Chapter 4

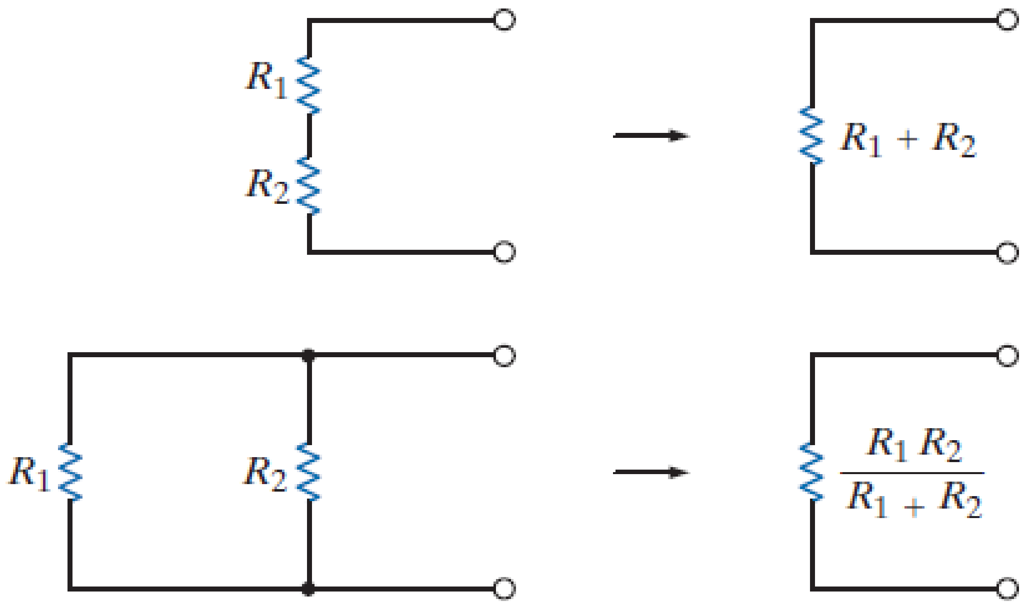
Additional Analysis Techniques

Learning goals

- By the end of this chapter, the students should be able to:
- Describe the concepts of linearity and equivalence.
- Analyze electric circuits using the principle of superposition.
- Calculate the Thévenin equivalent circuit for a linear circuit.
- Calculate the Norton equivalent circuit for a linear circuit.
- Apply source transformation appropriately.
- Apply the maximum power transfer theorem to determine the optimal load resistance for a particular circuit

Equivalence

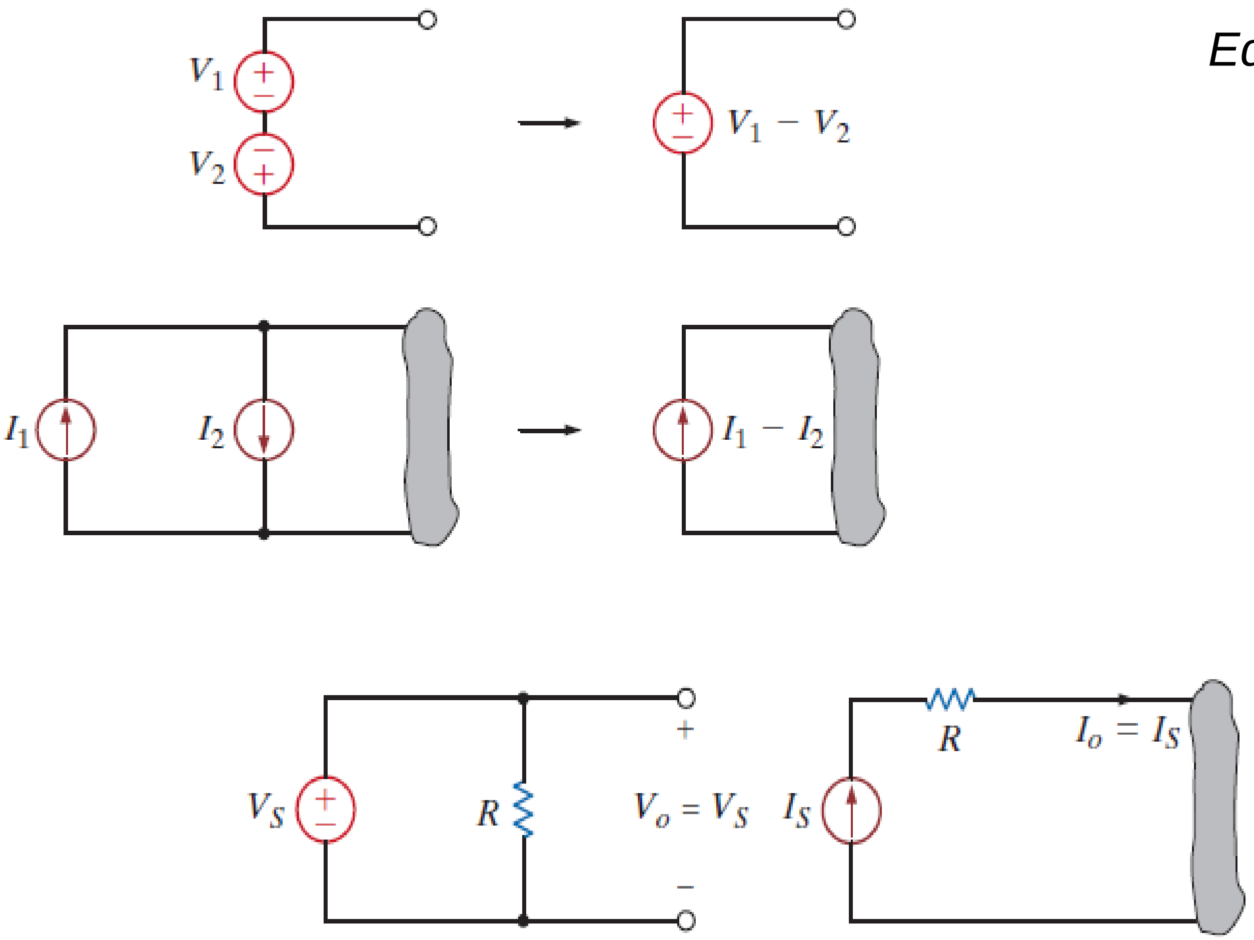
- It is important to note that a series connection of current sources or a parallel connection of voltage sources is forbidden unless the sources are pointing in the same direction and have exactly the same values.



Equivalent circuit forms

Equivalence

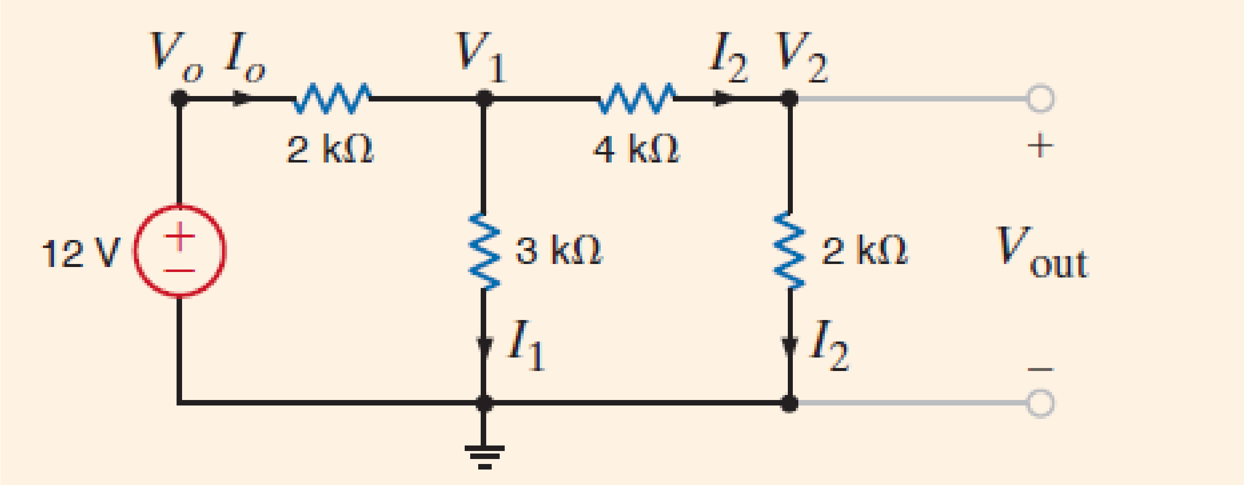
Equivalent circuit forms



Linearity

- All the circuits we have analyzed thus far have been linear circuits, which are described by a set of linear algebraic equations.
- Linearity requires both additivity and homogeneity (scaling). It can be shown that the circuits that we are examining satisfy this important property. The following example illustrates one way in which this property can be used.

Linearity



- For the circuit shown in Fig. above, we wish to determine the output voltage V'_{out}.
- However, rather than approach the problem in a straightforward manner and calculate I'_0 then I'_1 then I'_2, and so on, we will use linearity and simply assume that the output voltage is $V_{out} = 1V$.

