Linear Algebra

LU Decomposition

Automotive Intelligence Lab.

Contents

LU Decomposition

Introduction of Triangular Matrix

- **Matrix in which values of terms above or below diagonal elements based on main diagonal are all 0.**
- **Lower triangular matrix**
  - Matrix whose $\boxed{\text{upper}}$ diagonal terms are all *0*.
- **Upper triangular matrix**
  - Matrix whose $\boxed{\text{below}}$ diagonal terms are all *0*.

$$\boldsymbol{L} = \begin{bmatrix} l_{1,1} & 0 & 0 & 0 & 0 \\ l_{2,1} & l_{2,2} & 0 & 0 & 0 \\ l_{3,1} & l_{3,2} & \ddots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & 0 \\ l_{n,1} & l_{n,2} & \cdots & l_{n,n-1} & l_{n,n} \end{bmatrix}$$

Lower triangular matrix

$$\boldsymbol{U} = \begin{bmatrix} u_{1,1} & u_{1,2} & u_{1,3} & \cdots & u_{1,n} \\ 0 & u_{2,2} & u_{2,3} & \cdots & u_{2,n} \\ 0 & 0 & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \ddots & u_{n-1,n} \\ 0 & 0 & 0 & 0 & u_{n,n} \end{bmatrix}$$

Upper triangular matrix

Remind Solving Simultaneous Equations

- **Elementary matrix**
  - Row multiplication
  - Row switching
  - Row addition

- **If we think again about process of obtaining solution through elementary row operations,**
  - Equation at bottom leaves only expression for last unknown.
  - Equation above it leaves only last two unknowns, thereby eliminating the unknowns.
  - Obtain value of last unknown from bottom equation.
  - Substitute into equation above it to obtain value of next unknown...
  - You can see that it is possible to obtain values of unknowns one by one in this order.

- **This process is called back substitution.**
  - Because it calculates from last unknown to first unknown.

$$\begin{cases} x + y + z = 6 \\ 2x + 3y - z = 5 \\ 2x + 3y + 3z = 17 \end{cases} \Rightarrow \begin{array}{l} r_2 \to r_2 - 2r_1 \\ r_3 \to r_3 - 2r_1 \\ r_3 \to r_3 - r_2 \end{array} \Rightarrow \begin{cases} x + y + z = 6 \\ 0x + y - 3z = -7 \\ 0x + 0y + 4z = 12 \end{cases} \Rightarrow \begin{array}{l} z = 3 \\ \Rightarrow r_2 \to y = 2 \\ \Rightarrow r_1 \to x = 1 \end{array}$$

Back substitution

Simultaneous Equations represented as Matrix

- **Let's find solution using matrix.**
  - Perform elementary row operations.

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 2 & 3 & -1 & 5 \\ 2 & 3 & 3 & 17 \end{array}\right]$$

$$r_2 \to r_2 - 2r_1$$
$$r_3 \to r_3 - 2r_1$$
$$r_3 \to r_3 - r_2$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 1 & -3 & -7 \\ 0 & 0 & 4 & 12 \end{array}\right]$$

Representation of simultaneous equations as augmented matrix

Back Substitution represented as Matrix

- **If elementary row operations are expressed using elementary matrices,**
  - They can be summarized as Eq 1..
- **If you think about meaning of elementary matrices attached to left of original $[\boldsymbol{A}|\boldsymbol{b}]$ matrix,**
  - They are elementary row operations as Eq 2..

$$\begin{bmatrix}1&0&0\\0&1&0\\0&-1&1\end{bmatrix}\begin{bmatrix}1&0&0\\0&1&0\\-2&0&1\end{bmatrix}\begin{bmatrix}1&0&0\\-2&1&0\\0&0&1\end{bmatrix}\left[\begin{array}{ccc|c}1&1&1&6\\2&3&-1&5\\2&3&3&17\end{array}\right]=\left[\begin{array}{ccc|c}1&1&1&6\\0&1&-3&-7\\0&0&4&12\end{array}\right]$$

Eq 1. Representation of elementary matrix as elementary matrix

$$\underbrace{\begin{bmatrix}1&0&0\\0&1&0\\0&-1&1\end{bmatrix}}_{r_3\to(r_3-r_2)}\underbrace{\begin{bmatrix}1&0&0\\0&1&0\\-2&0&1\end{bmatrix}}_{r_3\to(r_3-2r_1)}\underbrace{\begin{bmatrix}1&0&0\\-2&1&0\\0&0&1\end{bmatrix}}_{r_2\to(r_2-2r_1)}\underbrace{\left[\begin{array}{ccc|c}1&1&1&6\\2&3&-1&5\\2&3&3&17\end{array}\right]}_{[\boldsymbol{A}|\boldsymbol{b}]}=\left[\begin{array}{ccc|c}1&1&1&6\\0&1&-3&-7\\0&0&4&12\end{array}\right]$$

Eq 2. Perform back substitution with final result obtained through elementary matrix operations.

Convert Coefficient Matrix A into Upper Triangular Matrix

- **Let's try something a little different by applying this idea.**
  - If we multiply elementary matrix in same way for matrix $\boldsymbol{A}$,
    - Only has equation coefficients instead of $[\boldsymbol{A}|\boldsymbol{b}]$.
  - we can obtain form without augmenting matrix on right side.
    - The result will be in form of $\boxed{\text{upper}}$ triangular matrix introduced earlier.

$A = LU.$

$$\left(\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}}_{r_3 \to (r_3 - r_2)} \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}}_{r_3 \to (r_3 - 2r_1)} \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{r_2 \to (r_2 - 2r_1)} \right) \underbrace{\begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & -1 \\ 2 & 3 & 3 \end{bmatrix}}_{\boldsymbol{A}} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & -3 \\ 0 & 0 & 4 \end{bmatrix}$$

Convert coefficient matrix $\boldsymbol{A}$ into upper triangular matrix through elementary matrix operations.

