ECON 310: Dynamic Macroeconomics

Spring 2025

Fulbright University Vietnam

Model: Stochastic Growth Model with Endogenous Labor.

The social planner's problem is

$$\max_{\{c_t\}_{t=1}^{\infty}, \{n_t\}_{t=1}^{\infty}, \{k_{t+1}\}_{t=1}^{\infty}} \mathbb{E}_0 \sum_{t=1}^{\infty} \beta^t \left[\frac{c_t^{1-\sigma}}{1-\sigma} + \gamma \frac{(1-n_t)^{1+\frac{1}{\nu}}}{1+\frac{1}{\nu}} \right],$$

$$\text{s.t. } y_t = c_t + i_t \ ,$$

$$y_t = A_t k_t^{\alpha} n_t^{1-\alpha} \ ,$$

$$k_{t+1} = (1-\delta)k_t + i_t \ ,$$

$$\log(A_{t+1}) = \mu + \rho \log(A_t) + \epsilon_{t+1} \ ,$$

$$1 = n_t + l_t \ ,$$

$$c_t > 0, k_t > 0 \ ,$$

where $|\rho| < 1$, $\epsilon_{t+1} \sim \mathcal{N}(0, \sigma_e^2)$, and the constraints are for $t = 1, 2, ...,$. The rationale for the constraints that are similar to those in the stochastic growth model. The second to the last line is the time constraint: given an endowment of one unit of time, part of it can go to labor n_t and the rest will go to leisure l_t. Consumers derive utility from leisure but must supply labor to produce the final good. The choice of utility functions for consumption and leisure are assumptions based on the literature; other functions exist and will be discussed when discussing households in greater detail.

Recursive Formulation.

The recursive formulation, combining the constraints into one, is

$$V_t(k_t, A_t) = \max_{c_t, n_t, k_{t+1}} \frac{c_t^{1-\sigma}}{1-\sigma} + \gamma \frac{(1-n_t)^{1+\frac{1}{\nu}}}{1+\frac{1}{\nu}} + \beta \mathbb{E}_t[V_{t+1}(k_{t+1}, A_{t+1})] \ ,$$

$$\text{s.t. } A_t k_t^{\alpha} n_t^{1-\alpha} = c_t + k_{t+1} - (1-\delta)k_t \ \text{ for } \ t = 1, 2, ... \ ,$$

$$\log(A_{t+1}) = \mu + \rho \log(A_t) + \epsilon_{t+1} \ ,$$

$$1 = n_t + l_t \ ,$$

$$c_t > 0, k_t > 0 \ .$$

Because of the labor supply choice, the choice space has now expanded to include this. Substituting the combined constraint into the utility function gives

$$V_t(k_t, A_t) = \max_{n_t, k_{t+1}} \frac{\left(A_t k_t^\alpha n_t^{1-\alpha} + (1-\delta)k_t - k_{t+1}\right)^{1-\sigma}}{1-\sigma} + \gamma \frac{(1-n_t)^{1+\frac{1}{\nu}}}{1+\frac{1}{\nu}} + \beta \mathbb{E}_t[V_{t+1}(k_{t+1}, A_{t+1})] \ ,$$

$$\text{s.t. } \log(A_{t+1}) = \mu + \rho \log(A_t) + \epsilon_{t+1} \ ,$$

$$1 = n_t + l_t \ ,$$

$$k_t > 0 \ ,$$

simplifying the problem slightly.

Set up directories and paths.

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The directory is C:/Users/xmgb/Dropbox/02_FUV/teaching/spring_2025/dynamic_macro/code/vfi_sgml_mat
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Set the parameters and generate the state space.

Calls: model.m.

As before, we cannot directly compute the steady state capital because of the productivity fluctuations, but we can still build the grid for k_t around the deterministic steady state.

Solve the model using Value Function Iteration.

Calls: solve.m, and model.m.

Note that the allocation of labor and leisure is a static decision. From (5.23) in Adda and Cooper, the intratemporal condition is

$$(A_t k_t^\alpha n_t^{1-\alpha} + (1-\delta)k_t - k_{t+1})^{-\sigma} A_t(1-\alpha)k_t^\alpha n_t^{-\alpha} = -\gamma(1-n_t)^{\frac{1}{\nu}} \ .$$

We have pre-determined grids for k_t and A_t, and we use the same grid for k_{t+1} as k_t. Given these and the equation above, we can n_t by using a numerical solver to minimize

$$f(n_t) = (A_t k_t^\alpha n_t^{1-\alpha} + (1-\delta)k_t - k_{t+1})^{-\sigma} A_t(1-\alpha)k_t^\alpha n_t^{-\alpha} + \gamma(1-n_t)^{\frac{1}{\nu}} \ ,$$

and obtain optimal labor supply given any combination of the state variables and any potential choice of k_{t+1} associated with these. Crucially, we can do this outside of the value function iteration algorithm because the decision is static. Once we have the optimal n_t, we can treat this as a parameter for each combination of k_t, k_{t+1}, and A_t.

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------------Solving for Labor Supply.------------

-----------Labor Supply Done.------------

-----------Beginning Value Function Iteration.------------

Iteration: 25.
Iteration: 50.
Iteration: 75.
Iteration: 100.
Iteration: 125.
Iteration: 150.
Iteration: 175.
Iteration: 200.
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Iteration: 225.
Iteration: 250.
Iteration: 275.
Iteration: 300.
Iteration: 325.
Iteration: 350.
Iteration: 375.
Iteration: 400.
Iteration: 425.