Linearity

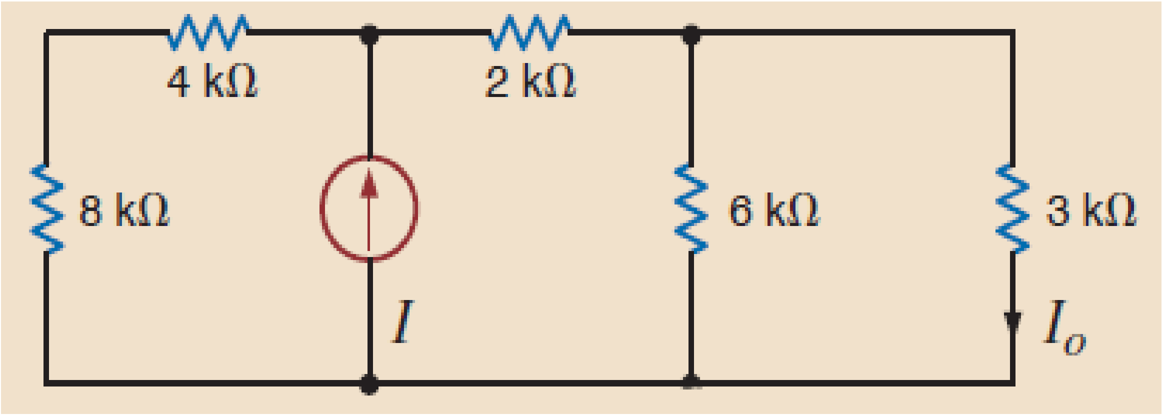
- This assumption will yield a value for the source voltage. We will then use the actual value of the source voltage and linearity to compute the actual value of V'_{out}.
- $Assuming\ V_{out} = V_2 = 1V$
- Then $I_2 = \frac{V_2}{2k} = 0.5\,mA$
- V_1 can be calculated as $V_1 = 4kI_2 + V_2 = 3V$
- Hence $I_1 = \frac{V_1}{3k} = 1mA$

Linearity

- Applying KCL, $I_0 = I_1 + I_2 = 1.5\,mA$
- $V_0 = 2kI_0 + V_1 = 6V$
- Therefore the assumption that $V_{out} = 1V$ produced a source voltage of 6V. However, since the actual source voltage is 12V, the actual output voltage is 1V(12/6)=2V.

Linearity

- ***Exercise***
- Use linearity and the assumption that $I_0 = 1mA$ to compute the correct current I_0 in the circuit in fig. below if I = 6 mA. (3mA)



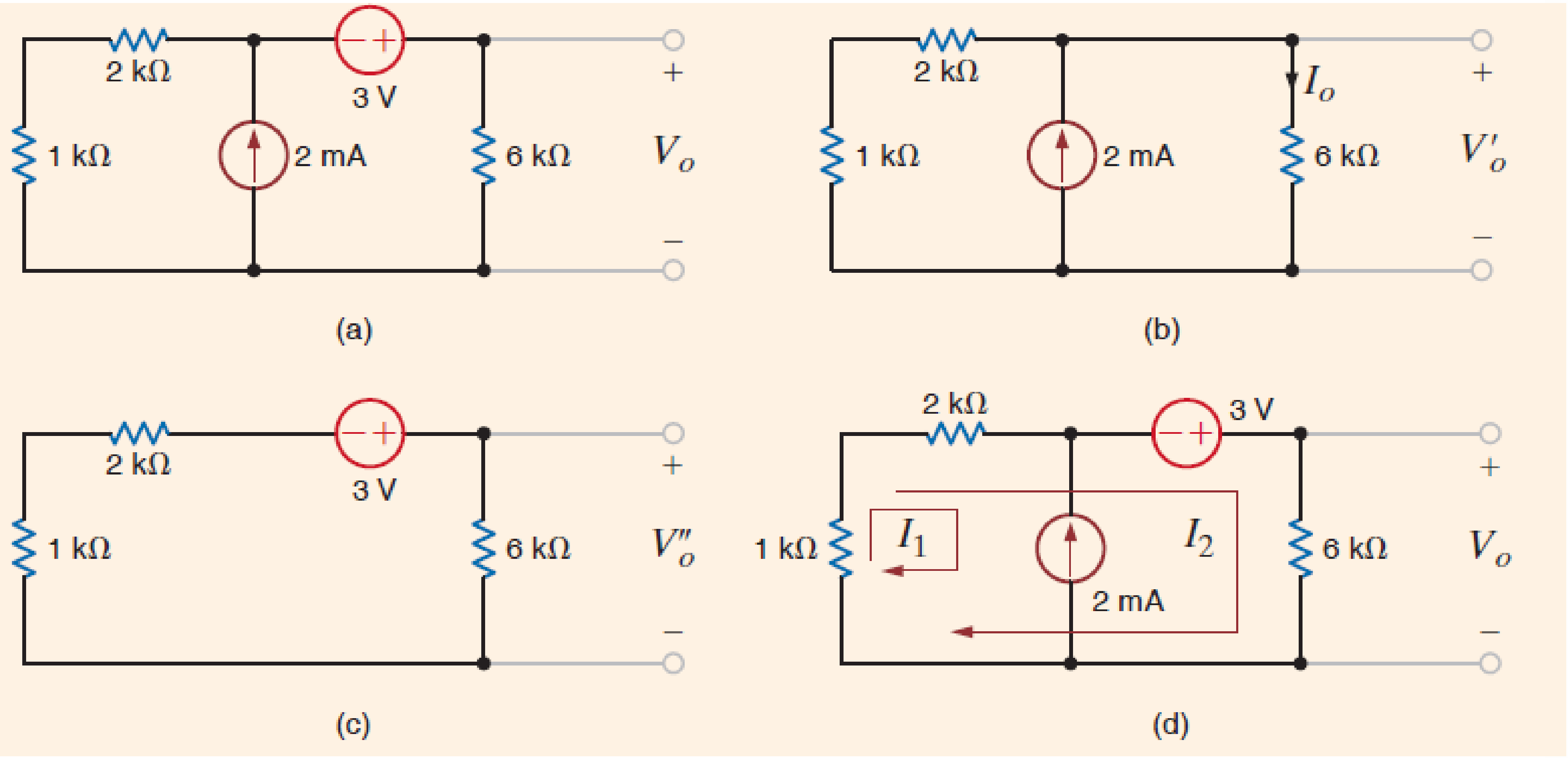
Superposition

- *The principle of superposition states that* "In any linear circuit containing multiple independent sources, the current or voltage at any point in the network may be calculated as the algebraic sum of the individual contributions of each source acting alone"
- When determining the contribution due to an independent source, any remaining voltage sources are made zero by replacing them with short circuits, and any remaining current sources are made zero by replacing them with open circuits.

Superposition

- Although superposition can be used in linear networks containing dependent sources, it is not useful in this case since the dependent source is never made zero.
- Let us use superposition to find V_o in the circuit in Fig. a.

Superposition



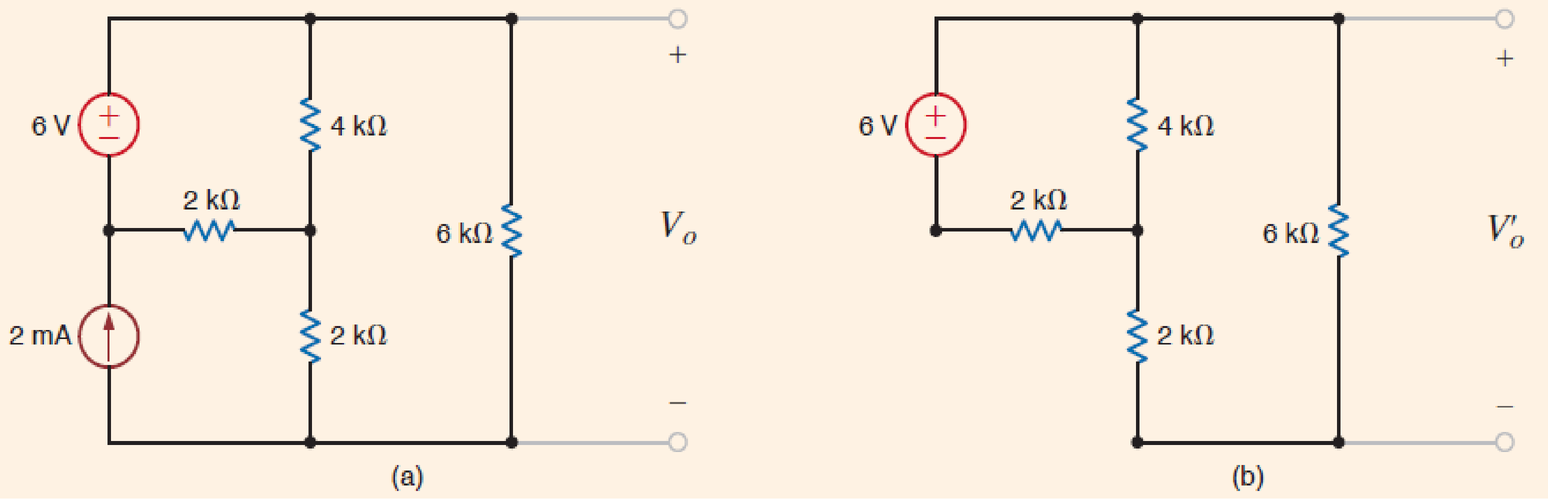
Superposition

- Considering the 2-mA source (b).
- Using current division
- $I_0 = (2 \times 10^{-3}) \left(\frac{1k+2k}{1k+2k+6k}\right) = \frac{2}{3} mA, \quad and\ V_0' = I_0(6k) = 4V$
- Considering the 3-V source (c). using voltage division
- $V_0'' = 3 \left(\frac{6k}{1k+2k+6k}\right) = 2V$
- $V_0 = V_0' + V_0'' = 6V$

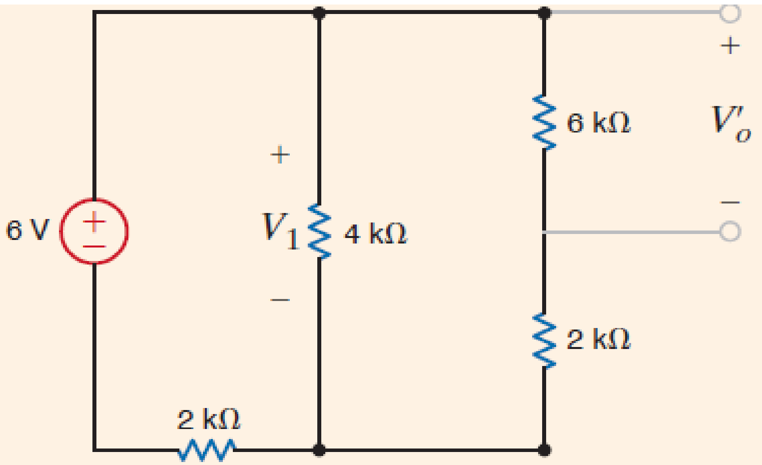
Exercise: Use nodal analysis to prove the result

