Black-Scholes Equations

1 The Black-Scholes Model

Up to now, we only consider hedgings that are done upfront. For example, if we write a naked call (see Example 5.2), we are exposed to unlimited risk if the stock price rises steeply. We can hedge it by buying a share of the underlying asset. This is done at the initial time when the call is sold. We are then protected against any steep rise in the asset price. However, if we hold the asset until expiry, we are not protected against any steep dive in the asset price. So is there a hedging that is really riskless?

The answer was given by Black and Scholes, and also by Merton in their seminal papers on the theory of option pricing published in 1973. The idea is that a writer of a naked call can protect his short position of the option by buying *a certain amount* of the stock so that the loss in the short call can be exactly offset by the long position in the stock. This is standard in hedging. The question is how many stocks should he buy to minimize the risk? By adjusting the *proportion* of the stock and option *continuously* in the portfolio during the life of the option, Black and Scholes demonstrated that investors can create a *riskless* hedging portfolio where all market risks are eliminated. In an efficient market with no riskless arbitrage opportunity, any portfolio with a zero market risk must have an expected rate of return equal to the riskless interest rate. The Black-Scholes formulation establishes the equilibrium condition between the expected return on the option, the expected return on the stock, and the riskless interest rate. We will derive the formula in this chapter.

Since the publication of Black-Scholes' and Merton's papers, the growth of the field of derivative securities has been phenomenal. The Black-Scholes equilibrium formulation of the option pricing theory is attractive since the final valuation of the option prices from their model depends on a few observable variables except one, the volatility parameters. Therefore the accuracy of the model can be ascertained by direct empirical tests with market data. When judged by its ability to explain the empirical data, the option pricing theory is widely acclaimed to be the most successful theory not only in finance, but in all areas of economics. In recognition of their pioneering and fundamental contributions to the pricing theory of derivatives, Scholes and Merton received the 1997 Nobel Prize in Economics. Unfortunately, Black was unable to receive the award since he had already passed away then.

To begin with the Black-Scholes model, let us state the list of assumptions underlying the Black-Scholes model.

i) The asset price follows the geometric Brownian motion discussed in Chapter 6. That is,
$$dS(t) = \mu S(t)dt + \sigma S(t)dX(t). \tag{1}$$

ii) The risk-free interest rate r and the asset volatility σ are known functions.

iii) There are no transaction costs.

iv) The asset pays no dividends during the life of the option.

v) There are no arbitrage possibilities.

vi) Trading of the asset can take place continuously.

vii) Short selling is permitted.

viii) We can buy or sell any number (not necessarily an integer) of the asset.

We note that the Black-Scholes model can be applied to asset models other than (1), such as jump-diffusion models, but it may be difficult to derive explicit formulas then, as we do have for geometric Brownian motions. However, this should not discourage their use, since an accurate numerical solution is usually quite straightforward. We will relax assumptions (ii)–(iv) in the next chapter. For example, (iv) can be dropped if the dividends are known beforehand. They can be paid either at discrete intervals or continuously over the life of the option. We will discuss them in the next chapter.

2   Derivation of the Black-Scholes Differential Equation

Suppose that we have an option whose value $V(S, t)$ depends only on S and t. It is not necessary at this stage to specify whether V is a call or a put; indeed, V can be the value of a whole portfolio of different options although for simplicity we can think of a simple call or put. Using Itô's lemma (Theorem 7.1) and noting that $S(t)$ follows (1), we can write

$$dV = \left(\mu S \frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + \frac{\partial V}{\partial t}\right) dt + \sigma S \frac{\partial V}{\partial S} dX_t. \tag{2}$$

This gives the stochastic process followed by V. Note that by (2) we require V to have at least one t derivative and two S derivatives. Next we construct a portfolio consisting of longing one option and shorting a number Δ of the underlying asset. Here if $\Delta < 0$, we are in fact buying Δ amount of the underlying asset. For example, if we have bought one put option, we may want to buy certain amount of stock to minimize the risk. In that case, $\Delta < 0$.

The Black-Scholes idea is first to find this proportion Δ so that the portfolio becomes deterministic. Note that the value of this portfolio is

$$\Pi(t) = V - \Delta S. \tag{3}$$

The change in the value of this portfolio in one time-step dt is

$$d\Pi(t) = dV - \Delta dS, \tag{4}$$

where we assume Δ is held fixed during the time-step. Substituting (1) and (2) into (4), we find

$$d\Pi(t) = \left(\mu S \frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + \frac{\partial V}{\partial t} - \mu \Delta S\right) dt + \sigma S \left(\frac{\partial V}{\partial S} - \Delta\right) dX_t. \tag{5}$$

Note that there are two terms in the right hand side. The first term is deterministic while the second term is stochastic as it involves the standard Wiener process X_t. But if we choose $\Delta = \partial V/\partial S$, then the stochastic term is zero, and (5) becomes

$$d\Pi = \left(\frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + \frac{\partial V}{\partial t} \right) dt. \tag{6}$$

Thus choosing

$$\Delta = \frac{\partial V}{\partial S} \tag{7}$$

reduces the stochastic expression into a deterministic expression.

We now appeal to the concepts of arbitrage and the assumption of no transaction costs. The return on an amount Π invested in riskless assets would see a growth of $r\Pi dt$ in a time dt. In fact, if $d\Pi$ were greater than this amount, $r\Pi dt$, an arbitrager could make a guaranteed riskless profit by borrowing an amount Π to invest in the portfolio. The return for this risk-free strategy would be greater than the cost of borrowing. Conversely, $d\Pi$ were less than $r\Pi dt$, then the arbitrager would short the portfolio and invest Π in the bank. Either way the arbitrager would make a riskless, no cost, instantaneous profit. The existence of such arbitrageurs with the ability to trade at low cost ensures that the return on the portfolio and on the riskless account are more or less equal. Thus, we should have $d\Pi = r\Pi dt$, and hence by (6),

$$r\Pi dt = \left(\frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + \frac{\partial V}{\partial t} \right) dt. \tag{8}$$

Now replace Π in (8) by $V - \Delta S$ as given in (3), and replace Δ by $\partial V/\partial S$ as given in (7), and then divide both sides by dt. We arrive at

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0. \tag{9}$$

This is the *Black-Scholes partial differential equation.* It is hard to over-emphasize the fact that, under the assumptions stated earlier, any derivative security whose price depends only on the current value of S and on t, and which is paid for up-front, must satisfy the Black-Scholes equation. Many seemingly complicated option valuation problems, such as exotic options, become simple when looked at in this way.

