String Search using Compressed Indexes

Simon Gog and Andrew Turpin

Department of Computing and Information Systems
The University of Melbourne

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Lecture objectives

At the end of this lecture you should:

- know what the string searching problem is
- know what the suffix array data structure is
- understand the connection between the *BWT* and suffix array
- know how to do backward search in the *BWT*
- know what a wavelet tree is and how it implements *rank*()

Problem Definition

Given a string T and a pattern P over an alphabet Σ of constant size σ. Let $n = |T|$ be the length of T, and $m = |P|$ be the length of P and $n \gg m$.

Example

$T = abracadabrabarbara\$$
$P = bar$
$\Sigma = \{\$, a, b, c, d, r\}, \sigma = 6, n = 18, m = 3$

Problem: String search

- Does P occur in T? (Existence query)
- How often does P occur in T? (Count query)
- Where does P occur in T? (Locate query)

Simple Solutions

- Check for each i in $i \in \{0, ..., n-m-1\}$ if $T[i..i+m-1] = P$.

```
1   for (size_t i=0; i<n; ++i){
2     bool match=true;
3     for (size_t j=0; j<m && match; ++j)
4       match = (i+j < n && T[i+j] == P[j]);
5     if (match)
6       return true;
7   }
8   return false;
```

- Time complexity: $\mathcal{O}(n \cdot m)$ comparisons

Improved solution

- Knuth, Morris, and Pratt precomputed a table of size m which allows to shift the pattern by possibly more than one position in case of a mismatch and get complexity: $\mathcal{O}(n+m)$
- This solution is optimal in the online scenario, in which we are not allowed to pre-process T (online scenario), but not in ...

our scenario

We are allowed to pre-compute an index structure I for T and use I for the string search.

- I should be small
- Time complexity of matching independent of n

First attempt: Suffix Arrays (1)

i	$SA[i]$	$T[SA[i]..n-1]T[0..SA[i]-1]$
18	18	$abracadabrabarbara
17	17	a$abracadabrabarbar
10	10	abarbara$abracadabr
7	7	abrabarbara$abracad
0	0	abracadabrabarbara$
3	3	acadabrabarbara$abr
5	5	adabrabarbara$abrac
15	15	ara$abracadabrabarb
12	12	arbara$abracadabrab
14	14	bara$abracadabrabar
11	11	barbara$abracadabra
8	8	brabarbara$abracada
1	1	bracadabrabarbara$a
4	4	cadabrabarbara$abra
6	6	dabrabarbara$abraca
16	16	ra$abracadabrabarba
9	9	rabarbara$abracadab
2	2	racadabrabarbara$ab
13	13	rbara$abracadabraba

- First sort suffixes of T. (quicksort: $\mathcal{O}(n^2 \log n)$, best algorithms: $\mathcal{O}(n)$)
- Storing all suffixes takes $n^2 \log \sigma$ bits space. Only store starting positions of suffixes in SA ($n \log n$ bits).
- Question: How fast can we search using T and SA?

First attempt: Suffix Arrays (2)

- The suffixes are *ordered* in SA. We can use *binary search*!
- Start with the empty string ϵ which matches all prefixes (i.e. the interval $[sp_0..ep_0] = [0..n-1]$) of suffixes in SA.
- Then use binary search to determine the interval $SA[sp_j..ep_j]$ in $SA[sp_{j-1}..ep_{j-1}]$ so that all suffixes start with $P[0..j-1]$ for all $j \in [1..m]$.
- P occurs in T if $[sp_m..ep_m]$ is not empty.
- If P occurs the count query can be answered by $ep_m - sp_m + 1$.
- Time complexity: $\mathcal{O}(m \cdot \log n)$, space $\mathcal{O}(n \log n + n \log \sigma)$

First attempt: Suffix Arrays, Example

i	$SA[i]$	$T[SA[i]..n-1]T[0..SA[i]-1]$
0	18	$abracadabrabarbara
1	17	a$abracadabrabarbar
2	10	abarbara$abracadabr
3	7	abrabarbara$abracad
4	0	abracadabrabarbara$
5	3	acadabrabarbara$abr
6	5	adabrabarbara$abrac
7	15	ara$abracadabrabarb
8	12	arbara$abracadabrab
9	14	bara$abracadabrabar
10	11	barbara$abracadabra
11	8	brabarbara$abracada
12	1	bracadabrabarbara$a
13	4	cadabrabarbara$abra
14	6	dabrabarbara$abraca
15	16	ra$abracadabrabarba
16	9	rabarbara$abracadab
17	2	racadabrabarbara$ab
18	13	rbara$abracadabraba

- Search for *bar*.
- Step 1: *b* interval [9..12]
- Step 2: *ba* interval [9..10]
- Step 2: *bar* interval [9..10]

First attempt: Suffix Arrays, Example

i	$SA[i]$	$T[SA[i]..n-1]T[0..SA[i]-1]$
0	18	$abracadabrabarbara
1	17	a$abracadabrabarbar
2	10	abarbara$abracadabr
3	7	abrabarbara$abracad
4	0	abracadabrabarbara$
5	3	acadabrabarbara$abr
6	5	adabrabarbara$abrac
7	15	ara$abracadabrabarb
8	12	arbara$abracadabrab
9	14	bara$abracadabrabar
10	11	barbara$abracadabra
11	8	brabarbara$abracada
12	1	bracadabrabarbara$a
13	4	cadabrabarbara$abra
14	6	dabrabarbara$abraca
15	16	ra$abracadabrabarba
16	9	rabarbara$abracadab
17	2	racadabrabarbara$ab
18	13	rbara$abracadabraba

- Search for *bar*.
- Step 1: *b* interval [9..12]
- Step 2: *ba* interval [9..10]
- Step 2: *bar* interval [9..10]

First attempt: Suffix Arrays, Example

i	$SA[i]$	$T[SA[i]..n-1]T[0..SA[i]-1]$
0	18	$abracadabrabarbara
1	17	a$abracadabrabarbar
2	10	abarbara$abracadabr
3	7	abrabarbara$abracad
4	0	abracadabrabarbara$
5	3	acadabrabarbara$abr
6	5	adabrabarbara$abrac
7	15	ara$abracadabrabarb
8	12	arbara$abracadabrab
9	14	bara$abracadabrabar
10	11	barbara$abracadabra
11	8	brabarbara$abracada
12	1	bracadabrabarbara$a
13	4	cadabrabarbara$abra
14	6	dabrabarbara$abraca
15	16	ra$abracadabrabarba
16	9	rabarbara$abracadab
17	2	racadabrabarbara$ab
18	13	rbara$abracadabraba