Superposition

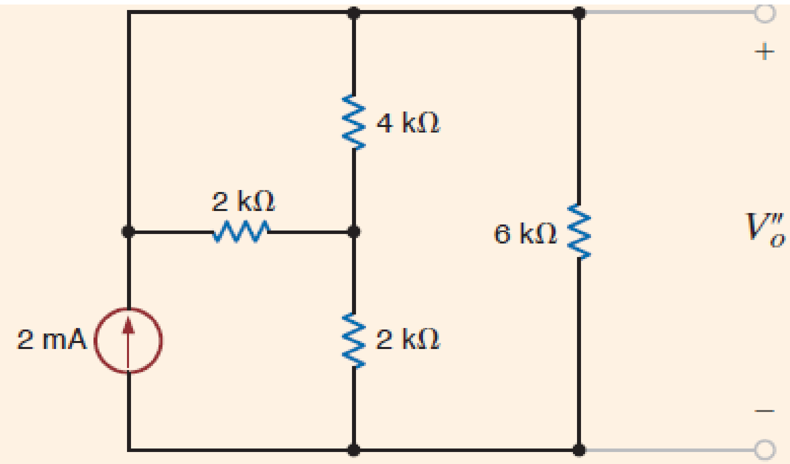
- **Example 2**
- Consider now the network in Fig. a. Let us use superposition to find V_0'.



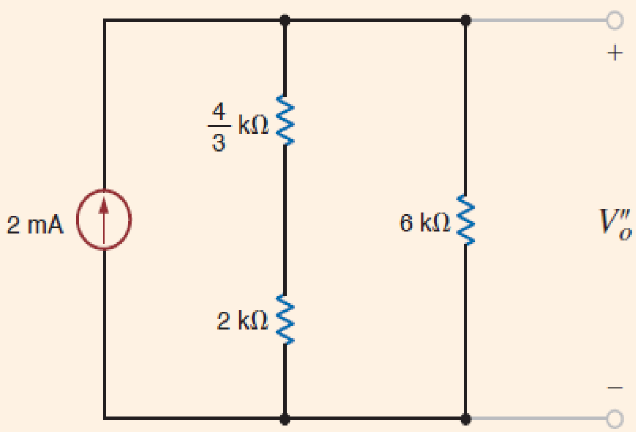
Superposition



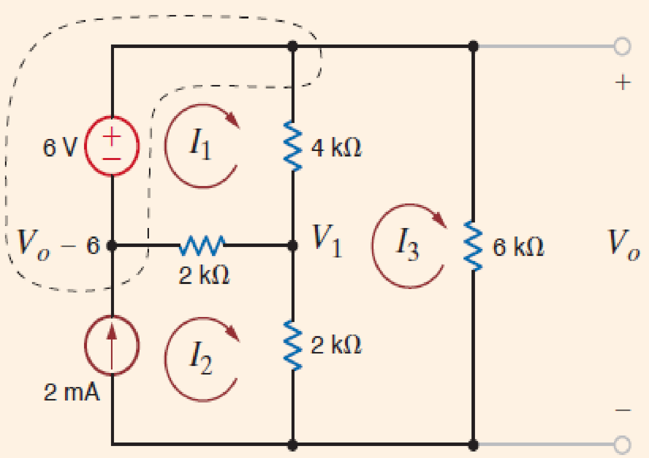
(c)



(d)



(e)



(f)

Superposition

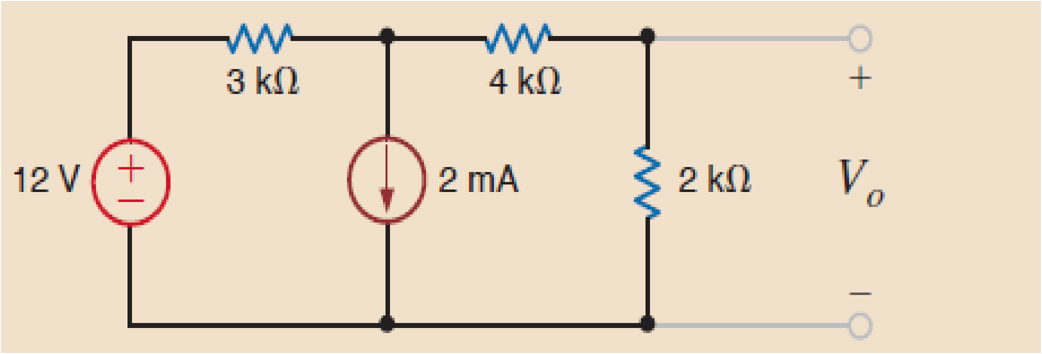
- Considering 6-V source (b, c)
- By voltage division
- $V_1 = 6\left(\frac{\frac{8}{3}k}{\frac{8}{3}k+2k}\right) = \frac{24}{7}V$
- Applying voltage division again
- $V_0' = V_1\left(\frac{6k}{6k+2k}\right) = \frac{18}{7}V$
- Considering 2-mA source (4.3 d, e)
- $V_0'' = (2 \times 10^{-3})\left(\frac{10}{3}k//6k\right) = 30/7V$
- $V_0 = V_0' + V_0'' = \frac{48}{7}V$

Superposition

- A nodal analysis of the network can be performed using fig. f. The equation for the supernode is
- $-2 \times 10^{-3} + \frac{(V_0-6)-V_1}{2k} + \frac{V_0-V_1}{4k} + \frac{V_0}{6k} = 0$
- The equation for the node labeled V_1 is
- $\frac{V_1-V_0}{4k} + \frac{V_1-(V_0-6)}{2k} + \frac{V_1}{2k} = 0$
- Solving the two equations gives $V_0 = \frac{48}{7} V$
- **Exercise:** Refer to figure f and use mesh analysis to obtain the result.

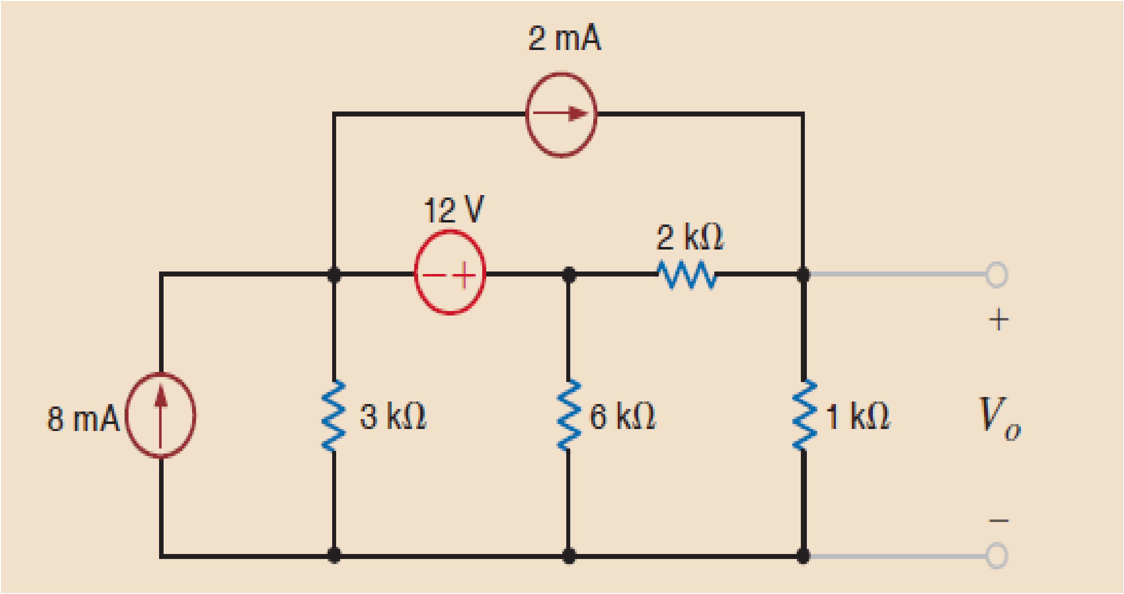
Superposition

- **Exercise**
- Compute V_0 in circuit below using superposition. (4/3 V)



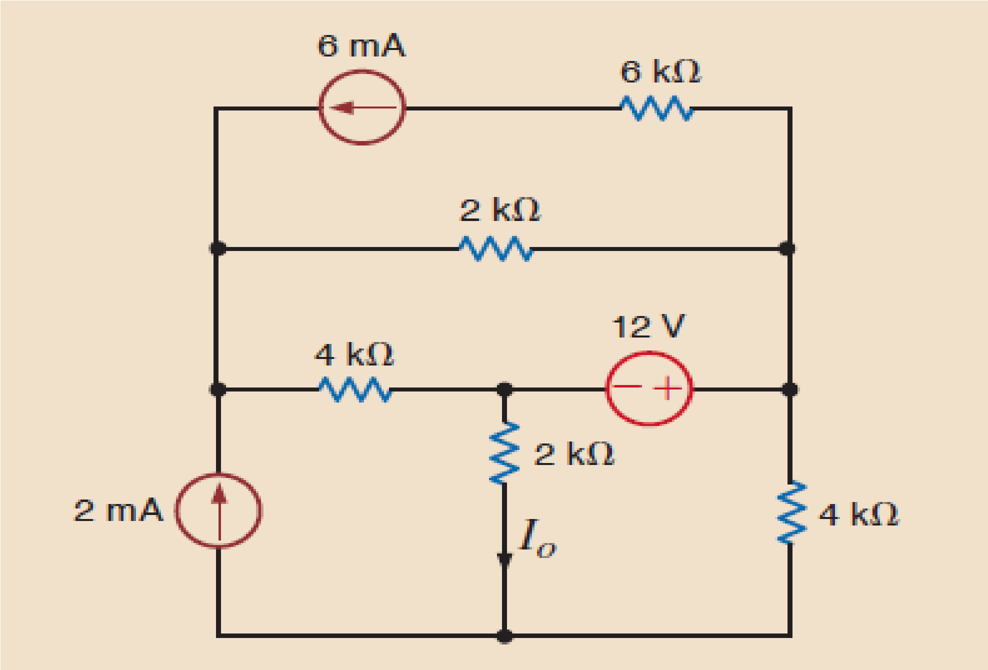
Superposition

- Compute V_0 in circuit below using superposition. (5.6 V)



Superposition

- Find I_0 in Fig. below using superposition.(-2/3 mA)



Thévenin's and Norton's Theorems

- Suppose that we are given a circuit and that we wish to find the current, voltage, or power that is delivered to some resistor of the network, which we will call the load.
- *Thévenin's theorem* tells us that we can replace the entire network, exclusive of the load, by an equivalent circuit that contains only an independent voltage source in series with a resistor in such a way that the current–voltage relationship at the load is unchanged.

