Chapter 10
Fuzzy Portfolio Selection Models: A Numerical Study

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Abstract In this chapter we analyze the numerical performance of some possibilistic models for selecting portfolios in the framework of risk-return trade-off. Portfolio optimization deals with the problem of how to allocate wealth among several assets, taking into account the uncertainty involved in the behavior of the financial markets. Different approaches for quantifying the uncertainty of the future return on the investment are considered: either assuming that the return on every individual asset is modeled as a fuzzy number or directly measuring the uncertainty associated with the return on a given portfolio. Conflicting goals representing the uncertain return on and risk of a fuzzy portfolio are analyzed by means of possibilistic moments: interval-valued mean, downside-risk, and coefficient of skewness. Thus, several nonlinear multi-objective optimization problems for determining the efficient frontier could appear. In order to incorporate possible trading requirements and investor's wishes, some constraints are added to the optimization problems, and the effects of their fulfillment on the corresponding efficient frontiers are analyzed using a data set from the Spanish stock market.

10.1 Introduction

The portfolio selection problem deals with finding an optimal strategy for building satisfactory portfolios. From Markowitz's seminal work many different modeling approaches have been developed in order to propose suitable investment strategies [44]. Concerning uncertainty quantification, it is assumed that these decision problems can be modeled either by using the stochastic tools provided by probability theory or by using soft computing approaches based on fuzzy set theory [65].

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The decision problem under uncertainty must be then approached using different mathematical tools, and from an optimization point of view both linear and nonlinear programming and different meta-heuristic techniques can be used for solving the portfolio selection problem.

Modern portfolio selection theory usually deals with two opposite concepts: risk aversion and maximizing return. The main point of this modeling approach is how risk and asset profitability are defined and measured. Following Markowitz's proposal classic models consider the return on an asset as a random variable and its profitability is defined as the mathematical expectation of that random variable, while risk is measured by means of the variance. Since the formulation of the mean–variance model, a variety of enlarged and improved models have been developed in several directions. One dealt with alternative portfolio selection models either by modifying the risk measure (mean–semivariance model [45], mean-absolute deviation model [31], mean–downside risk models [54, 55]) or by adding higher probability moments to best represent the uncertainty on the returns [30, 34]. Others dealt with the introduction of factors influencing stock prices based on the Capital Asset Pricing Model [42, 50] and derivative methodologies.

In order to identify the best portfolio for a given level of desired return, Levy and Markowitz [39] propose extending the classic portfolio selection models into multi-criteria decision-making models. A straightforward approach to selecting an optimal portfolio is minimizing the risk and maximizing the expected return simultaneously, that is considering a bi-objective optimization problem. From then on, multi-objective programming techniques have been applied for portfolio selection, and the solutions are usually obtained using scalar optimization by aggregation of the objectives into a single one [3, 57]. Some multi-objective approaches also allow the incorporation of higher-order moments as several alternative criteria for portfolio selection [10, 34, 56, 62].

Another approach to uncertainty quantification is based on fuzzy set theory, which also provides a framework for the analysis of decisions about investment under imperfect knowledge of future market behavior. Different elements and characteristics of the portfolio selection problem can be fuzzified such as the following papers reflect. Watada [63] uses fuzzy sets to introduce the vague goals of the decision makers for the expected rate of return and risk in the mean–variance model. Ramaswamy [48] applies fuzzy decision theory to selecting optimal portfolios with targets above the risk-free rate, taking into account only market risk under different scenarios of market behavior. Tanaka and Guo [58] use possibility distributions to model uncertainty in the returns, allowing the incorporation of expert knowledge by means of a possibility grade, which reflects the degree of similarity between the future state of stock markets and the state of previous periods. Arenas et al. [2] propose a fuzzy goal programming approach for portfolio selection based on a factor model, also taking into account the liquidity of the investment. León et al. [37] present a fuzzy interactive algorithm for selecting optimal portfolios that uses a modification of Zimmermann's method [67] for solving multi-objective decision problems; the authors also describe a fuzzy optimization scheme to manage unfeasible instances of the portfolio selection problem.

In this chapter we analyze the performance of certain possibilistic multi-objective portfolio selection models with the main objective of providing the investor with sufficient tools to address the portfolio selection problem. Discrete constraints representing trading requirements and investors' preferences are introduced by means of diversification and restricted cardinality conditions. The optimization scheme goals are to minimize the risk of the investment while maximizing the expected return and skewness. Then, a multi-objective evolutionary algorithm finds efficient portfolios that also meet the investors' wishes. The uncertainty in the expected income is represented by modeling the returns on the individual assets as trapezoidal fuzzy numbers, or alternatively, directly computing the trapezoidal fuzzy number that approximates to the return on a given portfolio, without requiring the estimation of the joint distribution of the returns on the assets. The expected return on the investment are approximated by using interval-valued possibilistic means, the risk being measured as a fuzzy downside risk. In order to analyze whether the introduction of skewness would significantly improve the quality of chosen portfolios, the relationship between the fuzzy downside risk and the possibilistic coefficient of skewness is considered, also with respect to the observation that increasing diversification could lead to a loss of skewness. Our approaches for selecting efficient portfolios are applied to a selection of assets from the Spanish Stock Market.

The rest of the chapter is organized in the following manner. The next section presents a brief summary of portfolio optimization models. In Sect. 10.3 we include some definitions and basic results of fuzzy sets and possibilistic moments, which are used to represent the uncertainty of returns. Section 10.4 describes the formulation of the portfolio selection problem as a possibilistic multi-objective programming problem. Section 10.5 shows that a satisfactory solution can be easily obtained using a meta-heuristic procedure. Suitable comparisons are reported, also showing the influence of the presence of discrete constraints and the effect of considering skewness as an effective goal of the optimization scheme. Section 10.6 gives the conclusions.

10.2 Portfolio Optimization Models

Let us present some portfolio modeling approaches, paying special attention to the optimization techniques that have been used for selecting optimal portfolios.

10.2.1 Portfolio Selection Models Based on Probability Theory

Markowitz shows how rational investors can construct optimal portfolios under conditions of uncertainty using both probability theory and optimization techniques. In the mean–variance (MV) portfolio approach the return on any portfolio is quantified as its expected value and its risk is quantified as its variance. In a standard formulation we have the following quadratic programming problem, for a given expected return ρ:

$$
\begin{aligned}
(\mathrm{MV})\ \text{Min}\ & \textstyle\sum_{j=1}^{n}\sum_{i=1}^{n}\sigma_{ij}x_ix_j \\
\text{s.t.}\ & \textstyle\sum_{j=1}^{n} x_j E(R_j) = \rho \\
& \textstyle\sum_{j=1}^{n} x_j = 1 \\
& x_j \geq 0,
\end{aligned}
$$

where x_j is the proportion of the portfolio held in the asset jth, R_j is the random variable representing the return on asset jth, and σ_{ij} is the covariance between R_i and R_j, for $i, j = 1, \ldots, n$. The solution for this quadratic program for different values of ρ allows identifying the set of efficient portfolios (see [4], for a complete report on nonlinear optimization). It is usual to plot the pairs (σ, ρ) to represent the portfolio efficient frontier where σ is the standard deviation.

A number of researchers have introduced alternative measures of risk for portfolio planning, and in many cases these measures are linear, leading to a corresponding simplification in the computational model. According to Konno and Yamazaki [31], when the returns on the assets are multivariate normally distributed, the above portfolio selection problem is equivalent to the mean-absolute deviation model (MAD), which minimizes the sum of absolute deviations from the averages associated with the x_j choices:

$$
\begin{aligned}
(\mathrm{MAD})\ \text{Min}\ & E(|\textstyle\sum_{j=1}^{n} R_j x_j - E(\sum_{j=1}^{n} R_j x_j)|) \\
\text{s.t.}\ & \textstyle\sum_{j=1}^{n} x_j E(R_j) \geq M\rho \\
& \textstyle\sum_{j=1}^{n} x_j = M \\
& 0 \leq x_j \leq u_j,
\end{aligned}
$$

where M is the total fund and u_j represents the maximum amount of the total fund which can be invested in the asset jth. This modeling approach permits avoidance of one of the main drawbacks associated with the solution of the MV model: the input problem of estimating $2n + n(n-1)/2$ parameters. Moreover, it can be easily converted into a finite linear optimization problem by replacing its objective function with:

$$
\begin{aligned}
\text{Min}\ & (1/T)\textstyle\sum_{k=1}^{T} y_k \\
\text{s.t.}\ & y_k + \textstyle\sum_{j=1}^{n}(r_{kj} - E(R_j))x_j \geq 0 \quad k = 1, \ldots, T \\
& y_k - \textstyle\sum_{j=1}^{n}(r_{kj} - E(R_j))x_j \geq 0 \quad k = 1, \ldots, T,
\end{aligned}
$$

where the returns on the assets over T periods are given and r_{kj} denotes the return on the jth asset at the time k, for $k = 1, \ldots, T$. The corresponding linear programming problem (LMAD) gives portfolios which involve fewer nonzero components and hence reduces the numerous small transactions that are likely to appear in the MV model.

