Linear Algebra

Eigen value decomposition : Part2

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Contents

Eigendecomposition of Singular Matrices

Wrong Idea about Egiendecomposition of Singular Matrices

- **Students often get idea.**
  - Singular matrices cannot be eigen decomposed
  - Eigenvectors of singular matrix must be unusual somehow.
- **That idea is completely wrong!**
  - Eigendecomposition of singular matrices is pefectly fine.

Code Exercise of Eigendecomposition of Singular Matrix

Code Exercise (13_03)

- This rank-2 matrix has one zero-valued eigenvalue with nonzeros eigenvector.
- Explore eigendecomposition of other reduced-rank random matrices by using example code.

```
% Clear workspace, command window, and close all figures
clc; clear; close all;

% Define the matrix
A = [1 4 7; 2 5 8; 3 6 9];

% Calculate the matrix rank
rankA = rank(A);

% Eigen decomposition
[V, D] = eig(A);

% Display the results
disp('Rank =')
disp(rankA);
disp('Eigenvalues:');
disp(diag(D));
disp('Eigenvectors:');
disp(V);

% Optionally round eigenvalues and eigenvectors for display
disp('Rounded Eigenvalues:');
disp(round(diag(D).',2)); % Round and transpose for horizontal display
disp('Rounded Eigenvectors:');
disp(round(V, 2));
```

MATLAB code of eigendecomposition of singular matrix

One Special Property of Eigendecomposition of Singular Matrices

- **At least one eigenvalue is guaranteed to be zero.**
  - That doesn't mean that number of nonzero eigenvalues equals rank of matrix.
  - True for $\boxed{\text{Singular Values}}$
    - Scalar values from the SVD (Singular Value Decomposition)
  - Not for $\boxed{\text{eigen values}}$
    - But if matrix is singular, then at least one eigenvalue equals zero.
- **Converse is also true.**
  - Every full-rank matrix has zero zero-valued eigenvalues.
- **Why this happens**
  - Singular matrix already has nontrivial null space.
    - $\lambda = 0$ provides nontrivial solution to $(\boldsymbol{A} - \lambda \boldsymbol{I})\boldsymbol{v} = 0$.

이미 singular

- **Main take-homes of this section**
  - Eigendecomposition is valid for reduced-rank matrices.
  - Presence of at least one zero-valued eigenvalue indicates reduced-rank matrix.

Quadratic Form, Definiteness, and Eigenvalues

Quadratic Form and Definiteness

- **Quadratic form and definiteness are intimidating terms.**
  - Don’t worry.
  - They are both straightforward concepts that provide gateway to advanced linear algebra and applications.
  - advanced linear algebra technique such as
    - Principal components analysis (PCA)
    - Monte Carlo simulations
  - Integrating MATLAB code into your learning will give you huge advantage over learning about these concepts.
    - Compared to traditional linear algebra textbooks.

Quadratic Form of Matrix

- **Consider Eq 1..**
  - Pre- and postmultiply square matrix by same vector $\boldsymbol{w}$ and get scalar.
    - Notice: This multiplication is valid only for square matrices.
- **This is called** $\boxed{\text{quadratic form.}}$ **on matrix $\boldsymbol{A}$.**

- **Which matrix and which vector do we use?**
  - Idea of quadratic form
    - To use one specific matrix.
    - To set of all possible vectors.
      - Appropriate size
  - Important point
    - Signs of α for all possible vectors.

$$\boldsymbol{w}^T \boldsymbol{A} \boldsymbol{w} = \alpha$$

Eq 1. Quadratic form of matrix

Example of Quadratic Form of Matrix

- **For this particular matrix as Eq 1.**
  - There is no possible combination of x and y that can give negative answer.
    - Even when x or y is negative value.
      - Because squared terms $(2x^2 \ and \ 3y^2) \gg$ cross-term $(4xy)$.
  - α can be nonpositive.
    - α comes from $\boldsymbol{w}^T \boldsymbol{A} \boldsymbol{w} = \alpha$.
    - Only when $x = y = 0$.
      - In remaining cases, α is always positive.
- **That is not trivial result of quadratic form.**
  - Eq 2. can have positive or negative α depending on values of x and y.
  - $[x \quad y] \rightarrow [-1 \quad 1]$: Negative quadratic form result.
  - $[x \quad y] \rightarrow [-1 \quad -1]$: Positive. quadratic form result

$$[x \quad y] \begin{bmatrix} 2 & 4 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 2x^2 + (0+4)xy + 3y^2$$