Thévenin's and Norton's Theorems

- *Norton's theorem* is identical to the preceding statement except that the equivalent circuit is an independent current source in parallel with a resistor.
- Note that this is a very important result. It tells us that if we examine any network from a pair of terminals, we know that with respect to those terminals, the entire network is equivalent to a simple circuit consisting of an independent voltage source in series with a resistor or an independent current source in parallel with a resistor.

Thévenin's and Norton's Theorems

- **Applying Thévenin's Theorem**
- **Step 1.** Remove the load and find the voltage across the open-circuit terminals.
- **Step 2.** Determine the Thévenin equivalent resistance of the network at the open terminals with the load removed. Three different types of circuits may be encountered in determining the resistance,
- If the circuit contains only independent sources, they are made zero by replacing the voltage sources with short circuits and the current sources with open circuits. R_{Th} is then found by computing the resistance of the purely resistive network at the open terminals.

Thévenin's and Norton's Theorems

- If the circuit contains only dependent sources, an independent voltage or current source is applied at the open terminals and the corresponding current or voltage at these terminals is measured. The voltage/current ratio at the terminals is the Thévenin equivalent resistance. Since there is no energy source, the open-circuit voltage is zero in this case.
- If the circuit contains both independent and dependent sources, the open-circuit terminals are shorted and the short-circuit current between these terminals is determined. The ratio of the open-circuit voltage to the short-circuit current is the resistance

Thévenin's and Norton's Theorems

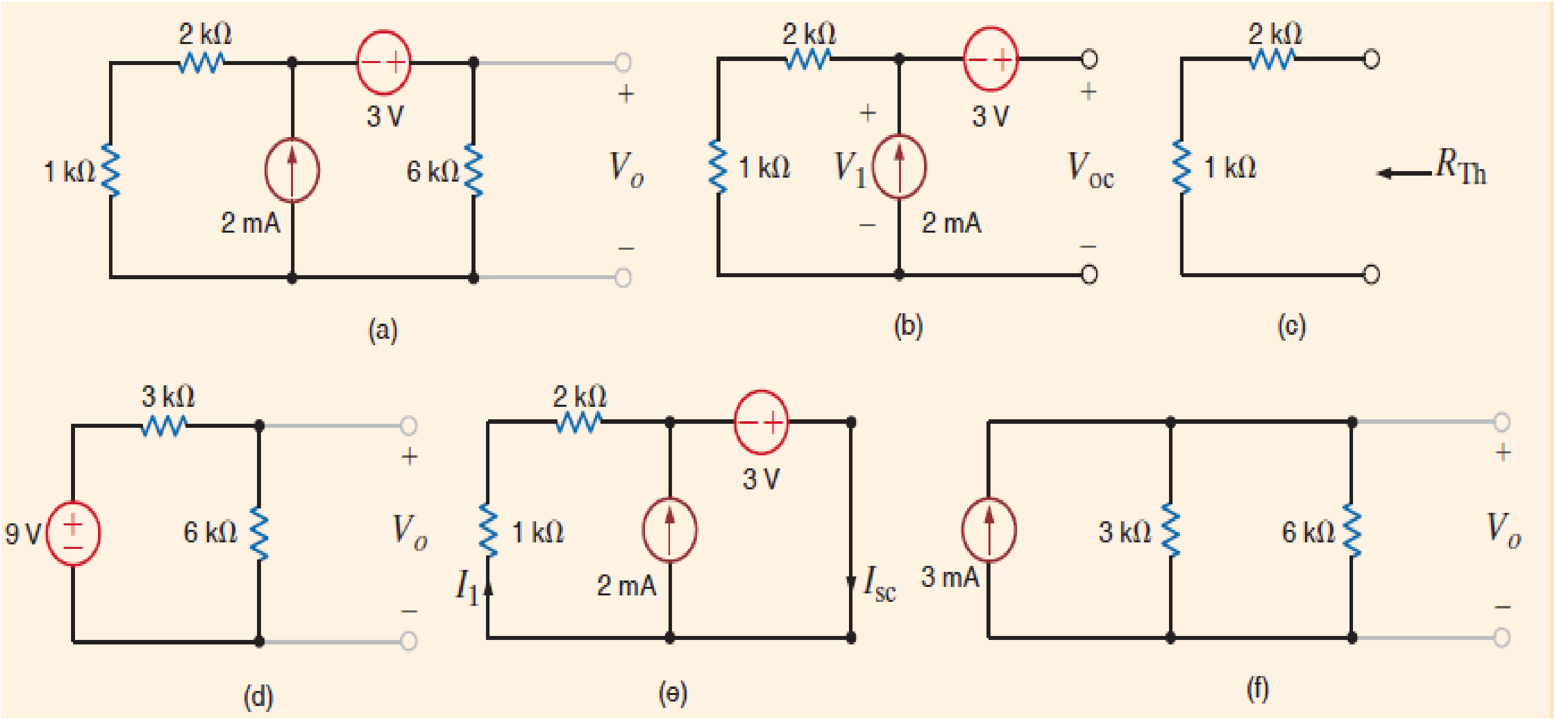
- **Step 3.** If the load is now connected to the Thévenin equivalent circuit, consisting of in series with the desired solution can be obtained.

Applying Norton's Theorem

- The problem-solving strategy for Norton's theorem is essentially the same as that for Thévenin's theorem with the exception that we are dealing with the short-circuit current instead of the open-circuit voltage.

Thévenin's and Norton's Theorems

- **Circuits Containing only Independent Sources**



Thévenin's and Norton's Theorems

- **Circuits Containing only Independent Sources**
- **Example**
- Let us use Thévenin's and Norton's theorems to find V_0 in the network in figure above. Its redrawn in figure a.
- To determine the Thévenin equivalent, we break the network at the 6kΩ load as shown in Fig. b.
- KVL indicates that the open-circuit voltage
- $V_{os} = 3V + V_1$
- $V_1 = (2 \times 10^{-3})(1k + 2k) = 6V$
- $V_{os} = 9V.$

Thévenin's and Norton's Theorems

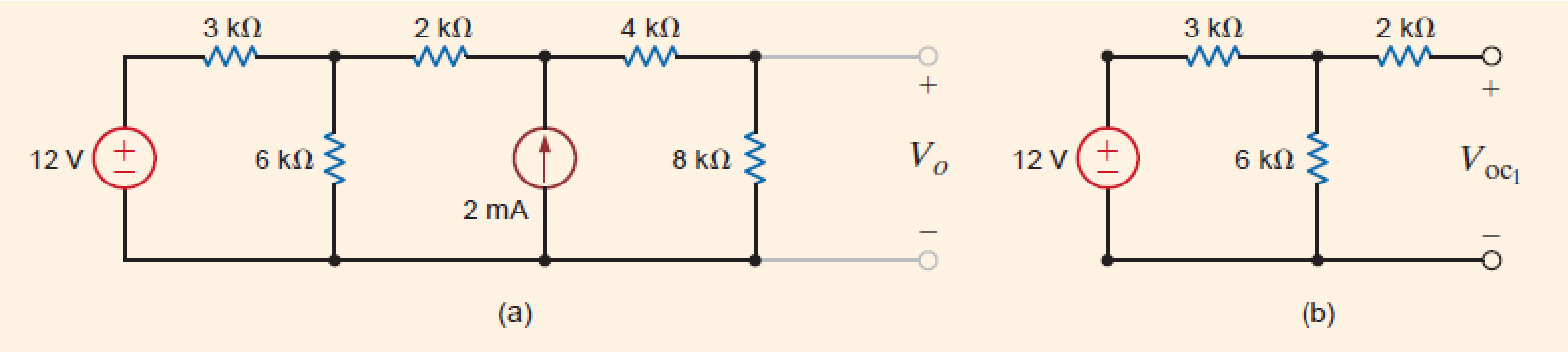
- **Circuits Containing only Independent Sources**
- Therefore, By making both sources zero, we can find the Thévenin equivalent resistance, using the circuit in Fig c.
- $R_{Th} = 3k\Omega$
- Figure d shows the Thévenin's equivalent circuit.
- Using a simple voltage divider, we find that $V_0 = 6V$.

Thévenin's and Norton's Theorems

- **Circuits Containing only Independent Sources**
- To determine the Norton equivalent circuit at the terminals of the load, we must find the short-circuit current as shown in Fig. e. Note that the short circuit causes the 3-V source to be directly across (i.e., in parallel with) the resistors and the current source. Therefore, $I_1 = \frac{3}{1k+2k} = 1mA$
- Then, using KCL, $I_{sc} = 3mA$ .We have already determined R_{Th}, and, therefore, connecting the Norton equivalent to the load results in the circuit in Fig. f.

