Linear Algebra

Eigendecomposition: Part1

Automotive Intelligence Lab.

Contents

Interpretations of Eigenvalues and Eigenvectors

Interpretations in Geometry

Special combination of a matrix and a vector

- Matrix *stretched* vector but did not $\boxed{\text{rotate}}$ that vector.
  - That vector: **eigenvector** of matrix
  - Amount of stretching: **eigenvalue**
- Eigenvectors point in same direction.
  - Before and after post multiplying the matrix.
- In Fig 1.,
  - $\boldsymbol{v_1}, \boldsymbol{v_2}$: eigen vectors.
- In Fig 2.,
  - $\boldsymbol{w_1}, \boldsymbol{w_2}$: not eigen vectors.

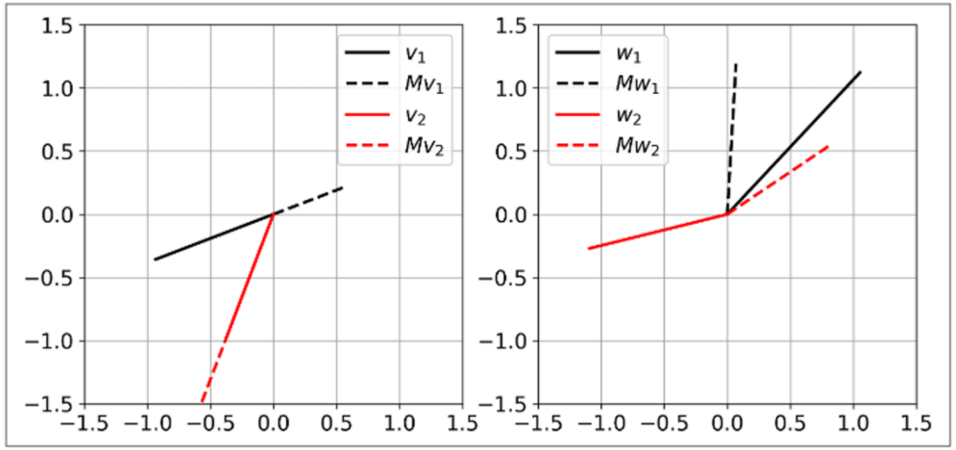


Fig 1.

Fig 2.

Geometric meaning of eigenvector

- Matrix-vector multiplication acts like scalar-vector multiplication.
- Write eigenvalue equation as:

$$\boldsymbol{Av} = \boldsymbol{\lambda v}$$

Eigenvalue equation

- Equation doesn't say that matrix equals the scala.
  - It says that the $effect$ of matrix on the vector is same as the $effect$ of the scalar on that same vector.

Principal Components Analysis

Implement Principal Component Analysis(PCA) on statistical data.

- To identify important patterns or structures.
  - We will practice PCA later!

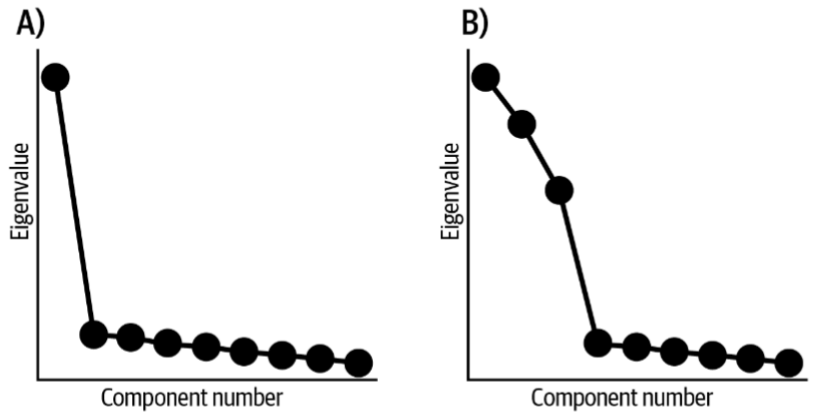
Role of eigenvalue in PCA

- Eigenvalue play a crucial role in PCA.
  - Represent variance of each principal component.
  - The larger the eigenvalue, the more variance(information) the principal component captures.