Before moving on, we make three remarks about the derivation we have just seen.

(i) By definition, the "delta" $\Delta = \partial V/\partial S$ is the amount of assets that we need to hold to get a riskless hedge. For example, at expiry T, $c(S,T) = \max\{S - E, 0\}$. Hence $\Delta = 1$ if $S > E$ and $\Delta = 0$ if $S < E$. That means we need to hold 1 stock if $S > E$ as the buyer will come to exercise the option; and if $S < E$, there is no need to hold any stock. The value of Δ is therefore of fundamental importance in both theory and practice, and we will return to it repeatedly. It is a measure of the correlation between the movements of the option or other derivative products and those of the underlying asset.

(ii) Second, the linear differential operator given by

$$\mathcal{L}_{BS} = \frac{\partial}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2}{\partial S^2} + rS\frac{\partial}{\partial S} - r \tag{10}$$

has a financial interpretation as a measure of the difference between the return on a hedged option portfolio (the first two terms) and the return on a bank deposit

(the last two terms)—see (8). Although this difference must be identically zero for a European option in order to avoid arbitrage, we see later that this need not be so for an American option.

(iii) Third, the Black-Scholes equation (9) does not contain the drift parameter μ of the underlying asset. Hence the price of the options will be independent of how rapidly or slowly an asset grows. The price will depend on the volatility σ however. A consequence of this is that two people may have quite different views on μ, yet still agree on the value of an option. We will return to this in Section 5.

3   Boundary and Final Conditions for European Options

Equation (9) is the first partial differential equation (PDE) that we have derived in this course. We now introduce a few basic points in the theory of PDE so that we are aware of what we are trying to achieve.

By deriving the partial differential equation (9) for a quantity such as an option price, we have made an enormous step towards finding its value—we just need to solve the equation. Sometimes this involves solution by numerical means if exact formula cannot be found. However, a partial differential equation on its own generally has many solutions; for example the simple differential equation $dy^2(s)/ds^2 = 1$ already has infinitely many solutions: $y(s) = \frac{1}{2}s^2 + \alpha s + \beta$ for any α and β. In our case, the values of puts, calls and S itself all satisfy the Black-Scholes equation. The value of an option should be unique (otherwise, arbitrage possibilities would arise), and so, to pin down the solution, we must also impose *boundary conditions*. A boundary condition specifies the behavior of the required solution at some part of the solution domain. As an example, to determine a particular solution for $dy^2(s)/ds^2 = 1$, we need two boundary conditions to pin down the parameters α and β. Suppose that $y(0) = 1$ and $y(2) = 7$. Then we have $\alpha = 3$ and $\beta = 1$, and the particular solution satisfying the boundary conditions is $y(s) = \frac{1}{2}s^2 + 3s + 1$.

The most frequent type of partial differential equation in financial problems is the parabolic equation. A parabolic equation for a function $V(S,t)$ is a specific relationship between V and its partial derivatives with respect to the independent variables S and t. In the simplest case, the highest derivative with respect to S is a second-order derivative, and the highest derivative with respect to t is only a first-order derivative. Thus (9) comes into this category. If the equation is linear and the signs of these particular derivatives are the same, when they appear on the same side of the equation, then the equation is called *backward parabolic*; otherwise it is called *forward parabolic*. Equation (9) is backward parabolic. The simplest type of forward parabolic equation is the *heat equation*:

$$\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial x^2}, \quad -\infty < x < \infty, \ \tau > 0. \tag{11}$$

Here $u(x, \tau)$ measures the temperature of a metal rod at the position x and time τ.

Once we have decided that our partial differential equation is of the parabolic type, we can make general statements about the sort of boundary conditions that lead to a unique solution. For the heat equation, which is first order in t and second order in x, we need one condition in t and two conditions in x. Typically, we have $u(x, 0) = u_0(x)$ for $-\infty < x < \infty$, i.e. we know the temperature of the rod at $t = 0$. Also, we need to know $u(\infty, t) = u_+(t)$ and $u(-\infty, t) = u_-(t)$, the temperature of the rod at the two ends for all time $t > 0$. Once these are given, we can solve the equation (11) and

obtain $u(x, t)$ for all $-\infty < x < \infty$ and $t > 0$. For the Black-Scholes equation, we must impose two conditions in S, as we have a $\partial^2 V/\partial S^2$ term in the equation, but only one in t, as we only have a $\partial V/\partial t$ term in it. For example, we could specify that

$$V(S, t) = V_a(t) \quad \text{on } S = a,$$

and

$$V(S, t) = V_b(t) \quad \text{on } S = b.$$

where $V_a(t)$ and $V_b(t)$ are two given functions of t.

If the equation is of backward type, we must also impose a *final* condition such as

$$V(S, T) = V_T(S).$$

where V_T is a known function. We then solve for V in the region $t < T$. That is, we solve *backwards in time*, hence the name. If the equation is of forward type, we impose an *initial* condition on $t = 0$, say, and solve in the region $t > 0$, in the forward direction. Of course, we can change from backward to forward by the simple change of variables $\tilde{t} = -t$. This is why both types of equations are mathematically equivalent and it is common to transform backward equations into forward equations before any analysis. It is important to remember, however, that the parabolic equation cannot be solved in the wrong direction; that is, we should not impose an initial condition on a backward equation or a final condition on a forward equation.

For the heat equation (11), which is a forward equation, we need to specify an initial condition, $u(x, 0)$, i.e. the initial temperature distribution of the rod. Then we can compute the temperature distribution of the rod $u(x, t)$ for any time $t > 0$. However, it is physically impossible that with a given final temperature distribution $u(x, T)$, one can compute $u(x, t)$ for any $t < T$. For the Black-Scholes equation in (9), which is a backward parabolic equation, we must specify final and boundary conditions, for otherwise the partial differential equation does not have a unique solution. For the moment we restrict our attention to a *vanilla European call* $c(S, t)$, with exercise price E and expiry date T.