- Search for *bar*.
- Step 1: *b* interval [9..12]
- Step 2: *ba* interval [9..10]
- Step 2: *bar* interval [9..10]

First attempt: Suffix Arrays, Example

i	$SA[i]$	$T[SA[i]..n-1]T[0..SA[i]-1]$
0	18	$abracadabrabarbara
1	17	a$abracadabrabarbar
2	10	abarbara$abracadabr
3	7	abrabarbara$abracad
4	0	abracadabrabarbara$
5	3	acadabrabarbara$abr
6	5	adabrabarbara$abrac
7	15	ara$abracadabrabarb
8	12	arbara$abracadabrab
9	14	bara$abracadabrabar
10	11	barbara$abracadabra
11	8	brabarbara$abracada
12	1	bracadabrabarbara$a
13	4	cadabrabarbara$abra
14	6	dabrabarbara$abraca
15	16	ra$abracadabrabarba
16	9	rabarbara$abracadab
17	2	racadabrabarbara$ab
18	13	rbara$abracadabraba

- Search for *bar*.
- Step 1: *b* interval [9..12]
- Step 2: *ba* interval [9..10]
- Step 2: *bar* interval [9..10]

Second attempt: Using BWT

i	$SA[i]$	BWT	$T[SA[i]..n-1]T[0..SA[i]-1]$
0	18	a	$abracadabrabarbara
1	17	r	a$abracadabrabarbar
2	10	r	abarbara$abracadabr
3	7	d	abrabarbara$abracad
4	0	$	abracadabrabarbara$
5	3	r	acadabrabarbara$abr
6	5	c	adabrabarbara$abrac
7	15	b	ara$abracadabrabarb
8	12	b	arbara$abracadabrab
9	14	r	bara$abracadabrabar
10	11	a	barbara$abracadabra
11	8	a	brabarbara$abracada
12	1	a	bracadabrabarbara$a
13	4	a	cadabrabarbara$abra
14	6	a	dabrabarbara$abraca
15	16	a	ra$abracadabrabarba
16	9	b	rabarbara$abracadab
17	2	b	racadabrabarbara$ab
18	13	a	rbara$abracadabraba

- $BWT[i] = T[SA[i]-1]$, for $SA[i] > 0$
- $BWT[i] = T[n-1]$, for $SA[i] = 0$
- I.e. $BWT[i]$ is the character preceding suffix $SA[i]$

Second attempt: Using BWT

i	BWT	$T[SA[i]..n-1]T[0..SA[i]-1]$
0	a	$abracadabrabarbara
1	r	a$abracadabrabarbar
2	r	abarbara$abracadabr
3	d	abrabarbara$abracad
4	$	abracadabrabarbara$
5	r	acadabrabarbara$abr
6	c	adabrabarbara$abrac
7	b	ara$abracadabrabarb
8	b	arbara$abracadabrab
9	r	bara$abracadabrabar
10	a	barbara$abracadabra
11	a	brabarbara$abracada
12	a	bracadabrabarbara$a
13	a	cadabrabarbara$abra
14	a	dabrabarbara$abraca
15	a	ra$abracadabrabarba
16	b	rabarbara$abracadab
17	b	racadabrabarbara$ab
18	a	rbara$abracadabraba

Add an array C containing the left border of each character interval:

$	a	b	c	d	r	r+1
0	1	9	13	14	15	19

- Operation $rank(i, X, BWT)$ returns how often character $X \in \Sigma$ occurs in the prefix $BWT[0..i-1]$.
- Now search backwards for *bar*.

Second attempt: Using BWT

i	BWT	$T[SA[i]..n-1]T[0..SA[i]-1]$
0	a	$abracadabrabarbara
1	r	a$abracadabrabarbar
2	r	abarbara$abracadabr
3	d	abrabarbara$abracad
4	$	abracadabrabarbara$
5	r	acadabrabarbara$abr
6	c	adabrabarbara$abrac
7	b	ara$abracadabrabarb
8	b	arbara$abracadabrab
9	r	bara$abracadabrabar
10	a	barbara$abracadabra
11	a	brabarbara$abracada
12	a	bracadabrabarbara$a
13	a	cadabrabarbara$abra
14	a	dabrabarbara$abraca
15	a	ra$abracadabrabarba
16	b	rabarbara$abracadab
17	b	racadabrabarbara$ab
18	a	rbara$abracadabraba

C

$	a	b	c	d	r
0	1	9	13	14	15

- Now search backwards for *bar*.
- Initial interval:
  $[sp_0, ep_0] = [0..n-1]$
- Determine interval for r:
  $sp_1 = C[r] + rank(sp_0, r, BWT)$
  $ep_1 = C[r] + rank(ep_0 + 1, r, BWT) - 1$

Second attempt: Using BWT

i	BWT	$T[SA[i]..n-1]T[0..SA[i]-1]$
0	a	$abracadabrabarbara
1	r	a$abracadabrabarbar
2	r	abarbara$abracadabr
3	d	abrabarbara$abracad
4	$	abracadabrabarbara$
5	r	acadabrabarbara$abr
6	c	adabrabarbara$abrac
7	b	ara$abracadabrabarb
8	b	arbara$abracadabrab
9	r	bara$abracadabrabar
10	a	barbara$abracadabra
11	a	brabarbara$abracada
12	a	bracadabrabarbara$a
13	a	cadabrabarbara$abra
14	a	dabrabarbara$abraca
15	a	ra$abracadabrabarba
16	b	rabarbara$abracadab
17	b	racadabrabarbara$ab
18	a	rbara$abracadabraba

C					
$	a	b	c	d	r
0	1	9	13	14	15

- Now search backwards for *bar*.
- Initial interval:
  $[sp_0, ep_0] = [0..n-1]$
- Determine interval for r:
  $sp_1 = 15 + rank(0, r, BWT)$
  $ep_1 = 15 + rank(19, r, BWT)$

Second attempt: Using BWT

i	BWT	$T[SA[i]..n-1]T[0..SA[i]-1]$
0	a	$abracadabrabarbara
1	r	a$abracadabrabarbar
2	r	abarbara$abracadabr
3	d	abrabarbara$abracad
4	$	abracadabrabarbara$
5	r	acadabrabarbara$abr
6	c	adabrabarbara$abrac
7	b	ara$abracadabrabarb
8	b	arbara$abracadabrab
9	r	bara$abracadabrabar
10	a	barbara$abracadabra
11	a	brabarbara$abracada
12	a	bracadabrabarbara$a
13	a	cadabrabarbara$abra
14	a	dabrabarbara$abraca
15	a	ra$abracadabrabarba
16	b	rabarbara$abracadab
17	b	racadabrabarbara$ab
18	a	rbara$abracadabraba