Eq 1. First example of quadratic form of matrix

$$[x \quad y] \begin{bmatrix} -9 & 4 \\ 3 & 9 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = -9x^2 + (3+4)xy + 9y^2$$

Eq 2. Second example of quadratic form of matrix

Scalar for All Possible Vectors

- **How can you possibly know whether quadratic form will produce positive?**
  - Or negative, or zero-valued
- **Key**
  - Full-rank eigenvectors matrix spans all of $\mathbb{R}^M$.
  - Therefore, Every vector in $\mathbb{R}^M$ can be expressed.
    - As some linear weighted combination of eigenvectors.
- **Then, start from eigenvalue equation and left-multiply by eigenvector to return to quadratic form as Eq 1..**

부호

$$\boldsymbol{Av} = \lambda\boldsymbol{v}$$
$$\boldsymbol{v}^T\boldsymbol{Av} = \lambda\boldsymbol{v}^T\boldsymbol{v}$$
$$\boldsymbol{v}^T\boldsymbol{Av} = \lambda||\boldsymbol{v}||^2$$

Eq 1. Return to quadratic form

Key of Return to Quadratic Form

- **In Eq 1., Final equation is key.**
  - Note, $||\boldsymbol{v}^T\boldsymbol{v}||$ is strictly positive.
    - Vector magnitudes cannot be negative.
    - Ignore zeros vector.
  - Sign of right-hand side of equation is determined entirely by eigenvalue λ.
- **That equation uses only one eigenvalue and its eigenvector.**
  - But we need to know about any possible vector.

$$\boldsymbol{A}\boldsymbol{v} = \lambda\boldsymbol{v}$$
$$\boldsymbol{v}^T\boldsymbol{A}\boldsymbol{v} = \lambda\boldsymbol{v}^T\boldsymbol{v}$$
$$\boldsymbol{v}^T\boldsymbol{A}\boldsymbol{v} = \lambda||\boldsymbol{v}||^2$$

Eq 1. Return to quadratic form

Insight of Return to Quadratic Form

- **If equation is valid for each eigenvector-eigenvalue pair,**
  - It is valid for any combination of eigenvector-eigenvalue pairs as Eq 1..
- **In other words**
  - Set any vector $\boldsymbol{u}$ to be some linear combination of $\boxed{\text{eigen vectors}}$.
  - Set some scalar ζ to be that same linear combination of eigenvalues.
- **Anyway, it doesn't change principle**
  - Sign of right-hand side (quadratic form) is determined by sign of eigenvalues.

$$\boldsymbol{v}_1^T \boldsymbol{A} \boldsymbol{v}_1 = \lambda_1 \|\boldsymbol{v}_1\|^2$$
$$\boldsymbol{v}_2^T \boldsymbol{A} \boldsymbol{v}_2 = \lambda_2 \|\boldsymbol{v}_2\|^2$$
$$(\boldsymbol{v}_1 + \boldsymbol{v}_2)^T \boldsymbol{A} (\boldsymbol{v}_1 + \boldsymbol{v}_2) = (\lambda_1 + \lambda_2) \|(\boldsymbol{v}_1 + \boldsymbol{v}_2)\|^2$$
$$\boldsymbol{u}^T \boldsymbol{A} \boldsymbol{u} = \zeta \|\boldsymbol{u}\|^2$$

Eq 1. Valid for each eigenvector-eigenvalue pair

Think about Equation under Different Assumption about sign of λ

- **All eigenvalues are positive.** → quadratic form을 만들면.
  - Right-hand side of equation is always positive.
  - $\boldsymbol{v}^T\boldsymbol{A}\boldsymbol{v}$ is always positive for any vector $\boldsymbol{v}$.
- **Eigenvalues are positive or zero.**
  - $\boldsymbol{v}^T\boldsymbol{A}\boldsymbol{v}$ is nonnegative
  - $\boldsymbol{v}^T\boldsymbol{A}\boldsymbol{v}$ will equal zero when $\lambda = 0$.
    - $\lambda = 0$ happens when matrix is singular.
- **Eigenvalues are negative or zero.**
  - Quadratic form result will be zero or negative.
- **Eigenvalues are negative.**
  - Quadratic form result will be negative for all vectors.

