MODULE III

FUNCTIONS

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Definition 1

Let A and B be sets. A function f from A to B is a rule that assigns exactly one element of B to each element of A.

If b is the unique element of B assigned by the function f to the element a of A, we write $f(a) = b$. i.e. if f is a function from A to B write $f : A \to B$.

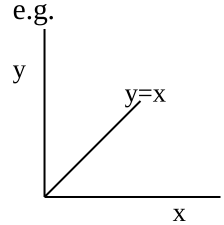
There are generally 4 ways to represent a function:

i. Verbally: by a description in words e.g. $p(t)$ is the population of the world at time t.

ii. Algebraically – by a formula e.g. $f(x) = x^2$

iii. Visually – by a graph

e.g.



iv. Numerically – by a table of values

e.g..

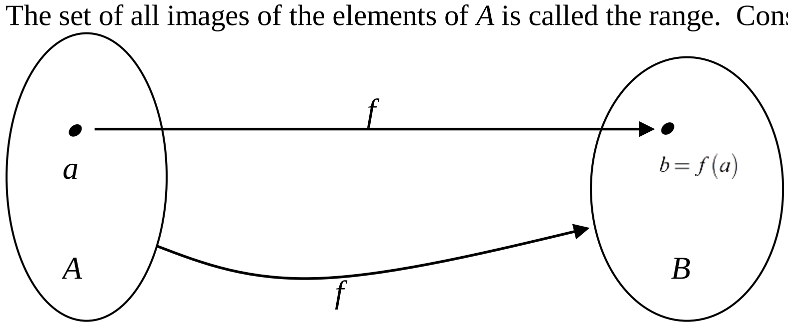
x	1	2	3
x^2	1	4	9

Definition 2

If f is a function from A to B then A is called the Domain of f and B the Codomain of f.

If $f(a) = b$, "b" is called the image of "a" and "a" is called the pre-image of "b".

The set of all images of the elements of A is called the range. Consider the figure below



The function f maps A to B.

i.e. $f : A \to B$

A is the domain of f

B is the codomain of f.

A is the pre-image of b.

B is the image of a

$f(a) = b$ is the range i.e. set of all images of elements of set A, more precisely all the elements in B constitute the range.

Example 1
The squaring function from the et of integers to the set of integers assigns to each <u>integer</u> number x its square x^2.
i.e. $f(x)=x^2$ e.g. $f(3)=3^2=9$, $f(4)=4^2=16$ etc
The domain of the function $f(x)=x^2$ is the set of all integers. The codomain is the set of all positive integers. (Since all squares are positive). The range is the set of all positive integers which are perfect squares.
(i.e. 0, 1, 4, 9, 16, …)

Example 2
For the function $f(x)=2x^2+3x-1$ evaluate

i. $f(2)$
ii. $f(a)$
iii. $f(a+h)$

Solution
i. $f(2)=2(2)^2+3(2)-1=8+6-1=13$

ii. $f(a)=2a^2+3a-1$

iii. $f(a+h)=2(a+h)^2+3(a+h)-1$

$$=2(a^2+2ah+h^2)+3(a+h)-1$$

$$=2a^2+4ah+2h^2+3a+3h-1$$

Addition and multiplication of functions
Definition
Let f_1 and f_2 be functions from set A to the set of real numbers. Then the sum f_1+f_2 and the product f_1f_2 are also functions from the set A to the set of reals defined by

$$(f_1+f_2)(x)=f_1(x)+f_2(x) \quad \text{[sum]}$$

and

$$(f_1f_2)(x)=f_1(x)f_2(x) \quad \text{[product]}$$

Example
Let f_1 and f_2 be functions from the set of real numbers to the set of real numbers i.e. from $\mathbb{R}$ to $\mathbb{R}$ such that $f_1(x)=x^2$ and $f_2(x)=x-x^2$. Find f_1+f_2 and f_1f_2

Solution
$$(f_1+f_2)(x)=f_1(x)+f_2(x)=x^2+(x-x^2)=x$$

$$(f_1f_2)(x)=x^2(x-x^2)=x^3-x^4$$

Exercise

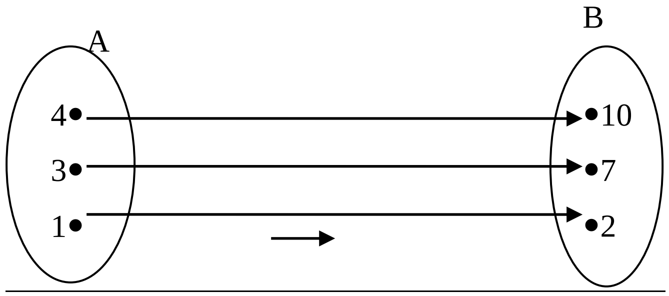
If $f(x) = x^2 + 1$ and $g(x) = x + 2$ are functions from the set of real numbers to the set of real numbers i.e. from $\mathbb{R}$ to $\mathbb{R}$. Find $f+g$ and *fg*.