Model solved in 32.8438 seconds.
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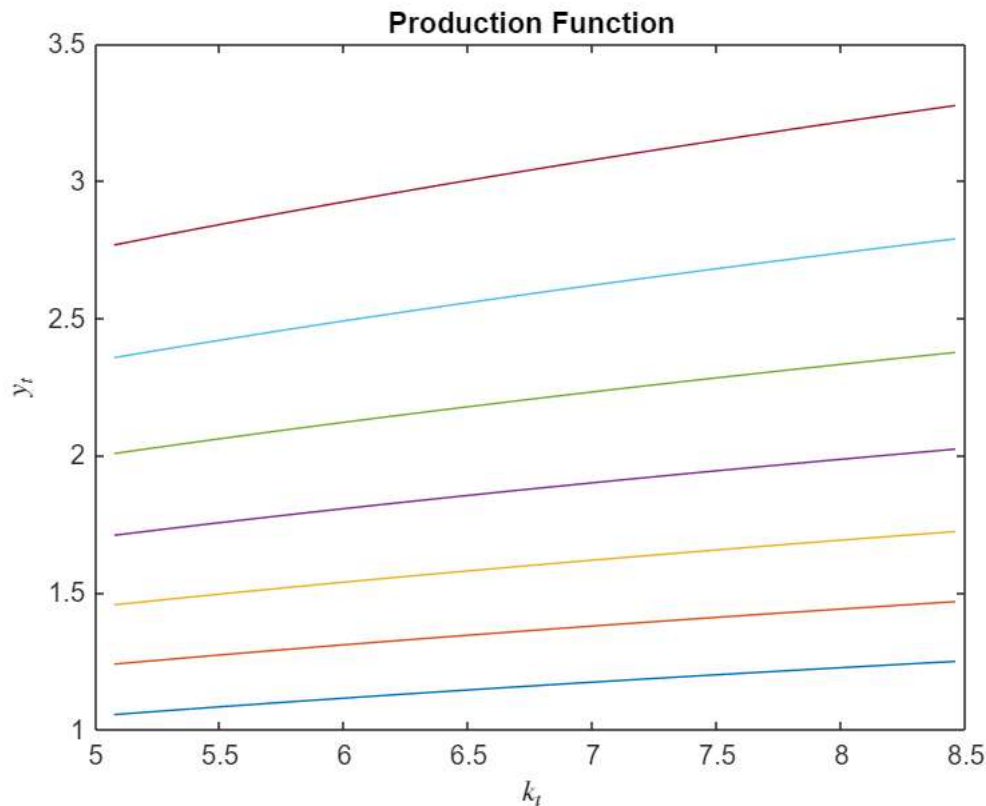