Definiteness

- **Characteristic of square matrix**
- **Defined by signs of eigenvalues of matrix.**
  - Same thing as signs of quadratic form results.
- **Implication**
  - Invertibility of matrix as well as advanced data analysis methods.
    - Such as generalized eigendecomposition
      - Used in multivariate linear classifiers and signal processing.

Categories of Definiteness

- There are 5 categories in definiteness.
- Five categories as shown in Table 1.
  - + and – signs indicate signs of eigenvalues.
  - Depends
    - Matrix can be invertible or singular depending on numbers in matrix.
    - Not on definiteness category.

Category	Quadratic form	Eigenvalues	Invertible
Positive definite	Positive	+	Yes
Positive semidefinite	Nonnegative	+ and 0	No
Indefinite	Positive and negative	+ and -	Depends
Negative semidefinite	Nonpositive	- and 0	No
Negative definite	Negative	-	Yes

Table 1. Definiteness categories

$A^T A$ is Positive (Semi)definite

- **Specific matrix is guaranteed to be positive definite or positive semidefinite.**
  - Expressed as product of matrix and its transpose.
  - That is, $\boldsymbol{S} = \boldsymbol{A}^T \boldsymbol{A}$
  - Combination of these two categories is often written as $\boxed{\text{Positive (semi) definite}}$.
- **All data covariance matrices are positive (semi)definite.**
  - Because covariance matrices defined: $\boldsymbol{A}^T \boldsymbol{A}$
    - where data matrix: $\boldsymbol{A}$
  - All covariance matrices have nonnegative eigenvalues.
- **Case1: When data matrix is full-rank,**
  - If data is stored as observations by features,
    - Full column-rank
  - Eigenvalues will be all positive.
- **Case2: If data matrix is reduced-rank,**
  - At least one zero-valued eigenvalue

Proof of $A^T A$

- **Proof that S is positive (semi)definite.**
  - Writing out its quadratic form.
  - Applying some algebra manipulations.
- **In Eq 1..**
  - Transition from first to second equation simply involves moving parentheses around.
    - Such "proof by parentheses" is common in linear algebra.

$$
\begin{aligned}
\boldsymbol{w}^T S \boldsymbol{w} &= \boldsymbol{w}^T (A^T A) \boldsymbol{w} \\
&= (\boldsymbol{w}^T A^T)(A\boldsymbol{w}) \\
&= (A\boldsymbol{w})^T (A\boldsymbol{w}) \\
&= ||A\boldsymbol{w}||^2
\end{aligned}
$$

Eq 1. Proof that S is positive (semi)definite by parentheses

Point of Proof of $A^T A$

- **Quadratic form of $A^T A$ equals $\|matrix\|^2 * vector$.**
- **Characteristic of magnitudes**
  - Cannot be negative.
  - Can be zero
    - Only when vector is zero.
- **If $Aw = \mathbf{0}$ for nontrivial $\mathbf{w}$,**
  - Then $\boldsymbol{A}$ is singular.
- **Notice**
  - Although all $\boldsymbol{A}^T\boldsymbol{A}$ matrices are symmetric, not all symmetric matrices can be expressed as $\boldsymbol{A}^T\boldsymbol{A}$.
    - Matrix symmetry on its own does not guarantee positive (semi)definiteness.
      - Because not all symmetric matrices can be expressed as product of matrix and its transpose.

Importance of Quadratic Form and Definiteness

- **Importance in data science.**
  - Because some linear algebra operations are applied only to well-endowed matrices.
    - Cholesky decomposition
      - Create correlated datasets in Monte Carlo simulations.
  - Importance in optimization problems.
    - $\boxed{\text{Gradient descent}}$
      - Because guaranteed minimum to find
- **In your never-ending quest to improve your data science prowess,**
  - You might encounter technical papers.
    - Use abbreviation SPD (Symmetric Positive Definite).