The final condition of a call is just its payoff at T:

$$c(S, T) = \max(S - E, 0), \quad \text{for all } S \geq 0. \tag{12}$$

This is the final condition for our partial differential equation. Our 'spatial' or asset-price boundary conditions are applied at zero asset price, $S = 0$, and as $S \to \infty$. We can see from (1) that if S is ever zero, then dS is also zero, and therefore S can never change. Since if $S = 0$ at expiry, the payoff will be zero. Thus the call option is worthless on $S = 0$ even if there is a long time to expiry. Hence on $S = 0$ we have

$$c(0, t) = 0 \quad \text{for all } t \geq 0. \tag{13}$$

Finally as $S \to \infty$, it becomes ever more likely that the option will be exercised and the magnitude of the exercise price becomes less and less important. Thus, as $S \to \infty$, the value of the option becomes that of the asset minus the exercise price we need to pay to exchange for the asset. Hence we have for all $t > 0$,

$$c(S, t) \sim S - Ee^{-r(T-t)}, \quad \text{as } S \to \infty. \tag{14}$$

Note that the second term accounts for the discounted exercise price. For a European call option, without the possibility of early exercise, Black-Scholes equations (9)

together with the boundary conditions (12)–(14) can be solved exactly to give the Black-Scholes value of a European call option. We will do that in §4 for European calls and puts. In Figure 1, we give the solution domain (the domain where we want to solve the call option value) and the boundary conditions.
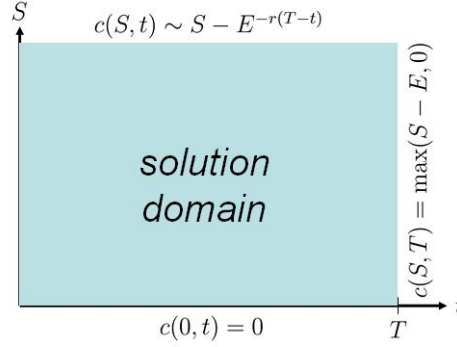

Figure 1. Solution domain of European call options and the boundary conditions.

For a *vanilla European put* option $p(S,t)$, the final condition is the payoff

$$p(S,T) = \max(E - S, 0), \quad \text{for all } S \geq 0. \tag{15}$$

For the boundary conditions, we have already mentioned that if S is ever zero, then it must remain zero. In this case the final payoff for a put is known with certainty to be E. To determine $p(0,t)$, we simply have to calculate the present value of an amount E received at time T. Assuming that interest rates are constant we find the boundary condition at $S = 0$ to be

$$p(0,t) = Ee^{-r\tau}, \quad \text{for all } t \geq 0. \tag{16}$$

As $S \to \infty$, the option is unlikely to be exercised and so for $t > 0$, we have

$$p(S,t) \to 0, \quad \text{as } S \to \infty. \tag{17}$$

The boundary conditions for vanilla American options are more difficult and will be left to the next chapter.

You can easily check that $V(S,t) = S$ itself is a solution to the Black-Scholes equation (9). But what are the boundary conditions for $V(S,t)$? In fact, $V(S,t) \equiv S$ for all S and t and hence we have: when $S = 0$, $V(0,t) = 0$ for all t; when $S \to \infty$, $V(S,t) = S$ for all t; and for $t = T$, we have $V(S,T) = S(T)$ for all S. Does that mean by solving the Black-Scholes equation, we can get the stock price $S(t)$ for all t? The answer is no because we don't know the exact value of the boundary conditions. For example, we don't know $S(T)$ at time T, when we are at $t = 0$. Hence the Black-Scholes solution $V(S,t) = S$ is something that is of no use to us.

4   Solution of the Black Scholes Equation

In this section, we give the formulas for European calls and puts. We verify that the formulas we give are the solution of the Black-Scholes equation.

Theorem 1. *The value of the vanilla European call is given by*

$$c(S,t) = c(S(t), E, T-t, r) = SN(d_1) - Ee^{-r(T-t)}N(d_2), \tag{18}$$

where

$$N(d) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{d} e^{-\frac{1}{2}s^2} ds, \tag{19}$$

the cumulative distribution function for the standard normal distribution,

$$d_1 = \frac{\ln(S/E) + (r + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}} \quad \text{and} \quad d_2 = \frac{\ln(S/E) + (r - \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}. \tag{20}$$

Proof. We first check that $c(S,t)$ in (18) really satisfies the Black-Scholes equation (9). We first note that for $\omega = t$ or S, we have

$$\frac{\partial N(d_i)}{\partial \omega} = \frac{\partial N(d_i)}{\partial d_i}\frac{\partial d_i}{\partial \omega} = \frac{1}{\sqrt{2\pi}}\frac{\partial}{\partial d_i}\int_{-\infty}^{d_i} e^{-\frac{s^2}{2}}ds \cdot \frac{\partial d_1}{\partial \omega} = \frac{e^{-\frac{d_i^2}{2}}}{\sqrt{2\pi}}\frac{\partial d_i}{\partial \omega}.$$

We can check that

$$\frac{\partial d_1}{\partial t} = \frac{d_1}{2(T-t)} - \frac{1}{\sqrt{T-t}}\left(\frac{r}{\sigma} + \frac{\sigma}{2}\right) \quad \text{and} \quad \frac{\partial d_2}{\partial t} = \frac{d_2}{2(T-t)} - \frac{1}{\sqrt{T-t}}\left(\frac{r}{\sigma} - \frac{\sigma}{2}\right).$$

Hence we have

$$\frac{\partial c}{\partial t} = S\frac{\partial N(d_1)}{\partial d_1}\frac{\partial d_1}{\partial t} - rEe^{-r(T-t)}N(d_2) - Ee^{-r(T-t)}\frac{\partial N(d_2)}{\partial d_2}\frac{\partial d_2}{\partial t}$$

$$= \frac{Se^{-\frac{d_1^2}{2}}}{\sqrt{2\pi}}\left[\frac{d_1}{2(T-t)} - \frac{1}{\sqrt{T-t}}\left(\frac{r}{\sigma} + \frac{\sigma}{2}\right)\right] - rEe^{-r(T-t)}N(d_2)$$

$$- \frac{Ee^{-r(T-t)}e^{-\frac{d_2^2}{2}}}{\sqrt{2\pi}}\left[\frac{d_2}{2(T-t)} - \frac{1}{\sqrt{T-t}}\left(\frac{r}{\sigma} - \frac{\sigma}{2}\right)\right]. \tag{21}$$