L^{-1}

Inverse of Elementary Matrix

- **Inverse matrices of elementary matrices have very simple form.**
  - Row multiplication as Eq 1.
  - Row addition as Eq 2.
  - Elementary matrix that changes order of rows as Eq 3.

$$\boldsymbol{E} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & s & 0 \\ 0 & 0 & 1 \end{bmatrix} \rightarrow \boldsymbol{E}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/s & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Eq 1. Relationship between row multiplication matrix and its inverse matrix

$$\boldsymbol{E} = \begin{bmatrix} 1 & 0 & 0 \\ s & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \rightarrow \boldsymbol{E}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -s & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Eq 2. Relationship between elementary matrix that performs row addition and its inverse matrix

$$\boldsymbol{E} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \rightarrow \boldsymbol{E}^{-1} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

Eq 3. Relationship between elementary matrix and its inverse matrix that performs function of changing order of rows

Multiply by Inverse of Elementary Matrix

- **If you multiply inverse matrices of elementary matrices that were multiplied in front of coefficient matrix A in order,**
  - You can rewrite matrix A as follows.

$r_3 \rightarrow (r_3 - r_2)$ $\quad$ $r_2 \rightarrow (r_2 - 2r_1)$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \underset{A}{\begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & -1 \\ 2 & 3 & 3 \end{bmatrix}} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & -3 \\ 0 & 0 & 4 \end{bmatrix}$$

$r_3 \rightarrow (r_3 - 2r_1)$

Multiplying inverse matrix of elementary matrix

$$\underset{A}{\begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & -1 \\ 2 & 3 & 3 \end{bmatrix}} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & -3 \\ 0 & 0 & 4 \end{bmatrix}$$

Multiplying inverse matrix of elementary matrix

LU Decomposition

- **If calculate inverse matrices and combine them into one matrix through matrix multiplication,**
  - They can be combined into lower triangular matrix as shown in equation below.

$$\overset{A}{\begin{bmatrix}1 & 1 & 1\\ 2 & 3 & -1\\ 2 & 3 & 3\end{bmatrix}} = \begin{bmatrix}1 & 0 & 0\\ -2 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}^{-1}\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ -2 & 0 & 1\end{bmatrix}^{-1}\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & -1 & 1\end{bmatrix}^{-1}\begin{bmatrix}1 & 1 & 1\\ 0 & 1 & -3\\ 0 & 0 & 4\end{bmatrix}$$

$$= \begin{bmatrix}1 & 0 & 0\\ 2 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 2 & 0 & 1\end{bmatrix}\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 1 & 1\end{bmatrix}\begin{bmatrix}1 & 1 & 1\\ 0 & 1 & -3\\ 0 & 0 & 4\end{bmatrix}$$

$$= \underset{L}{\begin{bmatrix}1 & 0 & 0\\ 2 & 1 & 0\\ 2 & 1 & 1\end{bmatrix}}\underset{U}{\begin{bmatrix}1 & 1 & 1\\ 0 & 1 & -3\\ 0 & 0 & 4\end{bmatrix}}$$

Matrix A represented as product of lower triangular matrix and upper triangular matrix

Example of LU Decomposition

Code Exercise (10_05)

- Function for LU decomposition is in MATLAB.

$$\begin{bmatrix} 2 & 2 & 4 \\ 1 & 0 & 3 \\ 2 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ {}^{1}/_{2} & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 2 & 4 \\ 0 & -1 & 1 \\ 0 & 0 & -3 \end{bmatrix}$$

Numerical example of LU decomposition

```
% Clear workspace, command window, and close all figures
clc; clear; close all;

% Matrix A
A = [2 2 4; 1 0 3; 2 1 2];

% LU decomposition
[L,U] = lu(A);

% Display the results
disp("L")
disp(L);
disp("U")
disp(U);
```

MATLAB code of LU decomposition

Use of Permutation Matrix

- **For some matrices,**
  - LU decomposition may not be possible immediately without row swap.
  - Consider LU decomposition, which also includes row swap operations.
    - Consider matrix $\boldsymbol{A}$ as shown below.
    - Final output of this type of matrix cannot be upper triangular matrix.
      - By using only row addition or row scaling among elementary lower triangular matrices.
    - Because first and second elements in first row are already set to 0.
    - Therefore, rows of $\boldsymbol{A}$ must be changed and started.
      - In order to be able to use only elementary matrix corresponding to row addition and row scaling of lower triangular matrix.

$$\boldsymbol{A} = \begin{bmatrix} 0 & 0 & 3 \\ 1 & 1 & 1 \\ 2 & 3 & -1 \end{bmatrix}$$

Example of matrix

LU Decomposition including Row Operations

- **First, let's replace rows 1 and 3 and then consider LU decomposition.**
  - Then, $\boldsymbol{A}$ is multiplied by matrix $\boldsymbol{P}_{13}$.