Concerning the performance of the above modeling approaches, Simaan [51] stated that although the minimization of mean-absolute deviation is close to the MV formulation, they lead to two different efficient sets. This divergence is probably due to the fact that each model utilizes different sample statistics and consequently relies on a different set drawn from the sample. On the other hand, since the normality assumption is rarely verified in practice, different portfolios can be obtained when the above models are applied. Júdice et al. [29] use MV and LMAD models in real-life capital markets for analyzing the stability of the selected portfolios and their expected return, and conclude that no one model is superior to the other. Papahristodoulou and Dotzauer [47] also compare them with out-of-sample data from shares traded in the Stockholm Stock Exchange and the results show that the MV model yields higher utility levels and higher degrees of risk aversion in very similar computing times.

Dissatisfaction with the traditional notion of variance as a measure of investment risk may be due to the fact that it makes no distinction between gains and losses. Thus, Markowitz [45] proposed the use of the semi-variance instead of the variance:

$$\mathrm{SV}(x) = \mathrm{E}\left(\left(\max\left\{0, \mathrm{E}(\sum_{j=1}^{n} R_j x_j) - \sum_{j=1}^{n} R_j x_j\right\}\right)^2\right) \tag{10.1}$$

From then on, several optimization models which consider only the downside risk of a portfolio have been introduced. In particular, if risk is measured by means of the mean-absolute semi-deviation, as proposed in Speranza [55], the following risk function appears:

$$\mathrm{SD}(x) = \mathrm{E}\left(\left|\min\left\{0, \sum_{j=1}^{n} R_j x_j - \mathrm{E}(\sum_{j=1}^{n} R_j x_j)\right\}\right|\right) \tag{10.2}$$

which can be easily evaluated in contrast with the complexity of computing the semi-variance of a given portfolio. Konno et al. [32] review the performance of different lower partial measures of risk by applying linear programming techniques, which also allows the efficient resolution of large-scale instances.

Recently, a lot of research has been undertaken with the purpose of constructing efficient portfolios using meta-heuristic techniques [1, 14, 43] and for developing decision support system (DSS) strategies for assessing the investors during the process of decision making [64, 69].

10.2.2 Fuzzy Approaches for Portfolio Selection

Portfolio selection models use parameters, goals, and constraints whose characteristics and values are imprecise in a certain sense. This imperfect knowledge can be introduced by means of fuzzy quantities in very different ways. Let us mention some of them.

The portfolio selection problem under uncertainty can be transformed into a problem of decision making in a fuzzy environment by modeling the investors' aspiration levels for the expected return on and risk of through suitable membership functions of a fuzzy set. For instance, the fuzzy portfolio selection model in Watada [63], based on the mean–variance model, assigns a logistic membership function to the goal of expected return, as follows:

$$\mu_E\left(\sum_{j=1}^{n} x_j E(R_j)\right) = \frac{1}{1+\exp[-\beta_1(\sum_{j=1}^{n} x_j E(R_j) - E_M)]}, \tag{10.3}$$

where β_1 is the positive shape parameter of the logistic function and E_M is the point in the support of the membership function with a value of 0.5 (analogously for the membership function of the goal associated with risk: β_2 and V_M). Since goals and constraints can be represented as a fuzzy set, the corresponding decision problem can be defined as the intersection of goals and constraints. Therefore, if we consider a full compensation approach, it would be more appropriate to define the membership function of the fuzzy decision by taking the maximum degree of membership achieved by any of the fuzzy sets representing objectives or constraints. The portfolio selection problem can then be stated as

$$\begin{array}{lll} \text{(FMV)} & \text{Max} & \lambda \\ & \text{s.t.} & \lambda + (\exp[-\beta_1(\sum_{j=1}^{n} x_j E(R_j) - E_M)])\lambda \leq 1 \\ & & \lambda + (\exp[\beta_2(\sum_{j=1}^{n} \sigma_{ij} x_i x_j - V_M)])\lambda \leq 1 \\ & & \sum_{j=1}^{n} x_j = 1 \\ & & x_j \geq 0, \end{array}$$

where λ is the degree of satisfaction of the solution of the above nonlinear programming problem.

Another approach to managing uncertainty is based on possibility distributions which are associated with fuzzy variables [20, 66]. Tanaka and Guo [58], for instance, use exponential possibility distribution to build a possibility portfolio model which integrates the historical data set of returns on individual assets and experts' experience and judgment. Two possibility distributions (upper and lower) are identified from the given possibility degrees for data that reflect two opposing expert opinions. Portfolio selection problems based on upper and lower possibility distributions are formalized as quadratic programming problems minimizing the spreads of possibility portfolios subject to the given center returns (r_c), as follows:

$$\begin{array}{lll} \text{(TG)} & \text{Min} & x^t D_A x \\ & \text{s.t.} & c^t x = r_c \\ & & \sum_{j=1}^{n} x_j = 1 \\ & & x_j \geq 0. \end{array}$$

The estimation of the matrix D_A is made using a linear programming problem, where the data are (r_k, h_k), $k = 1, \ldots, T$ and $r_k = (r_{1k}, \ldots, r_{nk})^t$ is the vector of the returns on n assets during the kth period, h_k being an associated possibility grade given by expert knowledge to reflect the degree of similarity between the future state of stock markets and the state of the kth sample. The authors show that the spread of the portfolio return based on a lower possibility distribution is smaller than the spread of the portfolio return based on an upper possibility distribution for the same center value. More information about this possibilistic model and other approaches to portfolio selection based on fuzzy set theory can be found in [22].

In contrast with the above approaches other authors propose the incorporation of fuzzy numbers to directly represent the uncertainty of the future returns on the assets [12,38,46]. In Carlsson et al. [12], the rates of return on securities are modeled by possibility distributions, and the return on, $E(r_p)$, and risk of the portfolio $\sigma^2(r_p)$ are, respectively, quantified using the possibilistic mean and variance previously defined in [11]. The authors find an exact optimal solution to the following portfolio selection problem under trapezoidal possibility distributions:

$$\begin{aligned} (\text{CFM})\ \text{Max}\ & E(r_p) - 0.005A\sigma^2(r_p) \\ \text{s.t.}\ & \textstyle\sum_{j=1}^{n} x_j = 1 \\ & x_j \geq 0, \end{aligned}$$

where A is an index of the investors' aversion to risk. Alternatively, assuming that the downside risk is a more realistic description of investor preferences, because this risk function only penalizes the non-desired deviations, some fuzzy models for portfolio selection have been proposed [61]. The fuzzy downside risk function evaluates the mean-absolute semi-deviation with respect to the total return $\tilde{R}_P(x)$ as follows [38]:

$$\mathrm{w}_P(x) = \mathrm{E}(\max\{0, \mathrm{E}(\tilde{R}_P(x)) - \tilde{R}_P(x)\}) \tag{10.4}$$

where the total return on the fuzzy portfolio is a convex linear combination of the individual asset returns, that is: $\tilde{R}_P(x) = \sum_{j=1}^{n} x_j \tilde{R}_j$ and $\tilde{R}_j$ are LR-fuzzy numbers, for $j = 1, 2, \ldots, n$. Then, the fuzzy mean–downside risk portfolio selection problem can be stated in the following way:

$$\begin{aligned} (\text{MDR})\ \text{Min}\ & \mathrm{E}(\max\{0, \mathrm{E}(\tilde{R}_P(x)) - \tilde{R}_P(x)\}) \\ \text{s.t.}\ & \textstyle\sum_{j=1}^{n} x_j \tilde{R}_j \succeq \tilde{r} \\ & \textstyle\sum_{j=1}^{n} x_j = 1 \\ & l_j \leq x_j \leq u_j, \end{aligned}$$

where $l_j \geq 0$ represents the minimum amount of the total fund which can be invested in asset jth. The calculation of the fuzzy expected return and downside risk depends on both the characteristics of the LR-fuzzy numbers which represent the individual returns and the definition of the average of a fuzzy number. Also, for satisfying the fuzzy constraint, different approaches can be used (see, for instance, [59]). Using interval-valued expectations the above MDR model provides optimal portfolios by applying linear optimization, linear semi-infinite optimization, or linear interval programming, for different modeling approaches.