Scree plot

- graph of the eigenvalues of the dataset's covariance matrix

- A) 1 component accounts for most of the variance system.
  - All other components account for very little variance.
- B) 3 major subcategories.
  - Other components expect 3 major categories account for very little variance.



Example of Scree plot

Interpretations in Noise Reduction

- **Most datasets contain noise.**

- **Noise**
  - Refers to Variance. in a dataset either unexplained or unwanted.

- **Method to reducing random noise**
  - Many ways to attenuate or eliminate noise,
    but optimal reduction strategy depends on origin of the noise or characteristics of signal
  - Method with eigenvalues and eigenvectors,
    - Identify eigenvalues and eigenvectors of a system.
    - And "project out" directions in the data space associated **with small eigen-values**.

- **Meaning of "projecting out" a data dimension**
  - Reconstruct dataset after setting some eigenvalues to zero which eigenvalues below some threshold.

Interpretations in Dimension Reduction(Data Compression)

- **It is beneficial to $compress$ data before transmitting it.**
  - Compression: Reduce the size of data while having minimal impact on the quality of the data.

- **One way to dimension-reduce a dataset**
  - Take its eigendecomposition
    - Drop eigenvalues and eigenvectors associated with small directions in data space.
    - Transmit only relatively larger eigenvector-value pairs.

- **All of the data compression idea is same!**
  - Decompose dataset into a set of basis vectors.
    - Basis vectors that capture the most important features of data.
  - Reconstruct a high-quality version of the original data.

Finding Eigenvalues

Find Eigenvalues Using MATLAB

- **To eigendecompose a square matrix…,**
  - First, find eigenvalues first.
  - Then, use each eigenvalue to find its corresponding eigenvector.
  - Super easy in MATLAB.
    - Just use function **eig()**
    - Eigenvalues of the matrix below are -0.37 and 5.37.

```
% Clear workspace, command window, and close all figures
clc; clear; close all;

% Define the matrix
matrix = [1 2; 3 4];

% Get the eigenvalues
evals = eig(matrix);

% Display the eigenvalues
disp('Eigenvalues of the matrix:');
disp(evals);
```

MATLAB code to find eigenvalues

- **Probably you have question…!**
  - How are the eigenvalues of a matrix identified?

Method to Find Eigenvalues of Matrix

λ: Scalar

Av:

- **Do some simple arithmetic!**

$$\boxed{\boldsymbol{Av} = \lambda \boldsymbol{v}}$$
$$\boldsymbol{Av} - \lambda \boldsymbol{v} = \boldsymbol{0}$$
$$(\boldsymbol{A} - \lambda \boldsymbol{I})\boldsymbol{v} = \boldsymbol{0}$$

$$v \to \boxed{A} \to \lambda v$$

Reorganize eigenvalue equation

- **First equation**
  - Repeat of eigenvalue equation.
- **Second equation**
  - Simply subtracted right-hand side to set equation of right-hand side to the zeros vector.
- **Third equation**
  - Left-hand side of second equation has two vector terms.
    - Both of which involve $\boldsymbol{v}$ so that factor out the vector.
  - After that, it leaves us with the subtraction of a matrix and a scalar $\boldsymbol{A} - \lambda$.
    - Matrix-scalar subtraction is not a defined operation in linear algebra.
    - So, $shift$ matrix by λ.
    - $\lambda \boldsymbol{I}$ is sometimes called a scalar matrix.

Meaning of Eigenvalue Equation

- **Eigenvector is in the null space of the matrix shifted by its** $\boxed{\text{eigen value}}$**.**

$$(A - \lambda I)v = 0$$

Reorganized eigenvalue equation

$(A - \lambda I) \times V = 0.$

none-singular 0

singular $\boxed{x}$

- **Remember…,**
  - Ignore trivial solutions in linear algebra which means don't consider $v = 0$ to be an eigenvector.

- **Matrix shifted by its eigenvalue is singular.**
  - Because only singular matrices have a **nontrivial null space**.

- **What else do we know about singular matrices?**
  - Know that their Determinant is **zero**!
  - Hence, we can write as below:

$$det(A - \lambda I) \ = 0$$

Determinant of $A - \lambda I$

Key to Finding Eigenvalues: Determinant

- **Shift matrix by unknown eigenvalue λ.**
  - Set its determinant to $\boxed{\text{zero}}$, and solve for λ.