$A \xrightarrow{x} LU$

$P_{13} A \rightarrow LU$

$$\boldsymbol{P}_{13} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

$$\boldsymbol{P}_{13}\boldsymbol{A} = \begin{bmatrix} 2 & 3 & -1 \\ 1 & 1 & 1 \\ 0 & 0 & 3 \end{bmatrix}$$

Result of $P_{13}A$

Result of LU Decomposition including Row Operations

- **Perform $r_2 \rightarrow r_2 - (^1/_2)r_1$.**
  - Result is an upper triangular matrix.
- **Thus, consider that you can take inverse matrix of elementary row operations and write it as follows.**

$P_{13}A$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ -1/2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 3 & -1 \\ 1 & 1 & 1 \\ 0 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 2 & 3 & -1 \\ 0 & -1/2 & 3/2 \\ 0 & 0 & 3 \end{bmatrix}$$

Result is an upper triangular matrix

$$\boldsymbol{P}_{13}\boldsymbol{A} = \begin{bmatrix} 1 & 0 & 0 \\ -1/2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 2 & 3 & -1 \\ 0 & -1/2 & 3/2 \\ 0 & 0 & 3 \end{bmatrix}$$

$$= \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 1/2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{\boldsymbol{L}} \underbrace{\begin{bmatrix} 2 & 3 & -1 \\ 0 & -1/2 & 3/2 \\ 0 & 0 & 3 \end{bmatrix}}_{\boldsymbol{U}} = LU$$

$\Rightarrow P_{13}A = LU.$

Representation of LU decomposition

PLU Decomposition

- **When performing LU decomposition by changing** row **order of matrix A to be decomposed in advance.**
- **Since inverse matrix of row permutation matrix is** itself
  - Original coefficient matrix A can be decomposed as follows.

$$\boldsymbol{A} = \begin{bmatrix} 0 & 0 & 3 \\ 1 & 1 & 1 \\ 2 & 3 & -1 \end{bmatrix}$$

$$\boldsymbol{P}_{13}\boldsymbol{L}\boldsymbol{U} = \underset{\boldsymbol{P}}{\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}} \underset{\boldsymbol{L}}{\begin{bmatrix} 1 & 0 & 0 \\ 1/2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}} \underset{\boldsymbol{U}}{\begin{bmatrix} 2 & 3 & -1 \\ 0 & -1/2 & 3/2 \\ 0 & 0 & 3 \end{bmatrix}}$$

Result of decomposition of coefficient matrix A

Code Exercise of PLU decomposition

Code Exercise (10_06)

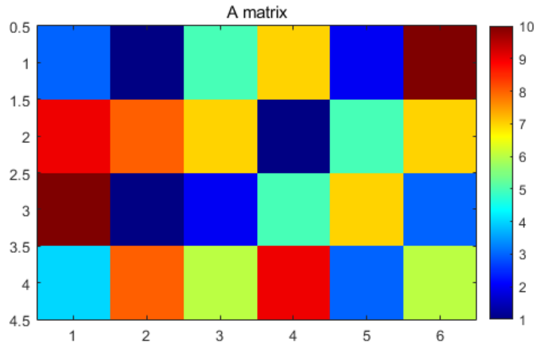
- Use MATLAB function $lu()$.

```
% Clear workspace, command window, and close all figures
clc; clear; close all;

% Matrix A
A = [3 1 5 7 2 10;
     9 8 7 1 5 7;
     10 1 2 5 7 3;
     4 8 6 9 3 6];

% LU decomposition
[L,U,P] = lu(A);

% Verify the equality of A and transpose(P)*L*U
A2 = P' * L * U;

% Visualize the results
figure;
imagesc(A); % Display the matrix as a color image
title('A matrix');
colorbar; % Show a color scale
colormap jet; % Use the jet color map
axis equal tight;  % Adjust axes to fit the data

figure;
imagesc(P');
title('transpose P Matrix');
colorbar;
colormap jet;
axis equal tight;

figure;
imagesc(L);
title('L Matrix');
colorbar;
colormap jet;
axis equal tight;

figure;
imagesc(U);
title('U Matrix');
colorbar;
colormap jet;
axis equal tight;

figure;
imagesc(A2);
title('A2 Matrix');
colorbar;
colormap jet;
axis equal tight;
```

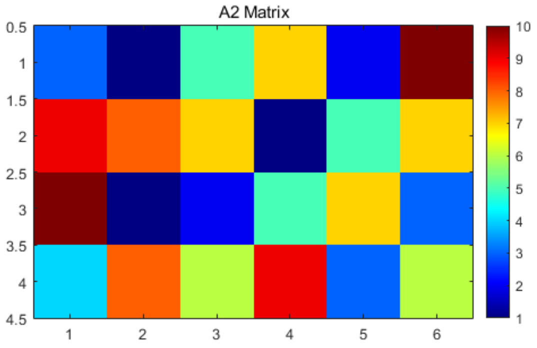
MATLAB code to verify LU decomposition with permutation matrix

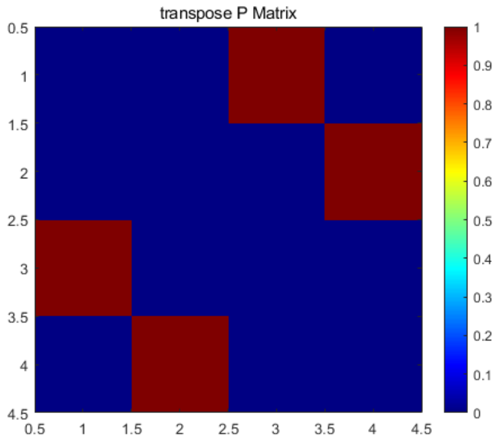
Visualization of results of PLU decomposition

