The University of Danang
University of Science and Technology

MATHEMATICS FOR COMPUTER SCIENCE

Chap 2. . Linear Algebra

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References

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Linear Algebra

- **Introduction**
- Eigenvectors and finding Eigenvectors
- Matrix Decompositions

Introduction

- When formalizing intuitive concepts, a common approach is to construct a set of objects (symbols) and a set of rules to manipulate these objects. This is known as an *algebra*.
- Linear algebra is the study of vectors and certain algebra rules to manipulate vectors.
  - The vectors many of us know from school are called "**geometric vectors**", which are usually denoted by a small arrow above the letter, e.g., $\vec{a}$, $\vec{b}$.
  - A bold letter is used to represent them, e.g., $\boldsymbol{a}$ and $\boldsymbol{b}$.
  - In general, vectors are special objects that can be added together and multiplied by scalars to produce another object of the same kind.

Introduction

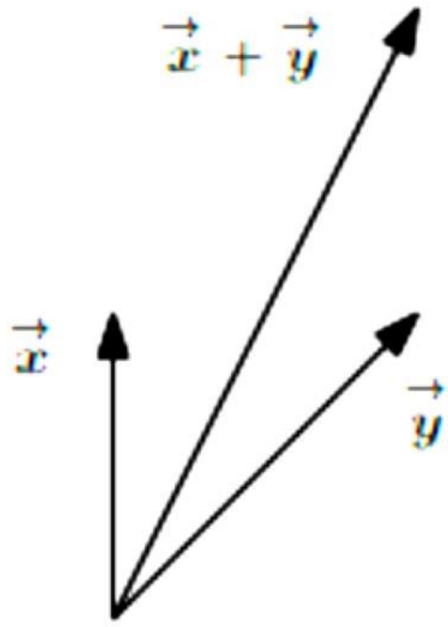
- From an abstract mathematical viewpoint, any object that satisfies these two properties can be considered a vector. Here are some examples of such vector objects:
  - **Geometric vectors**.
  - **Polynomials**
  - **Audio signals are vectors**
  - **Elements of $\mathbb{R}^n$ (tuples of n real numbers) are vectors**

Introduction

- Linear algebra focuses on the similarities between these vector concepts.
  - We can add them together and multiply them by scalars.
  - We will largely focus on vectors in $\mathbb{R}^n$ since most algorithms in linear algebra are formulated in $\mathbb{R}^n$.
  - We often consider data to be represented as vectors in $\mathbb{R}^n$.
  - We will focus on finite-dimensional vector spaces, in which case there is a 1:1 correspondence between any kind of vector and $\mathbb{R}^n$. When it is convenient, we will use intuitions about geometric vectors and consider array-based algorithms.
  - **Linear algebra plays an important role in machine learning and general Mathematics. The concept of a vector space and its properties underlie much of machine learning**

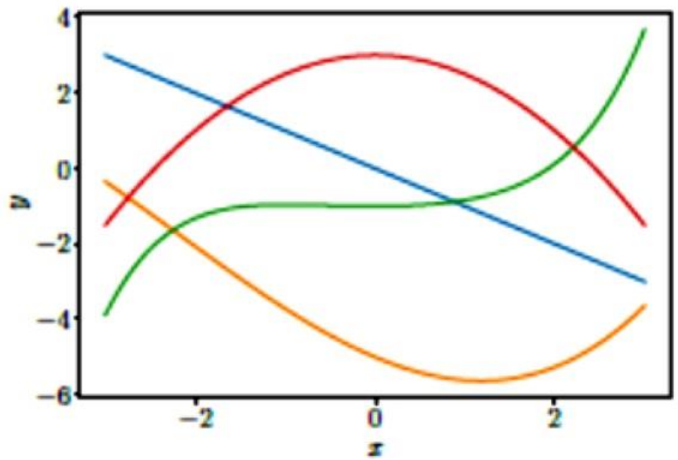
Introduction - vector objects

- **Geometric vectors**. This example of a vector may be familiar from high school mathematics and physics. Geometric vectors are directed segments, which can be drawn (at least in two dimensions).
  - Two geometric vectors $\vec{x}, \vec{y}$ can be added, such that $\vec{x} + \vec{y} = \vec{z}$ is another geometric vector.
  - Multiplication by a scalar $\lambda\vec{x}$ is also a geometric vector. In fact, it is the original vector scaled by λ
  - Interpreting vectors as geometric vectors enables us to use our intuitions about direction and magnitude to reason about mathematical operations.



Introduction - vector objects

- **Polynomials** are also vectors:
  - Two polynomials can be added together, which results in another polynomial
  - They can be multiplied by a scalar $\lambda \in \mathbb{R}$, and the result is a polynomial as well. Therefore, polynomials are (rather unusual) instances of vectors.
  - Note that polynomials are very different from geometric vectors. While geometric vectors are concrete "drawings", polynomials are abstract concepts.



Introduction - vector objects

- **Audio signals** are vectors.
  - Audio signals are represented as a series of numbers.
  - We can add audio signals together, and their sum is a new audio signal.
  - If we scale an audio signal, we also obtain an audio signal.