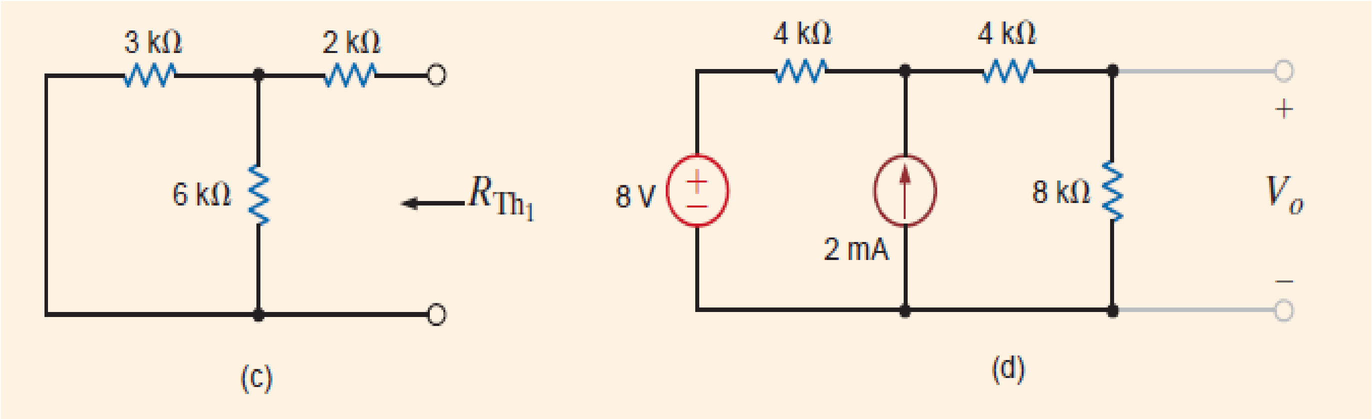
Thévenin's and Norton's Theorems

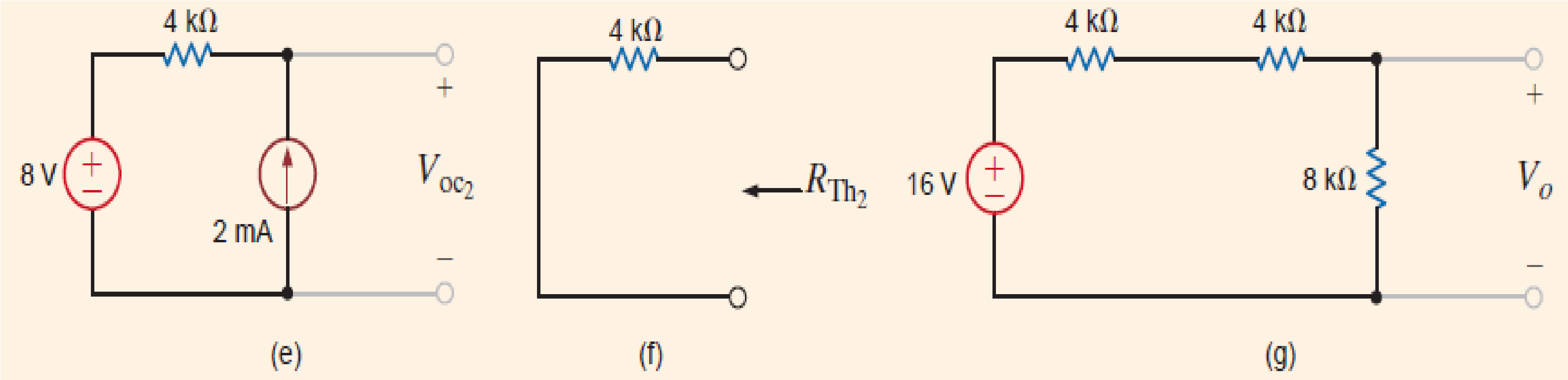
- **Circuits Containing only Independent Sources**
- Hence V_0, is equal to the source current multiplied by the parallel resistor combination, which is 6 V.
- **Example 2**
- Let us use Thévenin's theorem to find V_0 in the network in Fig. a.



Thévenin's and Norton's Theorems

- **Circuits Containing only Independent Sources**





Thévenin's and Norton's Theorems

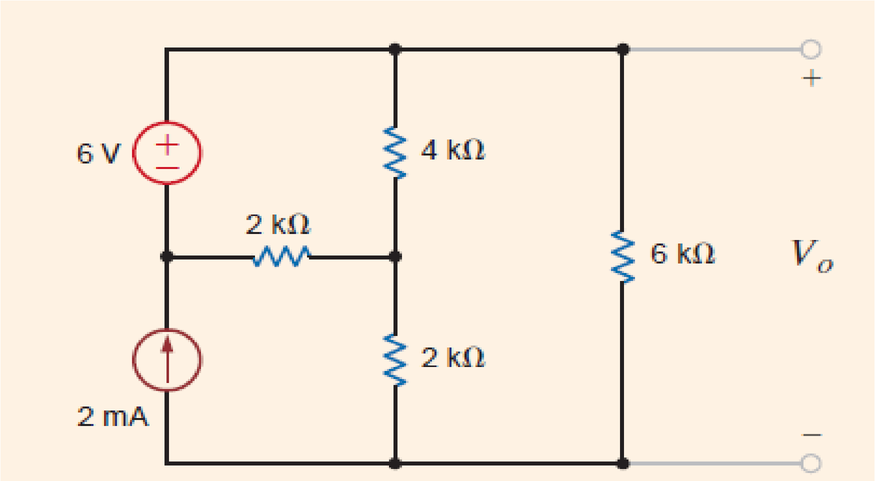
- **Circuits Containing only Independent Sources**
- If we break the network to the left of the current source, the open-circuit voltage V_{oc1} is as shown in Fig. b. Since there is no current in the 2kΩ resistor and therefore no voltage across it, V_{oc1} is equal to the voltage across the 6-k Ω resistor, which can be determined by voltage division as
- $V_{oc1} = 12\left(\frac{6k}{6k+3k}\right) = 8V$
- The Thévenin equivalent resistance R_{Th1}, is found from Fig. c as
- $R_{Th1} = 2k + 3k//6k = 4k\Omega.$

Thévenin's and Norton's Theorems

- **Circuits Containing only Independent Sources**
- Connecting this Thévenin equivalent back to the original network produces the circuit shown in Fig. d. We can now apply Thévenin's theorem again, and this time we break the network to the right of the current source as shown in Fig. e. In this case V_{oc2} is
- $V_{oc2} = (2m)(4k) + 8 = 16V$.
- And $R_{Th2} = 4k\Omega$.
- Connecting this Thévenin equivalent to the remainder of the network produces the circuit shown in Fig. g. Simple voltage division applied to this final network yields $V_o = 8V$.
- **Exercise:** Use Norton's theorem to solve the above example.

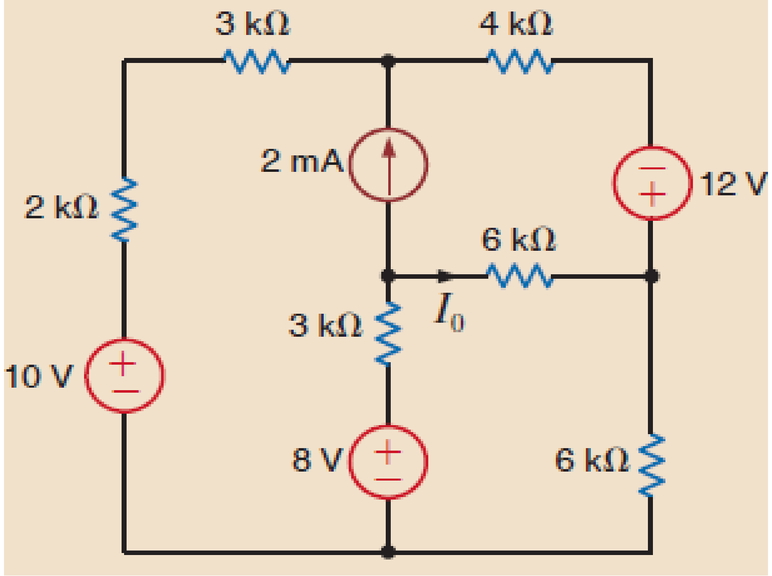
Thévenin's and Norton's Theorems

- **Exercise**
- Use the two theorems and solve V_0 in the circuit below. (48/7 V)



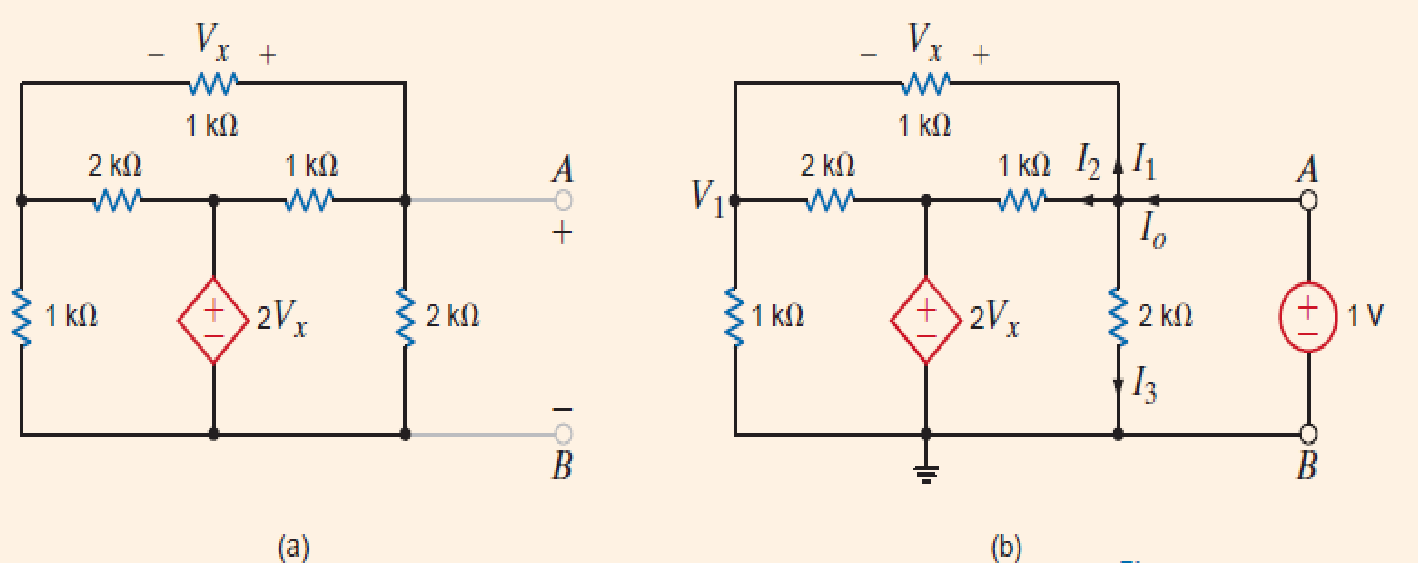
Thévenin's and Norton's Theorems

- Find I_0 using Norton's theorem in the figure below (-0.587mA)



Thévenin's and Norton's Theorems

- **Circuits Containing only Dependent Sources**



- We wish to determine the Thévenin equivalent of the network in Fig. a at the terminals A-B.
- Our approach to this problem will be to apply a 1V voltage source at the terminals A-B and then compute the current I_0 *and* $R_{th} = 1/I_0$.