- **Example of finding eigenvalues in 2×2 matrix**
  - You can apply quadratic formula to solve for two λ values.

$$\left|\begin{bmatrix} a & b \\ c & d \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right| = 0$$

$$\begin{vmatrix} a-\lambda & b \\ c & d-\lambda \end{vmatrix} = 0$$

$$(a-\lambda)(d-\lambda) - bc = 0$$

$$\lambda^2 - (a+d)\lambda + (ad - bc) = 0$$

(제어공학)
characteristic equation.

Process of finding eigenvalues

Examples to Finding Eigenvalues

Traditional way to find eigenvalues

- Subtract the unknown value lambda off the diagonals.
- Solve for the determinant is equal to zero.

Direct way to find eigenvalues

- Trace of matrix is equal to sum of the eigenvalues.
- Determinant of a matrix is equal to the product of the two eigenvalues.

Find the eigenvalues of $\begin{bmatrix} 3 & 1 \\ 4 & 1 \end{bmatrix}$

$$\det\left(\begin{bmatrix} 3-\lambda & 1 \\ 4 & 1-\lambda \end{bmatrix}\right) = (3-\lambda)(1-\lambda) - (1)(4)$$

$$= (3 - 4\lambda + \lambda^2) - 4$$

$$= \underbrace{\lambda^2 - 4\lambda - 1 = 0}$$

$$\lambda_1, \lambda_2 = \frac{4 \pm \sqrt{4^2 - 4(1)(-1)}}{2} = \frac{4 \pm \sqrt{20}}{2} = 2 \pm \sqrt{5}$$

traditional way to find eigenvalue

Known.

$$1)\quad \frac{1}{2}\mathrm{tr}\left(\begin{bmatrix} \boxed{a} & b \\ c & \boxed{d} \end{bmatrix}\right) = \frac{a+d}{2} = \frac{\lambda_1 + \lambda_2}{2} = m \quad \text{(mean)}$$

Known.

$$2)\quad \det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = ad - bc = \lambda_1 \lambda_2 = p \quad \text{(product)}$$

$$3)\quad \lambda_1, \lambda_2 = m \pm \sqrt{m^2 - p}$$

$$\begin{bmatrix} 8 & 4 \\ 2 & 6 \end{bmatrix} \quad m = 7, \quad p = 40$$

Direct way to find eigenvalue

Code Exercise to Find Eigenvalues

Code Exercise (13_01)

- Find the eigenvalues using different method.
- Use the 'direct way' in the previous slide.

```
% Clear workspace, command window, and close all figures
clc; clear; close all;

% 2x2 matrix : A
A = [1 2; 3 4];

% Calculate the trace of the matrix
trA = trace(A);

% Calculate the determinant of the matrix
detA = det(A);

% Calculate the eigenvalues using the direct way
lambda1 = trA/2 + sqrt((trA/2)^2-detA);
lambda2 = trA/2 - sqrt((trA/2)^2-detA);

% Display the eigenvalues
disp('Eigenvalues of the matrix:');
disp([lambda1 lambda2]);
```

MATLAB code to find eigenvalues using direct way

Logical Progression of Mathematical Concepts of Eigenvalue Equation

- **The matrix-vector multiplication acts like: $\boxed{\text{Scalar}}$-vector multiplication.**

- **Set eigenvalue equation to zeros vector, and factor out common terms.**
  - Eigen vector is null space of matrix shifted by eigenvalue.
  - Do not consider zeros vector to be an eigenvector.
    - Shifted matrix is singular.

$A v = \lambda v$

$A v - \lambda v = 0$

$\underbrace{(A - \lambda I)}_{\text{singular}} \underbrace{v}_{\neq 0} = 0.$ $\det(A - \lambda I) = 0.$

- **Set determinant of shifted matrix to $\boxed{\text{Zero}}$.**
  - Solve for unknown eigenvalue.

- **Determinant of an eigenvalue-shifted matrix set to $\boxed{\text{Zero.}}$.**
  - Called characteristic polynomial of the matrix.

- **$n\,th$ order polynomial has n solutions.**
  - Some of solutions might be complex-valued.
    - Called fundamental theorem of algebra.
  - Characteristic polynomial of an $M \times M$ matrix will have λ^M term.
    - $M \times M$ matrix will have M eigenvalues.