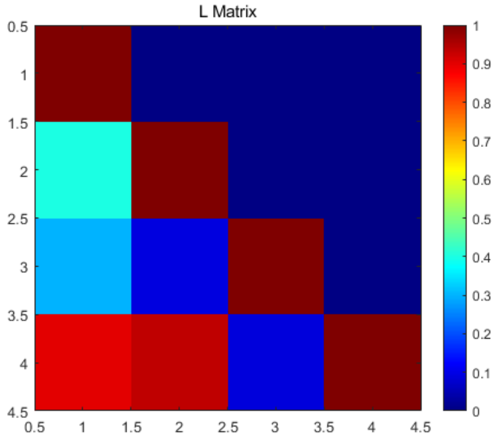
A

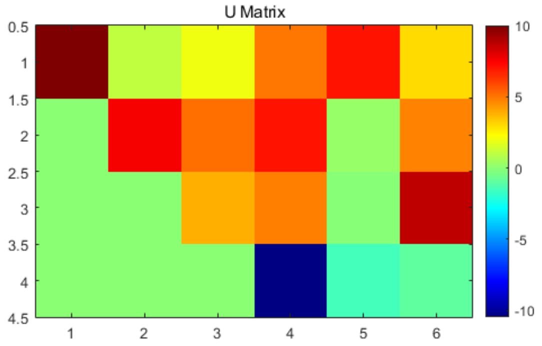


$A = P^T \times L \times U$









Visualization results of LU decomposition with permutation matrix

Use of LU Decomposition

Find solution to $Ax = b$

- If A is square matrix and can be decomposed as $A = LU$
  - You can think about it as follows.
    - Ux can also be thought of as a kind of column vector.
    - Therefore, replace it with column vector named $Ux = c$.
  - It becomes the same problem as $Lc = b$.

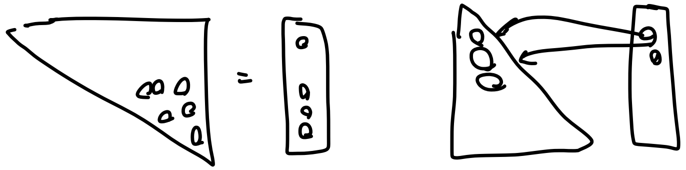
$$
\begin{aligned}
& Ax = b \\
\Rightarrow\ & (LU)x = b \\
\Rightarrow\ & L(Ux) = b \\
\Rightarrow\ & Lc = b
\end{aligned}
$$

Ux = C

Using LU decomposition to solve $Ax = b$

Characteristic of LU Decomposition

- **However, if you think about it carefully,**
  - L is lower triangular matrix.
  - Solution for lower triangular matrix can be easily obtained
    - By using forward substitution.
- **Then we solve problem as $Ux = c$, we will get answer to x.**
  - In this case, solution can be easily obtained
    - By using ______ substitution.



$Ux=c$. Backward

$Lc=b$

Example of LU Decomposition

- LU decomposition for matrix A is as Eq 1..
- In $\boldsymbol{Ax} = \boldsymbol{b}$,
  - If $\boldsymbol{b}$ is $[\mathbf{6}, \mathbf{5}, \mathbf{17}]^T$, $\boldsymbol{LUx} = \boldsymbol{b}$ is Eq 2..

$$\boldsymbol{A} = \boldsymbol{LU}$$

$$\Rightarrow \begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & -1 \\ 2 & 3 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 2 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & -3 \\ 0 & 0 & 4 \end{bmatrix}$$

Eq 1. LU decomposition for matrix $\boldsymbol{A}$

$$\overset{L}{\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 2 & 1 & 1 \end{bmatrix}} \overset{U}{\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & -3 \\ 0 & 0 & 4 \end{bmatrix}} \overset{x}{\begin{bmatrix} x \\ y \\ z \end{bmatrix}} = \overset{b.}{\begin{bmatrix} 6 \\ 5 \\ 17 \end{bmatrix}}$$

Eq 2. Substitute LU decomposition

Example of Shaping LU Decomposition

- **If $\boldsymbol{LUx} = \boldsymbol{b}$ is changed to $\boldsymbol{Lc} = \boldsymbol{b}$,**
  - It becomes Eq 1..
  - Then, we can easily know that $c_1 = 6, c_2 = -7, c_3 = 12$.
  - So considering that additional problem we need to solve is $\boldsymbol{Ux} = \boldsymbol{c}$.
    - Using Eq 2. and back-substitution, we can see that $z = 3, y = 2$, and $x = 1$.

$$\overset{L}{\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 2 & 1 & 1 \end{bmatrix}} \overset{U}{\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & -3 \\ 0 & 0 & 4 \end{bmatrix}} \overset{x}{\begin{bmatrix} x \\ y \\ z \end{bmatrix}} = \overset{b.}{\begin{bmatrix} 6 \\ 5 \\ 17 \end{bmatrix}} \Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 2 & 1 & 1 \end{bmatrix} \begin{bmatrix} \overset{6}{c_1} \\ \overset{-7}{c_2} \\ \overset{12}{c_3} \end{bmatrix} = \begin{bmatrix} 6 \\ 5 \\ 17 \end{bmatrix}$$

Result of $\boldsymbol{LUx} = \boldsymbol{b}$

$$\boldsymbol{Ux} = \boldsymbol{c} \Rightarrow \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & -3 \\ 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ -7 \\ 12 \end{bmatrix}$$

Result of $\boldsymbol{Ux} = \boldsymbol{c}$

Properties of Determinant

- **In same principle, if matrix A can be LU decomposed,**
  - You can consider Eq 1..
- **Eq 2. is established.**
  - Due to properties of determinant.

$$\boldsymbol{A} = \boldsymbol{L}\boldsymbol{U}$$

Eq 1. LU decomposition

$$det(\boldsymbol{A}) = det(\boldsymbol{L})det(\boldsymbol{U})$$

Eq 2. Property of determinant

Easy Way to Obtain Determinant

- **However, determinant of A can be easily obtained.**
  - since both L and U are triangular matrices, considering that determinant is calculated only by multiplying $\boxed{\text{diagonal}}$ components.
- **In other words, if L and U decomposed from A are the same as lower triangular matrix and upper triangular matrix,**
  - Determinant of A is the same as Eq 1..
  - It can be calculated simply.

$$det(\boldsymbol{A}) = \prod_{i=1}^{n} l_{i,i} \prod_{j=1}^{n} l_{j,j} = \prod_{i=1}^{n} l_{i,i} u_{i,i}$$

Eq 1. Determinant of $\boldsymbol{A}$

Gauss-Jordan Elimination

Introduction

- **In LU decomposition, row operation is not only possible on square matrix.**
  - How to create something similiar to an upper triangular matrix.
    - When number of expressions and variables are different ?
  - What if all the numbers on the diagonal elements are eliminated?
    - By taking a row operation.

$$\boldsymbol{Ax} = \boldsymbol{b}$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

$$\boldsymbol{Ux} = \boldsymbol{c}$$

$$\begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$$

$$\boldsymbol{E_3E_2E_1} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \boldsymbol{E_3E_2E_1} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

Process to get upper triangular matrix by basic row operations

LU Decomposition and REF, RREF

REF: Row-Echelon Form

RREF: $\boxed{\text{Reduced}}$ Row-Echelon Form

Performing a row operation on a rectangular matrix.