Thévenin's and Norton's Theorems

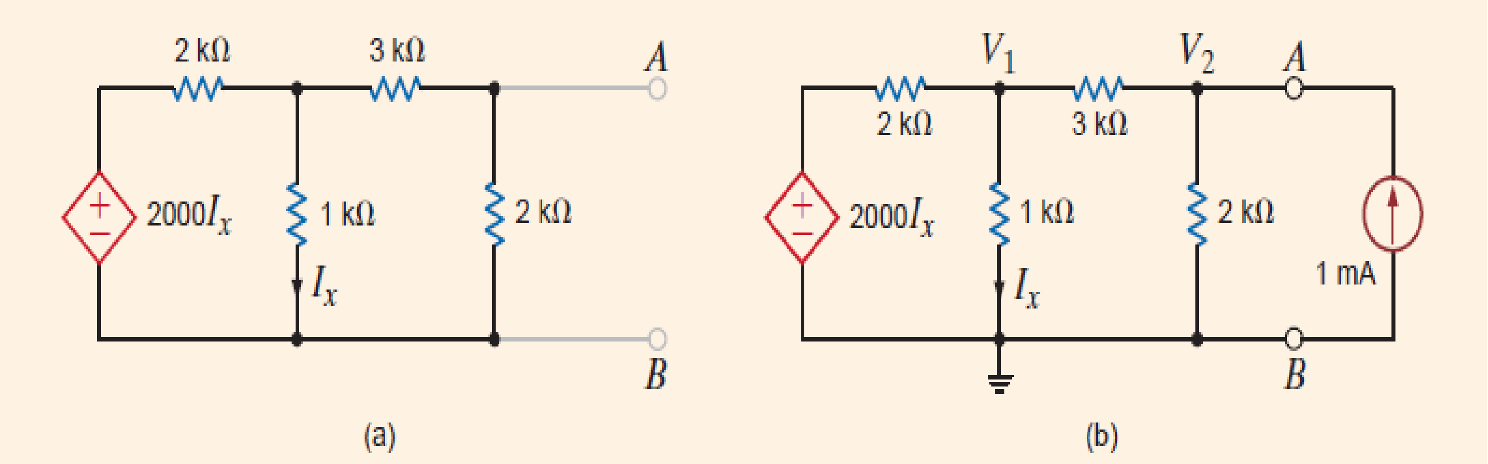
- **Circuits Containing only Dependent Sources**
- The equations for the network in Fig. b are as follows. KVL around the outer loop specifies that
- $V_1 + V_x = 1$
- KCL equation at node V_1 is
- $\frac{V_1}{1k} + \frac{V_1 - 2V_x}{2k} + \frac{V_1 - 1}{1k} = 0$
- $V_x = 3/7V$.

Thévenin's and Norton's Theorems

- **Circuits Containing only Dependent Sources**
- We can now compute the currents
- $I_1 = \frac{3}{7} mA, I_2 = \frac{1}{7} mA, I_3 = \frac{1}{2} mA$
- And thus $I_0 = I_1 + I_2 + I_3 = \frac{15}{14} mA$ and $R_{th} = \frac{1}{I_0} = \frac{14}{14} k\Omega$.

Thévenin's and Norton's Theorems

- **Circuits Containing only Dependent Sources**
- **Exercise:** Determine R_{th} at the terminals A-B for the network in Fig. below (10/7kΩ)

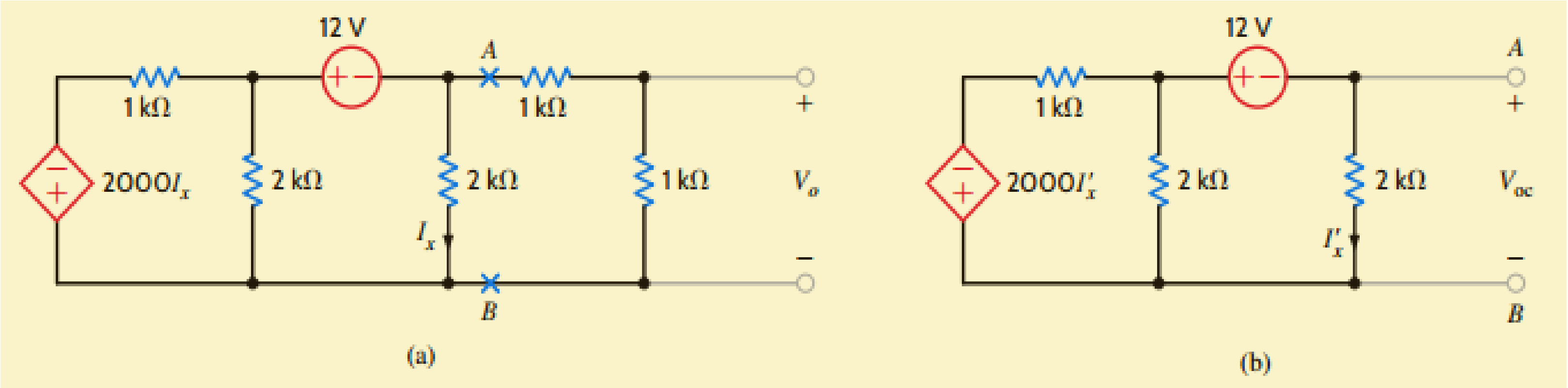


Thévenin's and Norton's Theorems

- **Circuits Containing both Independent and Dependent Sources**
- In these types of circuits we must calculate both the open-circuit voltage and short-circuit current to calculate the Thévenin equivalent resistance.
- Furthermore, we must remember that we cannot split the dependent source and its controlling variable when we break the network to find the Thévenin or Norton equivalent.

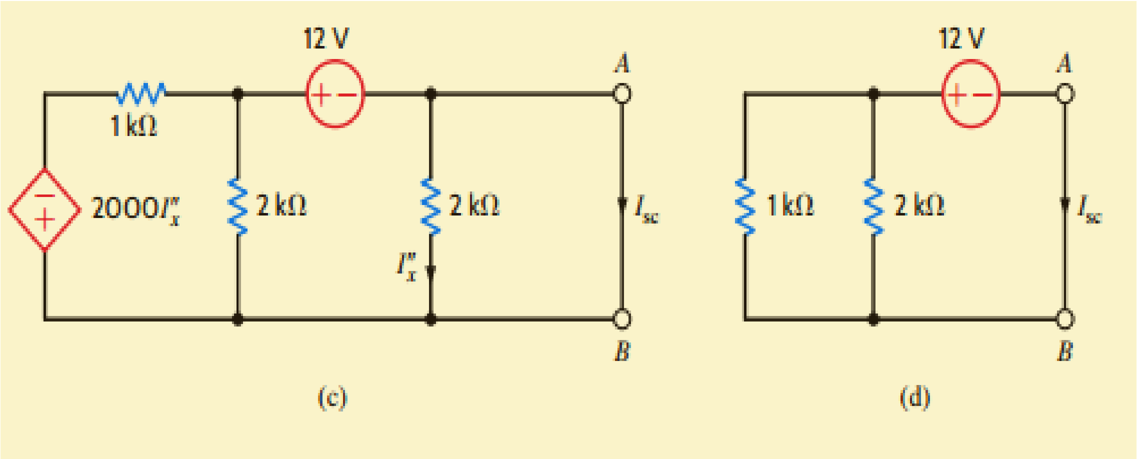
Thévenin's and Norton's Theorems

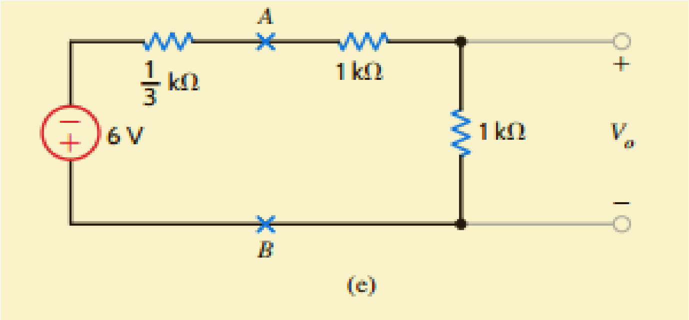
- **Circuits Containing both Independent and Dependent Sources**
- Let us use Thévenin's theorem to find V_o in the network in Fig. a.