Generalized Eigendecomposition

Eigendecomposition

- **Consider that Eq 1. is same as fundamental eigenvalue equation.**
  - This is obvious.
    - Because $\boxed{\boldsymbol{Iv} = \boldsymbol{v}.}$
    - Generalized eigendecomposition as Eq 2. involves replacing identity matrix with another matrix.
      - Not identity or zeros matrix

$$\boldsymbol{Av} = \lambda \boldsymbol{Iv}$$

Eq 1. Assumption equal to fundamental eigenvalue equation

$$\boldsymbol{Av} = \lambda \boldsymbol{Bv}$$

Eq 2. Generalized eigendecomposition

Generalized Eigendecomposition

- **It is also called simultaneous diagonalization of two matrices.**
- **Resulting (λ, v) pair is not eigenvalue / vector of A alone nor of B alone.**
  - Instead, two matrices share eigenvalue / vector pairs.
- **Conceptually, you can think of generalized eigendecomposition**
  - As "regular" eigendecomposition of product matrix as Eq 1..
- **Just conceptual**
  - In practice, does not require $\boldsymbol{B}$ to be invertible.
- **Not case**
  - Any two matrices can be simultaneously diagonalized.
  - If $\boldsymbol{B}$ is positive (semi)definite,
    - Diagonalization is possible.

$$\boldsymbol{C} = \boldsymbol{A}\boldsymbol{B}^{-1}$$
$$\boldsymbol{C}\boldsymbol{v} = \lambda\boldsymbol{v}^{-1}$$

Eq 1. "Regular" eigendecomposition of product matrix

Use Generalized Eigendecomposition in Data Science

- **Classification analysis**
- **In particular, fisher's linear discriminant analysis (LDA)**
  - Based on generalized eigendecomposition of two data covariance matrices.

Myriad Subtleties of Eigendecomposition

- **A lot of properties of eigendecomposition**
  - Sum of eigenvalues equals trace of matrix.
    - While product of eigenvalues equals determinant.
  - Not all square matrices can be diagonalized.
  - Some matrices have repeated eigenvalues.
    - Implications for their eigenvectors
  - Complex eigenvalues of real-valued matrices
    - Inside circle in complex plane.
- **Mathematical knowledge of eigenvalues runs deep.**
  - But this lecture provides essential foundational knowledge.
    - For working with eigendecomposition in applications.

Summary

Summary

- **Eigendecomposition identifies $\boldsymbol{M}$ scalar/vector pairs of an $\boldsymbol{M} \times \boldsymbol{M}$ matrix.**
  - It reflect special directions in the matrix.
  - And have myriad applications in data science.
    - As well as in geometry, physics, computational biology, and myriad other technical displines.
- **Eigenvalues can be found.**
  - Assuming that the matrix shifted by an unknown scalar λ is singular.
  - Setting its determinant to zero.
    - Called characteristic polynomial.
  - And solving for λs.
- **Eigenvectors can be found.**
  - By finding basis vector for the null space of $\boldsymbol{\lambda - shifted}$ matrix.
- **Meaning of diagonalizing a matrix.**
  - Represent matrix as $V^{-1}\Lambda V$.
    - V: matrix with eigenvectors in the columns.
    - Λ: diagonal matrix with eigenvalues in the diagonal elements.

Summary

- **Symmetric matrices have several special properties in eigendecomposition.**
  - In data science
    - All eigenvectors are pair-wise orthogonal.
      - Matrix of eigenvectors is orthogonal.
      - Inverse of eigenvectors matrix is its transpose.
- **Definiteness of matrix**
  - Signs of its eigenvalues
  - In data science
    - Positive (semi)definite
      - All eigenvalues are either nonnegative or positive.
  - Matrix times its transpose is always positive (semi)definite.
    - All covariance matrices have nonnegative eigenvalues.
- **Study of eigendecomposition**
  - Rich and detailed
  - Many fascinating subtleties, special cases, and applications

Code Exercises

A , A^{-1} Eigenvalue

- Interestingly, the eigenvectors of A^{-1} are the same as the eigenvectors of A while the eigenvalues are λ^{-1}. Prove that is the case by writing out the eigendecomposition of A and A^{-1}. Then illustrate it using a random full-rank 5×5 symmetric matrix.

```
% create the matrix
A = randn(5,5);
A = A' * A;

% compute its inverse
Ai = ;

% eigenvalues of A and Ai
eigvals_A = ;
eigvals_Ai = ;

% compare them (hint: sorting helps!)
disp('Eigenvalues of A:')
disp(sort(eigvals_A))

disp(' ')
disp('Eigenvalues of inv(A):')
disp(sort(eigvals_Ai))

disp(' ')
disp('Reciprocal of evals of inv(A):')
disp(sort(1./eigvals_Ai))
```

Sample code

Interesting property of random matrices

- One interesting property of random matrices is that their complex-valued eigenvalues are distributed in a circle with a radius proportional to the size of the matrix. To demonstrate this, compute 123 random 42×42 matrices, extract their eigenvalues, divide by the square root of the matrix size (42), and plot the eigenvalues on the complex plane, as in Figure below.