Recently, Huang [27] has introduced fuzzy models for portfolio selection under the assumption that the returns on the assets are random fuzzy variables; other approaches which deal with randomness and fuzziness simultaneously have also been proposed (see, for instance, [25, 33]).

The next section introduces definitions of and criteria for modeling return and risk using possibilistic moments and then looks into other related criteria which are based on fuzzy logic to represent the uncertainty of future returns on assets and portfolios.

10.3 Modeling Uncertainty

The concept of fuzzy sets was introduced by Zadeh in 1965, and since then it has been used for modeling uncertainty or impreciseness in data. Fuzzy set theory allows a more precise mathematical description to be given of what are normally vague statements, and has become an interesting tool when applied to decision problems (see, for instance, [5, 52, 68]). A deep and comprehensive treatment of fuzzy sets and their properties is provided, for instance, in [21].

10.3.1 Fuzzy Numbers and Fuzzy Arithmetic

Let us briefly recall some definitions and results which will be used in what follows.

Definition 10.1. Let X denote the universal set. A fuzzy set $\tilde{A}$ in X is characterized by a membership function $\mu_{\tilde{A}}(x)$ which associates a real number in the interval [0,1] with each point in X, where the value of $\mu_{\tilde{A}}(x)$ at x represents the *grade of membership* of x in $\tilde{A}$.

Definition 10.2. A fuzzy number $\tilde{A}$ is a fuzzy set defined on the set of real numbers $\Re$, characterized by means of a membership function $\mu_{\tilde{A}}(x)$ which is upper semi-continuous and satisfies the condition $\sup_{x\in\Re}\mu_{\tilde{A}}(x) = 1$, and whose α-cuts, for $0 \leq \alpha \leq 1$: $[\tilde{A}]^{\alpha} = \{x \in \Re : \mu_{\tilde{A}}(x) \geq \alpha\}$, are convex sets.

Definition 10.3. A fuzzy number is said to be a trapezoidal LR-fuzzy number, $\tilde{A} = (a_l, a_u, c, d)_{LR}$, if its membership function has the following form:

$$\mu_{\tilde{A}}(y) = \begin{cases} L(\frac{a_l-y}{c}) = \frac{y-(a_l-c)}{c} & \text{if } y \in [a_l - c, a_l] \\ 1 & \text{if } y \in [a_l, a_u] \\ R(\frac{y-a_u}{d}) = \frac{a_u+d-y}{d} & \text{if } y \in [a_u, a_u + d] \\ 0 & \text{otherwise} \end{cases},$$

where the reference functions $L, R : [0,1] \to [0,1]$ are linear, $[a_l, a_u]$ is the core of $\tilde{A}$, and the closure of the support of $\tilde{A}$, $\text{supp}(\tilde{A}) = \{y : \mu_{\tilde{A}}(y) > 0\}$, is exactly $[a_l - c, a_u + d]$.

The aggregation and ranking of positive linear combinations of LR-fuzzy numbers have been extensively dealt with when their reference functions are linear or all of them have the same shape, because it provides LR-fuzzy numbers of the same shape. Thus, using Zadeh's extension principle, the following arithmetical rules hold.

Theorem 10.1. *Let $\tilde{A} = (a_l, a_u, c_1, d_1)_{LR}$ and $\tilde{B} = (b_l, b_u, c_2, d_2)_{LR}$ be two LR-fuzzy numbers and let $\lambda \in \Re$ be a real number. Then,*

1. $\tilde{A} + \tilde{B} = (a_l + b_l, a_u + b_u, c_1 + c_2, d_1 + d_2)_{LR}$
2. $\lambda \tilde{A} = \begin{cases} (\lambda a_l, \lambda a_u, \lambda c_1, \lambda d_1)_{LR} & if\ \lambda \geq 0 \\ (\lambda a_u, \lambda a_l, |\lambda| d_1, |\lambda| c_1)_{LR} & if\ \lambda < 0 \end{cases}$,

where the addition and multiplication by a scalar is defined by the sup-min extension principle.

The above result cannot be applied to differently shaped LR-fuzzy numbers, where this aggregation is defined with respect to the α-level sets of the fuzzy number $\tilde{A}$ [20, 36].

10.3.2 Possibilistic Moments

The possibility measure of an event might be interpreted as the possibility degree of its occurrence under a possibility distribution. Let $\tilde{A}$ be a fuzzy number, the membership function values for every $x \in \Re$ can be interpreted as the degree of possibility of the statement "x is the value of $\tilde{A}$." Since the fuzzy numbers can be considered a special class of possibility distributions, the imprecise coefficients and vagueness and imprecision of data may be modeled by means of possibility distributions [66].

In this chapter the uncertainty regarding the returns of a given investment is modeled by means of fuzzy quantities for which different definitions of the average can be used. Let us remember some of them. In 1987 Dubois and Prade defined the mean value of a fuzzy number as a closed interval bounded by the expectations calculated from its lower and upper probability mean values [19]:

Definition 10.4. The interval-valued expectation of a fuzzy number $\tilde{A}$ is the following interval: $E(\tilde{A}) = [E_*(\tilde{A}), E^*(\tilde{A})]$, whose endpoints are $E_*(\tilde{A}) = \int_0^1 \inf \tilde{A}_\alpha \mathrm{d}\alpha$ and $E^*(\tilde{A}) = \int_0^1 \sup \tilde{A}_\alpha \mathrm{d}\alpha$, where $\inf \tilde{A}_\alpha$ and $\sup \tilde{A}_\alpha$ denote the left and right extreme points of the α-cut of $\tilde{A}$ for $0 \leq \alpha \leq 1$.

This mean interval definition provides the nearest interval approximation to the fuzzy number with respect to the metric introduced in [24], its width being equal to the width of $\tilde{A}$ in the Chanas sense [13]. We will use the midpoint of this mean interval as the crisp representation of the fuzzy expected return, which is denoted as $\bar{E}(\tilde{A})$. In order to incorporate the importance of α-level sets into the definition of mean value of a fuzzy quantity, Fullér and Majlender [23] introduce the concept of weighted possibilistic expectation of fuzzy numbers, extending the definition of interval-valued possibilistic mean and variance given by Carlsson and Fullér [11].

Definition 10.5. Let $\tilde{A}$ be a fuzzy number and $f(\alpha)$ be a weighted function. The f-weighted possibilistic mean of $\tilde{A}$ is the following interval: $M(\tilde{A}) = [M_*(\tilde{A}), M^*(\tilde{A})]$, where the endpoints are calculated as $M_*(\tilde{A}) = \int_0^1 f(\alpha)\inf\tilde{A}_\alpha \mathrm{d}\alpha$ and $M^*(\tilde{A}) = \int_0^1 f(\alpha)\sup\tilde{A}_\alpha \mathrm{d}\alpha$.

A variety of alternative criteria for the definition of higher order moments for the portfolio selection can be found in the literature. Recently, Saedifar and Pasha [49] have introduced new weighted possibilistic moments of fuzzy numbers and analyzed the properties of the nearest weighted possibilistic points which are usually used in the stage of defuzzification processes. Let us recall their definition of a 3rd weighted possibilistic moment, which will be useful in the context of measuring the asymmetry of LR-fuzzy numbers.

Definition 10.6. Let $\tilde{A}$ be a fuzzy number and $f(\alpha)$ be a weighted function. The 3rd weighted possibilistic moment about the weighted possibilistic mean value of $\tilde{A}$, $\bar{M}(\tilde{A})$, is

$$\mu_3(\tilde{A}) = \frac{1}{2}\int_0^1 f(\alpha)[\inf\tilde{A}_\alpha - \bar{M}(\tilde{A})]^3 \mathrm{d}\alpha + \frac{1}{2}\int_0^1 f(\alpha)[\sup\tilde{A}_\alpha - \bar{M}(\tilde{A})]^3 \mathrm{d}\alpha \quad (10.5)$$

where $\bar{M}(\tilde{A})$ is the midpoint of the f-weighted possibilistic mean interval.