Finding Eigenvectors

Find Eigenvectors Using MATLAB

$[A] \xrightarrow{eig}$ $V = \begin{bmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_n \\ | & | & & | \end{bmatrix}$, $\begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & \lambda_n \end{bmatrix}$

Finding eigenvectors is super easy in MATLAB.

- Most important thing to keep in mind
  - Eigenvectors are stored in columns of the matrix.
- Columns of the matrix $evecs$
  - Eigenvectors
  - Columns are same order as eigenvalues.
- Paired
  - Eigenvector in the first column of matrix $evecs$.
  - First eigenvalue in vector $evals$.
- People use variable names $\boldsymbol{L}$ & $\boldsymbol{V}$ or $\boldsymbol{D}$ & $\boldsymbol{V}$.
  - $\boldsymbol{V}$ matrix: each column i is eigenvector v_i.
  - $\boldsymbol{L}$ is for $\boldsymbol{\Lambda}$ (capital of λ)
  - $\boldsymbol{D}$ is for diagonal.
    - Eigen values are often stored in a diagonal matrix.
    - Reasons will be explained later in this chapter.

```
% Clear workspace, command window, and close all figures
clc; clear; close all;

% 2x2 matrix
matrix = [1 2; 3 4];

% Calculate the eigenvalues and eigenvectors
[vecs, vals] = eig(matrix);

% Display the eigenvalues and eigenvectors
disp('Eigenvalues:');
disp(diag(vals));  % Extracts and displays the eigenvalues from the diagonal matrix
disp('Eigenvectors:');
disp(vecs);  % Displays the eigenvectors
```

MATLAB code to find eigenvalues

Important question

- **Where** do eigenvectors come from and **how** do we find them?

Important Thing to Keep In Mind About Eigenvectors When Coding

- **Eigenvectors are stored in the columns of the matrix.**
  - Not in the rows.
  - Disastrous consequences in applications.
    - If accidentally using the rows instead of the columns of the eigenvectors matrix.

- **Remember common convention in linear algebra.**
  - Vectors are in column orientation.

Method to Find Eigenvector

- **Find vector v that is in the null space of matrix shifted by λ.**
  - In other words:

$$\boldsymbol{v_i} \in \boldsymbol{N}(\boldsymbol{A} - \boldsymbol{\lambda_i I})$$

Equation of eigenvector

- **Numerical example**

$$\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \longrightarrow \lambda_1 = 3\,, \lambda_2 = -1$$

Example of matrix and its eigenvalues

  - Focus on the first eigenvalue.
    - Shift the matrix by 3 (value of first eigenvalue).
    - Find a vector in its null space.

singular.

$$\begin{bmatrix} 1-3 & 2 \\ 2 & 1-3 \end{bmatrix} = \begin{bmatrix} -2 & 2 \\ 2 & -2 \end{bmatrix} \longrightarrow \begin{bmatrix} -2 & 2 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Find eigenvector of the matrix

    - $[1 \quad 1]$: an eigenvector of the matrix **associated with** an eigenvalue of 3.
    - How can we find null space vectors (eigenvectors of the matrix)?

Method to Find Null Space Vectors in Practice

- **Good way to conceptualize the solution**
  - Use **Gauss-Jordan** to solve a system of equations.
    - Coefficients matrix is λ shifted matrix.
    - Constants vector is zeros vector.

- **In implementation…,**
  - More stable numerical methods are applied for finding eigenvalues and eigenvectors.
    - Including $\boldsymbol{QR}$ **decomposition** and **Procedure** called the power method

Sign and Scale Indeterminacy of Eigenvectors

Return to numerical example in previous section

- Why $[1 \quad 1]$ was an eigenvector of matrix?
  - $[1 \quad 1]$: a basis for the null space of the matrix shifted by its eigenvalue of 3.
- Is $[1 \quad 1]$ unique eigenvector of matrix?
  - No, $[4 \quad 4]$ or $[-5.4 \quad -5.4]$ or …
  - Any scaled version of vector $[1 \quad 1]$ is a basis for that null space.
- If v is an eigenvector of a matrix, αv can also be eigenvector.
  - αv for any real-valued α except zero.

