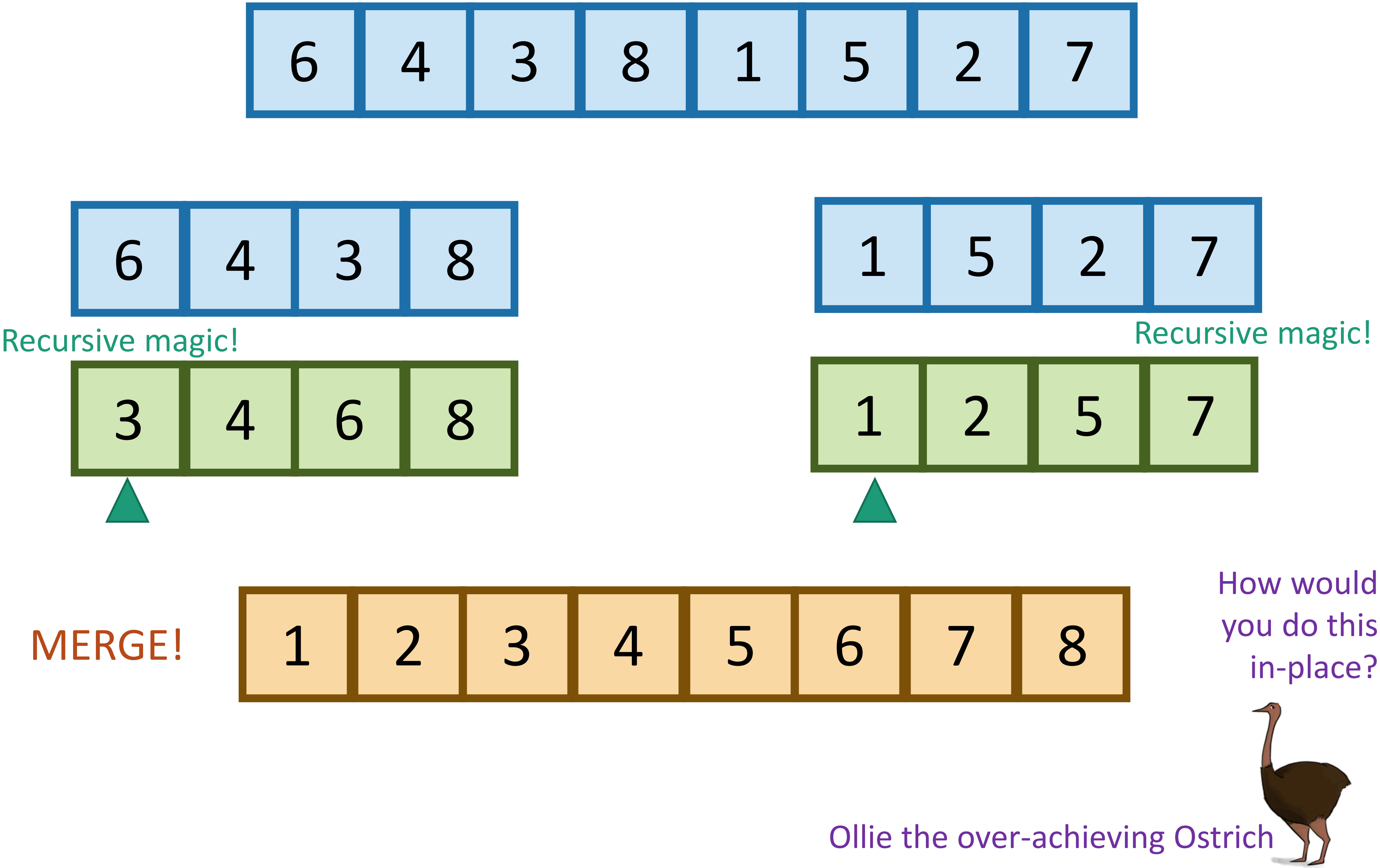
Can we do better?

- MergeSort: a divide-and-conquer approach
- Recall from last time:

Divide and Conquer:



MergeSort


6 4 3 8 1 5 2 7
6 4 3 8
1 5 2 7
Recursive magic!
Recursive magic!
3 4 6 8
1 2 5 7
MERGE!
1 2 3 4 5 6 7 8
How would you do this in-place?
Ollie the over-achieving Ostrich