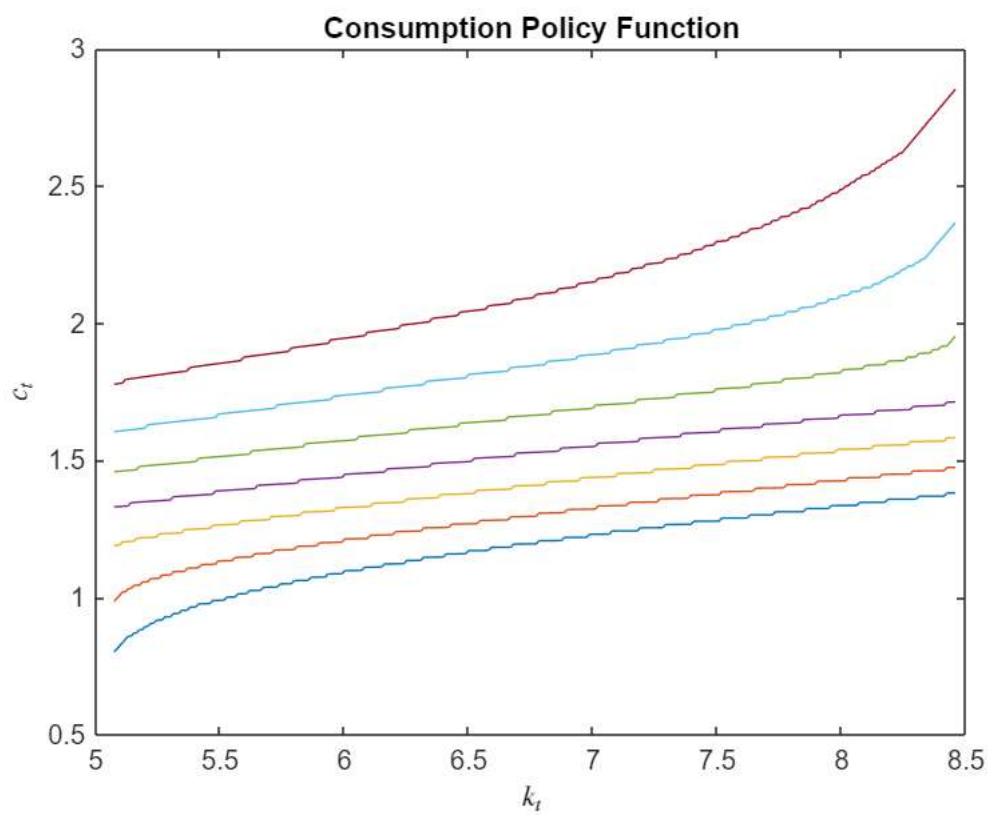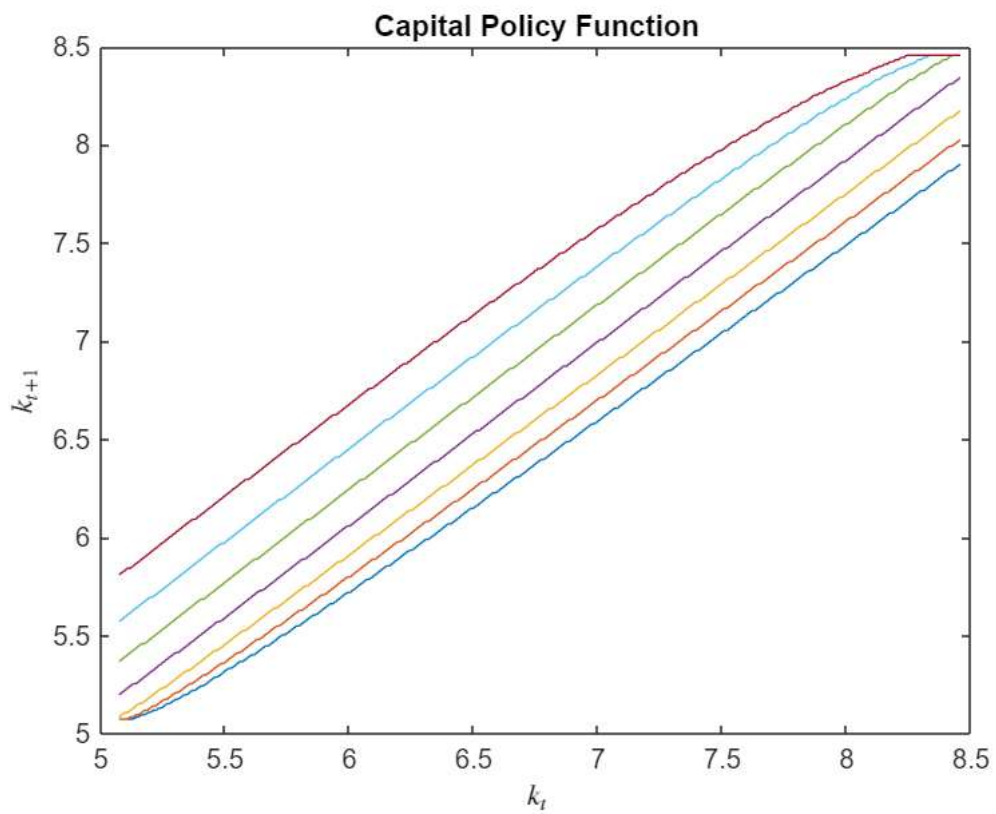
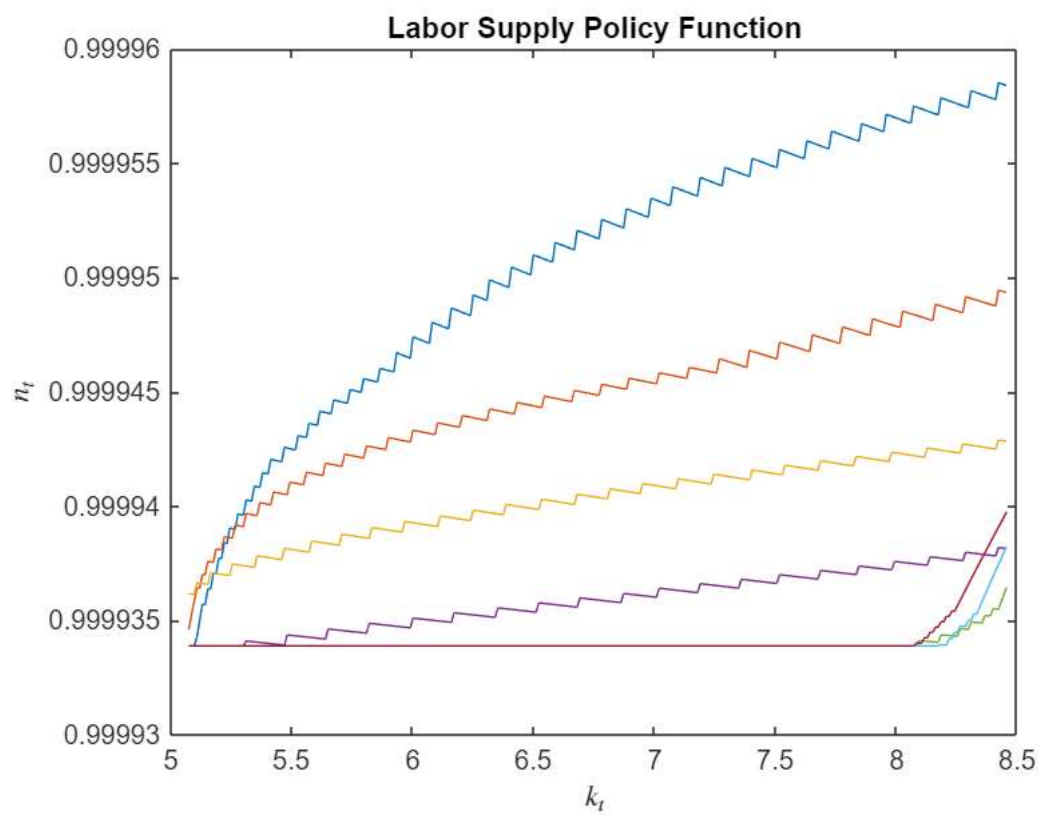