Indeed, eigenvectors are important because of their $\boxed{\text{direction.}}$.

- Not because of their magnitude

$$\begin{bmatrix}1 & 2\\2 & 1\end{bmatrix} \rightarrow \lambda_1 = 3\,, \lambda_2 = -1 \rightarrow \begin{bmatrix}1-3 & 2\\2 & 1-3\end{bmatrix} = \begin{bmatrix}-2 & 2\\2 & -2\end{bmatrix} \rightarrow \begin{bmatrix}-2 & 2\\2 & -2\end{bmatrix}\begin{bmatrix}1\\1\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$$

Previous example about eigenvalues and eigenvector

Questions About Infinity of Possible Null Space Basis Vectors

Is there one “best” basis vector?

- No “best” basis vector.
- But convenient to have eigenvectors that are **unit normalized**.
  - Euclidean norm of 1.
  - Particularly useful for symmetric matrices for reasons will be explained later in this chapter.

What is “$correct$” sign of an eigenvector?

- There is none.
- Can get **different eigenvector signs** from same matrix when using different software.
  - Python, Julia, Mathematica , ….
- There are principled ways for assigning a sign in applications.
  - Such as PCA.
  - But it is just common convention to facilitate interpretation.

Diagonalizing a Square Matrix

Make Equations Compact and Elegant

- **Eigenvalue equation lists one eigenvalue and one eigenvector.**
  - Means that an $M \times M$ matrix has M eigenvalue equations.

A

- **Nothing wrong with that series of equations…!**
  - But this equation sets are ugly.
  - Ugliness violates one of the principles of linear algebra which make equations compact and elegant.

$$\begin{pmatrix} \boldsymbol{A}\boldsymbol{v}_{\mathbf{1}} = \lambda_1 \boldsymbol{v}_{\mathbf{1}} \\ \vdots \\ \boldsymbol{A}\boldsymbol{v}_{\boldsymbol{M}} = \lambda_M \boldsymbol{v}_{\boldsymbol{M}} \end{pmatrix}$$

M eigenvalue equations of $M \times M$ matrix

- **Therefore, we need to transform this series of equations into one matrix equation for compact!**

Key Insight for Writing Out Matrix Eigenvalue Equation

- **Each $\boxed{\text{column.}}$ of the eigenvectors matrix is scaled by exactly one eigenvalue.**
  - Can implement this through post multiplication by a diagonal matrix.
  - Store eigenvalues in diagonal of a matrix instead of storing eigenvalues in a vector.

- **Form of diagonalization for a 3×3 matrix**
  - Using @ in place of numerical values in the matrix
  - In the eigenvectors matrix,
    - First subscript number corresponds to eigenvector.
    - Second subscript number corresponds to eigenvector element.
  - Take a moment to confirm!
    - Each eigenvalue scales all elements of its corresponding eigenvector and not any other eigenvectors.

$$Av_1 = \lambda_1 v_1$$
$$\vdots$$
$$Av_M = \lambda_M v_M$$

$$\begin{bmatrix} @ & @ & @ \\ @ & @ & @ \\ @ & @ & @ \end{bmatrix}\underbrace{\begin{bmatrix} v_{11} & v_{21} & v_{31} \\ v_{12} & v_{22} & v_{32} \\ v_{13} & v_{23} & v_{33} \end{bmatrix}}_{V} = \overbrace{\begin{bmatrix} v_{11} & v_{21} & v_{31} \\ v_{12} & v_{22} & v_{32} \\ v_{13} & v_{23} & v_{33} \end{bmatrix}}^{V}\overbrace{\begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}}^{\Lambda}$$

diagonal

$$= \begin{bmatrix} \lambda_1 v_{11} & \lambda_2 v_{21} & \lambda_3 v_{31} \\ \lambda_1 v_{12} & \lambda_2 v_{22} & \lambda_3 v_{32} \\ \lambda_1 v_{13} & \lambda_2 v_{23} & \lambda_3 v_{33} \end{bmatrix}$$

$\rightarrow AV = V\Lambda.$

Diagonalization for 3×3 matrix

Eigen Decomposition

- **Consider list of equivalent declarations of matrix eigenvalue equation as shown below.**

$$V^{-1}AV = V\Lambda$$
$$A = V\Lambda V^{-1}$$
$$\Lambda = V^{-1}AV$$

List of equivalent declarations

- **Code to return:**
  - Eigenvalues in a vector.