Thévenin's and Norton's Theorems

- **Circuits Containing both Independent and Dependent Sources**





Thévenin's and Norton's Theorems

- **Circuits Containing both Independent and Dependent Sources**
- To begin, we break the network at points A-B. Could we break it just to the right of the 12-V source? No! Why? The open-circuit voltage is calculated from the network in Fig. b.
- Note that we now use the source $2000I_x'$ because this circuit is different from that in Fig. a.
- KCL for the supernode around the 12-V source is
- $\frac{(V_{OC}+12)-(-2000I_x')}{1k}+\frac{V_{OC}+12}{2k}+\frac{V_{OC}}{2k}=0$

Thévenin's and Norton's Theorems

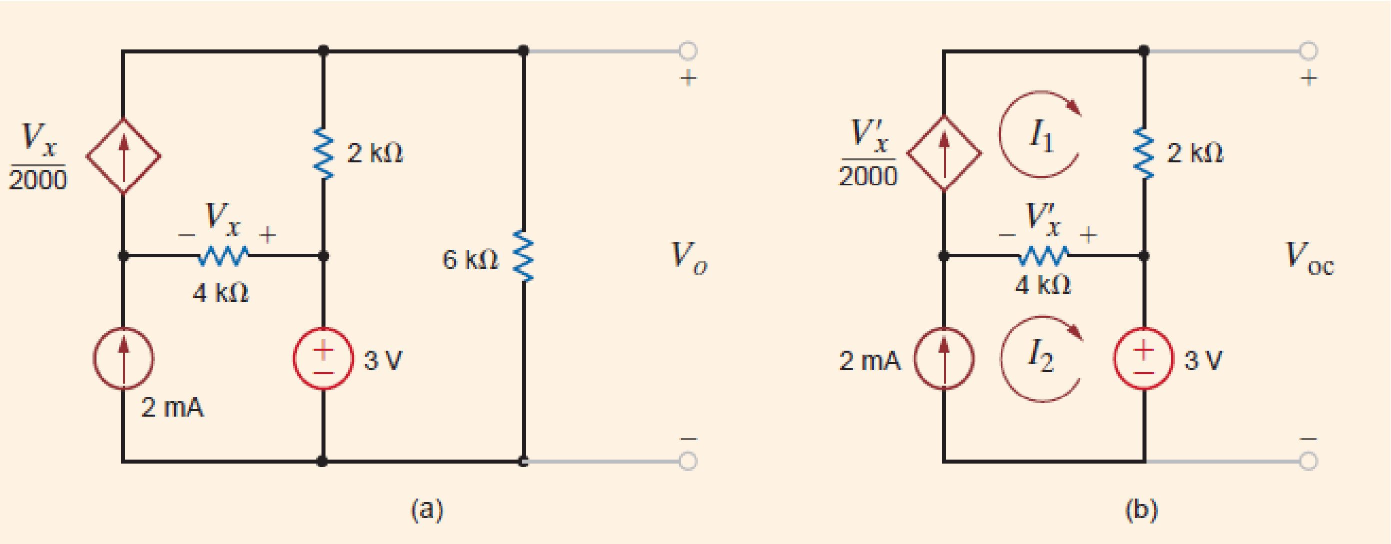
- **Circuits Containing both Independent and Dependent Sources**
- Where $I_x' = \frac{V_{oc}}{2k}$, yielding $V_{oc} = -6V$.
- I_{sc} can be calculated from the circuit in Fig. c. Note that the presence of the short circuit forces I_x'' to zero and, therefore, the network is reduced to that shown in Fig. d.
- Therefore
- $I_{sc} = -\frac{12}{\frac{2}{3}k} = -18mA$

Thévenin's and Norton's Theorems

- **Circuits Containing both Independent and Dependent Sources**
- Then $R_{th} = \frac{V_{oc}}{I_{sc}} = \frac{1}{3} k\Omega$.
- Connecting the Thévenin equivalent circuit to the remainder of the network at terminals A-B produces the circuit in Fig. e. At this point, simple voltage division yields
- $V_0 = (-6)\left(\frac{1k}{1k+1k+\frac{1}{3}k}\right) = -\frac{18}{7} V$

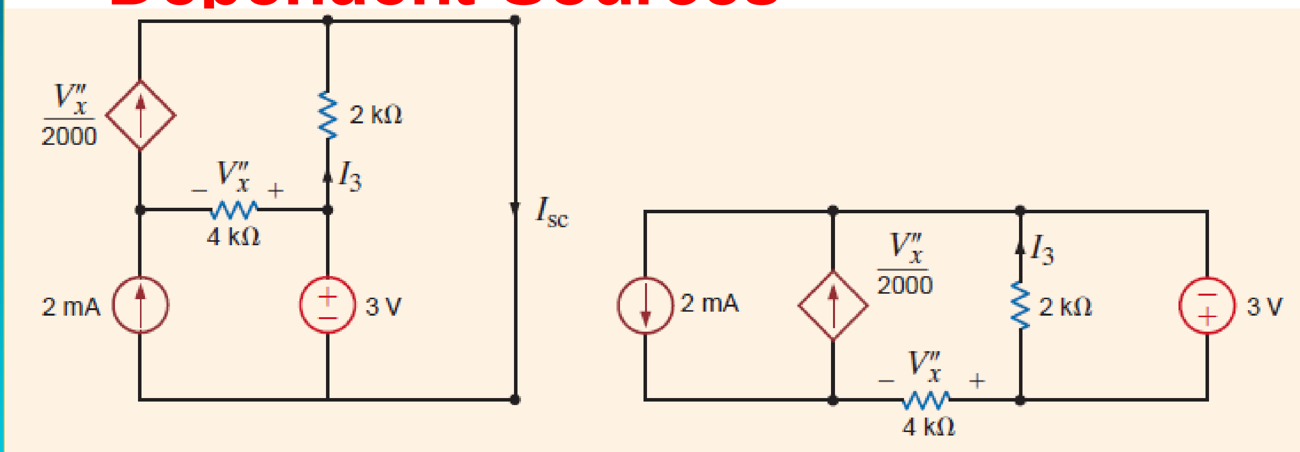
Thévenin's and Norton's Theorems

- **Circuits Containing both Independent and Dependent Sources**
- **Example 2**
- Let us use Thévenin's theorem to find V_o in the network in Fig. a.

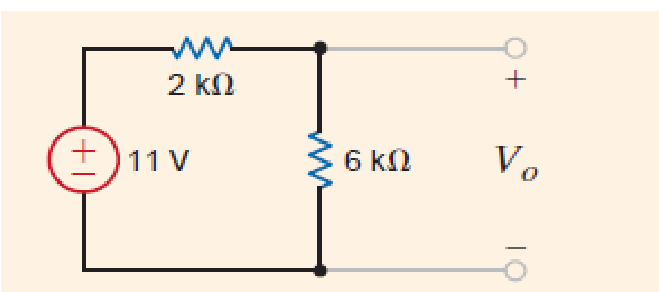


Thévenin's and Norton's Theorems

- **Circuits Containing both Independent and Dependent Sources**



(c) (d)



(e)

Thévenin's and Norton's Theorems

- **Circuits Containing both Independent and Dependent Sources**
- V_{oc} is determined from the network in Fig. b. Note that
- $I_1 = \frac{V_x'}{2k}$ $\quad I_2 = 2mA \quad and\ V_x' = 4k\left(\frac{V_x'}{2k} - 2m\right)$
- Solving these yields $I_1 = 4mA$ and hence $V_{oc} = 2kI_1 + 3 = 11V$
- I_{sc} is derived from the circuit in Fig. c. Note that if we collapse the short circuit, the network is reduced to that in Fig. d.

Thévenin's and Norton's Theorems

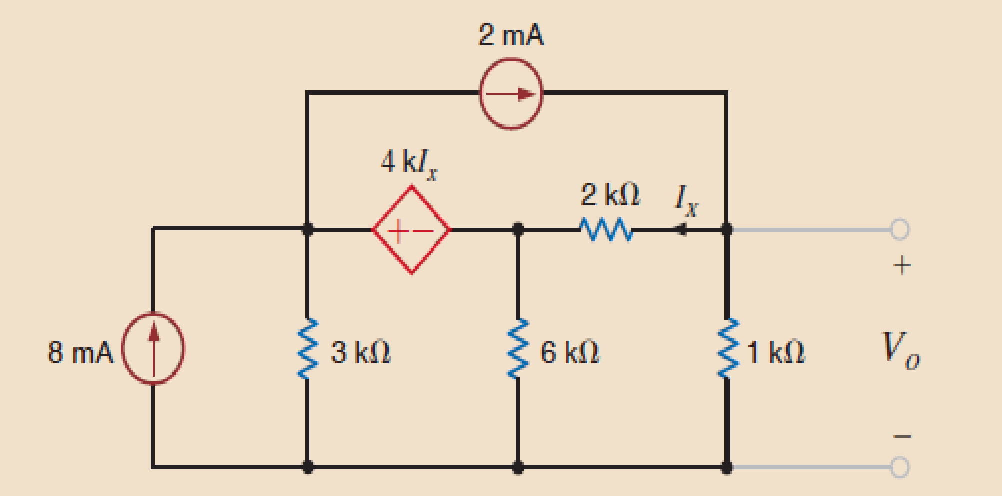
- **Circuits Containing both Independent and Dependent Sources**
- Although we have temporarily lost sight of I'_{sc} we can easily find the branch currents and they, in turn, will yield I'_{sc} .
- KCL at the node at the bottom left of the network is
- $\frac{V_x''}{4k} = \frac{V_x''}{2000} - 2 \times 10^{-3}$
- Or $V_x'' = 8V$
- $I_3 = \frac{3}{2k} = \frac{3}{2}mA$

Thévenin's and Norton's Theorems

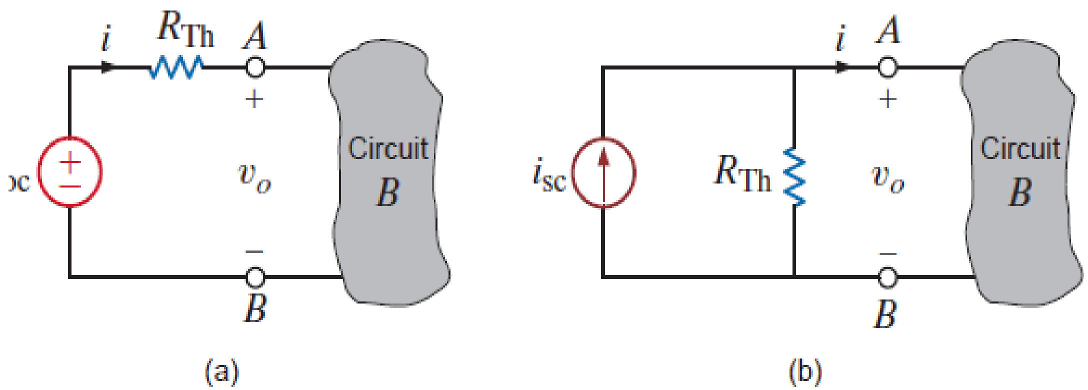
- **Circuits Containing both Independent and Dependent Sources**
- $I_{sc} = \frac{V_x''}{2000} + I_3 = \frac{11}{2} mA$
- Then $R_{Th} = \frac{V_{oc}}{I_{sc}} = 2k\Omega$.
- Connecting the Thévenin equivalent circuit to the remainder of the original network produces the circuit in Fig. e. Simple voltage division yields $V_o = \frac{33}{4} V$.