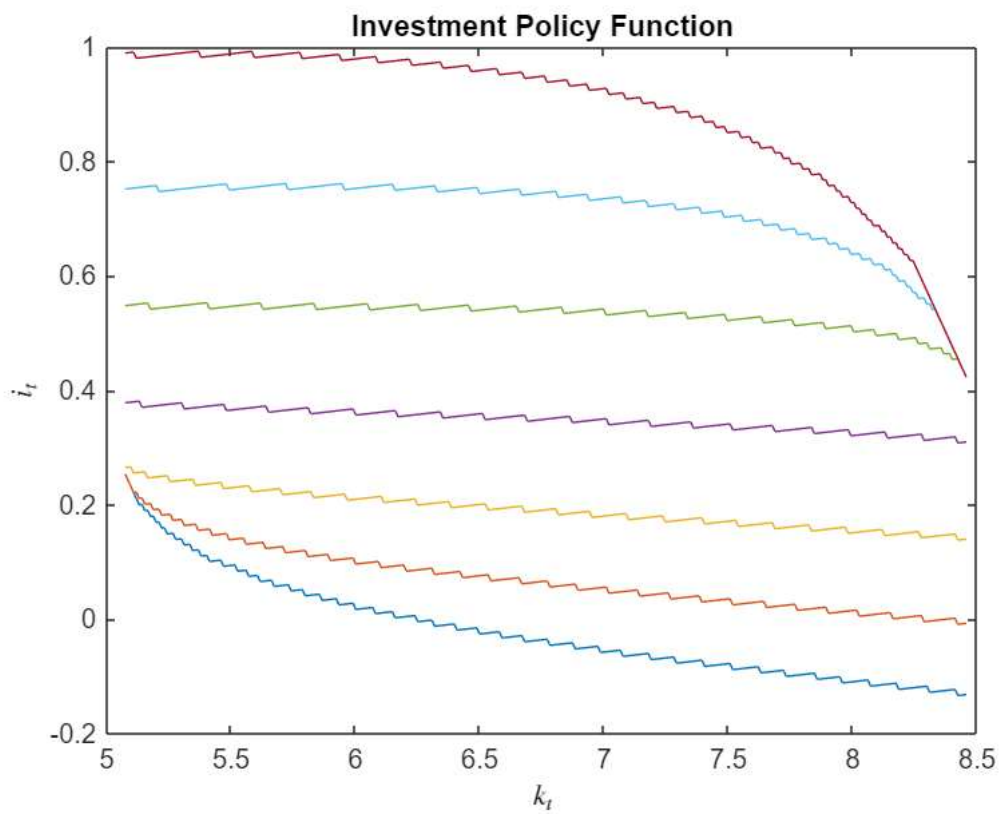## Simulate the model and plot the model functions and time series.