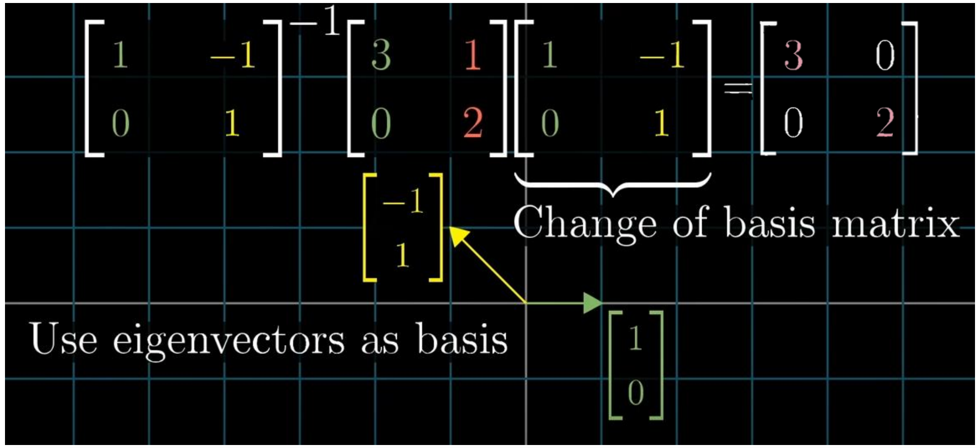
```
% Clear workspace, command window, and close all figures
clc; clear; close all;

% 2x2 matrix
matrix = [1 2; 3 4];

% Calculate the eigenvalues and eigenvectors
[vecs, vals] = eig(matrix);

% Display the D matrix
disp('D matrix:');
disp(vals);
```

MATLAB code to get D matrix

Example of Diagonalizing a Square Matrix

Diagonalizing matrix using eigen basis

- Use eigenvectors as basis.
- Take the coordinates of the two eigenvectors.
- Make those coordinates the columns of a matrix, known as the change of basis matrix.
- Put the change of basis matrix on its right and the inverse of the change of basis on its left.
- Result will be a matrix representing the same transformation,
  but from the perspective of the new basis vectors coordinate system.
- New matrix is guaranteed to be diagonal.
- Basis vectors which are also eigenvectors is called, **eigenbasis**



Example of Diagonalizing matrix

Special Awesomeness of Symmetric Matrices

Orthogonal Eigenvector

- **Symmetric matrices have orthogonal eigenvectors.**
  - All eigenvectors of symmetric matrix are pair-wise orthogonal.

- **Start with an example, then discuss implications of eigenvector orthogonality, finally show proof.**

Code Exercise of Orthogonal Eigenvector

Code Exercise (13_02)

- Three dot products are all zero.
  - Within computer rounding errors on order of 10^{-16}.
- Symmetric matrices were created as random matrix times its transpose.

```matlab
% Clear workspace, command window, and close all figures
clc; clear; close all;

% Create a random matrix and make it symmetric
A = randi([-3, 3], 3, 3);
A = A * A';  % Symmetric matrix

% Perform eigen decomposition
[V, D] = eig(A);

% Display the eigenvalues and eigenvectors
disp('Eigenvalues:');
disp(diag(D));
disp('Eigenvectors:');
disp(V);

% Calculate and display all pairwise dot products between eigenvectors
dot12 = dot(V(:,1), V(:,2));
dot13 = dot(V(:,1), V(:,2));
dot23 = dot(V(:,1), V(:,2));
disp('Dot product of first and second eigenvectors:')
disp(dot12);
disp('Dot product of first and third eigenvectors:')
disp(dot13);
disp('Dot product of second and third eigenvectors:')
disp(dot23);
```