C					
$	a	b	c	d	r
0	1	9	13	14	15

- Now search backwards for *bar*.
- Initial interval:
  $[sp_0, ep_0] = [0..n-1]$
- Determine interval for r:
  $sp_1 = 15 + 0$
  $ep_1 = 15 + rank(19, r, BWT) - 1$

Second attempt: Using BWT

i	BWT	$T[SA[i]..n-1]T[0..SA[i]-1]$
0	a	$abracadabrabarbara
1	r	a$abracadabrabarbar
2	r	abarbara$abracadabr
3	d	abrabarbara$abracad
4	$	abracadabrabarbara$
5	r	acadabrabarbara$abr
6	c	adabrabarbara$abrac
7	b	ara$abracadabrabarb
8	b	arbara$abracadabrab
9	r	bara$abracadabrabar
10	a	barbara$abracadabra
11	a	brabarbara$abracada
12	a	bracadabrabarbara$a
13	a	cadabrabarbara$abra
14	a	dabrabarbara$abraca
15	a	ra$abracadabrabarba
16	b	rabarbara$abracadab
17	b	racadabrabarbara$ab
18	a	rbara$abracadabraba

C					
$	a	b	c	d	r
0	1	9	13	14	15

- Now search backwards for *bar*.
- Initial interval:
  $[sp_0, ep_0] = [0..n-1]$
- Determine interval for r:
  $sp_1 = 15+0 = 15$
  $ep_1 = 15+4-1 = 18$

Second attempt: Using BWT

i	BWT	$T[SA[i]..n-1]\,T[0..SA[i]-1]$
0	a	$abracadabrabarbara
1	r	a$abracadabrabarbar
2	r	abarbara$abracadabr
3	d	abrabarbara$abracad
4	$	abracadabrabarbara$
5	r	acadabrabarbara$abr
6	c	adabrabarbara$abrac
7	b	ara$abracadabrabarb
8	b	arbara$abracadabrab
9	r	bara$abracadabrabar
10	a	barbara$abracadabra
11	a	brabarbara$abracada
12	a	bracadabrabarbara$a
13	a	cadabrabarbara$abra
14	a	dabrabarbara$abraca
15	a	ra$abracadabrabarba
16	b	rabarbara$abracadab
17	b	racadabrabarbara$ab
18	a	rbara$abracadabraba

C

$	a	b	c	d	r
0	1	9	13	14	15

- Now search backwards for *bar*.
- Interval: $[sp_1, ep_1] = [15..18]$
- Determine interval for *ar*:
  $sp_2 = C[a] + rank(sp_1, a, BWT)$
  $ep_2 = C[a] + rank(ep_1 + 1, a, BWT) - 1$

Second attempt: Using BWT

i	BWT	$T[SA[i]..n-1]T[0..SA[i]-1]$
0	a	$abracadabrabarbara
1	r	a$abracadabrabarbar
2	r	abarbara$abracadabr
3	d	abrabarbara$abracad
4	$	abracadabrabarbara$
5	r	acadabrabarbara$abr
6	c	adabrabarbara$abrac
7	b	ara$abracadabrabarb
8	b	arbara$abracadabrab
9	r	bara$abracadabrabar
10	a	barbara$abracadabra
11	a	brabarbara$abracada
12	a	bracadabrabarbara$a
13	a	cadabrabarbara$abra
14	a	dabrabarbara$abraca
15	a	ra$abracadabrabarba
16	b	rabarbara$abracadab
17	b	racadabrabarbara$ab
18	a	rbara$abracadabraba

C

$	a	b	c	d	r
0	1	9	13	14	15

- Now search backwards for *bar*.
- Interval: $[sp_1, ep_1] = [15..18]$
- Determine interval for *ar*:
  $sp_2 = 1 + rank(15, a, BWT)$
  $ep_2 = 1 + rank(ep_1, a, BWT)$

Second attempt: Using BWT

i	BWT	$T[SA[i]..n-1]T[0..SA[i]-1]$
0	a	$abracadabrabarbara
1	r	a$abracadabrabarbar
2	r	abarbara$abracadabr
3	d	abrabarbara$abracad
4	$	abracadabrabarbara$
5	r	acadabrabarbara$abr
6	c	adabrabarbara$abrac
7	b	ara$abracadabrabarb
8	b	arbara$abracadabrab
9	r	bara$abracadabrabar
10	a	barbara$abracadabra
11	a	brabarbara$abracada
12	a	bracadabrabarbara$a
13	a	cadabrabarbara$abra
14	a	dabrabarbara$abraca
15	a	ra$abracadabrabarba
16	b	rabarbara$abracadab
17	b	racadabrabarbara$ab
18	a	rbara$abracadabraba

C

$	a	b	c	d	r
0	1	9	13	14	15

- Now search backwards for *bar*.
- Interval: $[sp_1, ep_1] = [15..18]$
- Determine interval for *ar*:
  $sp_2 = 1 + rank(15, a, BWT)$
  $ep_2 = 1 + rank(ep_1, a, BWT)$

Second attempt: Using BWT

i	BWT	$T[SA[i]..n-1]T[0..SA[i]-1]$
0	a	$abracadabrabarbara
1	r	a$abracadabrabarbar
2	r	abarbara$abracadabr
3	d	abrabarbara$abracad
4	$	abracadabrabarbara$
5	r	acadabrabarbara$abr
6	c	adabrabarbara$abrac
7	b	ara$abracadabrabarb
8	b	arbara$abracadabrab
9	r	bara$abracadabrabar
10	a	barbara$abracadabra
11	a	brabarbara$abracada
12	a	bracadabrabarbara$a
13	a	cadabrabarbara$abra
14	a	dabrabarbara$abraca
15	a	ra$abracadabrabarba
16	b	rabarbara$abracadab
17	b	racadabrabarbara$ab
18	a	rbara$abracadabraba

C

$	a	b	c	d	r
0	1	9	13	14	15

- Now search backwards for *bar*.
- Interval: $[sp_1, ep_1] = [15..18]$
- Determine interval for *ar*:
  $sp_2 = 1 + 6$
  $ep_2 = 1 + rank(19, a, BWT) - 1$

Second attempt: Using BWT

i	BWT	$T[SA[i]..n-1]T[0..SA[i]-1]$
0	a	$abracadabrabarbara
1	r	a$abracadabrabarbar
2	r	abarbara$abracadabr
3	d	abrabarbara$abracad
4	$	abracadabrabarbara$
5	r	acadabrabarbara$abr
6	c	adabrabarbara$abrac
7	b	ara$abracadabrabarb
8	b	arbara$abracadabrab
9	r	bara$abracadabrabar
10	a	barbara$abracadabra
11	a	brabarbara$abracada
12	a	bracadabrabarbara$a
13	a	cadabrabarbara$abra
14	a	dabrabarbara$abraca
15	a	ra$abracadabrabarba
16	b	rabarbara$abracadab
17	b	racadabrabarbara$ab
18	a	rbara$abracadabraba