Throughout this chapter we work with the interval-valued expectation given in Definition 10.4, which means that we assume that all the α-cuts have the same weight: $f(\alpha) = 1$ for every $0 \leq \alpha \leq 1$, although the introduction of different weights for every α should be straightforward.

10.3.3 Fuzzy Risk and Expected Return on Portfolios

Let us consider a portfolio in which the total wealth has to be allocated to n risky assets. The vector $X = (x_1, \ldots, x_n)$ represents the proportions of the total investment devoted to each asset j^{th}, for $j = 1, \ldots, n$, and the components of vector X are restricted to the basic constraints of $\sum_{j=1}^n x_j = 1$ and $x_j \geq 0$, if the short-selling of the assets is not allowed. The uncertainty regarding its future return is modeled by means of fuzzy quantities based on the historical returns over T periods: $\{r_{tj}\}$, for $t = 1, \ldots, T$ and $j = 1, \ldots, n$. Therefore, associated with each rate of return there is a possibility distribution defined by the membership function of the corresponding fuzzy set. This approach allows the application of the above definitions of possibilistic moments in order to quantify the uncertainty associated with the future return on a portfolio, X.

Here we will analyze two different approaches to the measurement of the uncertainty on the return of a risky investment. Firstly, modeling the return on the individual assets using trapezoidal LR-fuzzy numbers $\tilde{R}_j$ and alternatively considering the returns on a given portfolio as the historical data set.

10.3.3.1 Fuzzy Returns on the Individual Assets

Let us denote the return on the jth asset by $\tilde{R}_j = (a_{lj}, a_{uj}, c_j, d_j)_{LR}$, a trapezoidal fuzzy number whose α-level cuts are $[\tilde{R}_j]^\alpha = [a_{lj} - c_j(1-\alpha), a_{uj} - dj(1-\alpha)]$, for $\alpha \in [0,1]$. The core and spreads of the fuzzy return on every asset j_0 are computed as functions of the sample percentiles of the data in the corresponding column: $\{r_{tj_0}\}_{t=1}^T$. Then, the total fuzzy return on the portfolio X is the following trapezoidal fuzzy number:

$$\begin{aligned}\tilde{R}_P(x) &= \textstyle\sum_{j=1}^n x_j \tilde{R}_j = \left(\sum_{j=1}^n a_{lj}x_j, \sum_{j=1}^n a_{uj}x_j, \sum_{j=1}^n c_j x_j, \sum_{j=1}^n d_j x_j\right)_{LR} \\ &= (P_l(x), P_u(x), C(x), D(x))_{LR}.\end{aligned}$$

This approach does not require the estimation of the joint possibility distribution of the return on the assets, which is not usually computable with any degree of confidence.

Since the interval-valued expectation remains additive in the sense of the addition of fuzzy numbers (Definition 10.4), we can easily compute the possibilistic moments of the total fuzzy return $\tilde{R}_P(x)$. It is easy to see that its interval-valued mean is

$$E(\tilde{R}_P(x)) = \left[P_l(x) - \tfrac{C(x)}{2}, P_u(x) + \tfrac{D(x)}{2}\right] \tag{10.6}$$

then as a scalar representative of this mean-interval we use its middle point:

$$\bar{E}(\tilde{R}_P(x)) = \frac{1}{2}\sum_{j=1}^n (a_{lj} + a_{uj})x_j + \frac{1}{4}\sum_{j=1}^n (d_j - c_j)x_j. \tag{10.7}$$

In terms of measuring the risk of the investment, we will use the fuzzy downside risk defined in (10.4). Then, applying the interval-valued expectation given in Definition 10.4, it can be found that this fuzzy interval risk is (see [61] for details):

$$\mathrm{w}_P(x) = \left[0, P_u(x) - P_l(x) + \frac{1}{3}(C(x) + D(x))\right] \tag{10.8}$$

and we use the length of this interval mean as a crisp representation of the investment risk:

$$\bar{\mathrm{w}}_P(x) = \sum_{j=1}^n \left(a_{uj} - a_{lj} + \frac{1}{2}(c_j + d_j)\right)x_j. \tag{10.9}$$

Delgado et al. [18], in the context of ranking fuzzy numbers, define the value $V(\tilde{B})$ and ambiguity $A(\tilde{B})$ of a fuzzy number $\tilde{B}$. It is easy to see that the value of $\tilde{R}_P(x)$, with respect to the reducing function $s(r) = r$, coincides with the expected return given by (10.7). Moreover, (10.9) is twice the ambiguity of $\tilde{R}_P(x)$. This fact reinforces the idea that these values are picking up the inexactness of the future return and they could be useful for selecting suitable sharing portfolios.

When the return on the portfolio is not symmetrically distributed around the mean, it is usually recommended to measure this asymmetry by using information about the 3rd moment of the possibility distribution. On the other hand, to obtain a relative measure of the asymmetry of the returns on fuzzy portfolios, we have recently introduced the following definition [60]:

Definition 10.7. Let $\tilde{R}_P(x)$ be the total return on a portfolio and $\mu_3(\tilde{R}_P(x))$ its 3rd possibilistic moment about the mean value $\bar{E}(\tilde{R}_P(x))$, then the coefficient of possibilistic skewness of $\tilde{R}_P(x)$ is defined as

$$S(\tilde{R}_P(x)) = \frac{\mu_3(\tilde{R}_P(x))}{(\bar{\mathrm{w}}_P(x))^3} \tag{10.10}$$

For trapezoidal fuzzy numbers we have also proved that the above coefficient of skewness can be calculated by means of the following ratio:

$$S(\tilde{R}_P(x)) = \frac{1}{16}\frac{D(x)^2 - C(x)^2}{(\bar{\mathrm{w}}_P(x))^2} \tag{10.11}$$

The values in (10.7) and (10.9), which are the crisp representation of the possibilistic expected return and downside risk of the total fuzzy return on a portfolio, have previously been used to define the goals and constraints of portfolio selection problems with fuzzy returns [38, 61].

10.3.3.2 Fuzzy Return on a Given Portfolio

The above approach does not incorporate the contemporary relationship of the returns on the individual assets into the portfolio composition, because their historical information has been independently analyzed. Since our main interest is to suitably model the returns on a given portfolio, we can directly consider its returns as the historical data set, instead of considering the individual returns on the assets as the data set [6, 7], therefore the main difference between these approaches appears in the modeling of uncertainty.

Now we propose to model the uncertainty about the future returns on a given portfolio $X = (x_1, \ldots, x_n)$ by using the information provided by the data set: $\{r_t(X)\}_{t=1}^T$, in such a way that the contemporary relationship among the individual returns is considered for each period $t = 1, \ldots, T$. We define this contemporary return on X as follows:

$$r_t(X) = \sum_{j=1}^{n} r_{tj} x_j. \tag{10.12}$$

Table 10.1 Yearly returns on five securities (1937–1954) from Markowitz's historical data

Year	Am. T.	A.T.T.	U.S.S.	C.C.	Frstn.
1937	−0.305	−0.173	−0.318	−0.065	−0.400
1938	0.513	0.098	0.285	0.238	0.336
1939	0.055	0.200	−0.047	−0.078	−0.093
1940	−0.126	0.030	0.104	−0.077	−0.090
1941	−0.280	−0.183	−0.171	−0.187	−0.194
1942	−0.003	0.067	−0.039	0.156	0.113
1943	0.428	0.300	0.149	0.351	0.580
1944	0.192	0.103	0.260	0.233	0.473
1945	0.446	0.216	0.419	0.349	0.229
1946	−0.088	−0.046	−0.078	−0.209	−0.126
1947	−0.127	−0.071	0.169	0.355	0.009
1948	−0.015	0.056	−0.035	−0.231	0.000
1949	0.305	0.038	0.133	0.246	0.223
1950	−0.096	0.089	0.732	−0.248	0.650
1951	0.016	0.090	0.021	−0.064	−0.131
1952	0.128	0.083	0.131	0.079	0.175
1953	−0.010	0.035	0.006	0.067	−0.084
1954	0.154	0.176	0.908	0.077	0.756

The sample percentiles of this data set define the core and spreads of the trapezoidal fuzzy number $\tilde{X} = (p_l, p_u, c, d)_{LR}$, which represents the uncertainty about the future returns on X. Then, we have the same possibilistic model and its measures of risk and return are obtained by using the corresponding definitions. Then, we have:

$$\bar{E}(\tilde{X}) = \frac{p_l + p_u}{2} + \frac{d-c}{4} \tag{10.13}$$

$$\bar{w}(\tilde{X}) = p_u - p_l + \frac{d+c}{2} \tag{10.14}$$

$$S(\tilde{X}) = \frac{1}{16}\frac{d^2 - c^2}{\bar{w}(\tilde{X})^2}. \tag{10.15}$$

Let us show the performance of these two fuzzy approaches for modeling uncertainty by using the set of historical data introduced by Markowitz [45].