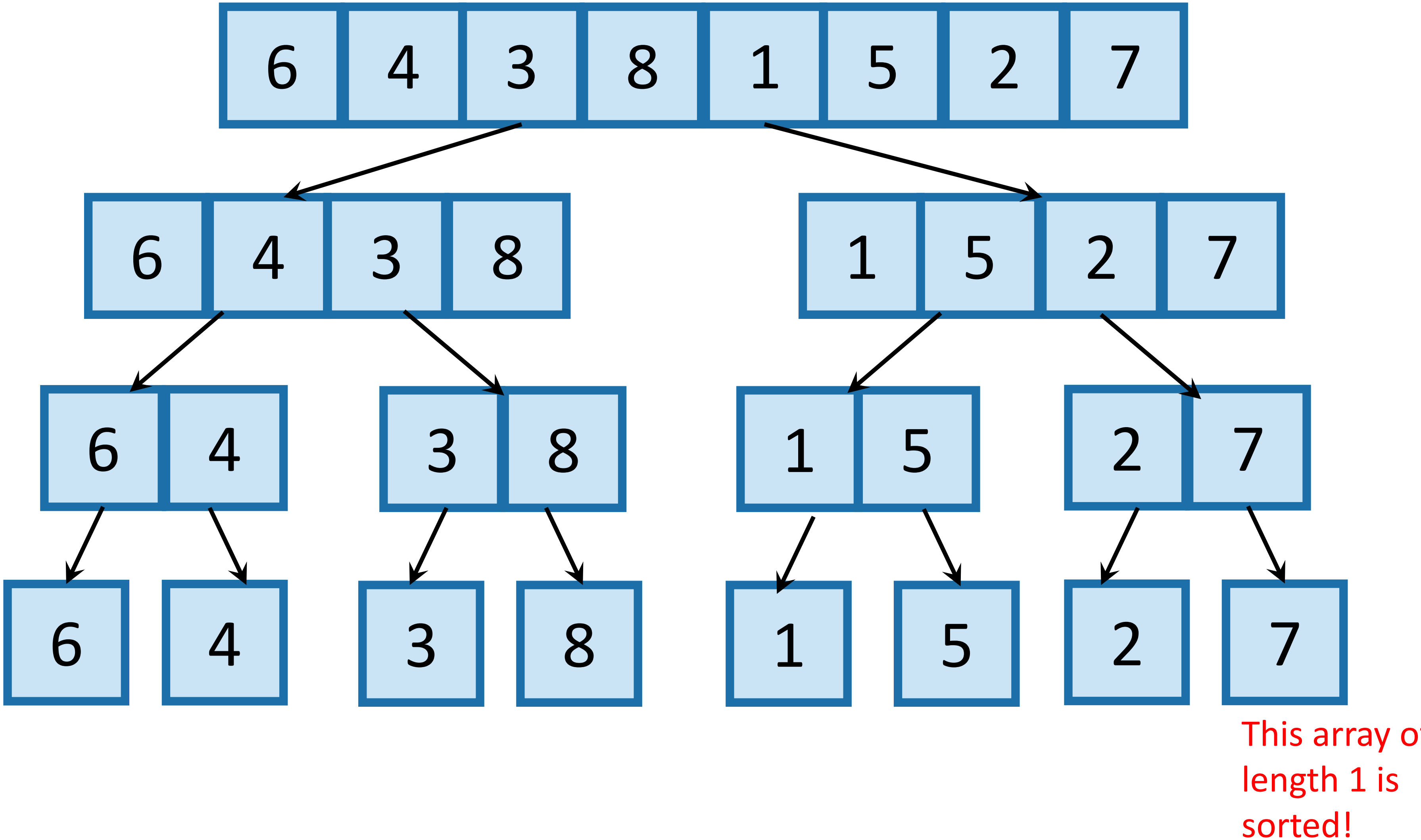
MergeSort Pseudocode

MERGESORT(A):

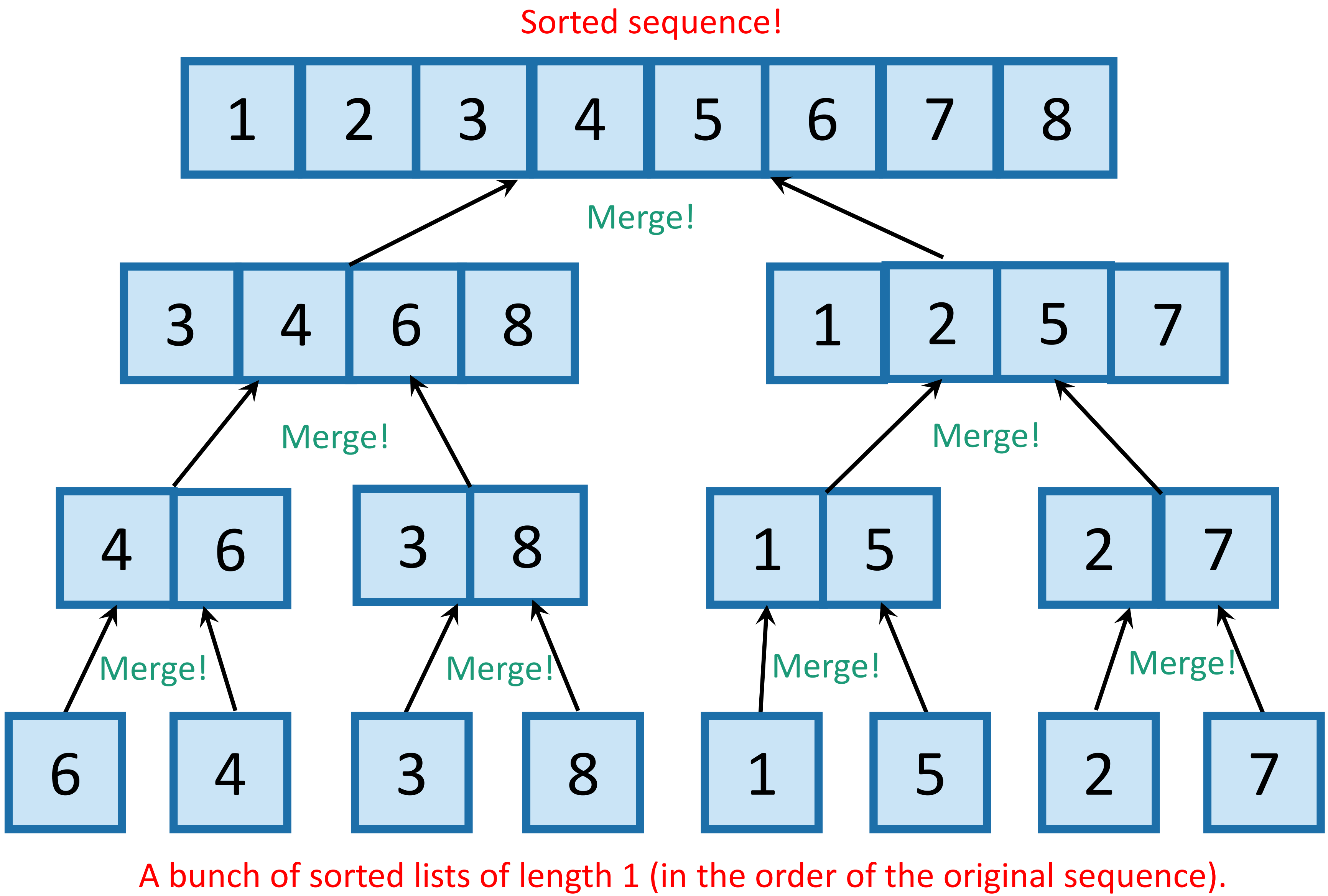
- n = length(A)
- **if** n $\leq$ 1: If A has length 1, It is already sorted!
  - **return** A
- L = MERGESORT(A[0 : n/2]) Sort the left half
- R = MERGESORT(A[n/2 : n]) Sort the right half
- **return** MERGE(L,R) Merge the two halves