C

$	a	b	c	d	r
0	1	9	13	14	15

- Now search backwards for *bar*.
- Interval: $[sp_1, ep_1] = [15..18]$
- Determine interval for *ar*:
  $sp_2 = 1+6 = 7$
  $ep_2 = 1+8-1 = 8$

Second attempt: Using BWT

i	BWT	$T[SA[i]..n-1]T[0..SA[i]-1]$
0	a	$abracadabrabarbara
1	r	a$abracadabrabarbar
2	r	abarbara$abracadabr
3	d	abrabarbara$abracad
4	$	abracadabrabarbara$
5	r	acadabrabarbara$abr
6	c	adabrabarbara$abrac
7	b	ara$abracadabrabarb
8	b	arbara$abracadabrab
9	r	bara$abracadabrabar
10	a	barbara$abracadabra
11	a	brabarbara$abracada
12	a	bracadabrabarbara$a
13	a	cadabrabarbara$abra
14	a	dabrabarbara$abraca
15	a	ra$abracadabrabarba
16	b	rabarbara$abracadab
17	b	racadabrabarbara$ab
18	a	rbara$abracadabraba

C

$	a	b	c	d	r
0	1	9	13	14	15

- Now search backwards for *bar*.
- Interval: $[sp_2, ep_2] = [7..8]$
- Determine interval for *bar*:
  $sp_3 = C[b] + rank(sp_2 - 1, b, BWT)$
  $ep_3 = C[b] + rank(ep_2, b, BWT) - 1$

Second attempt: Using BWT

i	BWT	$T[SA[i]..n-1]T[0..SA[i]-1]$
0	a	$abracadabrabarbara
1	r	a$abracadabrabarbar
2	r	abarbara$abracadabr
3	d	abrabarbara$abracad
4	$	abracadabrabarbara$
5	r	acadabrabarbara$abr
6	c	adabrabarbara$abrac
7	b	ara$abracadabrabarb
8	b	arbara$abracadabrab
9	r	bara$abracadabrabar
10	a	barbara$abracadabra
11	a	brabarbara$abracada
12	a	bracadabrabarbara$a
13	a	cadabrabarbara$abra
14	a	dabrabarbara$abraca
15	a	ra$abracadabrabarba
16	b	rabarbara$abracadab
17	b	racadabrabarbara$ab
18	a	rbara$abracadabraba

C

$	a	b	c	d	r
0	1	9	13	14	15

- Now search backwards for *bar*.
- Interval: $[sp_2, ep_2] = [7..8]$
- Determine interval for *bar*:
  $sp_3 = 9 + rank(7, b, BWT)$
  $ep_3 = 9 + rank(ep_1, b, BWT)$

Second attempt: Using BWT

i	BWT	$T[SA[i]..n-1]T[0..SA[i]-1]$
0	a	$abracadabrabarbara
1	r	a$abracadabrabarbar
2	r	abarbara$abracadabr
3	d	abrabarbara$abracad
4	$	abracadabrabarbara$
5	r	acadabrabarbara$abr
6	c	adabrabarbara$abrac
7	b	ara$abracadabrabarb
8	b	arbara$abracadabrab
9	r	bara$abracadabrabar
10	a	barbara$abracadabra
11	a	brabarbara$abracada
12	a	bracadabrabarbara$a
13	a	cadabrabarbara$abra
14	a	dabrabarbara$abraca
15	a	ra$abracadabrabarba
16	b	rabarbara$abracadab
17	b	racadabrabarbara$ab
18	a	rbara$abracadabraba

C

$	a	b	c	d	r
0	1	9	13	14	15

- Now search backwards for *bar*.
- Interval: $[sp_2, ep_2] = [7..8]$
- Determine interval for *bar*:
  $sp_3 = 9 + rank(7, b, BWT)$
  $ep_3 = 9 + rank(ep_1, b, BWT)$

Second attempt: Using BWT

i	BWT	$T[SA[i]..n-1]T[0..SA[i]-1]$
0	a	$abracadabrabarbara
1	r	a$abracadabrabarbar
2	r	abarbara$abracadabr
3	d	abrabarbara$abracad
4	$	abracadabrabarbara$
5	r	acadabrabarbara$abr
6	c	adabrabarbara$abrac
7	b	ara$abracadabrabarb
8	b	arbara$abracadabrab
9	r	bara$abracadabrabar
10	a	barbara$abracadabra
11	a	brabarbara$abracada
12	a	bracadabrabarbara$a
13	a	cadabrabarbara$abra
14	a	dabrabarbara$abraca
15	a	ra$abracadabrabarba
16	b	rabarbara$abracadab
17	b	racadabrabarbara$ab
18	a	rbara$abracadabraba

C

$	a	b	c	d	r
0	1	9	13	14	15

- Now search backwards for *bar*.
- Interval: $[sp_2, ep_2] = [7..8]$
- Determine interval for *bar*:
  $sp_3 = 9 + 0$
  $ep_3 = 9 + rank(9, b, BWT) - 1$

Second attempt: Using BWT

i	BWT	$T[SA[i]..n-1]T[0..SA[i]-1]$
0	a	$abracadabrabarbara
1	r	a$abracadabrabarbar
2	r	abarbara$abracadabr
3	d	abrabarbara$abracad
4	$	abracadabrabarbara$
5	r	acadabrabarbara$abr
6	c	adabrabarbara$abrac
7	b	ara$abracadabrabarb
8	b	arbara$abracadabrab
9	r	bara$abracadabrabar
10	a	barbara$abracadabra
11	a	brabarbara$abracada
12	a	bracadabrabarbara$a
13	a	cadabrabarbara$abra
14	a	dabrabarbara$abraca
15	a	ra$abracadabrabarba
16	b	rabarbara$abracadab
17	b	racadabrabarbara$ab
18	a	rbara$abracadabraba

C

$	a	b	c	d	r
0	1	9	13	14	15

- Now search backwards for *bar*.
- Interval: $[sp_2, ep_2] = [7..8]$
- Determine interval for *bar*:
  $sp_3 = 9+0 = 9$
  $ep_3 = 9+2-1 = 10$

Backward search summary

- Only C and a data structure R supporting the *rank* operation on *BWT* are required for existance and count queries.
- Space: $\sigma \log n$ bits for C + space for R
- Time: $\mathcal{O}(m \cdot t_{rank})$, where t_{rank} is time for one rank operation. Independent from n?
- Next: How can we implement *rank*?