- Same as obtaining upper triangular matrix through LU decomposition.
- Matrix in below figure: Row-Echelon matrix
  - Or called Row-Echelon matrix of given matrix.
  - ▲, -: non-zero elements.

$$\begin{bmatrix} \blacktriangle & - & - & - & - & - & - & - \\ 0 & \blacktriangle & - & - & - & - & - & - \\ 0 & 0 & \blacktriangle & - & - & - & - & - \\ 0 & 0 & 0 & \blacktriangle & - & - & - & - \\ 0 & 0 & 0 & 0 & 0 & 0 & \blacktriangle & - \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

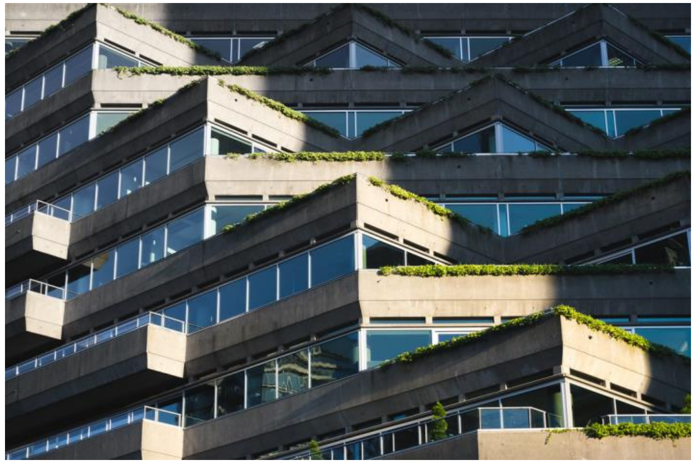
Form of Row-Echelon matrix

What is Echelon?

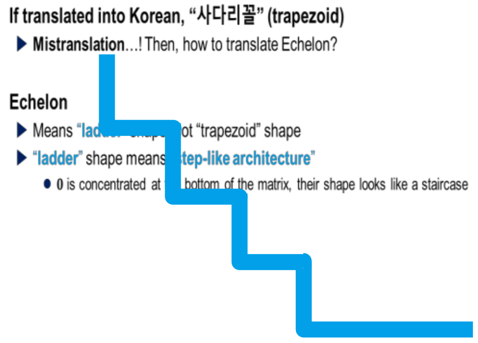
- **If translated into Korean, “사다리꼴” (trapezoid)**
  - ▶ **Mistranslation**…! Then, how to translate Echelon?

- **Echelon**
  - ▶ Means “**ladder**” shape, not “trapezoid” shape
  - ▶ “**ladder**” shape means “step-like **architecture**”
    - **0** is concentrated at the bottom of the matrix, their shape looks like a staircase

Trapezoid architecture



Step-like architecture



Form of Row-Echelon matrix

Three Characteristics of Row-Echelon Matrix

- **All non-zero rows are above row 0.**
  - ▶ Rows where all elements are **0** are at the $\boxed{\text{bottom}}$ of the matrix.

- **Leading coefficient in a non-zero row**
  - ▶ Always exists to the $\boxed{\text{right}}$ of the first non-zero entry in the row above.

- **All column entries under pivot are $\boxed{0}$.**
  - ▶ Pivot: Part where you step on the foot at the end of each step.

Pivots

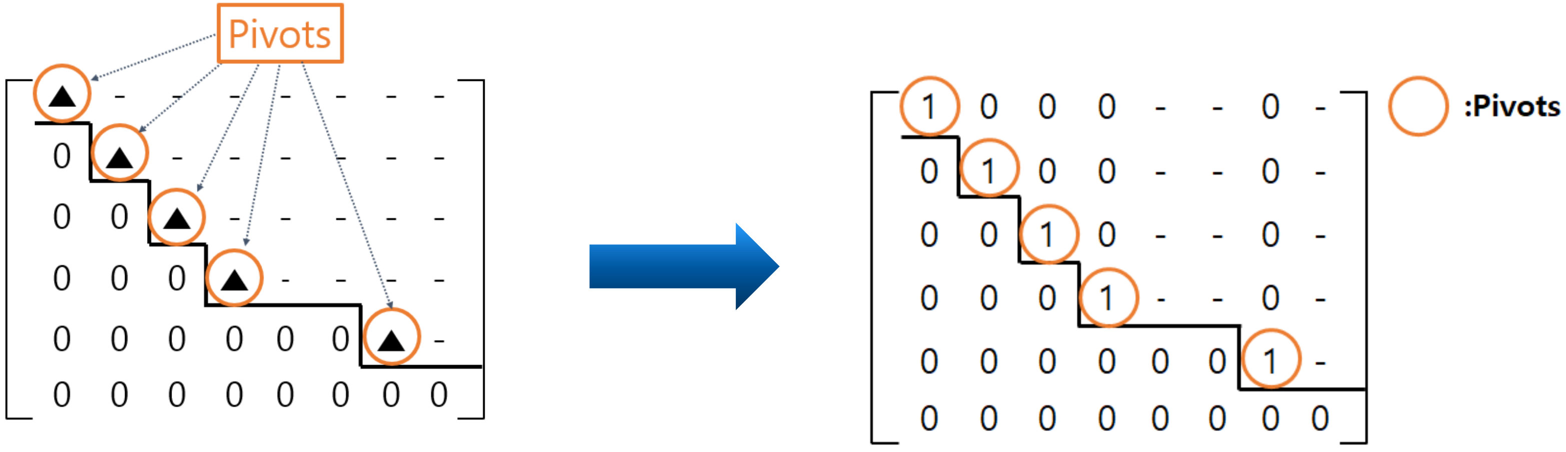
$$\begin{bmatrix} \blacktriangle & - & - & - & - & - & - & - \\ 0 & \blacktriangle & - & - & - & - & - & - \\ 0 & 0 & \blacktriangle & - & - & - & - & - \\ 0 & 0 & 0 & \blacktriangle & - & - & - & - \\ 0 & 0 & 0 & 0 & 0 & 0 & \blacktriangle & - \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Form of Row-Echelon matrix