What actually happens?

First, recursively break up the array all the way down to the base cases

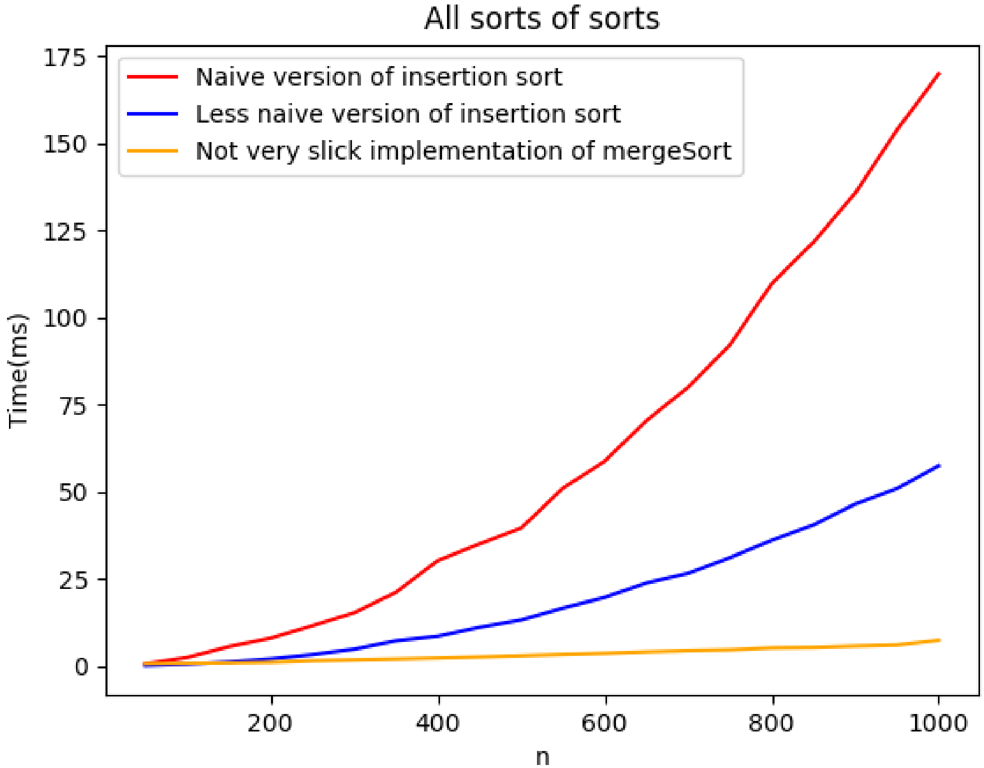


Then, merge them all back up!



Two questions

1. Does this work?
2. Is it fast?



Empirically:
1. Seems to work.
2. Seems fast.

It works

- Yet another job for…

Proof By Induction!

It works

Assume that n is a power of 2 for convenience.

- **Inductive hypothesis:**

  "In every the recursive call on an array of length at most i, MERGESORT returns a sorted array."

- **Base case (i=1):** a 1-element array is always sorted.
- **Inductive step:** Need to show: if the inductive hypothesis holds for k<i, then it holds for k=i.
- Aka, need to show that if L and R are sorted, then MERGE(L,R) is sorted.
- **Conclusion:** In the top recursive call, MERGESORT returns a sorted array.

- MERGESORT(A):
  - n = length(A)
  - **if** n $\leq$ 1:
    - **return** A
  - L = MERGESORT(A[1 : n/2])
  - R = MERGESORT(A[n/2+1 : n])
  - **return** MERGE(L,R)

Fill in the inductive step!

HINT: You will need to prove that the MERGE algorithm is correct, for which you may need…another proof by induction!

Assume that n is a power of 2 for convenience.

It’s fast

CLAIM:

MergeSort runs in time $O(n\log(n))$

- Proof coming soon.
- But first, how does this compare to InsertionSort?
    - Recall InsertionSort ran in time $O(n^2)$.

$O(n\log(n))$ vs. $O(n^2)$?

Aside:

All logarithms in this course are base 2

Quick log refresher

- Def: log(n) is the number so that $2^{\log(n)} = n$.
- Intuition: log(n) is how many times you need to divide n by 2 in order to get down to 1.

$32, \underbrace{16, 8, 4, 2, 1}_{\text{Halve 5 times}} \quad \Rightarrow \quad \log(32) = 5$

$64, \underbrace{32, 16, 8, 4, 2, 1}_{\text{Halve 6 times}} \quad \Rightarrow \quad \log(64) = 6$

log(128) = 7

log(256) = 8

log(512) = 9

….

log(**# particles in the universe**) < 280

- log(n) grows very slowly!