Also, since

$$\frac{\partial d_i}{\partial S} = \frac{1}{S\sigma\sqrt{T-t}}, \quad i = 1, 2,$$

we have

$$\frac{\partial c}{\partial S} = N(d_1) + S\frac{\partial N(d_1)}{\partial d_1}\frac{\partial d_1}{\partial S} - Ee^{-r(T-t)}\frac{\partial N(d_2)}{\partial d_2}\frac{\partial d_2}{\partial S}$$

$$= N(d_1) + \frac{e^{-\frac{d_1^2}{2}}}{\sqrt{2\pi}}\frac{1}{\sigma\sqrt{T-t}} - Ee^{-r(T-t)}\frac{e^{-\frac{d_2^2}{2}}}{\sqrt{2\pi}}\frac{1}{S\sigma\sqrt{T-t}}. \tag{22}$$

Differentiating it once more, we get

$$\frac{\partial^2 c}{\partial S^2} = \frac{e^{-\frac{d_1^2}{2}}}{S\sigma\sqrt{2\pi}\sqrt{T-t}} - \frac{d_1 e^{-\frac{d_1^2}{2}}}{S\sigma^2\sqrt{2\pi}(T-t)} + \frac{Ee^{-r(T-t)}e^{-\frac{d_2^2}{2}}}{S^2\sigma\sqrt{2\pi}\sqrt{T-t}} + \frac{Ee^{-r(T-t)}d_2 e^{-\frac{d_2^2}{2}}}{S^2\sigma^2\sqrt{2\pi}(T-t)}$$

$$= \frac{2}{S^2\sigma^2}\left\{\frac{Se^{-\frac{d_1^2}{2}}}{\sqrt{2\pi}}\left(\frac{\sigma}{2\sqrt{T-t}} - \frac{d_1}{2(T-t)}\right)\right.$$

$$\left. + \frac{Ee^{-r(T-t)}e^{-\frac{d_2^2}{2}}}{\sqrt{2\pi}}\left(\frac{\sigma}{2\sqrt{T-t}} + \frac{d_2}{2(T-t)}\right)\right\}. \tag{23}$$

By substituting (18), (21)–(23) into the left hand side of the Black-Scholes equation (9), we see that it is indeed identically equal to zero.

For the boundary condition (13), we first note that by (20), $d_1, d_2 \to -\infty$ as $S \to 0$. Obviously $N(-\infty) = 0$. Hence

$$c(0,t) = 0N(-\infty) - Ee^{-r(T-t)}N(-\infty) = 0.$$

For the boundary condition (14), we note again that $d_1, d_2 \to \infty$ as $S \to \infty$ where $N(\infty) = 1$. Hence

$$c(S,t) \to SN(\infty) - Ee^{-r(T-t)}N(\infty) \sim S - Ee^{-r(T-t)},$$

as $S \to \infty$.

Finally, we consider the final condition (12). At $t = T$, if $S > E$, then $d_1, d_2 \to \infty$. Hence $c(S,T) = S - E$. If $S < E$, then $d_1, d_2 \to -\infty$. Hence $c(S,T) = 0$. If $S = E$, by continuity, $c(S,T) = 0$. $\square$

Next we give the formula for European put options.

Theorem 2. *The value of the vanilla European put is given by*

$$p(S,t) = Ee^{-r(T-t)}N(-d_2) - SN(-d_1), \tag{24}$$

where d_1 and d_2 are given in (20).

Proof. One can of course verify that the formula (24) does satisfy the Black-Scholes equation and the boundary and final conditions for European puts as we did in the proof of Theorem 1. However, there is a better way to verify that. We can derive (24) immediately by using the put-call parity formula (see (4.7))

$$c(S,t) - p(S,t) = S - Ee^{-r(T-t)},$$

Theorem 1, and the identity $N(d) + N(-d) \equiv 1$ for any d. $\square$

We remark that although (18) and (24) seem to be closed-form solutions for the vanilla options, one still has to compute the integral $N(d_i)$ numerically by quadrature rules such as Simpson's rule or Gaussian rule.

Next we compute the deltas, the amount of the underlying asset that one should hold at any time t if one has short sell the option. Recall from (3) that the riskless portfolio is $\Pi(t) = V - \Delta S$. That is whenever we buy (or sell) one option, we have to short sell (or buy) Δ units of the underlying stock in order that the portfolio is riskless. Note that Δ is changing with time, and that means we have to do the hedging continuously. If one cannot compute Δ, one should not buy or sell the options.

Theorem 3. *The deltas of vanilla call and put options are*

$$\Delta_c(S,t) = N(d_1) \quad \text{and} \quad \Delta_p(S,t) = N(d_1) - 1.$$

Proof. From (22)

$$\Delta_c = N(d_1) + \frac{Ee^{-\frac{1}{2}d_1^2}}{S\sigma\sqrt{2\pi(T-t)}}\left(\frac{S}{E} - e^{-r(T-t)}e^{-\frac{1}{2}d_2^2 + \frac{1}{2}d_1^2}\right) = N(d_1).$$

The last equality can be established by noting that

$$\frac{S}{E} = e^{-r(T-t)-\frac{1}{2}d_2^2+\frac{1}{2}d_1^1} \iff \ln\left(\frac{S}{E}\right) + r(T-t) = \frac{1}{2}(d_1+d_2)(d_1-d_2),$$

which is indeed true by virtue of (20) and the fact that $d_2 = d_1 - \sigma\sqrt{T-t}$.

We can get Δ_p similarly or by using put-call parity. $\qquad\square$

By using the deltas to do the hedging, we see in (5) that the largest random component of the portfolio is eliminated. This hedging process is called *delta hedging*. Once can in fact hedge away higher order effect by knowing the following quantities.

Definition 4. The gamma, theta, vega and rho of a portfolio Π are defined respectively by

$$\Gamma = \frac{\partial^2 \Pi}{\partial S^2}, \quad \Theta = -\frac{\partial \Pi}{\partial t}, \quad \nu = \frac{\partial \Pi}{\partial \sigma}, \quad \rho = \frac{\partial \Pi}{\partial r}.$$

5   Pricing Options Using Risk Neutrality

An important observation of the Black-Scholes equation (9) is that it does not contain the drift parameter μ of the underlying asset. Hence we see in Theorems 1 and 2 that the price of vanilla European options is independent of how rapidly or slowly an asset grows. The only parameter from the stochastic differential equation (1) for the asset price that affects the option price is the volatility σ. A consequence of this is that two people may differ in their estimates for μ, yet still agree on the value of an option. Moreover, the risk preferences of investors are irrelevant: because the risk inherent in an option can all be hedged away, there is no return to be made over and above the risk-free return.