Characteristics of Reduced Row-Echelon Matrix

■ Reduced Row-Echelon Form (RREF)

- ▶ Make all pivots as (.
- ▶ Make the numbers above the pivots **0** as well.



Reduced Row-Echelon Form

Example of form of REF

- **Distinguish REF with 5 example matrices!**
  - Hint: Consider three characteristics of Row-Echelon matrix
  - −: non zero number

- **Is the first matrix in row-echelon form?**
  - $\boxed{\text{Yes.}}$, then why is it?
- **Is the second matrix in row-echelon form?**
  - $\boxed{\text{Yes}}$, then why is it?
- **Is the third matrix in row-echelon form?**
  - $\boxed{\text{No}}$, then why isn't it?
- **Is the fourth matrix in row-echelon form?**
  - $\boxed{\text{No}}$, then why isn't it?
- **Is the fifth matrix in row-echelon form?**
  - $\boxed{\text{No}}$, then why isn't it?

$$\begin{bmatrix} 1 & - & - & - \\ 0 & 1 & - & - \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Matrix 1

$$\begin{bmatrix} 0 & 3 & - & - & - & - & - \\ 0 & 0 & 2 & - & - & - & - \\ 0 & 0 & 0 & 0 & 0 & 5 & - \\ 0 & 0 & 0 & 0 & 0 & 0 & - \end{bmatrix}$$

Matrix 2

$$\begin{bmatrix} 1 & - & - & - \\ 0 & 2 & - & - \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Matrix 3

$$\begin{bmatrix} 0 & 1 & - & - \\ 1 & - & - & - \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Matrix 4

$$\begin{bmatrix} 3 & - & - & - \\ 0 & 4 & - & - \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Matrix 5

Example of form of RREF

- **Distinguish RREF with 3 example matrices!**
  - Hint: Consider definition of RREF

- **Is the first matrix in row-echelon form?**
  - Yes, then why is it?
- **Is the second matrix in row-echelon form?**
  - NO, then why isn't it?
- **Is the third matrix in row-echelon form?**
  - Yes., then why is it?

$$\begin{bmatrix} 1 & 0 & 3 & 2 \\ 0 & 1 & 4 & 5 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Matrix 1

$$\begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 4 & 1 & 5 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Matrix 2

$$\begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 1/4 & 5/4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Matrix 3

Get REF and RREF by Hand

- **Let's find REF and RREF with matrix A with elementary row operation**

$$\begin{bmatrix} 1 & 1 & 2 & 3 & 4 \\ 2 & 6 & 4 & 6 & 2 \\ 3 & 3 & 8 & 12 & 17 \end{bmatrix}$$

Matrix A

- **Step 1.**
  - $r_2 \rightarrow r_2 - 2r_1$
    $r_3 \rightarrow r_3 - 3r_1$

$$\begin{bmatrix} 1 & 1 & 2 & 3 & 4 \\ 0 & 4 & 0 & 0 & -6 \\ 0 & 0 & 2 & 3 & 5 \end{bmatrix} \longrightarrow \text{REF}$$

Result of Step 1.

- **Step 2.**
  - $r_2 \rightarrow \frac{1}{4} r_2$
    $r_2 \rightarrow \frac{1}{2} r_3$

$$\begin{bmatrix} 1 & 1 & 2 & 3 & 4 \\ 0 & 1 & 0 & 0 & -3/2 \\ 0 & 0 & 1 & 3/2 & 5/2 \end{bmatrix}$$

Result of Step 2.

- **Step 3.**
  - $r_1 \rightarrow r_1 - r_2$
    $r_1 \rightarrow r_1 - 2r_3$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 1/2 \\ 0 & 1 & 0 & 0 & -3/2 \\ 0 & 0 & 1 & 3/2 & 5/2 \end{bmatrix} \longrightarrow \text{RREF}$$

Result of Step 3.

$$\begin{bmatrix} 1 & 1 & 2 & 3 & 4 \\ 0 & 4 & 0 & 0 & -6 \\ 0 & 0 & 2 & 3 & 5 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 1 & 1 & 2 & 3 & 4 \\ 0 & 1 & 0 & 0 & -1.5 \\ 0 & 0 & 1 & 1.5 & 2.5 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 & -2.5 \\ 0 & 1 & 0 & 0 & -1.5 \\ 0 & 0 & 1 & 1.5 & 2.5 \end{bmatrix}$$

Get REF Using MATLAB

REF is not unique.