$O(n \log\ n)$ vs. $O(n^2)$?

- $\log(n)$ grows much more slowly than n
- $n \log(n)$ grows much more slowly than n^2

Punchline: A running time of O(n log n) is a lot better than $O(n^2)$!

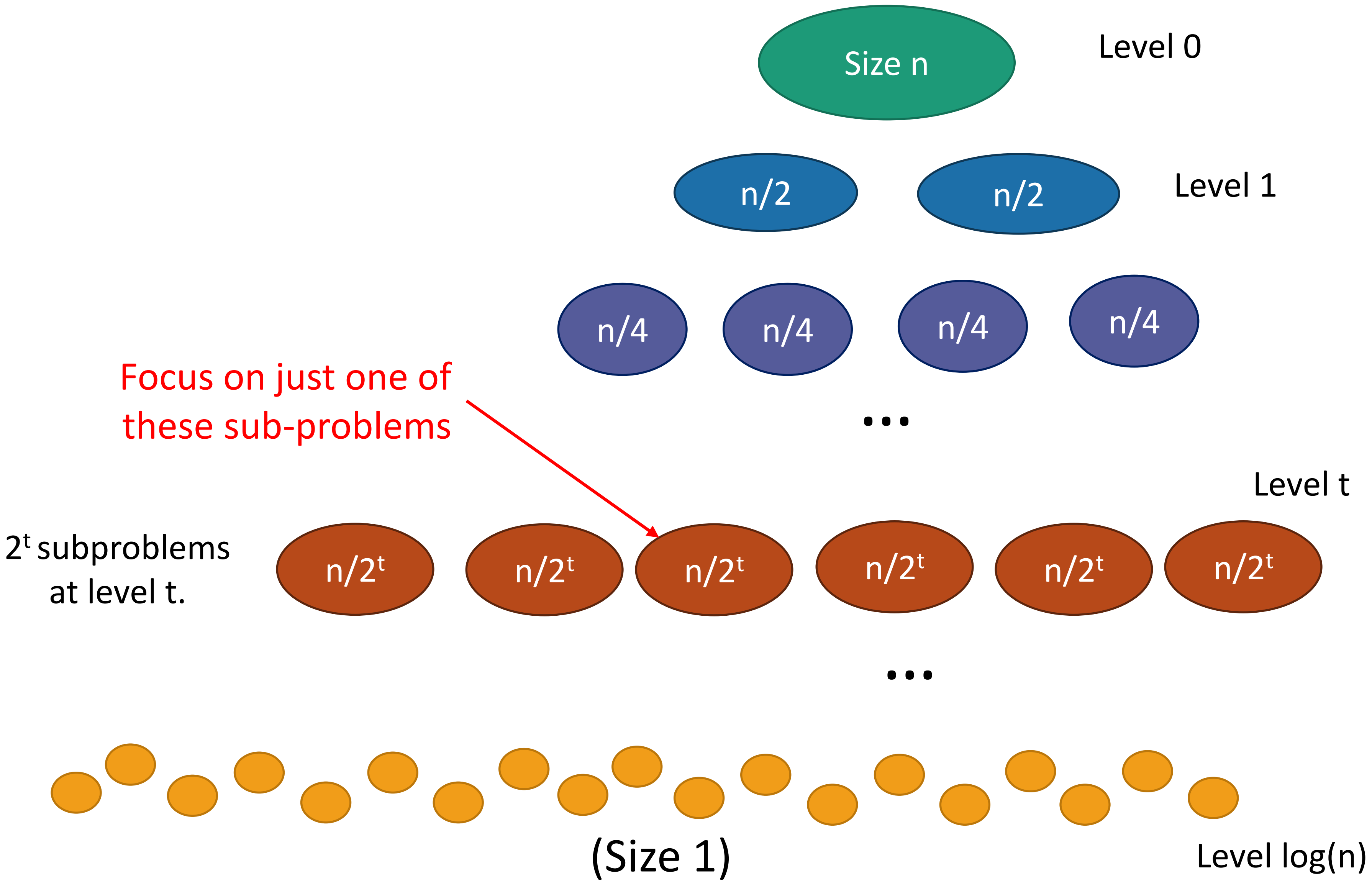
Assume that n is a power of 2 for convenience.

Now let's prove the claim

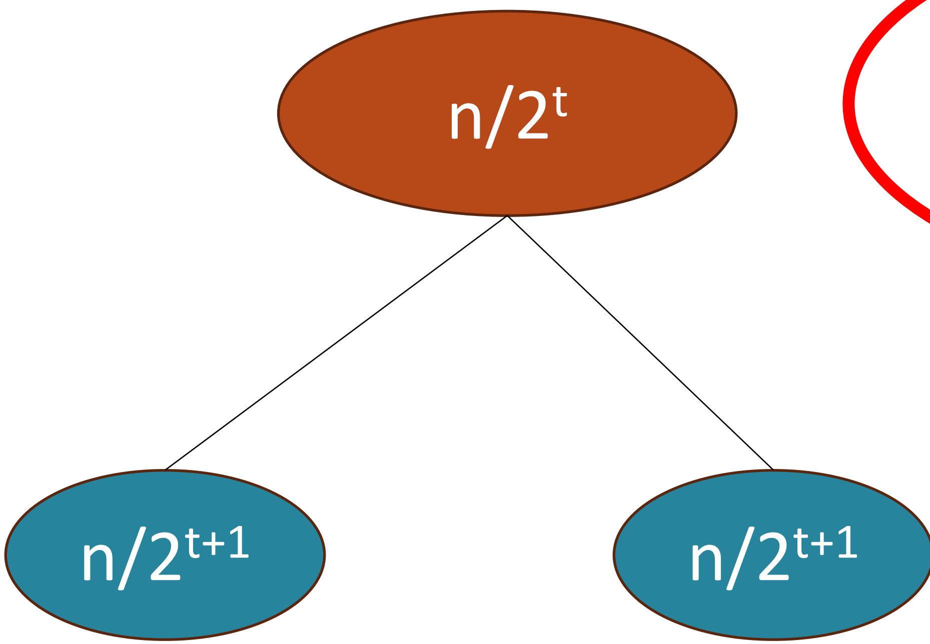
CLAIM:

MergeSort runs in time $O(n\log(n))$

Let's prove the claim



How much work in this sub-problem?