10.3.3.3 Numerical Example

Let us assume that an investor wants to distribute one unit of wealth among five securities (for instance: American Tobacco, A.T.T., United States Steel, Coca-Cola, and Firestone) from the Markowitz data set. Table 10.1 shows their yearly returns $\{r_{tj}\}$ from 1937 to 1954, for $t = 1,\ldots,18$ and $j = 1,\ldots,5$.

Table 10.2 Possibilistic moments of six portfolios built from Markowitz's historical data

Portfolio	$\bar{E}(\tilde{R}_P(x))$	$\bar{w}_P(x)$	$S(\tilde{R}_P(x))$	$\bar{E}(\tilde{X})$	$\bar{w}(\tilde{X})$	$S(\tilde{X})$
$X_1 = (0.1, 0.1, 0.4, 0.2, 0.2)$	0.121	0.552	0.041	0.108	0.483	0.030
$X_2 = (0.15, 0.15, 0.35, 0.2, 0.15)$	0.111	0.520	0.040	0.094	0.440	0.021
$X_3 = (0.1, 0.2, 0.3, 0.3, 0.1)$	0.099	0.481	0.036	0.089	0.383	0.010
$X_4 = (0.15, 0.25, 0.25, 0.25, 0.1)$	0.094	0.463	0.034	0.076	0.359	0.006
$X_5 = (0.1, 0.4, 0.2, 0.2, 0.1)$	0.090	0.427	0.028	0.075	0.322	−0.005
$X_6 = (0.35, 0.2, 0.15, 0.15, 0.15)$	0.089	0.469	0.031	0.064	0.370	−0.007

Table 10.2 shows the performance of certain portfolios for the above approaches to modeling uncertainty on future return. Firstly, for each asset j the core of the trapezoidal fuzzy returns is approximated using the interval $[q_{40}, q_{60}]$, q_k being the kth percentile of the sample $\{r_{tj}\}_{t=1}^{18}$, where the support is the interval between the minimum and maximum observed return. Once each trapezoidal fuzzy number $\tilde{R}_j$ has been built, we can evaluate the total fuzzy return $\tilde{R}_{P_i}(x)$ for each portfolio $\{X_i\}$ and their possibilistic moment values. On the other hand, we can also approximate the fuzzy returns on $\tilde{X}_i$ by using the same percentiles for the core and spreads from the sample $\{r_t(X)\}$ and explicitly evaluate their possibilistic moment values using (10.13)–(10.15).

The first columns in Table 10.2 show the possibilistic moments for the total return on the portfolios, by assuming fuzzy returns on individual assets, and the last three ones for the direct evaluation of the returns on given portfolios. Note that the value of possibilistic moments is usually greater when the uncertainty regarding the future returns on a given portfolio has been measured through the returns on individual assets. The results in Table 10.2 also show that for both approaches the possibilistic measures have a coherent behavior, that is higher returns and higher asymmetry values are associated with higher risk. Note also that portfolio X_4 dominates portfolio X_6 in both cases, the latter thus being inefficient. Moreover, since portfolios X_5 and X_6 have negative skewness they should be rejected if we are looking for portfolios with positive skewness, as will be stated in Sect. 10.4.

For the portfolios analyzed, the width of the support of $\tilde{R}_{P_i}(x)$ is greater than the width of the support of $\tilde{X}_i$, the cores being of similar length. This fact could mean that the fuzzy representation of the portfolio return is more imprecise when it is evaluated through the historical returns on the individual assets. Figure 10.1 shows the fuzzy representation of returns on the portfolios X_1 and X_5. It does not seem easy either to compare both fuzzy representations of uncertainty or to obtain conclusive results for the general statement of fuzzy portfolio selection problems.

In the next section we present certain multi-objective programs for portfolio selection based on the above possibilistic measures of return and risk. Without loss of generality they deal with the approach which directly builds the fuzzy return on a given portfolio, that is $\tilde{X}$.

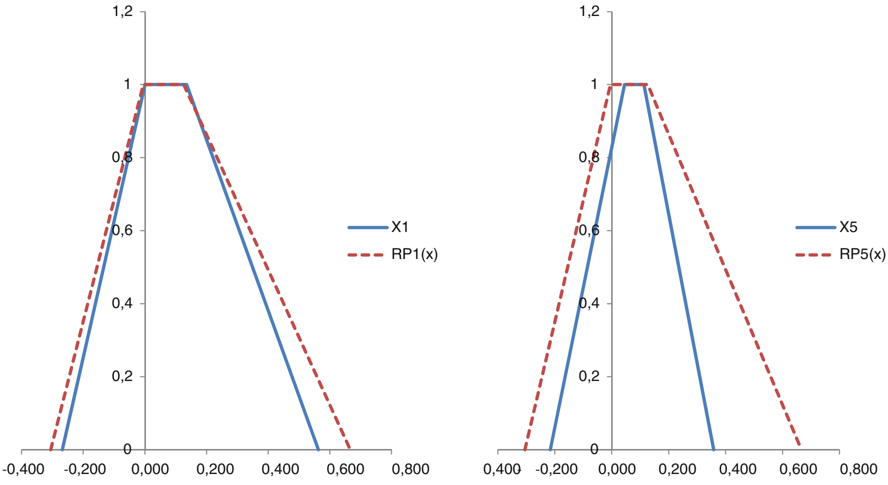

Fig. 10.1 Fuzzy representation of the return on given portfolios. The *left-hand graph* corresponds to portfolio X_1, and the *right-hand graph* to portfolio X_5. The *blue solid lines* correspond to $\tilde{X}_i$ and the *red dashed lines* correspond to $\tilde{R}_{P_i}(x)$, for $i = 1$ and 5, respectively

10.4 Multi-objective Possibilistic Models for Portfolio Selection

Recently, a few multi-criteria decision-making models have been proposed for determining appropriate portfolios in risk-return trade-off assuming a fuzzy representation of the uncertainty regarding future returns [8, 28, 40]. The multi-objective formulation allows consideration of the more realistic situation in which several conflicting goals competed in the allocation decision, providing both flexibility and a large set of choices for the decision maker: the Pareto-optimal set, whose elements are called efficient solutions. On the other hand, from a practical point of view, taking investors' preferences into account implies the inclusion of several types of constraints, which may change the feasible region in which the optimal portfolio must be selected and may also transform the type of optimization problem that must be solved in such a way that it could be considerably more difficult to solve the new problem than the original one. Therefore, the use of suitable mathematical programming techniques is necessary for solving the portfolio selection problem [56].

The mathematical formulation of a multiple criteria optimization problem is as follows:

$$\begin{aligned}(\text{MOP})\ \text{Max}\ & [f_1(x), \ldots, f_r(x)] \\ \text{s.t.}\ \ & x \in S,\end{aligned}$$

where f_i are deterministic functions and $S \in \mathfrak{R}^n$ is the feasible set. This problem also defines the objective feasible region $Z = \{z \in \mathfrak{R}^r : z = f(x),\ x \in S\}$ in the objective space $\mathfrak{R}^r$, r being the number of objectives (see, for instance, [16]).

In the optimization problems with multiple objectives, the set of solutions is composed of all those elements in the decision space S for which the corresponding objective vector cannot be improved in any dimension without another one deteriorating. To characterize the efficient solutions for the MOP problem we use the usual notion of Pareto optimality which determines how one alternative dominates another alternative. Let us recall some useful definitions.

Definition 10.8. Let $z^* \in Z$. Then z^* is a non-dominated solution for MOP if and only if there is no other $z \in Z$ such that $z_i \geq z_i^*$ for $i = 1, \ldots, r$, with strict inequality for at least one of them. Otherwise, $z^* \in Z$ is a dominated solution.