Introduction - vector objects

- **Elements of $\mathbb{R}^n$ (tuples of n real numbers)** are vectors
  - $\mathbb{R}^n$ is more abstract than polynomials, and it is the concept we focus on in this subject.
  - For instance, $a = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} \in \mathbb{R}^3$ is an example of a triplet of numbers. Adding two vectors $a,\ b \in \mathbb{R}^n$ component-wise results in another vector: $a + b = c \in \mathbb{R}^n$
  - Multiplying $a \in \mathbb{R}^n\ by\ \lambda \in \mathbb{R}$ results in a scaled vector $\lambda a \in \mathbb{R}^n$
  - Considering vectors as elements of $\mathbb{R}^n$ has an additional benefit that it loosely corresponds to arrays of real numbers on a computer.
  - **Many programming languages support array operations, which allow for convenient implementation of algorithms that involve vector operations.**

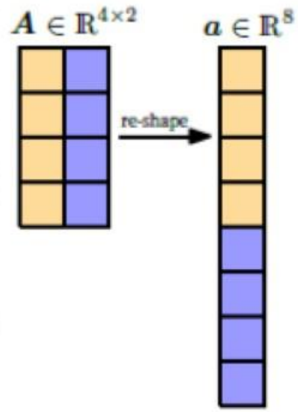
Introduction – Matrix

- Matrices
  - Matrices play a central role in linear algebra.
  - They can be used to compactly represent systems of linear equations, but they also represent linear functions (linear mappings)

Introduction – Matrix

- With $m; n \in N$ a real-valued $(m; n)$ *matrix* A is an $m \cdot n$-tuple of $elements\ a_{ij}$ $, i = 1, \ldots m, j = 1 \ldots n$, which is ordered according to a rectangular scheme consisting of m rows and n columns:

$$\boldsymbol{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}, \quad a_{ij} \in \mathbb{R}$$



  - By convention $(1; n)$ −matrices are called *rows* and $(m; 1)$ −matrices are called column *columns*. These special matrices are also called *row/column vectors*.
  - Rmn is the set of all real-valued $(m; n)$ −matrices. $A \in R^{m\times n}$ can be equivalently represented as a $\in R^{m\times n}$ by stacking all n columns of the matrix into a long vector

Introduction – Matrix operations

- **Matrix Addition and Multiplication:**
  - The sum of two matrices $A \in R^{m\times n}, B \in R^{mn}$ is defined as the elementwise sum, i.e.,

$$\boldsymbol{A}+\boldsymbol{B} := \begin{bmatrix} a_{11}+b_{11} & \cdots & a_{1n}+b_{1n} \\ \vdots & & \vdots \\ a_{m1}+b_{m1} & \cdots & a_{mn}+b_{mn} \end{bmatrix} \in \mathbb{R}^{m\times n}$$

  - For matrices$A \in R^{m\times n}, B \in \mathbb{R}^{n\times k}$, the elements of the product:

$C = AB \in \mathbb{R}^{m\times k}$ are computed as

$$c_{ij} = \sum_{l=1}^{n} a_{il} b_{lj}, \qquad i = 1, \ldots, m, \quad j = 1, \ldots, k$$

Introduction – Matrix operations

- **Inverse and Transpose:**
  - **Inverse:** Consider a square matrix $A \in \mathbb{R}^{n \times n}$. Let matrix $B \in \mathbb{R}^{n \times n}$ have the property that $AB = I_n = BA$. B is called the inverse of A and denoted by A^{-1}.
  - Unfortunately, not every matrix A possesses an inverse A^{-1}.
    - If this regular inverse does exist, A is called *regular/invertible/nonsingular*,
    - otherwise *singular/noninvertible*.

Introduction – Matrix operations

- **Inverse and Transpose:**
  - **Transpose:** For $A \in \mathbb{R}^{m\times n}$ the matrix $B \in \mathbb{R}^{m\times n}$ with $b_{ij} = a_{ji}$ is called the *transpose* of A. We write $B = A^T$
    - In general,A can be obtained by writing the columns of A as the rows of A
    - *Matrix $A \in \mathbb{R}^{n\times n}$ is symmetric if $A = A^T$*
    - *The **sum of symmetric matrices is always symmetric**. However, **their product is always defined, it is generally not symmetric***
- **Multiplication by a Scalar:** Matrices are multiplied by a scalar $\lambda \in \mathbb{R}$.
  - Let $A \in \mathbb{R}^{m\times n}$ and $\lambda \in \mathbb{R}$ then $\lambda A = K$, $K_{ij} = \lambda\, a_{ij}$. Practically, λ scales each element of A

Introduction – Matrix operations

- The determinant $\det(A)$ or $|A|$ of a square matrix A:
  - A number encoding certain properties of the matrix.
  - A matrix is invertible if and only if its determinant is nonzero.
  - The determinant of a product of square matrices equals the product of their determinants: $\det(AB) = \det(A) \cdot \det(B$
  - The determinant of a matrix tell us whether or not the matrix is invertible, can be used to find a formula for the inverse of a matrix, and give us a method for solving linear equations
  - If A has one row of zeros then $detA = 0$
  - If A is a triangular matrix then $detA$ is the product of the entries on the main diagonal

Introduction – Matrix operations

- The determinant $\det(A)$ or $|A|$ of a square matrix A:
  - Determinants are important concepts in linear algebra. A determinant is a mathematical object in the analysis and solution of systems of linear equations.
  - Determinants are only defined for square matrices $A \in \mathbb{R}^{n \times n}$, i.e., matrices with the same number of rows and columns.