Backward search summary

- Only C and a data structure R supporting the *rank* operation on *BWT* are required for existance and count queries.
- Space: $\sigma \log n$ bits for C + space for R
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- Next: How can we implement *rank*?

Simple solution for rank (first attempt)

- Remember $rank(i, X, A)$ is the count of how many times X occurs in $A[0..i-1]$
- Store for each $i \in [0..n-1]$ and for each $X \in \Sigma$ the answer for $rank(i, X, BWT)$.
- Pro: Constant time per rank. $\mathcal{O}(m)$ for string search!
- Con: Too much space ($\mathcal{O}(\sigma n \log n)$ bits).

Solution for rank (second attempt)

- We can solve *rank* on a bit vector in constant time using $n + o(n)$ bits as follows:
- Divide bit vector in superblocks of size $sbz = \log^2 n$ bits.
- (1) Store all values $rank(k \cdot sbz, 1)$.
- Divide superblocks in blocks of size $bz = \frac{\log n}{2}$ bits.
- (2) Store all values $rank(k \cdot sbz + k' \cdot bz, 1) - rank(k \cdot sbz, 1)$.
- (3) Pre-compute all answers of rank for bitvectors of size $\log n/2$.

Total space: n bits for bitvector +

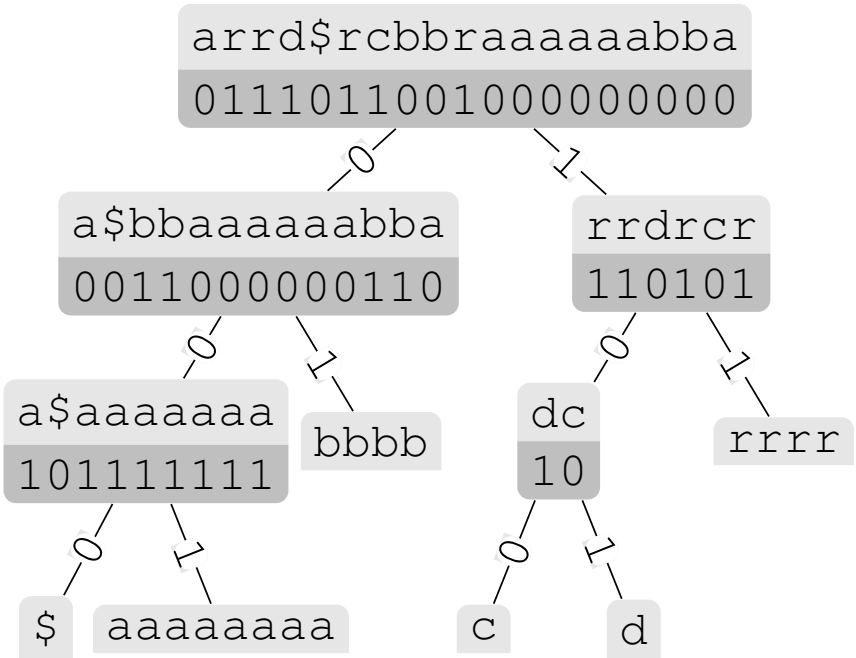
- (1) $n\lceil \frac{\log n}{\log^2 n} \rceil \in o(n)$ bits, (2) $n\lceil \frac{2 \log \log n}{\log n} \rceil \in o(n)$ bits
- (3) $2^{\log n/2} \log n/2 \log \log n/2 = \sqrt{n} \log n/2 \log \log n/2 \in o(n)$ bits

Solution for rank (second attempt)

Exercise: Implement the rank operation based on this schema.

Simple solution for rank (second attempt)

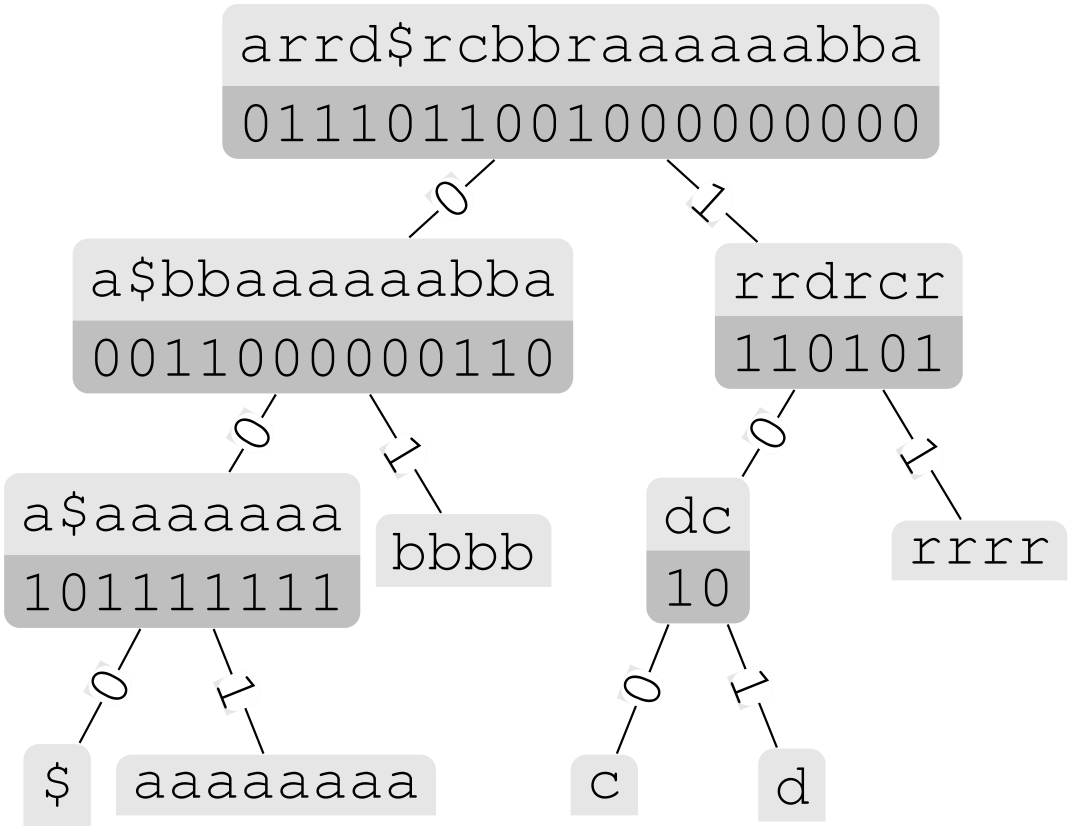
Use a wavelet tree to handle general alphabets:



Char	c	$codeword(c)$	freq
	$	000	1
	a	001	8
	b	01	4
	c	100	1
	d	101	1
	r	11	4

Depth: $\log \sigma$. Only bitvectors and pointers to bitvectors are stored.
Total space: $\approx n \log \sigma + 2\sigma \log n$

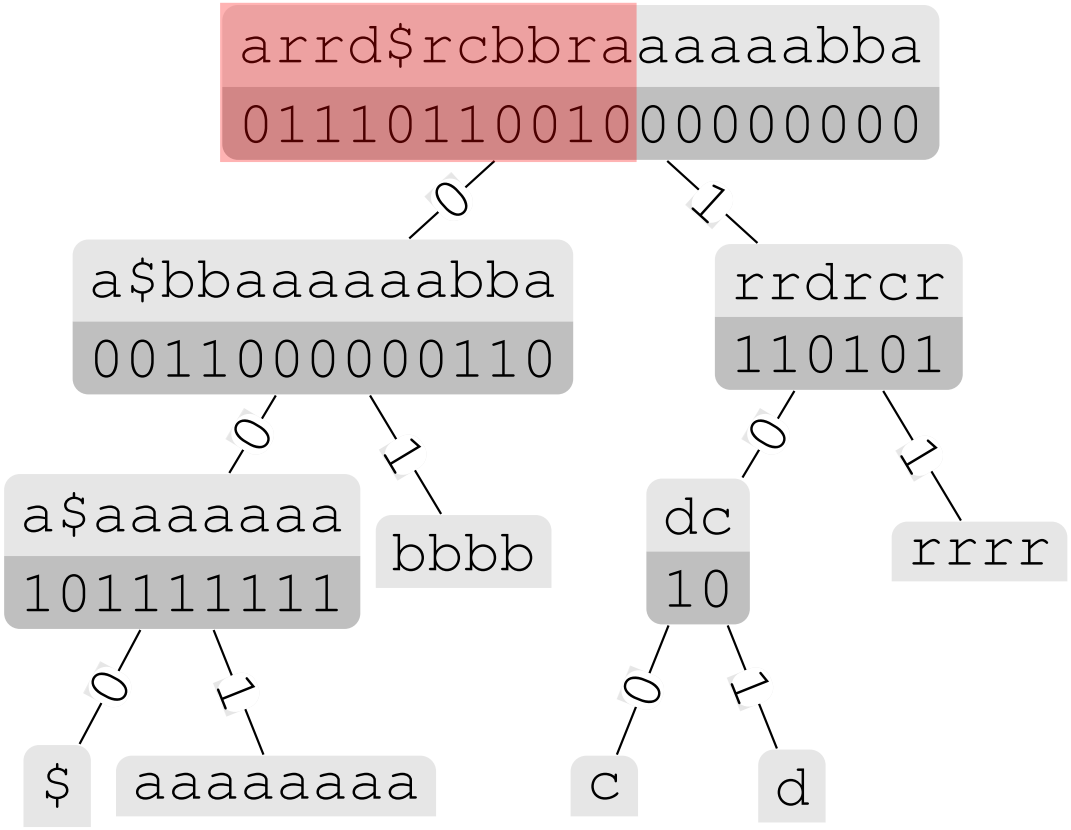
Wavelet Tree Example: Calculate Rank



a = 001

$rank(11, a, WT) = rank(rank(rank(11, 0, b_\epsilon) = 5, 0, b_0) = 3, 1, b_{00}) = 2$

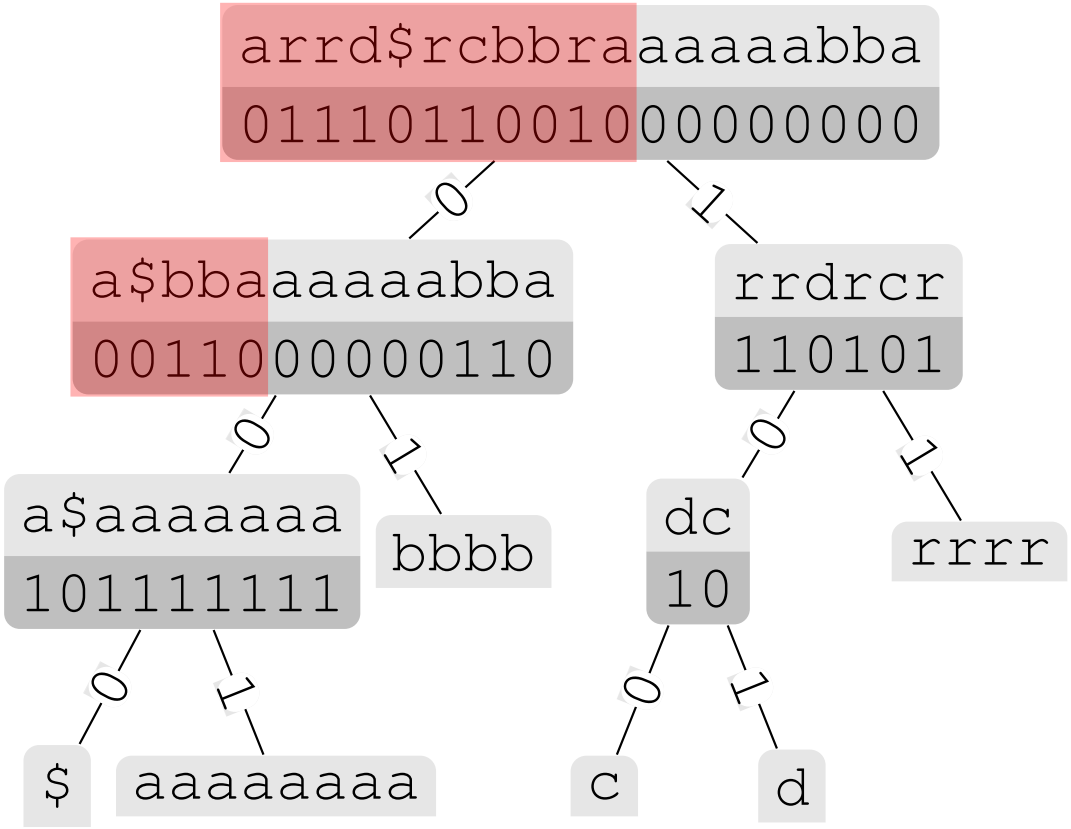
Wavelet Tree Example: Calculate Rank



a = 001

$rank(11, a, WT) = rank(rank(rank(11, 0, b_\epsilon) = 5, 0, b_0) = 3, 1, b_{00}) = 2$