- In MATLAB, result of REF can be different from answer obtained by hand.
- Even if the pivot value is not abbreviated when calculating the REF of a certain matrix, it is still treated as REF.

$$\begin{bmatrix} 1 & 1 & 2 & 3 & 4 \\ 2 & 6 & 4 & 6 & 2 \\ 3 & 3 & 8 & 12 & 17 \end{bmatrix}$$

Matrix A

$$\begin{bmatrix} 1 & 1 & 2 & 3 & 4 \\ 0 & 4 & 0 & 0 & -6 \\ 0 & 0 & 2 & 3 & 5 \end{bmatrix}$$

$REF(A)_1$

$$\begin{bmatrix} 1 & 1 & 2 & 3 & 4 \\ 0 & 2 & 0 & 0 & -3 \\ 0 & 0 & 2 & 3 & 5 \end{bmatrix}$$

$REF(A)_2$

```matlab
% Clear workspace, command window, and close all figures
clc; clear; close all;

% Matrix A
A = [1 1 2 3 4;2 6 4 6 2; 3 3 8 12 17];

% Calculate REF of A
[~,ref_A] = lu(A);

% Display the result
disp("A")
disp(A);
disp("ref_A")
disp(ref_A);
```

MATLAB code

```
A
     1     1     2     3     4
     2     6     4     6     2
     3     3     8    12    17

ref_A
    3.0000    3.0000    8.0000   12.0000   17.0000
         0    4.0000   -1.3333   -2.0000   -9.3333
         0         0   -0.6667   -1.0000   -1.6667
```

Answer of MATLAB

Get RREF Using MATLAB

RREF is unique.

- In MATLAB, result of RREF always same with answer obtained by hand.
- In MATLAB, we can obtain RREF using a function called $\boxed{}$.
- RREF is unique because it decomposes the pivot and eliminates all elements above the pivot.

$$\begin{bmatrix} 1 & 1 & 2 & 3 & 4 \\ 2 & 6 & 4 & 6 & 2 \\ 3 & 3 & 8 & 12 & 17 \end{bmatrix}$$

Matrix A

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 1/2 \\ 0 & 1 & 0 & 0 & -3/2 \\ 0 & 0 & 1 & 3/2 & 5/2 \end{bmatrix}$$

$RREF(A)$

```
% Clear workspace, command window, and close all figures
clc; clear; close all;

% Matrix A
A = [1 1 2 3 4;2 6 4 6 2; 3 3 8 12 17];

% Calculate RREF of A
rref_A = rref(A);

% Display the result
disp("A")
disp(A);
disp("rref_A")
disp(rref_A);
```

MATLAB code

```
A
     1     1     2     3     4
     2     6     4     6     2
     3     3     8    12    17

rref_A
    1.0000         0         0         0    0.5000
         0    1.0000         0         0   -1.5000
         0         0    1.0000    1.5000    2.5000
```

Answer of MATLAB

Application of RREF

- **When multiple types of b vectors in $Ax = b$,**
  - x can be obtained all at once by RREF.
- **Using augmented matrix and performing Gauss-Jordan elimination, it is possible to solve three equations at once!**

$$\begin{cases} 3x - z = 1 \\ x + 2y + 3z = 1 \\ 2x - y + z = 1 \end{cases} \quad \begin{cases} 3x - z = 2 \\ x + 2y + 3z = 2 \\ 2x - y + z = 2 \end{cases} \quad \begin{cases} 3x - z = 3 \\ x + 2y + 3z = 3 \\ 2x - y + z = 3 \end{cases}$$

3 equations

$$\begin{bmatrix} 3 & 0 & -1 \\ 1 & 2 & 3 \\ 2 & -1 & 1 \end{bmatrix}\begin{bmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 1 & 2 & 3 \end{bmatrix}$$

$[\boldsymbol{A}|\boldsymbol{B}]$

$$\left[\begin{array}{ccc|ccc} 3 & 0 & -1 & 1 & 2 & 3 \\ 1 & 2 & 3 & 1 & 2 & 3 \\ 2 & -1 & 1 & 1 & 2 & 3 \end{array}\right] \rightarrow \text{RREF} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & & & \\ 0 & 1 & 0 & & & \\ 0 & 0 & 1 & & & \end{array}\right]$$

Augmented matrix

$$\left[\begin{array}{ccc|ccc} 3 & 0 & -1 & 1 & 0 & 0 \\ 1 & 2 & 3 & 0 & 1 & 0 \\ 2 & -1 & 1 & 0 & 0 & 1 \end{array}\right]$$

Gauss – Jordan Elimination | *Make RREF Form*

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 2/5 & 4/5 & 6/5 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1/5 & 2/5 & 3/5 \end{array}\right]$$

Answer of equation

$$\begin{bmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{bmatrix}$$

Gauss-Jordan Elimination

Application of RREF: Inverse Matrix

How to get inverse matrix by Gauss-Jordan elimination?

- Apply the fact that we can use an augmented matrix.
- Matrix $\boldsymbol{B}$ must be multiplied by matrix $\boldsymbol{A}$, and matrix $\boldsymbol{I}$ should be obtained as a result.
  - It means matrix $\boldsymbol{B}$ is the inverse of the matrix $\boldsymbol{A}$.
    - $\boldsymbol{B} = \boxed{A^{-1}}$

$$\begin{cases} 3x - z = 1 \\ x + 2y + 3z = 1 \\ 2x - y + z = 1 \end{cases}$$

$$\begin{cases} 3x - z = 2 \\ x + 2y + 3z = 2 \\ 2x - y + z = 2 \end{cases}$$

$$\begin{cases} 3x - z = 3 \\ x + 2y + 3z = 3 \\ 2x - y + z = 3 \end{cases}$$

3 equations

$\boldsymbol{Set}\ [\boldsymbol{A}|\boldsymbol{I}]$

$$\left[\begin{array}{ccc|ccc} 3 & 0 & -1 & 1 & 0 & 0 \\ 1 & 2 & 3 & 0 & 1 & 0 \\ 2 & -1 & 1 & 0 & 0 & 1 \end{array}\right]$$