Definition 10.9. A decision vector $x \in S$ is said to be efficient for MOP if and only if $z = f(x)$ is a non-dominated solution.

The set of all efficient points is usually denoted by E and is called the efficient set. The inefficient points are those whose image in Z is a dominated solution. Note that when the goal is to minimize one objective f_i the definitions are analogous, taking into account that $\text{Min} f_i(x) = -\text{Max}(-f_i(x))$.

10.4.1 Possibilistic Mean–Downside Risk–Skewness Model

Here we propose to deal with certain multi-objective decision problems associated with the possibilistic mean–downside risk–skewness model (MDRS), where three objective functions corresponding to the crisp mean values of these possibilistic moments are considered. Note that the objective functions are nonlinear because they depend on the sample percentiles of the returns on the portfolio X. As is usual for selecting efficient portfolios, we propose to maximize the odd moments while minimizing the downside risk value. This multi-objective possibilistic portfolio selection problem can then be formulated as follows:

$$
\begin{aligned}
(\textbf{MDRS})\ \ & \text{Max } z_1 = \bar{E}(\tilde{X}) \\
& \text{Min } \ z_2 = \bar{w}(\tilde{X}) \\
& \text{Max } z_3 = S(\tilde{X}) \\
& \text{s.t.} \quad x \in S.
\end{aligned}
\tag{10.16}
$$

Therefore, a non-dominated portfolio must offer the highest level of expected return for a given level of risk and skewness and the lowest level of risk for a given level of return and skewness. On the other hand, an explicit characterization of the decision space S is needed in order to know where the feasible solutions must be found, and then some constraints must be included to incorporate investors' preferences into the model. For instance, concerning an investor's opinion, some of the following constraints should be added:

1. Limits to the budget to be invested in every asset.
   (a) *Lower bounds:* Indicate the minimum level below which an asset is not purchased and also imply an explicit diminution of the fund invested in the sharing portfolio.
   (b) *Upper bounds:* Limit the percentage of the budget in a given asset and it could imply more diversification in the investment.
2. *Portfolio size:* Explicit specification of the number of assets in the portfolio or an explicit rank-size.
3. Limits to the percentage that is invested in one group of assets $J \subset \{1,\ldots,n\}$, which are considered the basic unit of investment.
4. Achieving a given level of expected liquidity, and so on.

Note that the above constraints are implicitly related because of the requirement to invest of the total budget and they define different subsets in the n-simplex, requiring the introduction of binary and/or integer variables. Some heuristic algorithms have been proposed to deal with the optimization problems that incorporate these constraints into the mean–variance modeling approach [1, 14, 15].

In what follows we will consider finite upper and lower bounds for the asset weight and cardinality constraints, in such a way that the decision space is mathematically stated as follows:

$$S = \left\{ x \in \Re^n : \sum_{j=1}^{n} x_j = 1, l_j \le x_j \le u_j, l_j \ge 0, k_l \le c(X) \le k_u \right\}, \qquad (10.17)$$

where $c(X)$ is the number of positive proportions in portfolio X, that is $c(X) = \mathrm{rank}(\mathrm{diag}(X))$. It is well known that this cardinality constraint involves a quasi-concave function, $c(X)$, which implies that optimization problems with this feasible set are NP-hard [9]. Then, for solving the nonlinear multi-objective decision problem MDRS, we use a meta-heuristic procedure that independently manages the historical information about the returns on the assets and the investors' preferences.

In order to generate efficient portfolios taking into account the goals of the MDRS model, we will apply a multi-objective evolutionary algorithm which has been prepared to deal with two conflicting objective functions. We then deal with two alternative bi-objective optimization problems which incorporate the third goal as a constraint, in the following way:

$$\begin{aligned} (\mathrm{MDRS}_1)\ \mathrm{Min}\ & z_2 = \bar{w}(\tilde{X}) \\ \mathrm{Max}\ & z_3 = S(\tilde{X}) \\ \mathrm{s.t.}\ & \bar{E}(\tilde{X}) \ge \rho \\ & x \in S, \end{aligned} \qquad (10.18)$$

where ρ is a given expected return, which is usually the rate offered for risk-free investment, and

$$
\begin{array}{lll}
(\mathrm{MDRS}_2) & \mathrm{Max} & z_1 = \bar{E}(\tilde{X}) \\
 & \mathrm{Min} & z_2 = \bar{w}(\tilde{X}) \\
 & \mathrm{s.t.} & S(\tilde{X}) \geq \gamma \\
 & & x \in S.
\end{array}
\tag{10.19}
$$

In a previous work we dealt with a bi-objective optimization problem with the same goals as the MDRS_2 problem, but without the requirement of positiveness for the coefficient of asymmetry [7]. There we proposed a heuristic procedure for generating the approximate Pareto frontier which is the basis of our multi-objective evolutionary algorithm for solving the MDRS problem.

10.5 An Evolutionary Algorithm for Multi-objective Constrained Fuzzy Portfolio Selection

Evolutionary algorithms (EA) are population-based stochastic heuristic procedures based on the principles of natural selection. Starting with a random initial population, an evolutionary algorithm searches through a solution space by evaluating a set of possible candidates. After the best individuals are selected, new individuals are created for the next generation through random mutation and crossover. Then, the generational cycle is repeated a number of times until convergence.

The more popular evolutionary procedures are genetic algorithms (GA), originally proposed by Holland [26], which have been successfully applied in different fields of decision-making theory. Recently, concerning portfolio optimization some GA-based portfolio selection approaches have been proposed leading with MV and MAD models with additional constraints such as minimum transaction lots, cardinality size, buy-in thresholds, and transactions costs [15, 41, 53]. With respect to the evolutionary approaches for approximating the efficient frontier of multi-objective portfolio selection problems, Anagnostopoulos et al. [1] provide an interesting performance comparison among the more usual multi-objective evolutionary techniques.

10.5.1 *Description of the Algorithm*

Our proposal for solving the multi-objective optimization problems in (10.18) and (10.19) is to use a multi-objective evolutionary algorithm (MOEA), which has been specifically developed for dealing with admissible portfolios that meet (10.17) and for approximating their corresponding Pareto frontier. Let us describe the basis of our procedure, which follows the general framework outlined by Laumanns et al. [35] and incorporates a dominance method for sorting individuals based on the objective function values (see, for instance, [17]).

We use a standard real-valued vector to represent the proportions of the budget invested in the assets of a given portfolio: X. The procedure works with two populations of individuals. The first population, which is randomly generated in the initialization step, always has the same number of admissible individuals, $x \in S$, while the second one, which maintains the non-dominated solutions, attains different sizes depending on the number of non-dominated solutions in the current generation. At each generation, the quality of an individual is evaluated and the population is sorted according to each objective function values; the procedure rejects those individuals which do not meet the additional constraint. Once the non-dominated individuals have been identified we use them for building the current approximate Pareto frontier: the upper boundary of one generation.

Then, the algorithm selects the best solutions by measuring their distance to the current upper boundary and applies them over a mutation operator which slightly perturbs a pair of randomly selected proportions. Some learning rules are introduced for identifying which are the most interesting assets to being involved in the next generations. The individuals of the offspring population must be admissible portfolios and must contain all the non-dominated portfolios of the previous generation. In order to decide if the convergence has been met, the algorithm measures the distance between two successive upper boundaries, and stops if it is less than a given C. Then, the last approximate Pareto frontier contains the selected portfolios.

10.5.1.1 Experiment Settings

In our experiments we apply a generational genetic population strategy with a population of 500 individuals for each portfolio size, the number of securities in the portfolio varying between $k_l = 6$ and $k_u = 9$. We use an elitism mechanism that selects 20% of the better individuals from the current population. The selection mechanism prefers individuals that are better than other individuals in at least one objective value, i.e. that they are not dominated by another individual. The procedure then maintains the currently approximated Pareto-frontier, including all those individuals in the elite set. We apply a local mutation constant p_0 that takes values in the set $\{0.050, 0.020, 0.010, 0.001\}$. If the stopping criterion is not satisfied, for $C = 10^{-4}$, the algorithm builds 50 generations. These parameters were selected from preliminary experiments (see, for instance, [7]). The algorithm was implemented in R language (http://www.r-project.org/) and runs on a personal computer.