One To One And Onto Functions

Definition 1 – ONE TO ONE FUNCTION

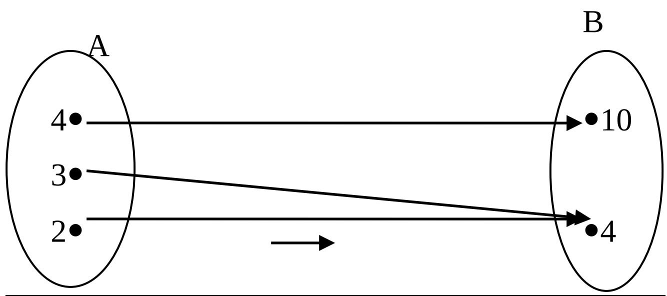
A function *f* is said to be one to one or injective if and only if $f(x) = f(y)$ implies that $x = y$ for all *x* and *y* in the domain of *f*. In other words a function within domain *A* is called a one to one function if no two elements of *A* have the same image in the codomain of *B*. Consider the function *f* and *g* below

<u>Figure I</u>



f is one to one since all the elements in *A* have unique image in *B*.

<u>Figure II</u>

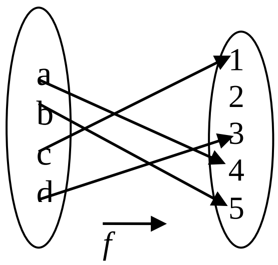


g is not one to one since 3 and 2 in *A* have the same image 4 in *B*.

Example 1

Determine whether the function *f* from the set $\{a, b, c, d\}$ to the set $\{1, 2, 3, 4, 5\}$ with $f(a) = 4$, $f(b) = 5$, $f(c) = 1$ and $f(d) = 3$ is one to one.

Solution



f is one to one since every element in the domain has a unique image.

Example 2

Determine whether the function $f(x) = x+1$ for $x = 0,1,2,3$ is one to one .

Solution

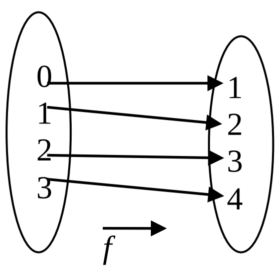
$f(x) = x+1$

$f(0) = 0+1 = 1$

$f(1) = 1+1 = 2$

$f(2) = 2+1 = 3$

$f(3) = 3+1 = 4$



The function is one to one.

Exercise

1. Determine whether the function $f(x) = x^3$ from $\mathbb{R}$ to $\mathbb{R}$ is one to one. Explain.
2. Is the function $f(x) = 3x+4$ from the set of integers to integers one to one? Why?.

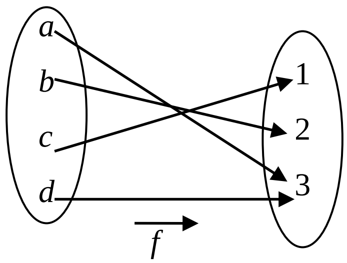
Definition 2 (ONTO FUNCTION)

A function *f* from *A* to *B* is called <u>onto</u> or surjective if and only if for every element $b \in B$ there is an element $a \in A$ with $f(a) = b$. i.e for <u>onto</u> functions every member of the codomain is the image of some element of the domain.

Example

Let *f* be a function from the set $\{a,b,c,d\}$ to the set $\{1,2,3\}$ defined by $f(a) = 3$, $f(b) = 2$, $f(c) = 1$ and $f(d) = 3$. Is *f* an onto function?

Solution



f is an onto function since all the three elements of the codomain are images of elements in the domain.
Note: for onto function, every element in the codomain must have a match in the domain.

Exercise
With an explanation determine whether the following functions from the set of integers to integers are one to one or onto.

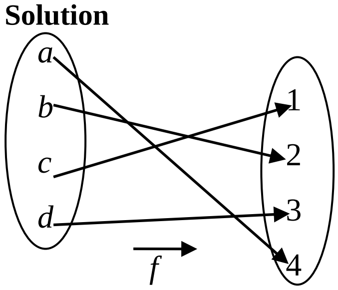
1. $f(x) = x^2$
2. $f(x) = x + 1$

Definition 3

One To One Correspondence
A function *f* is called a one to one correspondence or bijective if it is both one to one and onto. In other words a bijection is both an injection and a surjection.

Example
Let *f* be the function from the set $\{a, b, c, d\}$ to $\{1, 2, 3, 4\}$ with $f(a) = 4$, $f(b) = 2$, $f(c) = 1$ and $f(d) = 3$. Is *f* a one to one and onto function? Why?

Solution



f is a one to one and onto function.
Reasons

1. Its one to one since all the elements in the domain takes on distinct values in the codomain.
2. Its onto since all the four elements in the codomain are images of elements in the domain. Therefore *f* is a bijection.

Exercises
Determine whether each of the functions below from the set of real numbers to the set of real numbers i.e. from $\mathbb{R}$ to $\mathbb{R}$ is a one to one correspondence. Explain.