$\boldsymbol{Gauss-Jordan\ Elimination}$ $\boldsymbol{Make}\ [\boldsymbol{I}|\boldsymbol{A}^{-1}]$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/4 & 1/20 & 1/10 \\ 0 & 1 & 0 & 1/4 & 1/2 & -1/2 \\ 0 & 0 & 1 & -1/4 & 3/20 & 3/10 \end{array}\right] \boldsymbol{A}^{-1}$$

Agumented matrix

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Row Equivalent

- **Solution is still same.**
  - Even if row operation is applied.
  - Original matrix: A
    - REF of original matrix: U
    - RREF of original matrix: R
  - All of solution x is same.
    - $Ax = b\ , Ux = c, Rx = d$
    - c, d the vector b on the right side transformed
      - By changing the original matrices A and U.
  - A, U, R: row equivalent
    - No change in the row space.
      - Even if a row operation is performed

$$\begin{cases} 3x + 3y + z = 3 \\ 4x + 5y + 2z = 1 \\ 2x + 5y + z = 3 \end{cases}$$

$$\begin{bmatrix} 3 & 3 & 1 \\ 4 & 5 & 2 \\ 2 & 5 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 3 \end{bmatrix} \quad : Ax = b$$

$$\begin{bmatrix} 4 & 5 & 2 \\ 0 & 5/2 & 0 \\ 0 & 0 & -1/2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 5/2 \\ 3 \end{bmatrix} \quad : Ux = c$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 6 \end{bmatrix} \quad : Rx = \mathrm{d}$$

$$All\ of\ solution: x = 2, y = 1, z = -6$$

Row equivalent

Application of REF: Determination of Linear Dependence of rows

Review of linear dependent or independent

- If Eq 1. can be established with $\boldsymbol{c_1}$ and $\boldsymbol{c_2}$ other than $\mathbf{0}$, then two vectors $\boldsymbol{v_1}$ and $\boldsymbol{v_2}$ are linear ______.
  - If can't established, then linear ______.

$$\boldsymbol{c_1 v_1 + c_2 v_2 = 0}$$

Eq 1.

Obtaining REF or RREF

- Performed through a ______
- If some row elements becomes all zero,
  - That row could be obtained by a linear combination of other rows.
  - That row is **linearly dependent** with other rows.

$$A = \begin{bmatrix} 5 & 3 & 3 \\ 5 & 1 & 3 \\ 5 & 2 & 3 \end{bmatrix} \xrightarrow[\boldsymbol{r_3 \rightarrow r_3 - \frac{1}{2}r_1 - \frac{1}{2}r_2}]{\boldsymbol{To\ make\ REF}} A = \begin{bmatrix} 5 & 3 & 3 \\ 5 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix}$$

Third row eliminated through row operation

Summary

Summary

LU decomposition

- To decompose matrix $\boldsymbol{A} = \boldsymbol{L}\boldsymbol{U}$
  - $\boldsymbol{L}$: lower triangular matrix
  - $\boldsymbol{U}$: upper triangular matrix

Gauss-Jordan Elimination

- Row operation to make form of matrix:
  - Row echelon form(REF).
  - Reduced echelon form(RREF).

- Application of REF and RREF.
  - Can get solution from multiple equations.
  - Calculate matrix inverse.
  - Can determine linear dependence of rows.

Code Exercises

Solving Simultaneous Equations

- **Implement simultaneous equation in this lecture**
- **In Eq 1. you can get matrix B after solving simultaneous equation**
  - $r_2 \rightarrow (r_2 - 2r_1)$
  - $r_3 \rightarrow (r_3 - 2r_1)$
  - $r_3 \rightarrow (r_3 - r_2)$
- **Make 3 elementary matrix and multiply with matrix A for get matrix B**

$$\text{3 elementary matrix} \times \underbrace{\begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & -1 \\ 2 & 3 & 3 \end{bmatrix}}_{A} = \underbrace{\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & -3 \\ 0 & 0 & 4 \end{bmatrix}}_{B}$$

Eq 1.

```
% Clear workspace, command window, and close all figures
clc; clear; close all;

% Matrix A
A = [1 1 1;2 3 -1; 2 3 3];

% Matrix B
B = [1 1 1;0 1 -3;0 0 4];

% Implement three elementary Matrix
% like elem = [1 0 0;0 1 0;0 0 1];
% r_2 > r_2 - 2*r_1
elem_1 = ;
% r_3 > r_3 - 2*r_1
elem_2 = ;
% r_3 > r_3 - r_2
elem_3 = ;

% and multiply with matrix A
result = ;
% check that result are same
disp(B);
disp(result);
```

Sample code

LU Implementation

- **Implement LU decomposition in this lecture**
  - Do not use lu() function in MATLAB
- **You can use three elementary matrix in previous exercise**

$$\underset{A}{\begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & -1 \\ 2 & 3 & 3 \end{bmatrix}} = \mathbf{L} * \underset{U}{\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & -3 \\ 0 & 0 & 4 \end{bmatrix}}$$

Eq 1.

```
% Clear workspace, command window, and close all figures
clc; clear; close all;

% Matrix A
A = [1 1 1;2 3 -1; 2 3 3];

% Matrix B
U = [1 1 1;0 1 -3;0 0 4];

% Implement three elementary Matrix
% like elem = [1 0 0;0 1 0;0 0 1];
% r_2 > r_2 - 2*r_1
elem_1 = ;
% r_3 > r_3 - 2*r_1
elem_2 = ;
% r_3 > r_3 - r_2
elem_3 = ;
% get inverse matrix
elem_1_inv = ;
elem_2_inv = ;
elem_3_inv = ;
% and multiply with matrix A
L = ;
% display L & U
disp(L);
disp(U);
% Check that L * U is same with A
disp('--');
disp(L * U);
disp(A);
```

Sample code

THANK YOU
FOR YOUR ATTENTION