10.5.2 Numerical Results

In this section we report the results that we have obtained on randomly generated portfolios, X, whose risks and returns have been evaluated using a historical data set from the Spanish Stock Exchange in Madrid. We consider the weekly returns on 27 assets from the Spanish IBEX35 index between January 2007 and December 2009. We took the observations of the Wednesday prices as an estimate of the weekly prices. The sample returns r_{tj} for $t = 1, \ldots, 152$ and $j = 1, \ldots, 27$ are

$$r_{tj} = \frac{p_{(t+1)j} - p_{tj}}{p_{tj}}, \tag{10.20}$$

where p_{tj} is the price of asset jth on Wednesday of week tth. For every portfolio X we evaluate its weekly return using (10.12) and we assume a trapezoidal fuzzy representation of the uncertainty associated to its future weekly return. The core and support of $\tilde{X}$ are given by (q_{40}, q_{60}) and (q_5, q_{95}), respectively, q_h being the h percentile of the sample $\{r_t(X)\}$, and then its expected return, downside risk, and coefficient of skewness are obtained.

Let us assume that the diversification parameters are given by $l_j = 0$ and $u_j = 0.2$, for all j, and that the right-hand side values for the corresponding possibilistic moments are fixed as $\rho = 0.001$ and $\gamma = 0.01$ in (10.18) and (10.19), respectively.

In order to analyze the performance of our evolutionary algorithm we have solved numerous instances of the problems defined in (10.18) and (10.19), with different portfolio sizes. The number of positive proportions in portfolio k is set from 6 to 9. Note that because of the upper bound value, the minimum number of assets that compose a portfolio is 5. In addition, we decide to consider up to 9 assets for an admissible portfolio following the suggestion given in [15], which points out that investors should not consider k values above one-third of the total number of assets because of dominance relationships. Each configuration was randomly run 5 times for both multi-objective programs, using the same set of seeds. Let us show some of the results obtained.

10.5.2.1 Possibilistic Downside Risk–Skewness Model

Figure 10.2 shows the downside risk and possibilistic skewness for the first and final generations of 5 independent runs for (10.18), $k = 9$ being the number of assets in every portfolio. The figure clearly shows the generational improvement between these two generations of 2,500 portfolios and the discontinuities which will appear in the efficient frontier because of the cardinality constraint. Note that for the initial population, the algorithm randomly generates portfolios with negative skewness and very diversified risk values. Finally, it converges towards a frontier with an important decrease in risk.

The fact of having discontinuities and missing parts in the Pareto frontier is more clearly shown in Fig. 10.3, where the corresponding efficient frontiers for different portfolio sizes are represented, for k from 6 to 9. Since all of them have been obtained using populations of 2,500 portfolios, the plot shows the efficient frontiers of a set of 10,000 admissible portfolios. Note that the effect of portfolio size can be extremely hard if the investors decide to set this value to 6 assets. However, the dominance relationship is not clearly stated among portfolios of different sizes, at least with respect to risk and skewness.

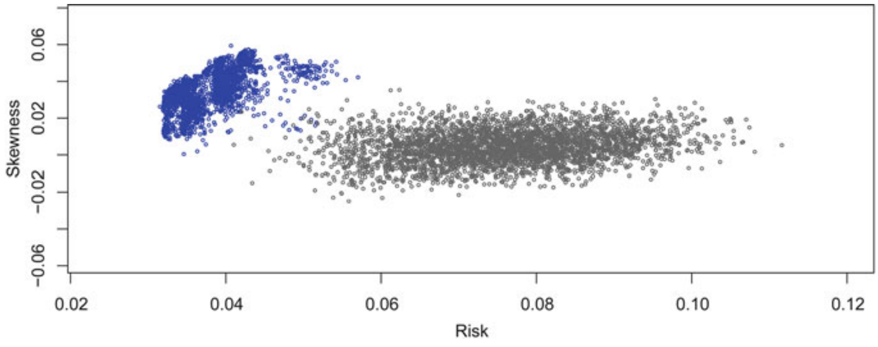

Fig. 10.2 Downside risk and possibilistic coefficient of skewness of the first and final generations obtained for $MDRS_1$ with $k = 9$. The *black points* correspond to the portfolios of the first generation, and the *blue points* to the final generation. Population size: 2,500 portfolios

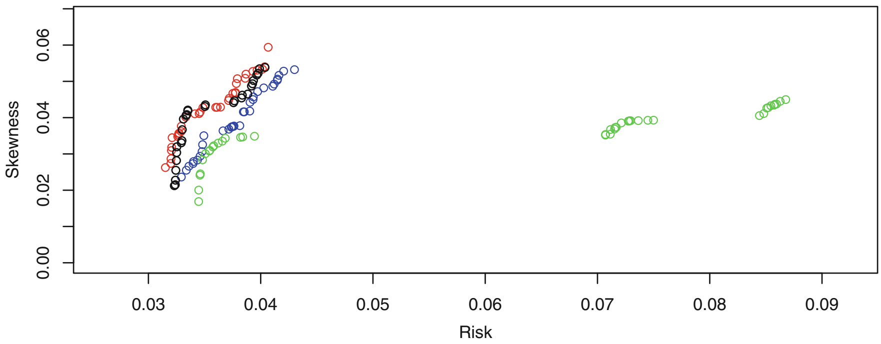

Fig. 10.3 Efficient frontiers of $MDRS_1$ for different portfolio sizes from $k = 6$ to $k = 9$. The *green points* correspond to $k = 6$, the *black points* to $k = 7$, the *blue points* to $k = 8$, and the *red ones* to $k = 9$

We therefore apply our procedure assuming that portfolios may alternatively contain 7, 8, or 9 assets. Figure 10.4 plots the coefficient of skewness and the downside risk corresponding to the 1,500 portfolios of the first and last generations. The final generation is now more explicitly defined, although its efficient frontier also presents a few discontinuities. However, there are not too many differences between the frontiers obtained by dealing with different portfolio sizes (population size: 10,000 portfolios) and this last frontier obtained by randomly building 1,500 portfolios at each generation.

Concerning the expected return of non-dominated portfolios (risk–skewness trade-off) shown in Fig. 10.4, their pairs of risk-return values are shown in Fig. 10.5. Note that some of the portfolios selected by solving $MDRS_1$ are not efficient in the risk-return trade-off.

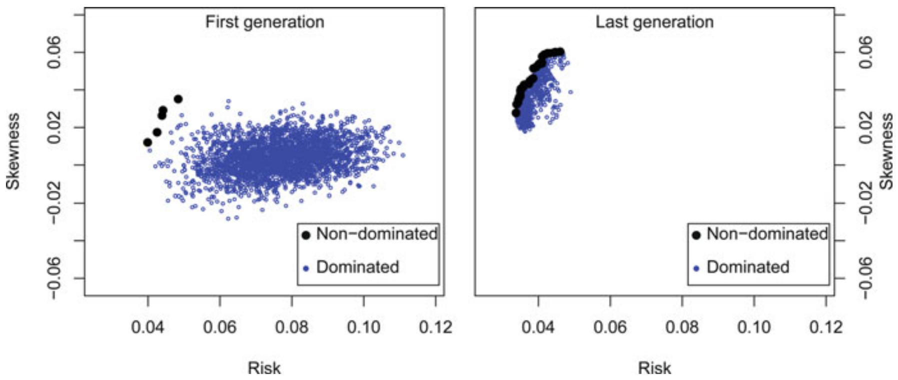

Fig. 10.4 Downside risk and possibilistic coefficient of skewness of the first and final generations obtained by solving $MDRS_1$ for k from 7 to 9. Population size: 1,500 portfolios

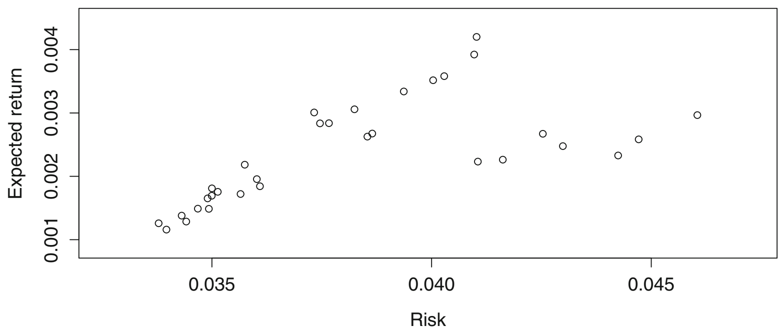

Fig. 10.5 Downside risk and expected return of the portfolios of the Pareto frontier of the final generation obtained by solving $MDRS_1$ for $k \in [7,9]$

On the other hand, since increasing the k value implies increasing diversification, it does not seem that this fact provides any loss of skewness, at least in this optimization framework. In fact, Fig. 10.3 shows that by increasing k we can obtain portfolios with greater values of the skewness coefficient.