MATLAB code of orthogonal eigenvectors

Property of Orthogonal Eigenvector

- **Dot product between any pair of eigenvectors is** zero **.**
  - While dot product of eigenvector with itself is nonzero.
  - Because not consider zeros vector to be eigenvector.
  - This can be written as Eq 1..
    - $\boldsymbol{D}$: Diagonal matrix with diagonals containing norms of eigenvectors

$$\boldsymbol{V}^T\boldsymbol{V} = \boldsymbol{D}$$

Eq 1. Property of orthogonal eigenvector

Direction VS Magnitude

- **Eigenvectors are important not magnitude but** $\boxed{\text{direction}}$**.**
  - Eigenvector can have any magnitude we want.
    - Except for magnitude of zero
- **Let's scale all eigenvectors so they have unit length.**
  - Question: If all eigenvectors are orthogonal and have unit length, what happens when we multiply eigenvectors matrix by its transpose?
  - Answer: As you know, it's Eq 1..
- **In other words, Eigenvectors matrix of symmetric matrix is orthogonal matrix!**

$$\boldsymbol{V}^T\boldsymbol{V} = \boldsymbol{I}$$

Eq 1. Multiply eigenvectors matrix with unit length by its transpose

Implication of Orthogonal Eigenvector

- **Multiple implications for data science.**
  - Eigenvectors are super easy to invert.
    - Simply transpose them.
- **Other implications of orthogonal eigenvectors for applications**
  - Such as principal components analysis
  - I will discuss later.

Proof of Orthogonal Eigenvector

Necessity

- Orthogonal eigenvectors of symmetric matrices is such important concept

Goal

- To show that $\boxed{\text{dot}}$ product between any pair of eigenvectors is zero.

Assumption

- Matrix $\boldsymbol{A}$ is symmetric.
- λ_1 and λ_2 are distinct eigenvalues of $\boldsymbol{A}$, with $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ as their corresponding eigenvectors.
  - λ_1 and λ_2 cannot equal each other.

Eigenvector Orthogonality Proof (1)

- **Try to follow each equality step from left to right of Eq 1..**
  - Pay attention to first and last terms.
    - Terms in middle are just transformations.
- **Eq 1. are written in Eq 2..**
  - Then subtracted to set to zero.

$$\lambda_1 \boldsymbol{v}_1^T \boldsymbol{v}_2 = (\boldsymbol{A}\boldsymbol{v}_1)^T \boldsymbol{v}_2 = \boldsymbol{v}_1^T \boldsymbol{A}^T \boldsymbol{v}_2 = \boldsymbol{v}_1^T \lambda_2 \boldsymbol{v}_2 = \lambda_2 \boldsymbol{v}_1^T \boldsymbol{v}_2$$

Eq 1. Proof of eigenvector orthogonality for symmetric matrices

$$\lambda_1 \boldsymbol{v}_1^T \boldsymbol{v}_2 = \lambda_2 \boldsymbol{v}_1^T \boldsymbol{v}_2$$

$$\lambda_1 \boldsymbol{v}_1^T \boldsymbol{v}_2 - \lambda_2 \boldsymbol{v}_1^T \boldsymbol{v}_2 = 0$$

Eq 2. Continuing eigenvector orthogonality proof

Eigenvector Orthogonality Proof (2)

- **Eq 1. can be factored out as Eq 2..**
  - Both terms contain dot product $\boldsymbol{v}_1^T\boldsymbol{v}_2$.
- **Eq 2. says that two quantities multiply to produce 0.**
  - One or both of those quantities must be zero.
    - $(\lambda_1 - \lambda_2)$ cannot equal zero.
      - Because we began from assumption that they are $\boxed{\text{zero}}$.
  - Therefore, $\boldsymbol{v}_1^T\boldsymbol{v}_2$ must equal zero.
    - Meaning: Two eigenvectors are orthogonal.

$$\lambda_1\boldsymbol{v}_1^T\boldsymbol{v}_2 - \lambda_2\boldsymbol{v}_1^T\boldsymbol{v}_2 = 0$$

Eq 1. Continuing eigenvector orthogonality proof

$$(\lambda_1 - \lambda_2)\boldsymbol{v}_1^T\boldsymbol{v}_2 = 0$$

Eq 2. Eigenvector orthogonality proof, part 3

Eigenvector Orthogonality Proof (3)