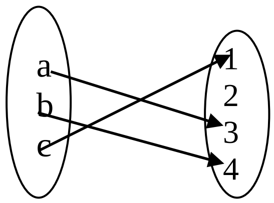
i. $f(x) = 2x + 1$

ii. $f\ \ x = x^2 + 1$

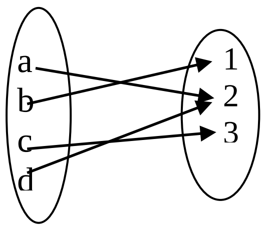
iii. $f\ \ x = x^3$

Summary of different types of correspondences

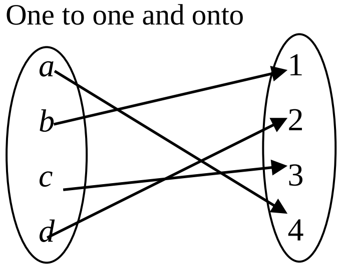
i. One to one but not onto



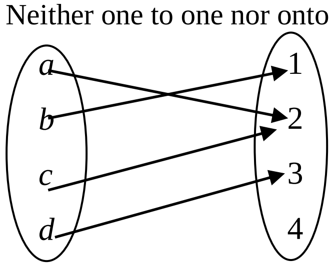
ii. Onto but not one to one



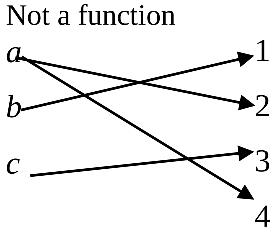
iii. One to one and onto



iv. Neither one to one nor onto



v. Not a function



Composition Of Functions

Definition

If f and g are functions of a variable x, then the composition of the functions f and g denoted fog is defined by

$$fog\left(x\right)=f\left(g\left(x\right)\right)$$

and the domain of fog is given by

$$Dom\left(fog\right)=\left\{X, g\left(x\right) : x\in X \text{ and } g\left(x\right)\in Dom\left(f\right)\right\}$$

Example

Let f and g be the functions from the set of integers defined by $f\left(x\right)=2x+3$ and $g\left(x\right)=3x+2$.

What is the composition of

i. f and g.

ii. g and f.

Solution

i. $fog\left(x\right)=f\left(g\left(x\right)\right)=f\left(3x+2\right)$

$$=2(3x+2)+3=6x+7$$

ii. $gof\left(x\right)=g\left(f\left(x\right)\right)=g\left(2x+3\right)$

$$=3\left(2x+3\right)+2=6x+11$$

Example 2

If $f\left(x\right)=x^2$ and $g\left(x\right)=x+1$, find fog.

Solution

$$fog=f\left(g\left(x\right)\right)=f\left(x+1\right)=\left(x+1\right)^2$$

Remarks

fog and gof are not equal hence the commutative property does not hold for composition of functions.

Exercise

Determine fog and gof given that

i. $f\left(x\right)=x^3$ and $g\left(x\right)=x^{\frac{1}{3}}$

ii. $f\left(x\right)=\dfrac{x-1}{x+1}$ and $g\left(x\right)=\dfrac{1}{x}$

iii. $f\left(x\right)=\sqrt{x}$ and $g\left(x\right)=x^2+1$

Inverse functions

Definition The functions f and g are said to be inverses of each other iff

i. $f\left(g\left(x\right)\right)=x$ for every x in the domain of g.

ii. $g(f(x)) = x$ for every x in the domain of f.

If g is the inverse of f then we denote g by f^{-1}, thus $f(f^{-1}(x)) = x$ and $f^{-1}(f(x)) = x$

Example 1

Show that the functions $f(x) = 2x+3$ and $g(x) = \frac{1}{2}(x-3)$ are inverses of each other.

Solution

$$f(g(x)) = f\left(\frac{1}{2}(x-3)\right) = 2\left(\frac{1}{2}(x-3)\right) + 3 = x$$

and

$$g(f(x)) = g(2x+3) = \frac{1}{2}((2x+3)-3) = x$$

Hence inverse

Example 2

Let f be the function from $\{a,b,c,d\}$ to $\{1,2,3\}$ such that $f(a) = 2$, $f(b) = 3$ and $f(c) = 1$. Find f^{-1}.

Solution

The inverse function f^{-1} reverses the correspondence given f such that $f^{-1}(1) = c$, $f^{-1}(2) = a$ and $f^{-1}(3) = b$.

Remarks: A function has an inverse iff it's a one to one correspondence.

Exercise

1. Show that the functions $f(x) = x^3$ and $g(x) = x^{\frac{1}{3}}$ are inverses of each other.
2. Show that the functions f and g are inverses of each other by showing that $f(g(x)) = x$.
   i. $f(x) = x^3 - 8$ and $g(x) = \sqrt[3]{x-8}$
   ii. $f(x) = \dfrac{x-5}{2x+3}$ and $g(x) = \dfrac{3x+5}{1-2x}$
3. Find the inverse of the functions
   i. $f(x) = 3x-2$
   ii. $f(x) = \dfrac{5-3x}{2}$