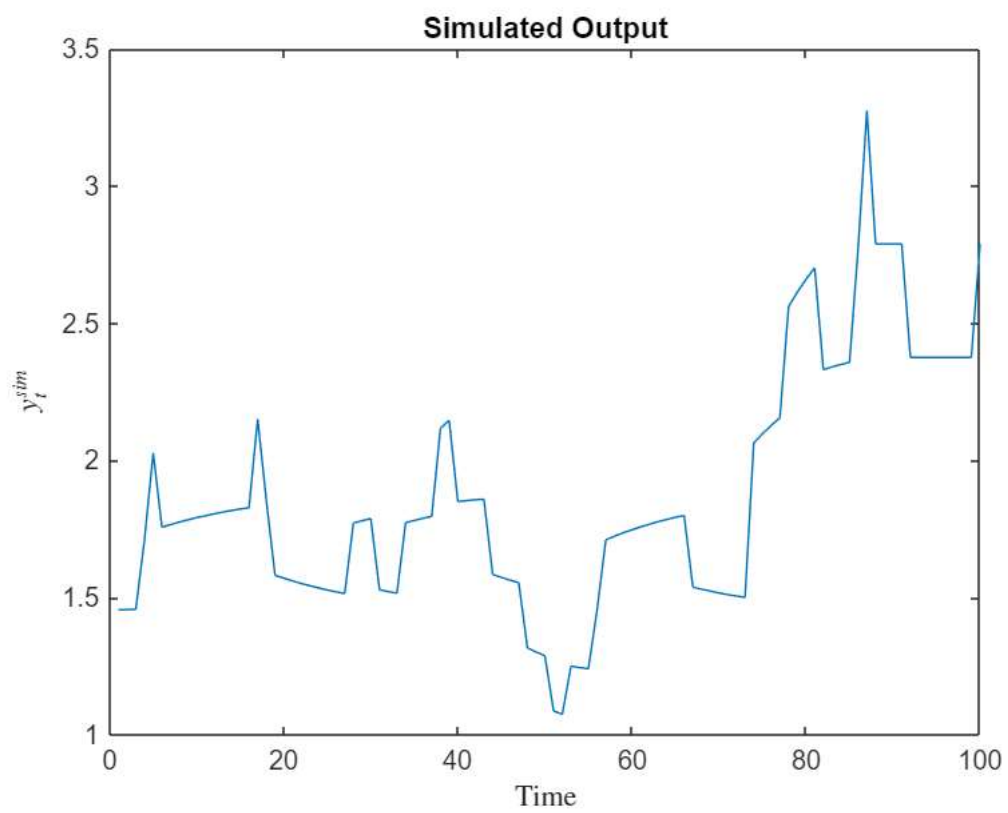
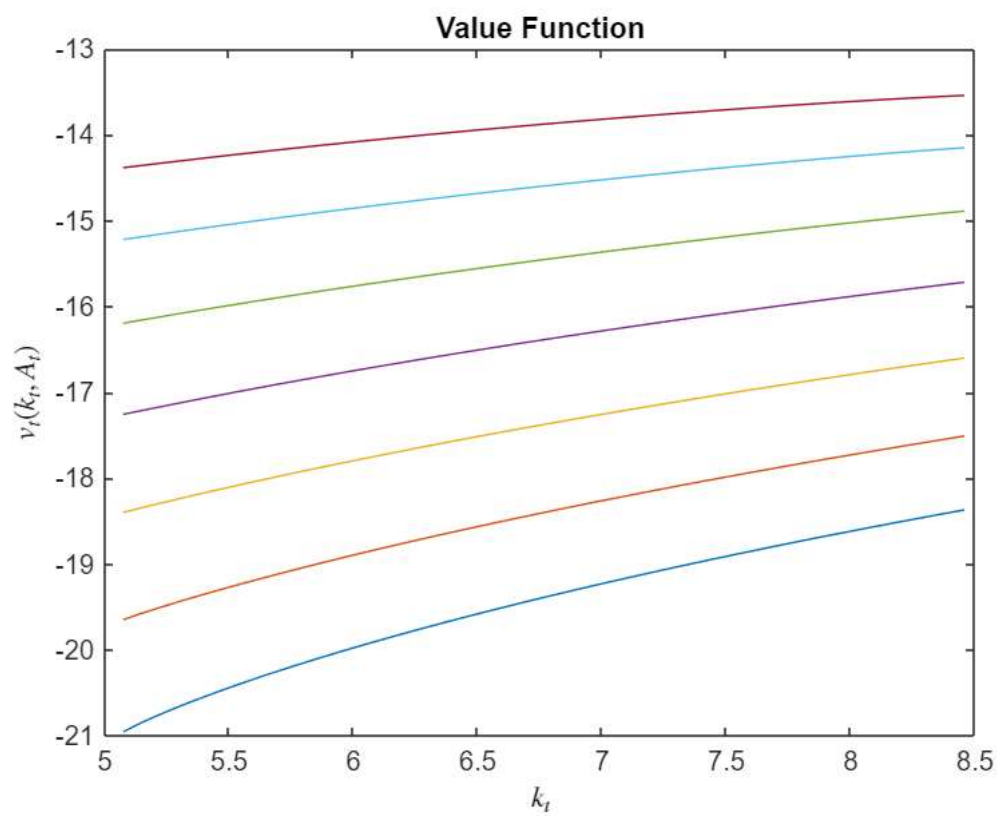Calls: simulate.m and my_graph.m.