- **Go back through Eq 1..**
  - Convince yourself that this proof fails for nonsymmetric matrices, when $\boldsymbol{A}^T \neq \boldsymbol{A}$.
  - Thus, eigenvectors of nonsymmetric matrix are not constrained to be orthogonal.
    - Linearly independent for all distinct eigenvalues.
      - But I will omit that discussion and proof.

$$\lambda_1 \boldsymbol{v}_1^T \boldsymbol{v}_2 = (\boldsymbol{A}\boldsymbol{v}_1)^T \boldsymbol{v}_2 = \boldsymbol{v}_1^T \underbrace{\boldsymbol{A}^T}_{\neq A.} \boldsymbol{v}_2 = \boldsymbol{v}_1^T \lambda_2 \boldsymbol{v}_2 = \lambda_2 \boldsymbol{v}_1^T \boldsymbol{v}_2$$

$$\lambda_1 \boldsymbol{v}_1^T \boldsymbol{v}_2 = \lambda_2 \boldsymbol{v}_1^T \boldsymbol{v}_2$$

$$\lambda_1 \boldsymbol{v}_1^T \boldsymbol{v}_2 - \lambda_2 \boldsymbol{v}_1^T \boldsymbol{v}_2 = 0$$

$$(\lambda_1 - \lambda_2) \boldsymbol{v}_1^T \boldsymbol{v}_2 = 0$$

Eq 1. Eigenvector orthogonality proof

Real-Valued Eigenvalues

- **Second special property of symmetric matrices**
  - Real-valued eigenvalues
  - Real-valued eigenvectors
- **Let me start by showing that matrices with all real-valued entries.**
  - Those have complex-valued eigenvalues.

```
% Clear workspace, command window, and close all figures
clc; clear; close all;

% Define the matrix A
A = [-3 -3 0; 3 -2 3; 0 1 2];

% Perform eigen decomposition
[V, D] = eig(A);

% Extract the eigenvalues from the diagonal matrix D
eigenvalues = diag(D);

% Display the eigenvalues as a column vector
disp('Eigenvalues:');
disp(eigenvalues);
```

```
Eigenvalues:
  -2.7447 + 2.8517i
  -2.7447 - 2.8517i
   2.4895 + 0.0000i
```

MATLAB code of egiendecomposition

In Code Exercise

- **3×3 matrix A**
  - Two complex eigenvalues and one real-valued eigenvalue.
    - Eigenvectors coupled to complex-valued eigenvalues
      - Themselves be complex-valued.
  - Nothing special
    - Because matrix A comes from random integers between -3 and +3.
- **Interestingly, complex-valued solutions come in conjugate pairs.**
  - If there is $\lambda_j = a + ib$, then there is $\lambda_k = a - ib$.
  - Their corresponding eigenvectors are also complex conjugate pairs.
- **I don't go into detail about complex-valued solutions, except to show you that complex solutions to eigendecomposition are straightforward.**
  - Straightforward: Mathematically expected
    - Interpreting complex solutions in eigendecomposition is far from straightforward.

Symmetric Matrix

- **Guarantee to have** $\boxed{\text{real-valued}}$ **eigenvalues.**
  - Also $\boxed{\text{real-valued}}$ eigenvectors.
- **Let me start by modifying previous example.**
  - Make matrix symmetric.

```
% Clear workspace, command window, and close all figures
clc; clear; close all;

% Define the matrix A
A = [-3 -3 0; -3 -2 1; 0 1 2];

% Perform eigen decomposition
[V, D] = eig(A);

% Extract the eigenvalues from the diagonal matrix D
eigenvalues = diag(D);

% Display the eigenvalues as a column vector
disp('Eigenvalues:');
disp(eigenvalues);
```

```
Eigenvalues:
   -5.5971
    0.2261
    2.3710
```

MATLAB code of eigendecomposition of symmetric matrix

Random Symmetric Matrix of Any Size

- **How to make**
  - Create random matrix.
  - Eigendecompositioning $A^T A$.
- **Where to use**
  - Confirm that eigenvalues are real-valued.

- **Guaranteed real-valued eigenvalues from symmetric matrices.**
  - It's fortunate
    - Because complex numbers are often confusing to work with.
- **In data science**
  - Lots of matrices are symmetric.
  - If you see complex eigenvalues in your data science applications,
    - It's possible that is problem with code or with data.

THANK YOU
FOR YOUR ATTENTION