The same conclusion is true for vanilla options as well as other derivative products. It is generally the case that if a portfolio can be constructed with a derivative product and the underlying asset in such a way that the random component can be eliminated—as was the case in our derivation of the Black-Scholes equation (9)—then the derivative product may be valued as if all the random walks involved are *risk-neutral*. This means that the drift parameter μ in the stochastic differential equation for the asset can be replaced by r wherever it appears. The option is then valued by calculating the present value of its expected return at expiry with this modification to the random walk.

To apply this option pricing idea to our geometric Brownian motion model, we can first replace (1) by

$$dS_t = rS_t dt + \sigma S_t dX_t.$$

It is a risk-neutral world: we pretend that the random walk for the return on S_t has drift r instead of μ. From this, we can calculate the probability density function of the future values of S_t. This is given by Theorem 7.2 with μ there replaced by r:

$$p_{S_t}(s) = \frac{1}{\sigma s\sqrt{2\pi t}} e^{-[\ln(s/S(0))-(r-\frac{1}{2}\sigma^2)t]^2/2\sigma^2 t}, \quad s \geq 0. \tag{25}$$

To evaluate the option price, we first calculate the expected payoff of the option at expiry. Suppose its payoff function at T for any S_t is given by $V(S_T)$. Then the expected payoff of the option at time T is

$$\mathbb{E}[V(S_T, T)] = \frac{1}{\sigma\sqrt{2\pi T}} \int_0^\infty \frac{V(s, T)}{s} e^{-[\ln(s/S_0)-(r-\frac{1}{2}\sigma^2)T]^2/2\sigma^2 T} ds.$$

The value of the option at present time ($t = 0$) is then obtained by discounting this amount of money at expiry back to current time:

$$\mathbb{E}[V(S_0, 0)] = e^{-rT}\mathbb{E}[V(S_T, T)] = \frac{e^{-rT}}{\sigma\sqrt{2\pi T}}\int_0^\infty \frac{V(s, T)}{s}e^{-[\ln(s/S_0)-(r-\frac{1}{2}\sigma^2)T]^2/2\sigma^2 T}ds. \tag{26}$$

One can verify that this solution indeed satisfies the Black-Scholes equation (9). (In fact, note the similarity between (26) and (35)). If the payoff function $V(S)$ is simple, such as in the case of binary options or vanilla options, one can integrate the integral to get the option price. If it is complicated, then one can use numerical quadrature rules or Monte Carlo methods to compute the integration.

Note that by replacing μ by r in our geometric Brownian motion model for S in (1), we do not mean that $\mu = r$. If it were correct, then all assets would have the same expected return as a bank deposit and no one would invest in the stock markets. It is just a trick to obtain the option price because we know that the value of the options does not depend on μ, and in a risk-neutral world, everything grows at a rate of r. We finally note that in the risk-neutral world, the asset price grows like:

$$S(t) = S_0 e^{(r-\frac{\sigma^2}{2})t+\sigma X_t}, \tag{27}$$

cf. (7.7) where we have μ instead of r. Thus instead of Corollary 7.3, we have the following corollary:

Corollary 5. *In the risk-neutral world, we have*

$$\mathbb{E}[S_t] = \int_{-\infty}^\infty sp_S(s)ds = S_0 e^{rt}, \tag{28}$$

$$\text{Var}[S_t] = S_0^2 e^{2rt}[e^{\sigma^2 t} - 1]. \tag{29}$$

I should emphasize again that this Corollary is true only when we are interested in computing the price of an option. However, if we want to compute for example the probability that the stock price S_T will be higher than the exercise price E at expiry T, we need to use μ and not r for the drift rate of S_t, i.e.

$$\text{Prob}\{S_T \geq E\} = \int_E^\infty p_{S_T}(s)ds = \frac{1}{\sigma\sqrt{2\pi T}}\int_E^\infty \frac{1}{s}e^{-[\ln(s/S_0)-(\mu-\frac{1}{2}\sigma^2)T]^2/2\sigma^2 T}ds. \tag{30}$$

One may ask: if μ does not appear in the Black-Scholes equation, and hence we can replace μ by r, why not replace μ by 0 to simplify the computation? The answer is no, since we know from the derivation of the Black-Scholes equation, in particular from (8) that in the risk-neutral world, everything grows with risk-free return rate r.

With (30), we can give a financial interpretation of the Black-Scholes formula in Theorem 1. The call option price in (18) can be rewritten as

$$c(S_0, 0) = e^{-rT}[S_0 e^{rT}N(d_1) - EN(d_2)].$$

Note that the term $N(d_2)$ is the probability that the option will be exercised in a risk-neutral world, i.e.,

$$\text{Prob}\{S_T \geq E\} = \frac{1}{\sigma\sqrt{2\pi T}}\int_E^\infty \frac{1}{s}e^{-[\ln(\frac{s}{S_0})-(r-\frac{\sigma^2}{2})T]^2/2\sigma^2 T}ds = N(d_2).$$

Therefore,

$$\mathbb{E}[E \cdot \mathbf{1}_{\{S_T \geq E\}}] = \int_E^\infty s p_{S_T}(s) ds = E N(d_2).$$

The term $S_0 e^{rT} N(d_1)$ is the expected value of a variable that is equal to S_T if $S_T \geq E$ and to zero otherwise in a risk-neutral world, i.e.,

$$\mathbb{E}[S_T \cdot \mathbf{1}_{\{S_T \geq E\}}] = \int_E^\infty s p_{S_T}(s) ds$$
$$= \frac{1}{\sigma\sqrt{2\pi T}} \int_E^\infty e^{-[\ln(\frac{s}{S_0})-(r-\frac{\sigma^2}{2})T]^2/2\sigma^2 T} ds = S_0 e^{rT} N(d_1).$$

Hence

$$c(S_0, 0) = e^{-rT} \mathbb{E}[(S_T - E)\mathbf{1}_{\{S_T \geq E\}}] = e^{-rT}\mathbb{E}[(S_T - E)^+],$$

which is just (26) with $V(S, t) = c(S, t)$.