The model functions behave in a similarly to those in the stochastic growth model. The difference now is that labor supply is negatively correlated with productivity. This is because when A_t is low, labor supply increases to boost output; when A_t is high, less labor is needed to produce a given level of output and more time can be spent on leisure. Given a Cobb-Douglas production function with constant returns to scale, the marginal product of labor is increasing with capital so n_t increases with k_t.
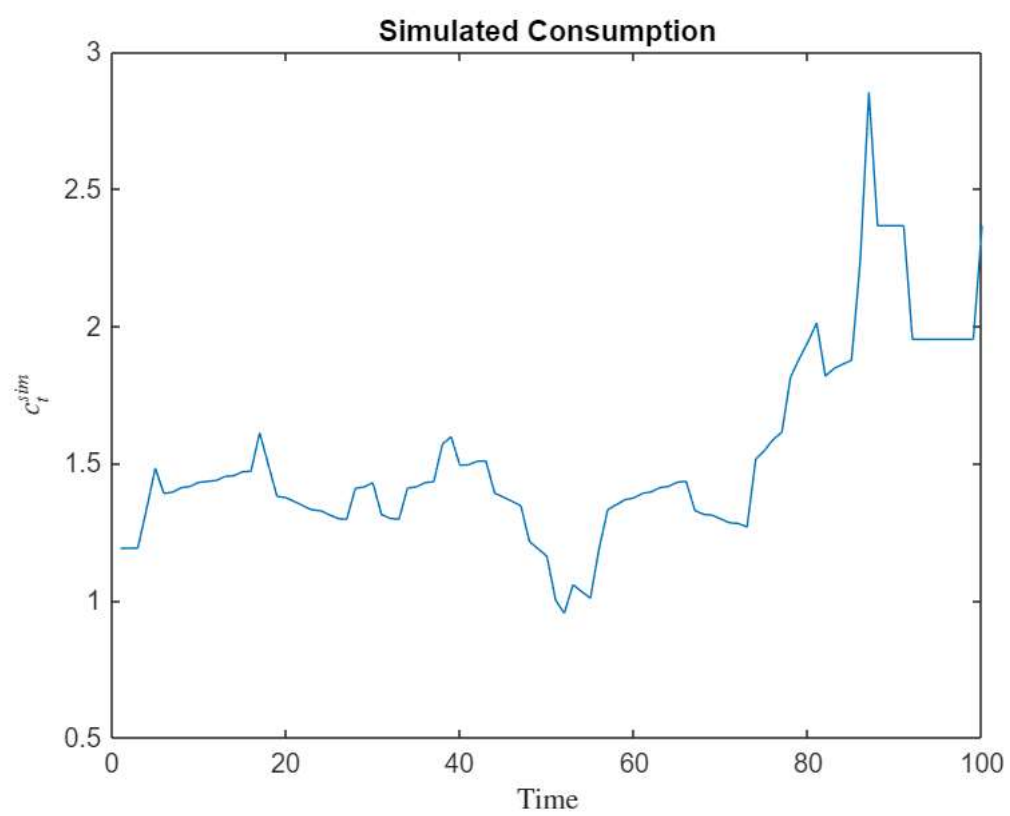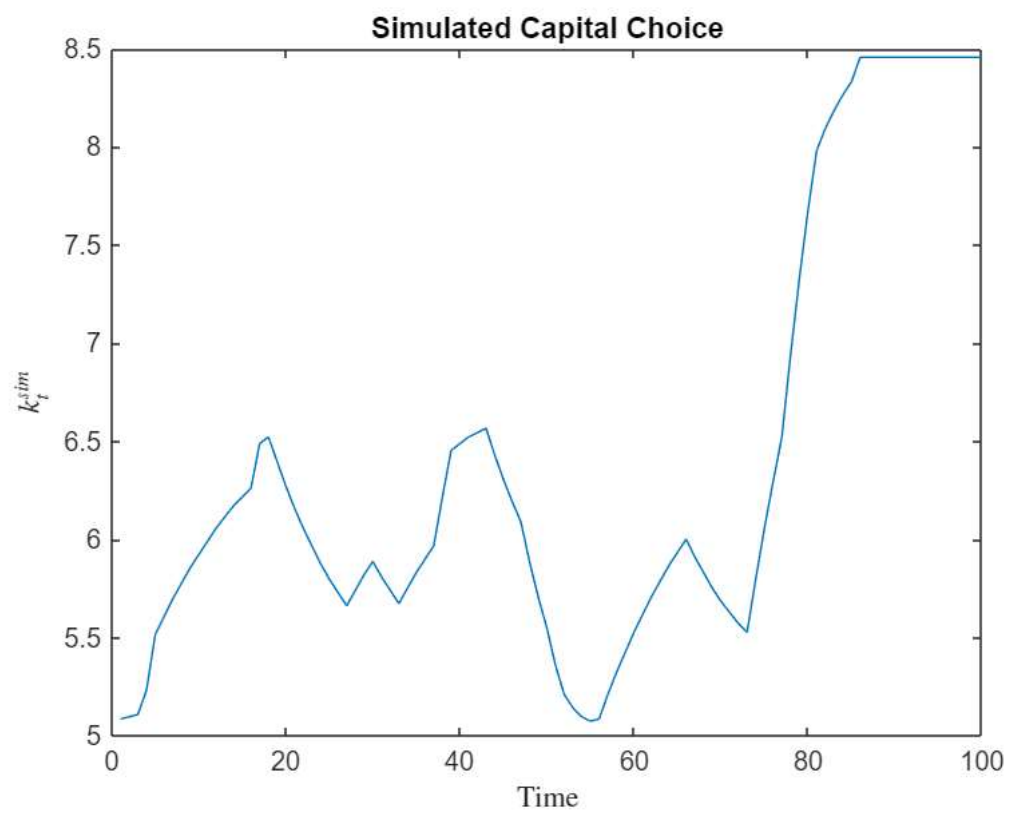
The time series data have similar qualitative results as in the model without labor supply. Labor itself is relatively constant (see the units of the graph).