10.5.2.2 Possibilistic Mean–Downside Risk Model

Analogously, we have performed the same experiments using the $MDRS_2$ problem and the results are presented in Fig. 10.5 for $k = 7$ and Fig. 10.6 for k from 6 to 9. Again, fixing the number of assets in the portfolio to 6 would not be recommended. In addition, it is also the most time-consuming experiment. Note that the rank of the expected return values has increased considerably with respect to the results obtained by solving $MDRS_1$. Now more risky portfolios with more expected benefits

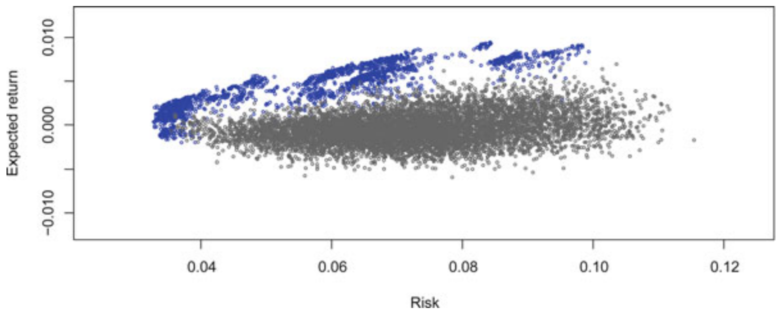

Fig. 10.6 Downside risk and expected return on the portfolios of the first and final generations obtained by solving $MDRS_2$ with $k = 7$. Population size: 2,500 portfolios

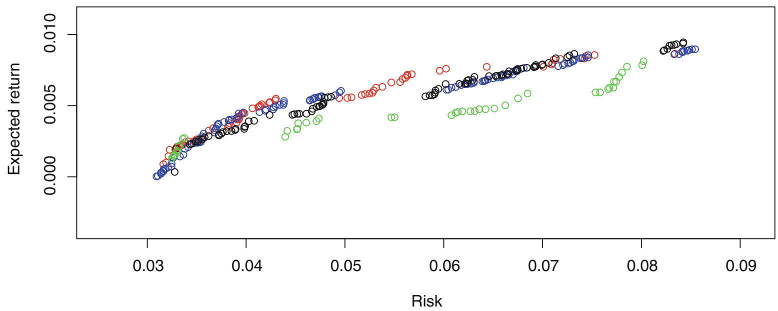

Fig. 10.7 Efficient frontiers of $MDRS_2$ for different portfolio sizes from $k = 6$ to $k = 9$. The *green points* correspond to $k = 6$, the *black points* to $k = 7$, the *blue points* to $k = 8$, and the *red ones* to $k = 9$

are provided for every experiment. However, what is not clearly established are the dominance relationships among the corresponding efficient frontiers where the number of assets in the portfolio is increased (Fig. 10.7).

Figure 10.8 jointly plots the downside risk of and expected return values on the 1,500 portfolios of the first and last generations obtained by applying our evolutionary algorithm to solve $MDRS_2$ for $k \in [7,9]$. Figure 10.9 shows the downside risk and possibilistic coefficient of skewness for the non-dominated portfolios of the last generation. Note that when the risk value increases, the portfolios obtained will be inefficient in the risk–skewness trade-off.

This result reinforces the idea that the inclusion of skewness in the portfolio optimization scheme can play an interesting role for investors with great risk aversion because it can provide them with alternative investment strategies. However, it does not seem to be useful if the investors want to assume bigger risks because in the risk–skewness trade-off the efficient frontier is obtained with lower risk values, which are related to portfolios with lower expected returns.

Concerning the use of our evolutionary algorithm for providing efficient and appropriate portfolios for the investors, the above results show the importance

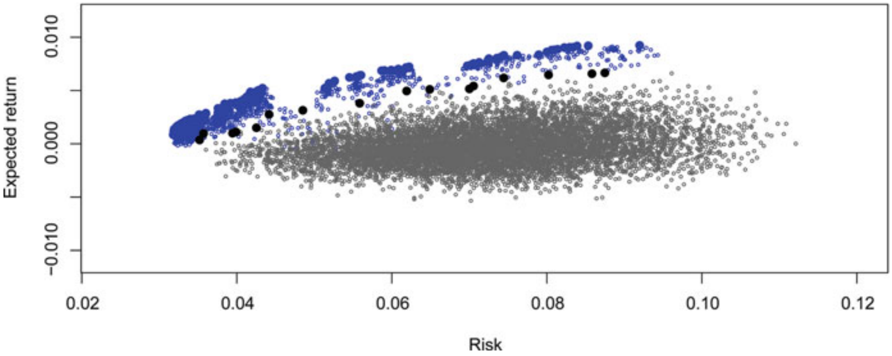

Fig. 10.8 Downside risk and expected return on the portfolios of the first and last generations with $k \in [7,9]$ and $MDRS_2$. The *black points* correspond to the first generation, and the *blue ones* to the last generation. Population size: 1,500 portfolios

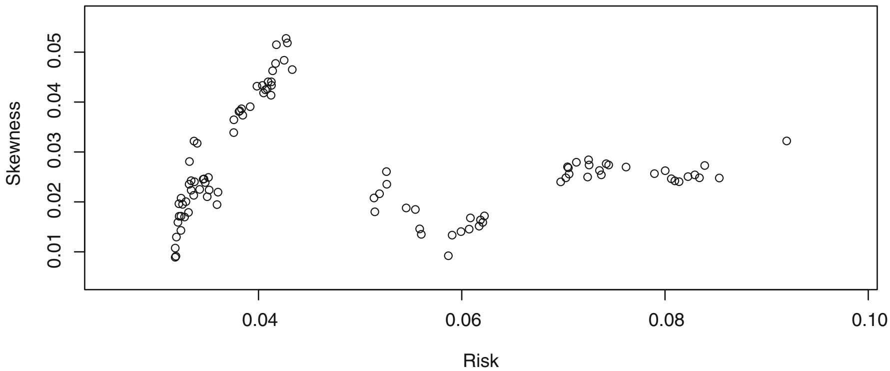

Fig. 10.9 Downside risk and possibilistic skewness of the portfolios of the Pareto frontier of the final generation obtained by solving $MDRS_2$ for $k \in [7,9]$

of suitably identifying their risk profiles. Then, different strategies based on the mean–downside risk–skewness model can be developed in order to obtain different investment proposals with portfolios satisfying the explicit preferences of the investors. This modeling approach can also be useful for providing information concerning the influence of bounds and cardinal sizes over the expected return on and risk of the investment, which could be useful to support her or his decision making.

10.6 Conclusions

Concerning quantification of the uncertainty regarding future returns on risky assets and given portfolios we propose to use LR-fuzzy numbers, whose possibilistic moments measure the risk and profitability of the investment. In order to directly approximate the contemporary relationship among the returns on the assets that compose a portfolio, we propose to consider the returns on a given portfolio as the historical data set instead of considering the individual returns on the assets as the data set.

We extend the mean–downside risk model into a mean–downside risk–skewness model using interval-valued expectations and higher possibilistic moments. We then formulate a new fuzzy portfolio selection problem in which trading requirements and investor preferences are introduced by means of discrete constraints. We propose to solve it by applying multi-objective optimization techniques based on evolutionary searches.

We present some numerical results for the possibilistic mean–downside risk–skewness model for trapezoidal fuzzy numbers. We use two different strategies for finding the efficient frontier and analyze the effect of introducing the possibilistic coefficient of skewness as a goal or as a constraint in the multi-objective optimization problems. Our multi-objective evolutionary algorithm is effective for solving these difficult optimization problems and for providing efficient frontiers that only take into account portfolios that meet investor preferences. Thus, different investment proposals suitably categorized by different risk tendencies can be proposed to the investors.

In our opinion, this multi-objective evolutionary algorithm could be a good strategy for finding suitable portfolios in the approximation Pareto frontier in those situations in which the description of the data set is also made with LR-fuzzy numbers of different shapes, because the analysis of the quality of a portfolio is only based on the specific uncertainty of every portfolio.

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