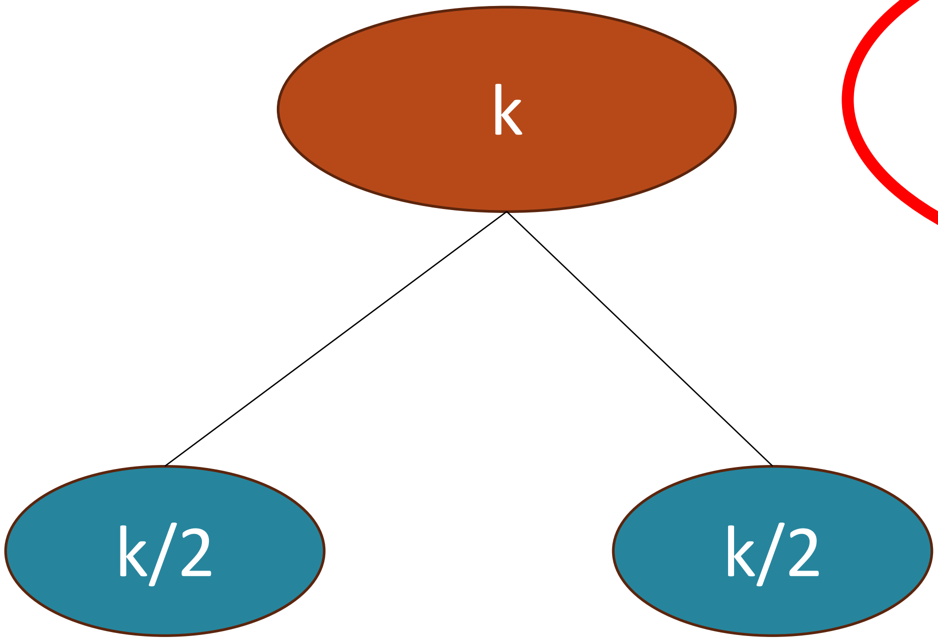
Time spent MERGE-ing the two subproblems

+

Time spent within the two sub-problems

How much work in this sub-problem?

Let $k = n/2^t$...

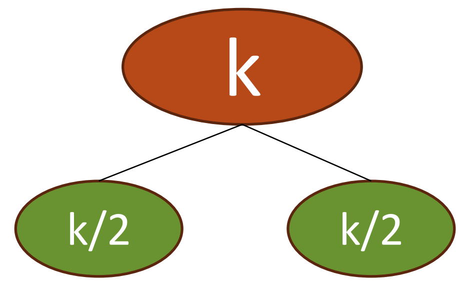


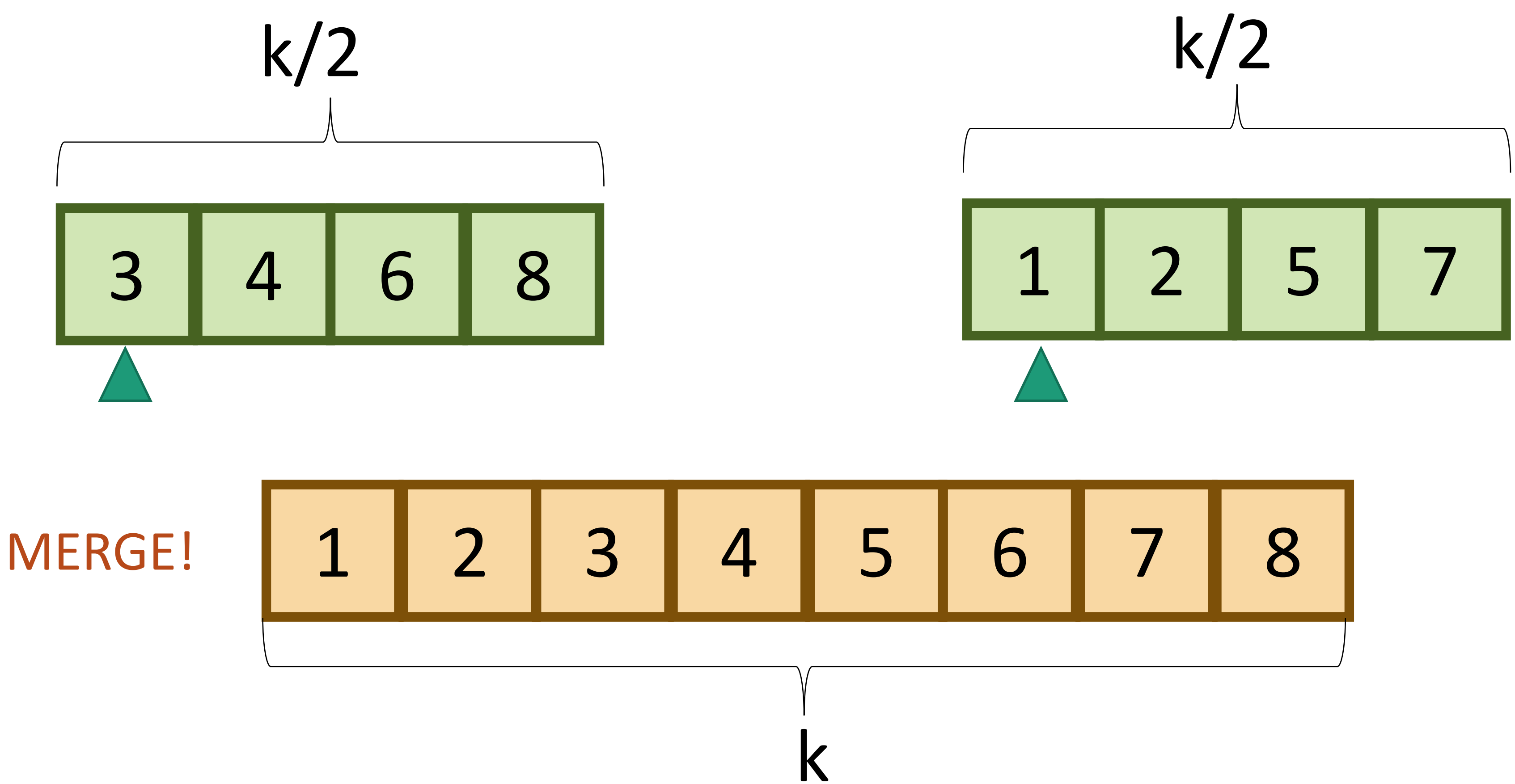
Time spent MERGE-ing the two subproblems