Appendix: Black-Scholes Formula by PDE Approach

One may wonder how did Black and Scholes get the formulas of the call and put options in Theorems 1 and 2 in the first place. Their idea is to transform the Black-Scholes equation (9) to the heat equation (11). Since the heat equation has been well-studied and its solution is well-known, one can just transform the solution of the heat equation back to obtain the solution to the Black-Scholes equation (9). In this Appendix, we go through this process once to derive the Black-Scholes formulas.

A1 Solution to the Heat Equation

The heat equation given in (11) is a forward parabolic equation that models the heat dissipation inside a metal rod. To solve it, we need one initial condition in time and two boundary conditions in space, see Figure 2.
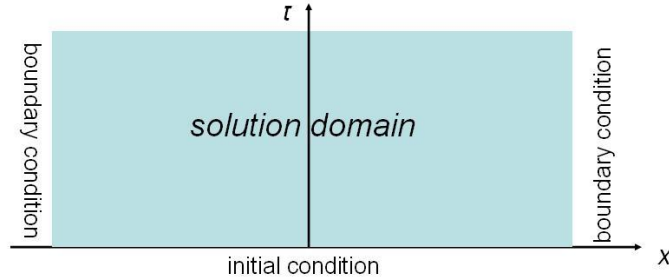

Figure 2. Solution domain of heat equation and the boundary conditions.

The boundary conditions we impose at both ends of the rod are

$$|u(x, \tau)| = O(e^{a|x|}), \text{ for some } a \text{ as } |x| \to \infty. \tag{31}$$

More precisely, as $|x| \to \infty$, we have $|u(x, \tau)| \leq \alpha_\tau e^{a|x|}$, for some positive constants a and α_τ where α_τ may depend on τ but not x. Note that it is a very relaxed condition on $u(x, t)$: we just need $u(x, t)$ to grow no faster than $e^{a|x|}$ when $x \to \infty$.

For the initial condition, we require

$$u(x,0) = u_0(x), \quad -\infty < x < \infty \tag{32}$$

where $u_0(x)$ is continuous and

$$|u_0(x)| = O(e^{b|x|}), \text{ for some } b > 0 \text{ as } |x| \to \infty . \tag{33}$$

i.e. as $|x| \to \infty$, $|u_0(x)| \le \beta e^{b|x|}$ for some positive β. We remark that once we know that (33) holds, we can further say that

$$|u_0(x)| \le \alpha e^{b|x|}, \quad \forall x \in \mathbb{R} \tag{34}$$

for some $\alpha > 0$. The reason is this: by (33), we know that there exists an $x_0 > 0$ such that when $|x| \ge x_0$, $|u_0(x)| \le \beta e^{b|x|}$. Let $M = \max |u_0(x)|$ in the interval $(-x_0, x_0)$. Then

$$|u_0(x)| \le \max(M, \beta) e^{b|x|}, \quad \forall x \in \mathbb{R}.$$

We will see that the Black-Scholes equation does satisfy these boundary and initial conditions when it is transformed into a heat equation, see for example (42). The solution to heat equation is well-known and is given below.

Theorem 6. *The solution to the heat equation (11) with boundary conditions (31) and initial condition (32) is given by*

$$u(x,\tau) = \frac{1}{2\sqrt{\pi\tau}} \int_{-\infty}^{\infty} u_0(s) e^{-(x-s)^2/4\tau} ds. \tag{35}$$

Proof. We first note that the integrand $\frac{1}{2\sqrt{\pi\tau}} e^{-(s-x)^2/4\tau}$ is the probability density function of $\mathcal{N}(x, 2\tau)$. Because of (33), the integral on the right hand side of (35) is well-defined. To show that (35) is indeed a solution, we first verify that it satisfies (11). In fact,

$$\frac{\partial u}{\partial \tau} = -\frac{1}{4\tau\sqrt{\pi\tau}} \int_{-\infty}^{\infty} u_0(s) e^{-(x-s)^2/4\tau} ds + \frac{1}{2\sqrt{\pi\tau}} \int_{-\infty}^{\infty} u_0(s) e^{-(x-s)^2/4\tau} \frac{(x-s)^2}{4\tau^2} ds,$$

and

$$\frac{\partial u}{\partial x} = -\frac{1}{2\sqrt{\pi\tau}} \int_{-\infty}^{\infty} u_0(s) e^{-(x-s)^2/4\tau} \frac{2(x-s)}{4\tau} ds.$$

Hence

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{2\sqrt{\pi\tau}} \int_{-\infty}^{\infty} u_0(s) e^{-(x-s)^2/4\tau} \frac{4(x-s)^2}{16\tau^2} ds - \frac{1}{2\sqrt{\pi\tau}} \int_{-\infty}^{\infty} u_0(s) e^{-(x-s)^2/4\tau} \frac{2}{4\tau} ds.$$

Thus (11) is satisfied. For those mathematically-conscientious, they can verify that the differentiation can be done inside the integral because the integral after differentiation is uniformly convergent as $u_0(x)$ grows slower than $O(e^{-bx^2})$.

To show that the boundary condition (31) holds, we just note that by (34),

$$|u(x,\tau)| \le \frac{1}{2\sqrt{\pi\tau}} \int_{-\infty}^{\infty} |u_0(s)| e^{-(x-s)^2/4\tau} ds$$

$$\le \frac{\alpha}{2\sqrt{\pi\tau}} \int_{-\infty}^{\infty} e^{b|s|} e^{-(x-s)^2/4\tau} ds$$

$$\le \frac{\alpha e^{b|x|}}{2\sqrt{\pi\tau}} \int_{-\infty}^{\infty} e^{b|\zeta|} e^{-\zeta^2/4\tau} d\zeta \le \beta_\tau e^{b|x|} = O(e^{b|x|}).$$

where α and β_τ are constants with β_τ depending on τ but not on x. Thus (31) holds.