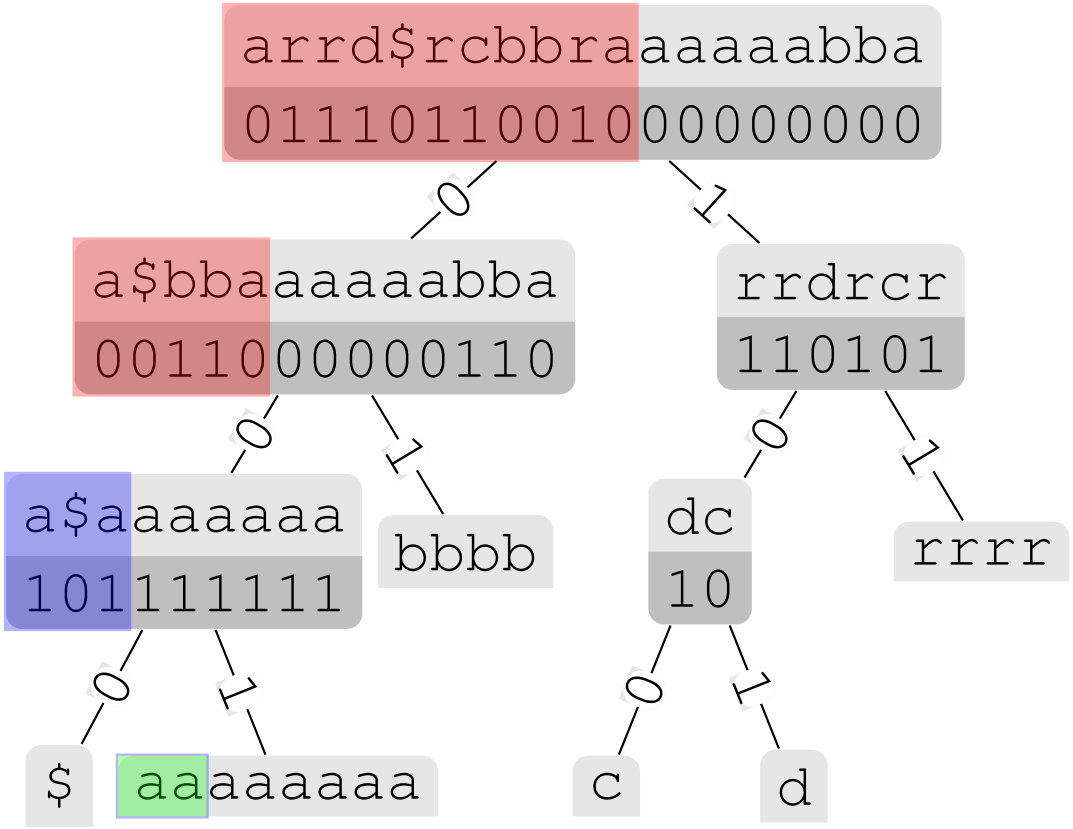
Wavelet Tree Example: Calculate Rank



a = 001

$rank(11, a, WT) = rank(rank(rank(11, 0, b_\epsilon) = 5, 0, b_0) = 3, 1, b_{00}) = 2$

Wavelet Tree Example: Calculate Rank



a = 001

$rank(11, a, WT) = rank(rank(rank(11, 0, b_\epsilon) = 5, 0, b_0) = 3, 1, b_{00}) = 2$

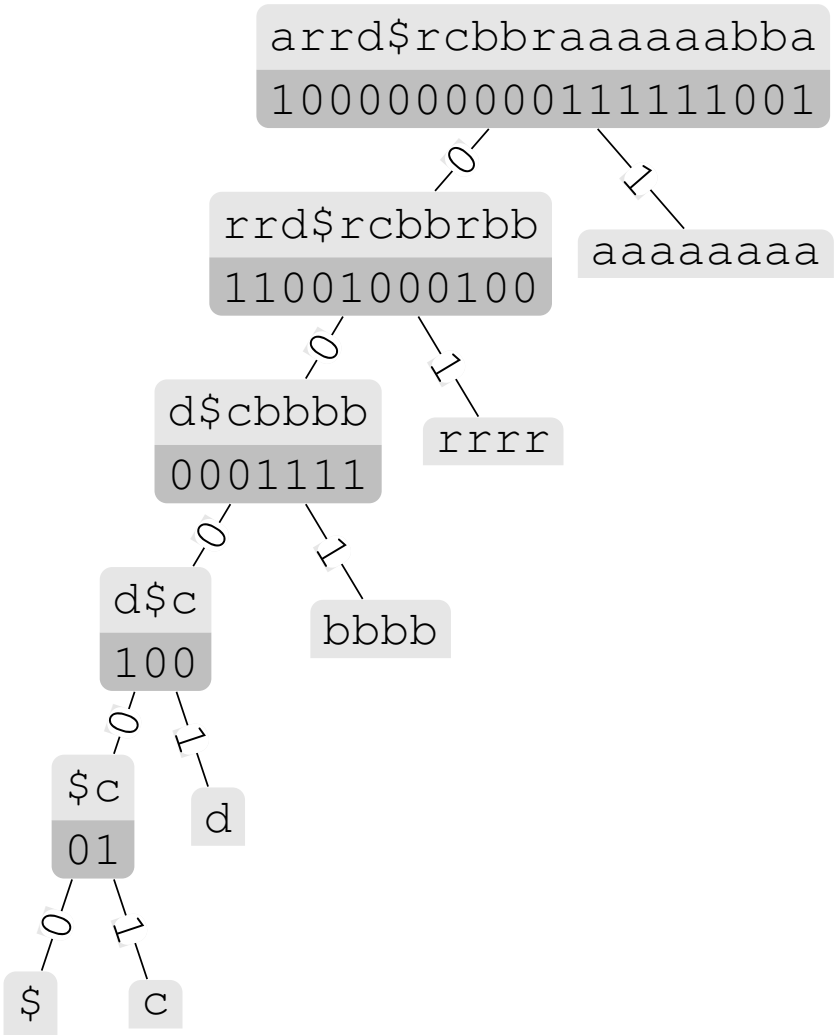
Pseudocode for rank on WT

$rank(i, c, WT)$

```
00  p ← b_ε
01  j ← 0
02  while not p! = codeword(c) do
03      if codeword(c)[j] = 0 then
04          i ← i − rank(i, 1, b_p)
05          p ← p0
06      else
07          i ← rank(i, 1, b_p)
08          p ← p1
09  return i
```

This code can also be used in a more space-efficient WT variant.