+

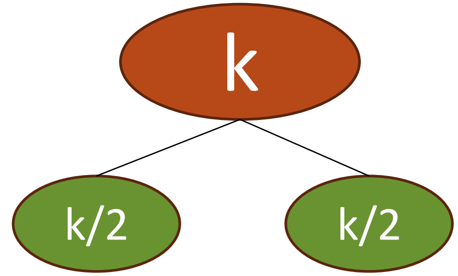
Time spent within the two sub-problems

How long does it take to MERGE?

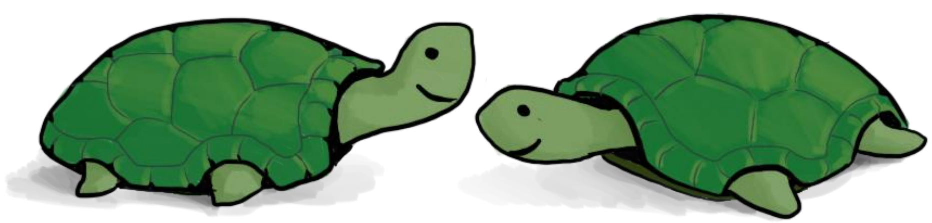




How long does it take to MERGE?



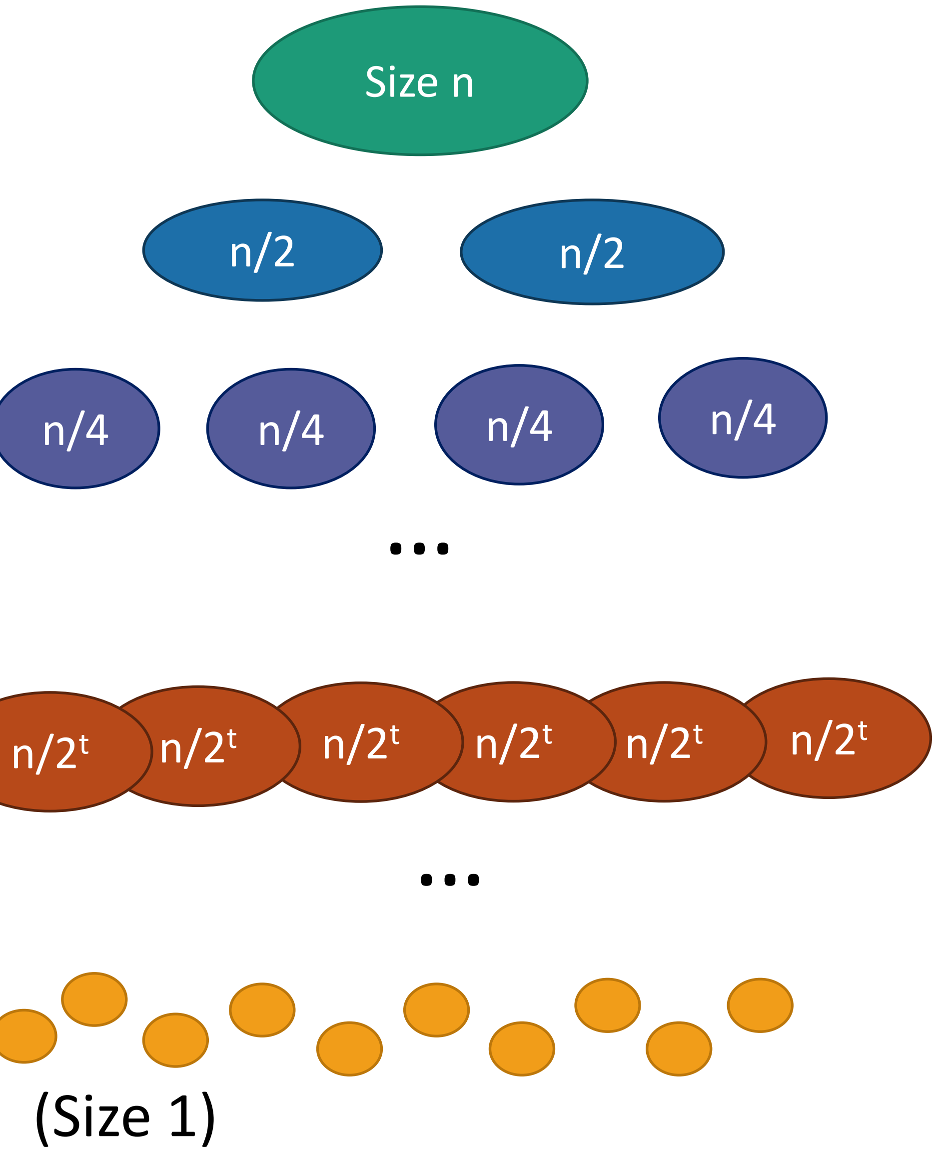
How long does it take to run MERGE on two lists of size k/2?