Finally we verify (32). Clearly we cannot just substitute $\tau = 0$ in (35). Instead we will show that

$$u(x,0) \equiv \lim_{\tau \to 0+} u(x,\tau) = u_0(x),$$

i.e. for all $\epsilon > 0$, there exists a $\delta > 0$ such that for all $\tau < \delta$, $|u(x,\tau) - u_0(x)| \leq \epsilon$. To prove that first note that

$$\frac{1}{2\sqrt{\pi\tau}} \int_{-\infty}^{\infty} e^{-(s-x)^2/4\tau} ds = 1,$$

as the integrand is the probability density function of $\mathcal{N}(x, 2\tau)$, see (6.1). Hence

$$|u(x,\tau) - u_0(x)| = \frac{1}{2\sqrt{\pi\tau}} \left| \int_{-\infty}^{\infty} [u_0(s) - u_0(x)] e^{-(s-x)^2/4\tau} ds \right|$$

$$\leq \frac{1}{2\sqrt{\pi\tau}} \int_{-\infty}^{\infty} |u_0(s) - u_0(x)| e^{-(s-x)^2/4\tau} ds.$$

For all $\epsilon > 0$, we choose δ such that $|u_0(s) - u_0(x)| \leq \epsilon$ whenever $|s - x| \leq \delta$. Then

$$|u(x,\tau) - u_0(x)|$$

$$\leq \frac{1}{2\sqrt{\pi\tau}} \left\{ \int_{|s-x|\leq\delta} \epsilon e^{-(s-x)^2/4\tau} ds + \int_{|s-x|\geq\delta} |u_0(s) - u_0(x)| e^{-(s-x)^2/4\tau} ds \right\}$$

$$\leq \frac{1}{2\sqrt{\pi\tau}} \left\{ \epsilon \int_{-\infty}^{\infty} e^{-(s-x)^2/4\tau} ds + \alpha \int_{|s-x|\geq\delta} [e^{b|x|} + e^{b|s|}] e^{-(s-x)^2/4\tau} ds \right\}$$

$$\leq \epsilon + \frac{\alpha e^{b|x|}}{2\sqrt{\pi\tau}} \int_{|s-x|\geq\delta} e^{-(s-x)^2/4\tau} ds + \frac{\alpha}{2\sqrt{\pi\tau}} \int_{|s-x|\geq\delta} e^{b|s|} e^{-(s-x)^2/4\tau} ds$$

$$\leq \epsilon + \frac{\alpha e^{b|x|}}{\sqrt{\pi}} \int_{|\zeta|\geq\delta/\sqrt{4\tau}} e^{-\zeta^2} d\zeta + \frac{\alpha e^{b|x|}}{2\sqrt{\pi\tau}} \int_{|\eta|\geq\delta} e^{b|\eta|} e^{-\eta^2/4\tau} d\eta.$$

The first integral can be made smaller than ϵ for all sufficiently small τ because $e^{-\zeta^2}/\sqrt{\pi}$ is the probability density function of $\mathcal{N}(0, 1/2)$, see (6.1). Hence we have $\int_{-\infty}^{\infty} e^{-\zeta^2} d\zeta = \sqrt{\pi}$. Therefore $\int_{|\zeta|\geq\delta/\sqrt{4\tau}} e^{-\zeta^2} d\zeta$, which is the tail of $\int_{-\infty}^{\infty} e^{-\zeta^2} d\zeta$, can be made as small as possible when $\tau \to 0$.

For the second integral, we note that if $\tau \leq \delta/8b$, then for all $|\eta| \geq \delta$, we have $8\tau b|\eta| \leq \delta|\eta| \leq \eta^2$. Hence $b|\eta| - \eta^2/4\tau \leq -\eta^2/8\tau$. Thus for τ sufficiently small,

$$|u(x,\tau) - u_0(x)| \leq 2\epsilon + \frac{\alpha e^{b|x|}}{2\sqrt{\pi\tau}} \int_{|\eta|\geq\delta} e^{-\eta^2/8\tau} d\eta$$

$$\leq 2\epsilon + \frac{\alpha\sqrt{2} e^{b|x|}}{\sqrt{\pi}} \int_{|\zeta|\geq\delta/\sqrt{8\tau}} e^{-\zeta^2} d\zeta \leq 3\epsilon. \quad \square$$

A2 Solution to the Black-Scholes Equation

Recall in §3 that the Black-Scholes equation and boundary conditions for a European call with value $c(S,t)$ are,

$$\begin{cases} \dfrac{\partial c}{\partial t} + \dfrac{1}{2}\sigma^2 S^2 \dfrac{\partial^2 c}{\partial S^2} + rS\dfrac{\partial c}{\partial S} - rc = 0, \\ c(0,t) = 0, \ c(S,t) \sim S - Ee^{-r(T-t)}, \ \text{as } S \to \infty, \\ c(S,T) = \max(S - E, 0). \end{cases} \tag{36}$$

To solve it, our idea is to transform (36) into the heat equation (11), and then use the formula (35) to get the solution.

There are two substitution steps involved. The first substitution step is to make the variables dimensionless, and also reverse the time. Let

$$S = Ee^x, \qquad t = T - \tau/\frac{1}{2}\sigma^2, \qquad c(S,t) = Ev(x,\tau). \tag{37}$$

Then we have

$$\frac{\partial v}{\partial \tau} = \frac{\partial \frac{1}{E}c}{\partial t}\frac{\partial t}{\partial \tau} = -\frac{1}{E}\frac{\partial c}{\partial t}\left(\frac{1}{\frac{1}{2}\sigma^2}\right),$$

$$\frac{\partial v}{\partial x} = \frac{\partial \frac{1}{E}c}{\partial S}\frac{\partial S}{\partial x} = \frac{1}{E}\frac{\partial c}{\partial S}Ee^x = \frac{\partial c}{\partial S}e^x,$$

and

$$\frac{\partial^2 v}{\partial x^2} = e^x\frac{\partial c}{\partial S} + e^x\frac{\partial^2 c}{\partial S^2}\frac{\partial S}{\partial x} = e^x\frac{\partial c}{\partial S} + e^x\frac{\partial^2 c}{\partial S^2}Ee^x.$$

Hence the PDE (36) becomes

$$\frac{\partial v}{\partial \tau} = \frac{\partial^2 v}{\partial x^2} + (k-1)\frac{\partial v}{\partial x} - kv, \tag{38}$$

where $k = r/\frac{1}{2}\sigma^2$. Note that (36) is transformed into a forward parabolic equation (38), and the final condition becomes an initial condition:

$$v(x,0) = \max(e^x - 1, 0). \tag{39}$$

The boundary conditions (13) and (14) will become

$$\begin{cases} v(x,\tau) = 0, & x \to -\infty, \\ v(x,\tau) \sim (e^x - e^{-r\tau}) \sim e^x, & x \to \infty. \end{cases} \tag{40}$$