Capital Policy Function

Consumption Policy Function

Investment Policy Function

Labor Supply Policy Function

Value Function

$v_t(k_t, A_t)$

k_t

Simulated Output

y_t^{sim}

Time

Simulated Capital Choice

Simulated Consumption

Simulated Investment

Simulated Labor Supply
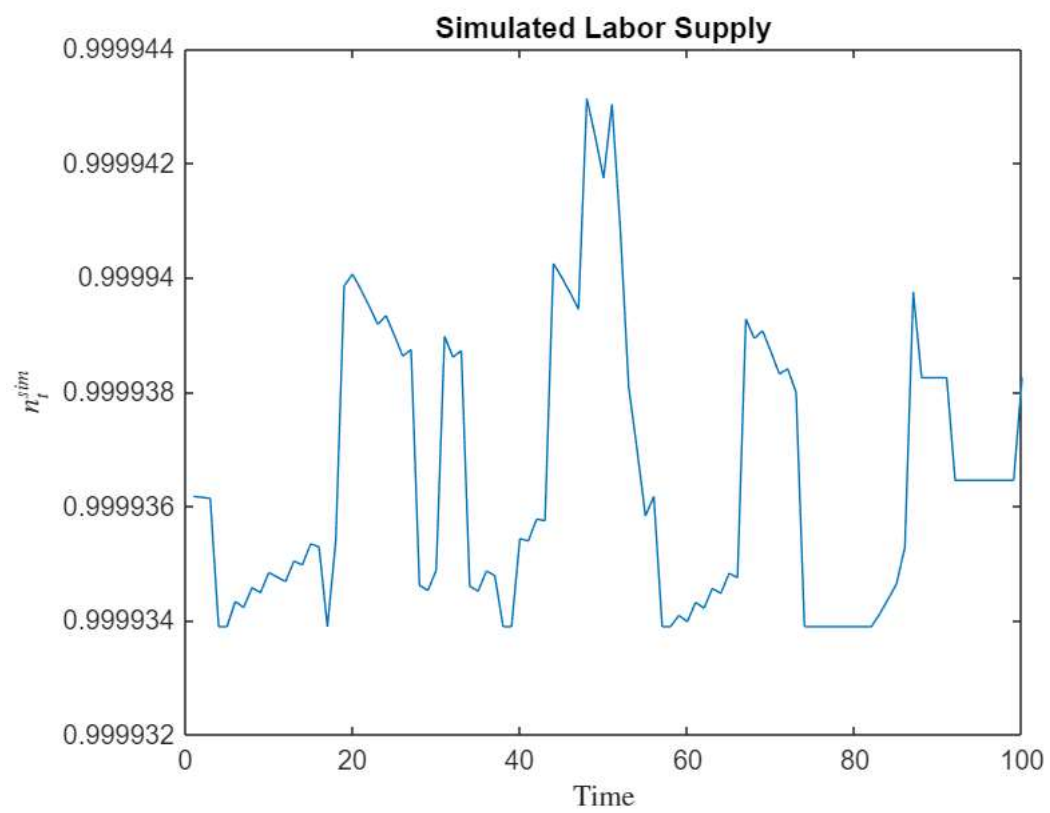
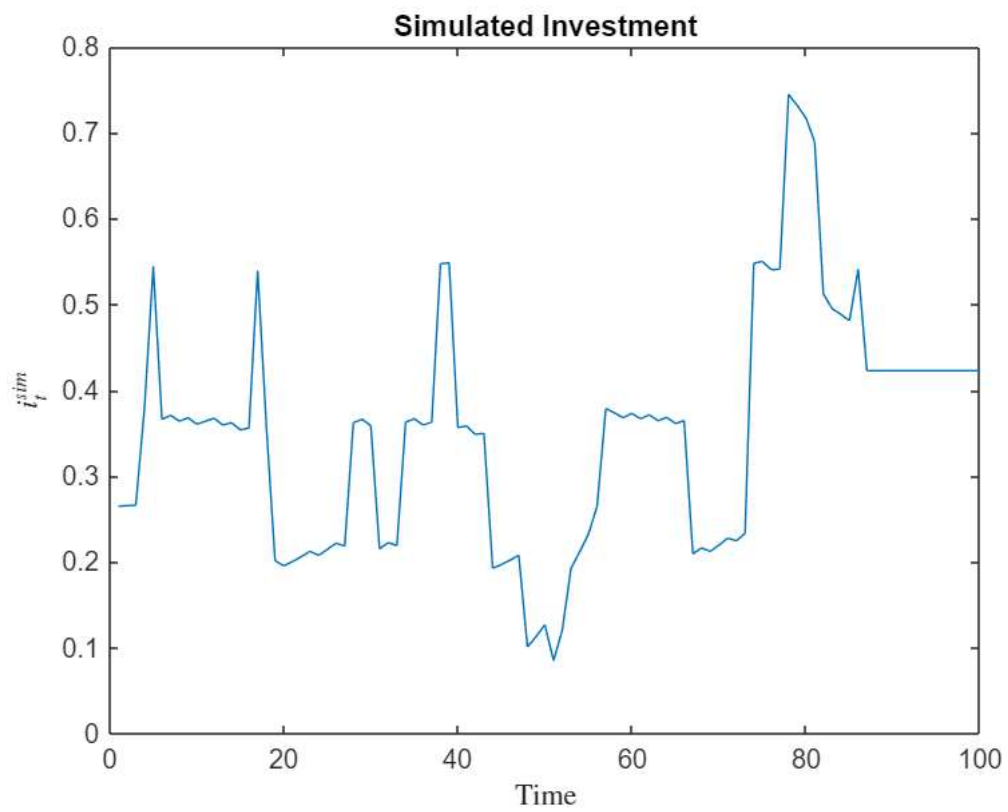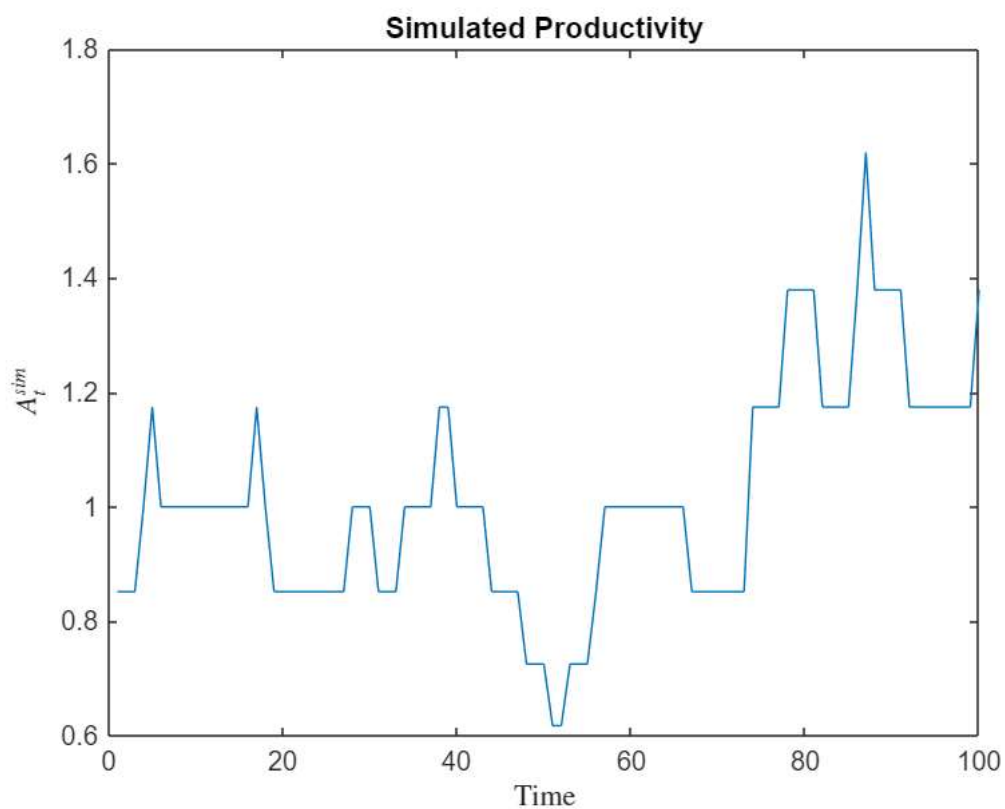**Simulated Utility**
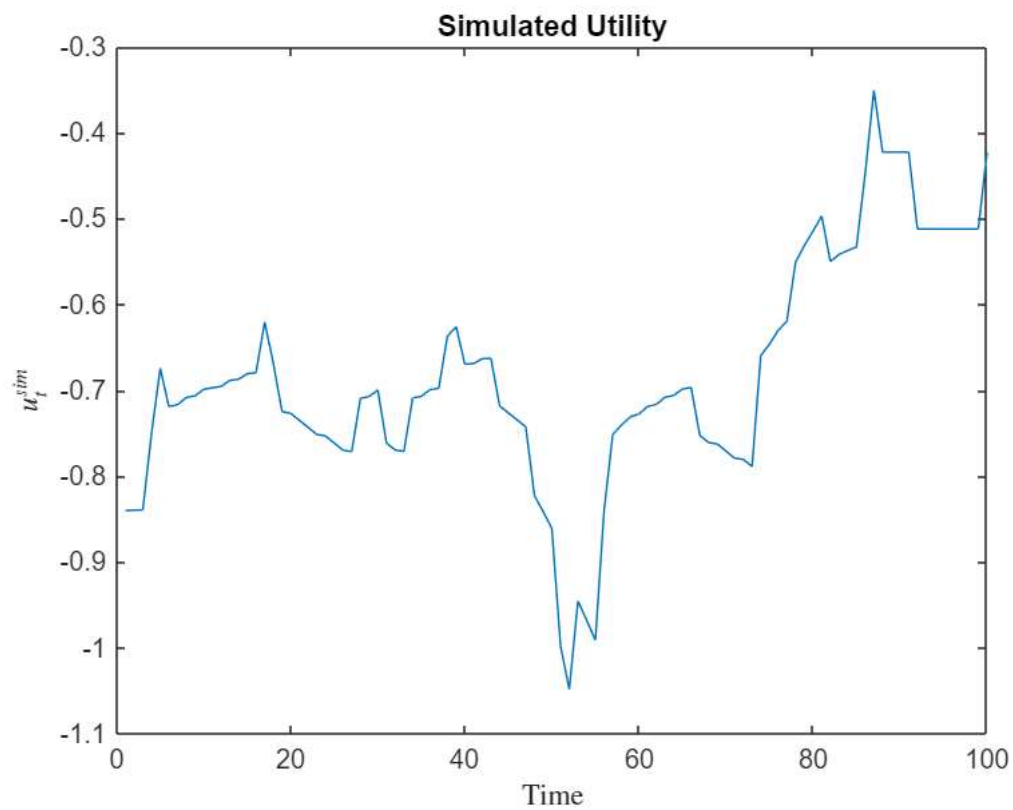

Simulated Productivity

Analysis.

To check the validity of the model, we compute statistics of interest and see how the data line up. In some cases, these are explicitly targeted by adjusting or estimating parameters that govern the behavior of the simulated data. In our case, we just set these exogenously from the literature, which in turn obtain

these from data or estimation. The model differs from the literature so this model is off from the data, see Table 5.1 in Adda and Cooper.

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The Std. Dev of Consumption Relative to Output is: 0.7747

The Std. Dev of Investment Relative to Output is: 0.3087

The Std. Dev of Labor Hours Relative to Output is: 0.0000

The Std. Dev of Real Wages Relative to Output is: 0.6801
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