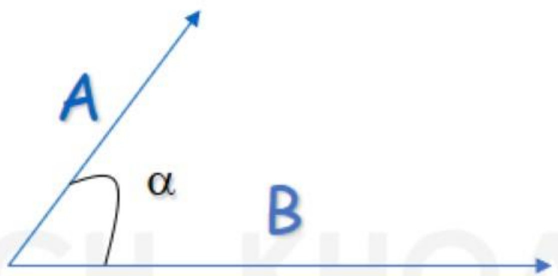
$$\det(\boldsymbol{A}) = \begin{vmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{vmatrix}$$

Introduction – Matrix operations

- The determinant $\det(A)$ or $|A|$ of a square matrix A:
  - Adding a multiple of any row to another row, or a multiple of any column to another column, does not change the determinant.
  - Interchanging two rows or two columns affects the determinant by multiplying it by -1.
  - Multiplying one row (or column) by a nonzero number k affects the determinant by multiplying it by k.

Introduction – Vector operations

- Inner (dot) Product: $v \cdot w$
  - The inner product is a **SCALAR**
  - $v \cdot w = (x_1, x_2) \cdot (y_1, y_2) = x_1 y_1 + x_2 y_2$
  - $v \cdot w = \|v\| \, \|v\| \cos(\alpha)$
  - $v \cdot w = 0 \iff v \perp w$
  - vectors $v \cdot w$ are "**columns**", then dot product is $w^T v$
  - $v \cdot w = (x_1, x_2) + (y_1, y_2) = (x_1 + y_1, x_2 + y_2)$
  - $\lambda \cdot v = \lambda(x_1, x_2) = (\lambda x_1, \lambda x_2)$



Introduction – Vector operations

- Dot Product
  - The dot product as a matrix multiplication

$$A \cdot B = A^T B = [a \quad b \quad c] \begin{bmatrix} d \\ e \\ f \end{bmatrix} = ad + be + cf$$

  - The dot product of a vector with itself $\|A\|^2 = A^T A = aa + bb + cc$
  - The dot product is also related to the angle between the two vectors

$$A \cdot B = \|A\| \, \|B\| \cos(\alpha)$$

Introduction – Vector operations

- Norms: the norm of a vector $\|x\|$
  - Absolutely homogeneous: $\lambda\|x\| = \|\lambda\|\|x\|$
  - Triangle inequality: $\|x + y\| \leq \|x\| + \|y\|$
  - Positive definite: $\|x\| \geq 0$ and $\|x\| = 0 \Leftrightarrow x = 0$
  - The Manhattan norm on $\mathbb{R}^n$ is defined for $x \in \mathbb{R}^n$ (is also called l_1 norm)

$$\|x\|_1 = \sum_{i=1}^{n} |x_i|,$$

where $\|\cdot\|$ is the absolute value, The Manhattan norm is also called l_1 norm

  - Euclidean Norm on $\mathbb{R}^n$ is defined for $x \in \mathbb{R}^n$ (is also called l_2 norm)

$$\|x\|_1 = \sqrt{\sum_{i=1}^{n} x_i^2} = \sqrt{x^T x}$$

Introduction – Vector operations

- Norms:
  - l_p norm of x is defined is ($p \geq 1$):

$$\|x\|_p = \sqrt[p]{\sum_{i=1}^{n} |x_i|^p}$$

  - l_∞ norm (maximum norm or uniform norm) of x is defined

$$\|x\|_p = \max\{|x_1|, |x_2|, \dots, |x_n|\}$$

Introduction – Vector operations

- Vector Addition:
  - $A + B = (x_1, x_2) + (y_1 + y_2) = (x_1 + x_2, y_1 + y_2)$
- Scalar Product:
  - $\lambda A = \lambda(x_1, x_2) = (\lambda x_1, \lambda x_2)$

Introduction - Matrix Norms

- Common matrix norms for a matrix A

column-sum norm $\|A\|_1 = \max_{1 \le j \le n} \sum_{i=1}^{n} |a_{ij}|$

Frobenius norm

row-sum norm $\|A\|_\infty = \max_{1 \le i \le n} \sum_{j=1}^{n} |a_{ij}|$

spectral norm (2 norm) $\|A\|_2 = (\mu_{\max})^{1/2}$

μ_{max} is the largest eigenvalue of $A^T A$.

Introduction – Vector operations

- Inner Products
  - Inner products allow for the introduction of intuitive geometrical concepts, such as the length of a vector and the angle or distance between two vectors.
  - A major purpose of inner products is to determine whether vectors are orthogonal to each other.
- *Cross product or vector product is an operation on two vectors in three-dimensional space*