The PDE (38) can be further simplified to eliminate the first order and constant terms. That's our second substitution step. Let

$$v(x,\tau) = e^{\alpha x + \beta \tau}u(x,\tau),$$

where α and β are two constants to be determined. Substituting this into (38) we obtain

$$\beta u + \frac{\partial u}{\partial \tau} = \alpha^2 u + 2\alpha\frac{\partial u}{\partial x} + \frac{\partial^2 u}{\partial x^2} + (k-1)\left(\alpha u + \frac{\partial u}{\partial x}\right) - ku.$$

Our idea is to choose α and β such that the terms u and $\partial u/\partial x$ are canceled out. This requires

$$\begin{cases} \beta = \alpha^2 + (k-1)\alpha - k \\ 0 = 2\alpha + (k-1), \end{cases}$$

or

$$\alpha = -\frac{1}{2}(k-1), \qquad \beta = -\frac{1}{4}(k+1)^2.$$

Then we have the required substitution:

$$v(x,\tau) = e^{-\frac{1}{2}(k-1)x - \frac{1}{4}(k+1)^2\tau} u(x,\tau). \tag{41}$$

With this substitution, the PDE in (36) becomes the heat equation (11) for the unknown function $u(x,\tau)$. Putting the substitution (41) into the initial condition (39), we obtain the required initial condition:

$$u(x,0) = u_0(x) = e^{\frac{1}{2}(k-1)x} v(x,0) = \max(e^{\frac{1}{2}(k+1)x} - e^{\frac{1}{2}(k-1)x}, 0). \tag{42}$$

As for the boundary conditions, by (40) and (41), $|u(x,\tau)| = O(e^{a|x|})$ for some $a > 0$ as $x \to -\infty$. For $x \to \infty$, by (40) and (41) again, we have $|u(x,\tau)| = O(e^{a|x|})$ for some $a > 0$. Thus (31) is satisfied. Hence by Theorem 6, the solution is given by (35) with $u_0(x)$ given by (42).

It remains to evaluate the integral on the right hand side of (35). Introducing the change of variable $y = (s-x)/\sqrt{2\tau}$ in (35), i.e. we try to normalize the variable distributed as $\mathcal{N}(x, 2\tau)$ by its mean and standard deviation, we have

$$
\begin{aligned}
&u(x,\tau) \\
&= \frac{1}{2\sqrt{\tau\pi}} \int_{-\infty}^{\infty} u_0(s) e^{-(s-x)^2/4\tau} dy \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u_0(y\sqrt{2\tau} + x) e^{-\frac{1}{2}y^2} dy \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \max\{e^{\frac{1}{2}(k+1)(x+y\sqrt{2\tau})} - e^{\frac{1}{2}(k-1)(x+y\sqrt{2\tau})}, 0\} e^{-\frac{1}{2}y^2} dy \\
&= \frac{1}{\sqrt{2\pi}} \int_{-x/\sqrt{2\tau}}^{\infty} e^{\frac{1}{2}(k+1)(x+y\sqrt{2\tau})} - e^{\frac{1}{2}(k-1)(x+y\sqrt{2\tau})} e^{-\frac{1}{2}y^2} dy \\
&= \frac{1}{\sqrt{2\pi}} \int_{-x/\sqrt{2\tau}}^{\infty} e^{\frac{1}{2}(k+1)(x+y\sqrt{2\tau})} e^{-\frac{1}{2}y^2} dy - \frac{1}{\sqrt{2\pi}} \int_{-x/\sqrt{2\tau}}^{\infty} e^{\frac{1}{2}(k-1)(x+y\sqrt{2\tau})} e^{-\frac{1}{2}y^2} dy \\
&\equiv I_1 - I_2. \tag{43}
\end{aligned}
$$

By completing the square in the exponent we can write I_1 as

$$
\begin{aligned}
I_1 &= \frac{1}{\sqrt{2\pi}} \int_{-x/\sqrt{2\tau}}^{\infty} e^{\frac{1}{2}(k+1)(x+y\sqrt{2\tau}) - \frac{1}{2}y^2} dy \\
&= \frac{e^{\frac{1}{2}(k+1)x}}{\sqrt{2\pi}} \int_{-x/\sqrt{2\tau}}^{\infty} e^{\frac{1}{4}(k+1)^2\tau} e^{-\frac{1}{2}(y - \frac{1}{2}(k+1)\sqrt{2\tau})^2} dy \\
&= \frac{e^{\frac{1}{2}(k+1)x + \frac{1}{4}(k+1)^2\tau}}{\sqrt{2\pi}} \int_{-x/\sqrt{2\tau} - \frac{1}{2}(k+1)\sqrt{2\tau}}^{\infty} e^{-\frac{1}{2}\rho^2} d\rho \\
&= e^{\frac{1}{2}(k+1)x + \frac{1}{4}(k+1)^2\tau} N(d_1), \tag{44}
\end{aligned}
$$

where

$$d_1 = \frac{x}{\sqrt{2\tau}} + \frac{1}{2}(k+1)\sqrt{2\tau}, \tag{45}$$

and $N(d_1)$ given in (19). The expression I_2 can be obtained similarly by replacing $(k+1)$ by $(k-1)$, i.e.

$$I_2 = e^{\frac{1}{2}(k-1)x + \frac{1}{4}(k-1)^2\tau} N(d_2),$$

where

$$d_2 = \frac{x}{\sqrt{2\tau}} + \frac{1}{2}(k-1)\sqrt{2\tau}.$$

Thus we have

Theorem 7. *The values of vanilla European calls and puts are given by (18) and (24) respectively.*

Proof. For (18), we just need to put (44)–(45) back into (43) and change everything back to the original variables S and t by using (41) and (37). For example,

$$c(S,t) = Ev(x,\tau) = Ee^{-\frac{1}{2}(k-1)x - \frac{1}{4}(k+1)^2\tau} u(x,\tau) = Ee^{-\frac{1}{2}(k-1)x - \frac{1}{4}(k+1)^2\tau}(I_1 - I_2).$$

But from (44)

$$Ee^{-\frac{1}{2}(k-1)x - \frac{1}{4}(k+1)^2\tau} I_1 = Ee^x N(d_1) = SN(d_1).$$

And we can work out similar expression for the second term in (18).

To get the European put option price in (24), using similar substitutions, we can also transform the final condition (15) into the initial condition for the heat equation:

$$u(x,0) = \max(e^{\frac{1}{2}(k-1)x} - e^{\frac{1}{2}(k+1)x}, 0).$$

Similarly, we can follow what we did above to find the solution. Of course, we can also resort to the put-call parity here.   □