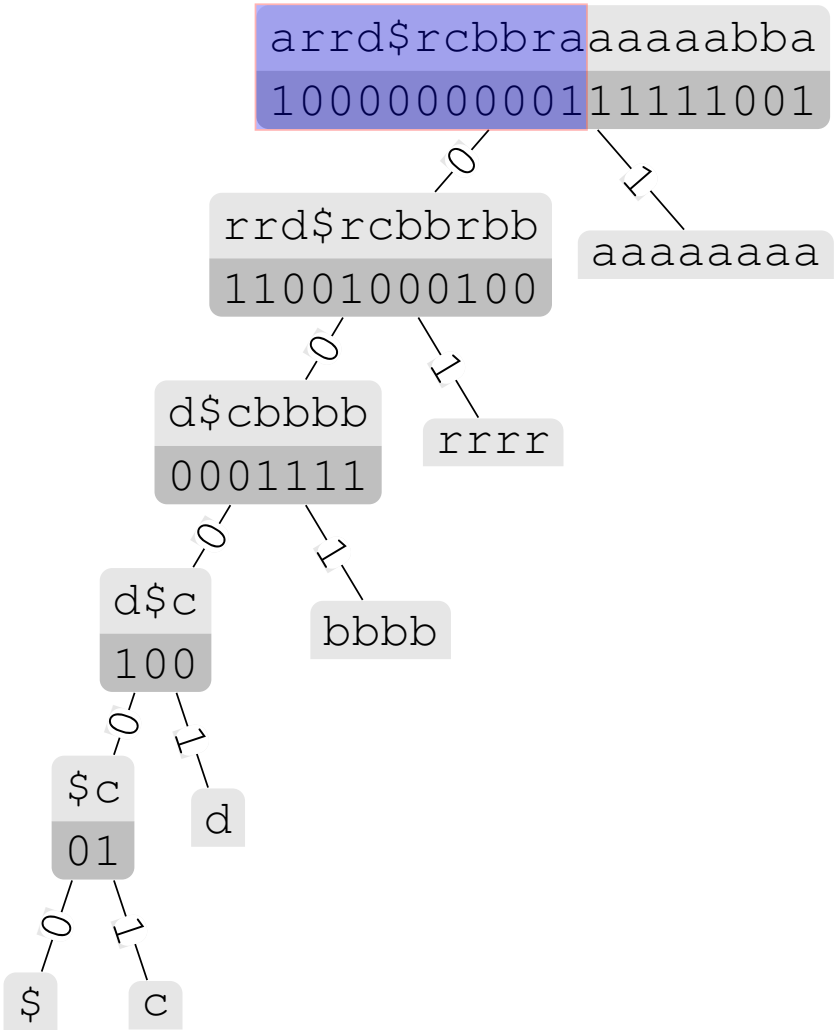
Huffman shaped wavelet tree



Char	c	$codeword(c)$
	$	00000
	a	1
	b	001
	c	00001
	d	0001
	r	01

Avg. depth: $H_0(BWT)$. Total space: $\approx nH_0 + 2\sigma \log n$

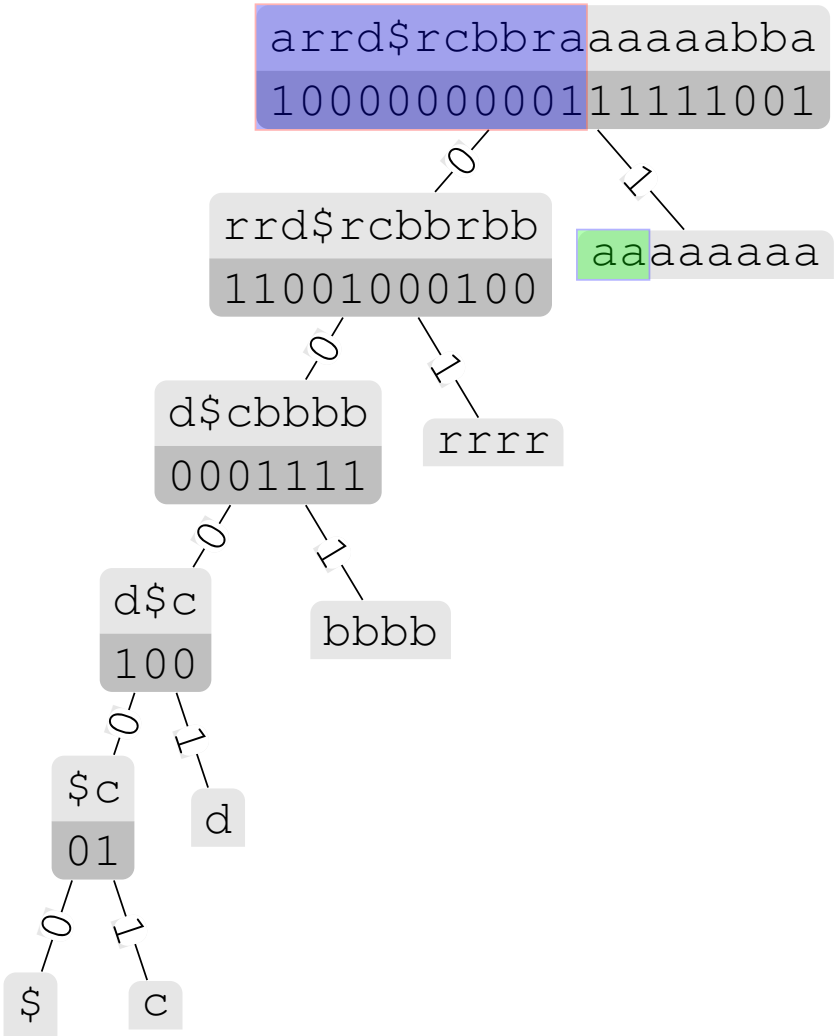
Huffman shaped wavelet tree



Char	c	$codeword(c)$
	$	00000
	a	1
	b	001
	c	00001
	d	0001
	r	01

$rank(11, a, WT) = rank(11, 1, b_\epsilon) = 2$

Huffman shaped wavelet tree



Char	c	$codeword(c)$
	$	00000
	a	1
	b	001
	c	00001
	d	0001
	r	01

$rank(11, a, WT) = rank(11, 1, b_\epsilon) = 2$

Summary

- Using a wavelet tree we can answer *rank*() in $\mathcal{O}(\log \sigma)$ time
- Using backward search in *BWT* we can count patterns in $\mathcal{O}(m \log \sigma)$ independent of n, the text size
- Space required for the index is just the wavelet tree and the C array: $\approx n \log \sigma + \sigma \log n$ bits or $\mathcal{O}(n + \sigma)$ words
- for locate queries we need the *SA* as well: same time, $n \log n$ additional bits.