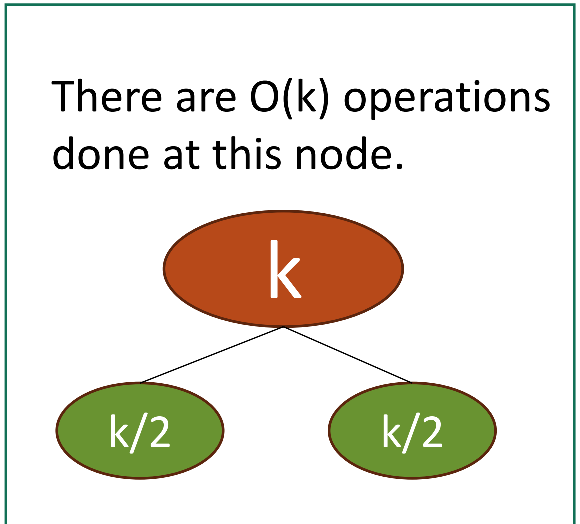


Think-Pair-Share Terrapins

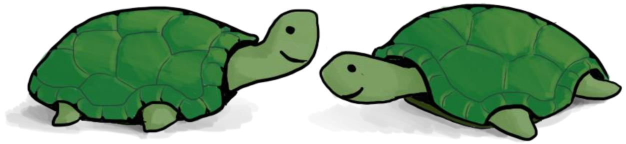
Answer: It takes time O(k), since we just walk across the list once.

Recursion tree

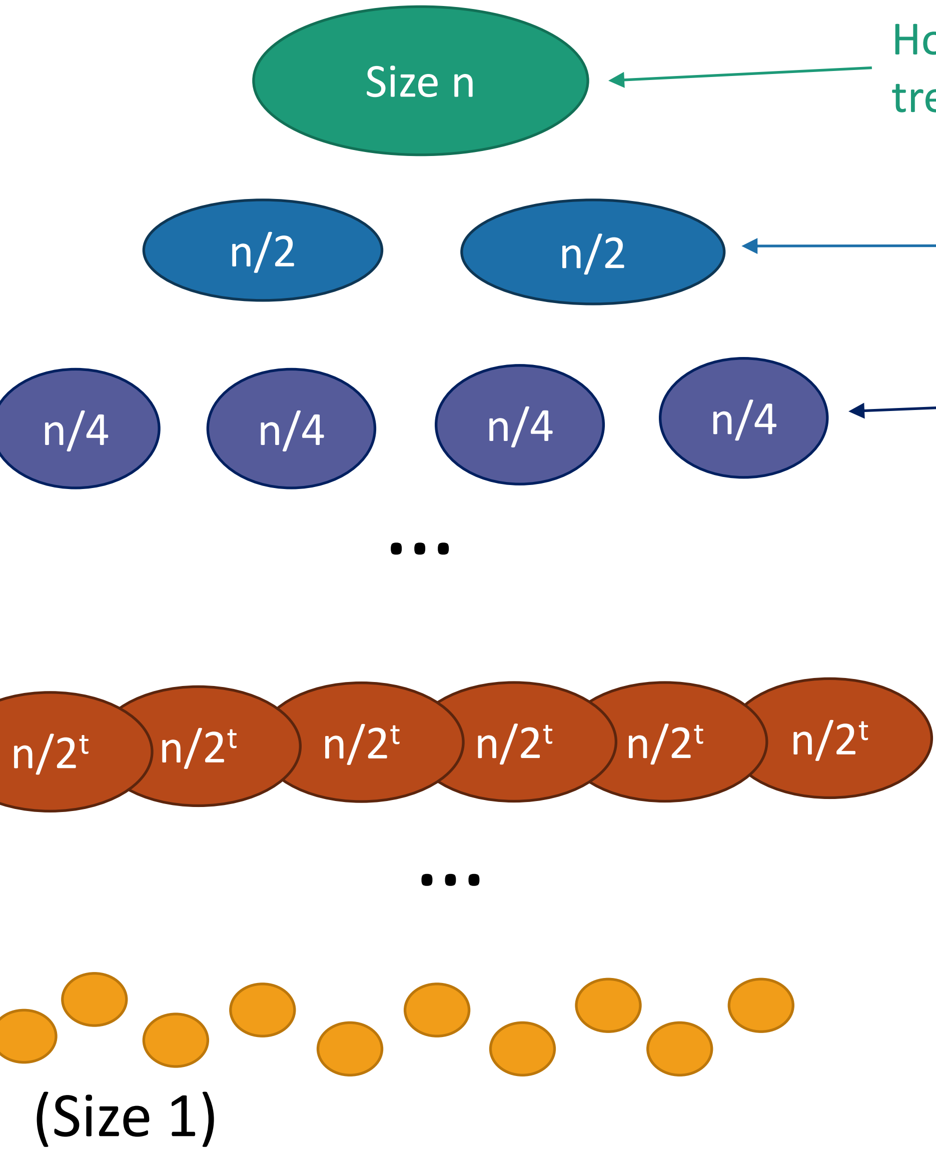




Recursion tree



Think, Pair, Share!