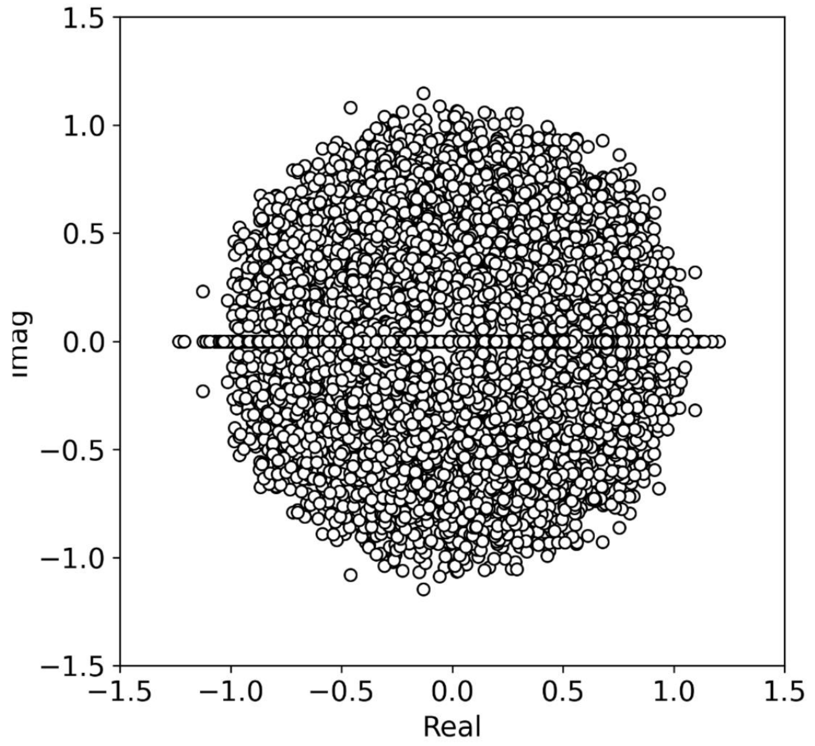
```
nIter = 123;
matsize = 42;
% fill this evals variable
evals = zeros(nIter, matsize);

% create the matrices and get their scaled eigenvalues
for i = 1:nIter
    % declear (matsize, matsise) sized matrix A in every iteration
    A = ; randn(matsize, matsize)
    evals(i, :) = ; eig(A)/sqrt(matsize)
end

% visualization
% and show in a plot
figure('Position', [100, 100, 600, 600]);

plot(real(evals(:)), imag(evals(:)), 'ko', 'MarkerFaceColor', 'w');
xlim([-1.5, 1.5]);
ylim([-1.5, 1.5]);
xlabel('Real');
ylabel('Imag');
```

Sample code



Method to Create Random Symmetric Matrices

- **Start by creating a 4×4 diagonal matrix with positive numbers on the diagonals(they can be, for example, the numbers 1,2,3,4). Then create a 4×4 Q matrix from the QR decomposition of a random-numbers matrix. Use these matrices as the eigenvalues and eigenvectors, and multiply them appropriately to assemble a matrix. Confirm that the assembled matrix is symmetric, and that its eigenvalues equal the eigenvalues you specified.**

```
% Create the Lambda matrix with positive values
Lambda = diag(rand(4,1) * 5);
randnMat = randn(4,4);

% create Q
;

% reconstruct to a matrix
A = ;

% the matrix minus its transpose should be zeros (within precision error)
result = ;

disp(result);

% sort(diag(Lambda)) and sort(eig(A)) disp same result
% print sorted diagonal of Lambda
disp('Sorted diagonal of Lambda:')
disp(sort(diag(Lambda)))

% print sorted eigenvalues of A
disp('Sorted eigenvalues of A:')
disp(sort(eig(A)))
```

Sample code

THANK YOU
FOR YOUR ATTENTION