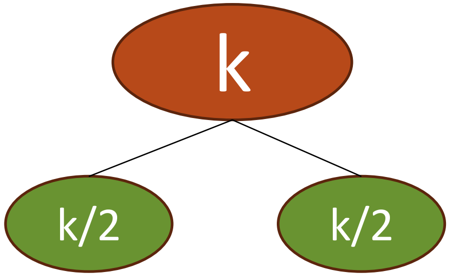
How many operations are done at this level of the tree? (Just MERGE-ing subproblems).

How about at this level of the tree? (between both n/2-sized problems)

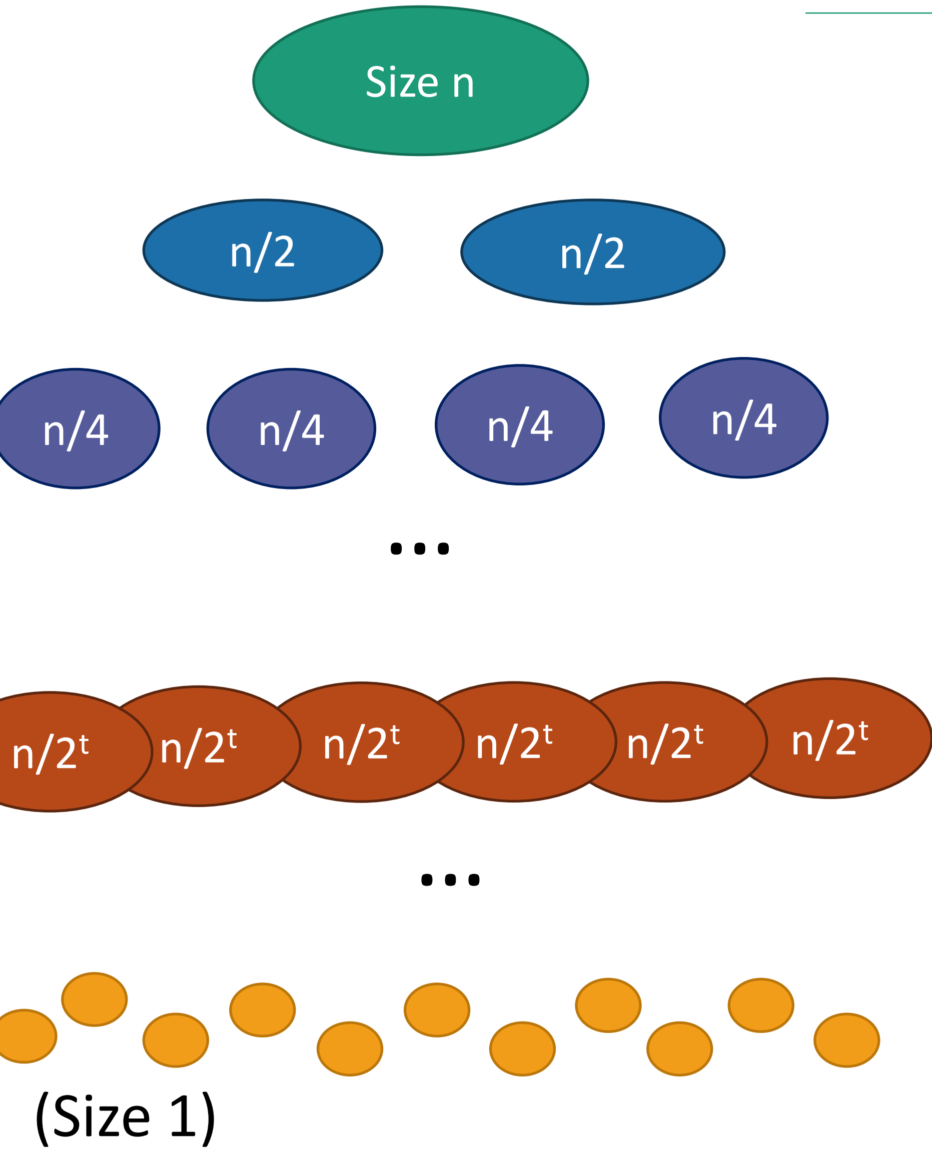
This level?

This level?

There are O(k) operations done at this node.



Recursion tree



Level	# problems	Size of each problem	Amount of work at this level
0	1	n	O(n)
1	2	n/2	O(n)
2	4	n/4	O(n)
...			
t	2^t	$n/2^t$	O(n)
...			
log(n)	n	1	O(n)

Total runtime…

- O(n) steps per level, at every level
- log(n) + 1 levels
- O(n log(n)) total!

That was the claim!

What have we learned?

- MergeSort correctly sorts a list of n integers in time $O(n \log(n))$.
- That’s (asymptotically) better than InsertionSort!

The Plan

- Karatsuba integer multiplication
- InsertionSort
  - Does it work?
  - Is it fast?
- MergeSort
  - Does it work?
  - Is it fast?

Wrap-Up

Recap

- InsertionSort runs in time $O(n^2)$
- MergeSort is a divide-and-conquer algorithm that runs in time $O(n \log(n))$

- How do we show an algorithm is correct?
  - Today, we did it by induction
- How do we measure the runtime of an algorithm?
  - Worst-case analysis
  - Asymptotic analysis
- How do we analyze the running time of a recursive algorithm?
  - One way is to draw a recursion tree.

Next time

- A more systematic approach to analyzing the runtime of recursive algorithms.