Thévenin's and Norton's Theorems

- **Circuits Containing both Independent and Dependent Sources**
- **Exercise**
- Find V_o in the fig. below using Thévenin's theorem. (6.29V)



Source Transformation/Exchange



- Thévenin and (b) Norton equivalent circuits.

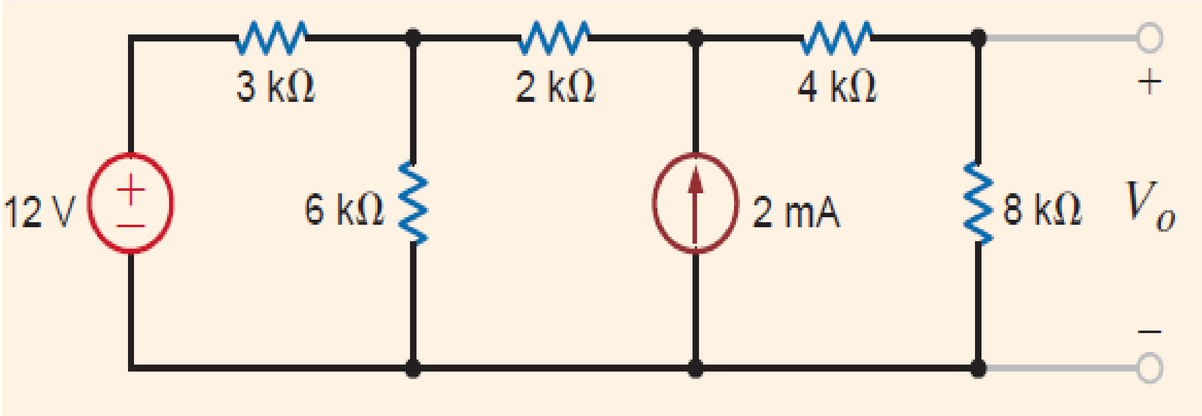
- If we have embedded within a network a current source *i* in parallel with a resistor *R*, we can replace this combination with a voltage source of value *v=iR* in series with the resistor *R*.

Source Transformation/Exchange

- The reverse is also true; that is, a voltage source v in series with a resistor *R* can be replaced with a current source of value $i = v/R$ in parallel with the resistor *R*. Parameters within the circuit (e.g., an output voltage) are unchanged under these transformations.
- The two equivalent circuits in *are equivalent only at the two external nodes*. For example, if we disconnect circuit B from both networks in the figure, the equivalent circuit in Fig. b dissipates power, but the one in Fig. a does not.

Source Transformation/Exchange

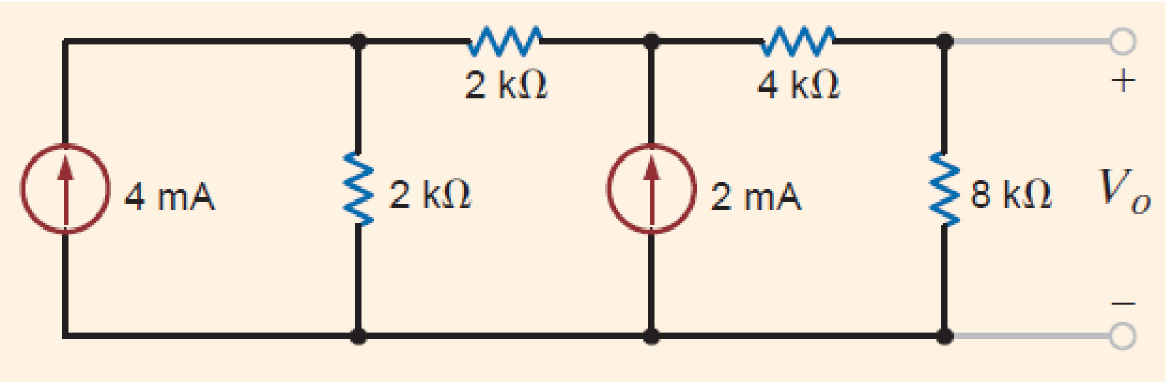
- Example
- Find V_0 in the circuit in Fig. a using the repeated application of source transformation



(a)

Source Transformation/Exchange

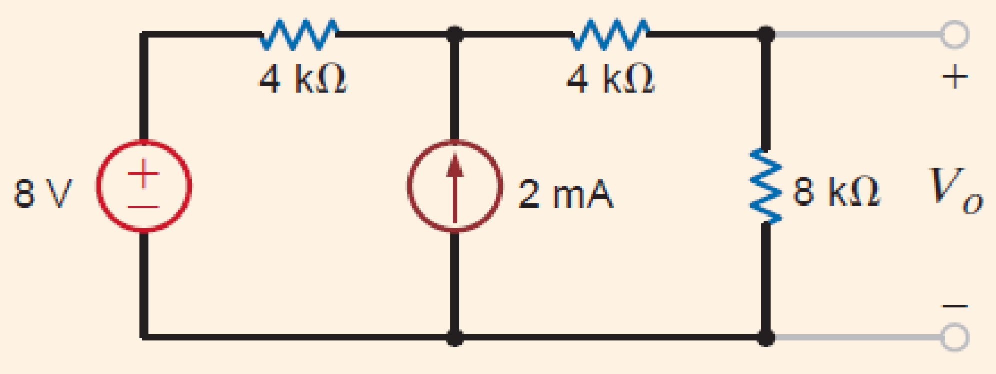
- From the left end of the network, the series combination of 12-V source and 3-kΩ is converted into a $4mA\ //3k\Omega$ resistor. The $3k\Omega//6k\Omega = 2k\Omega$ (Fig b)



(b)

Source Transformation/Exchange

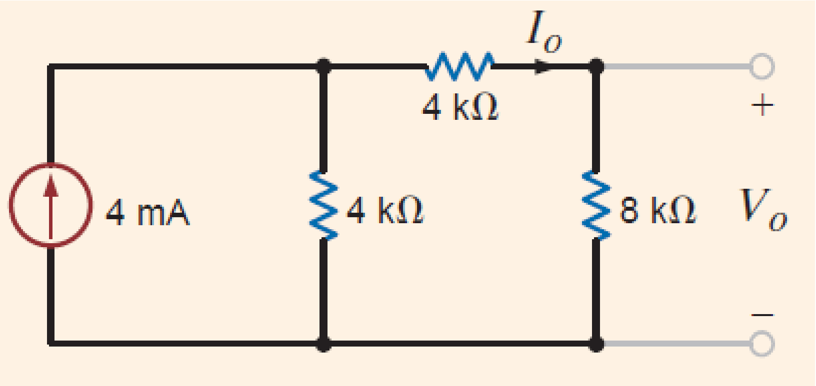
- The 4-mA source and 2-kΩ resistor are converted into on 8-V souce in series with this same 2-kΩ resistor.
- 2-kΩ in series with 2-kΩ =4 kΩ (Fig. c).



(c)

Source Transformation/Exchange

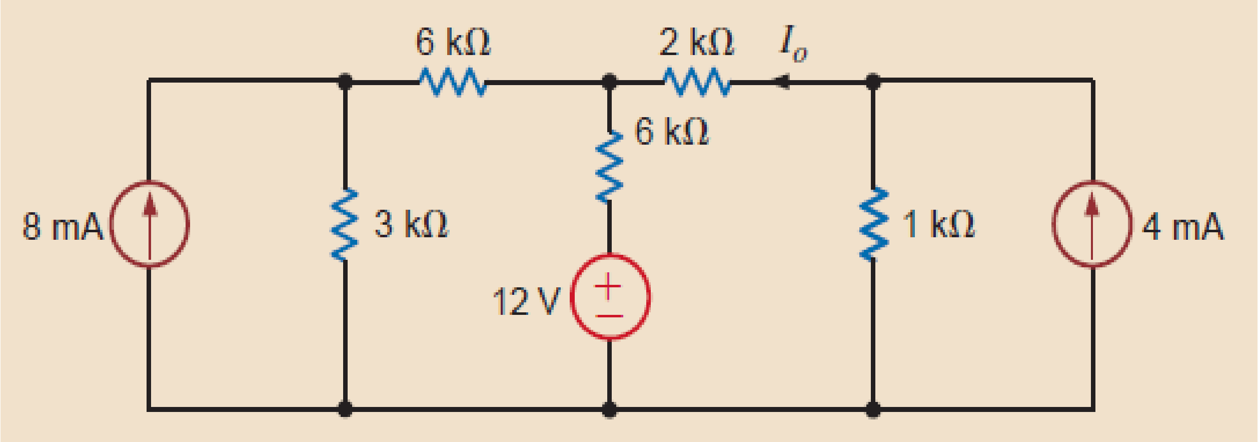
- The 8-V source and the 4-kΩ resistor are converted into a 2-mA // 4-kΩ.
- The resulting current source is combined with the other 2-mA source (Fig d).
- Using Current division
- $I_0 = (4 \times 10^{-3})\left(\frac{4k}{4k+4k+8k}\right) = 1mA$
- $V_0 = (1 \times 10^{-3})(8k) = 8V$



(d)

Source Transformation/Exchange

- Find the *Io* in Fig. E5.15 using source transformations.

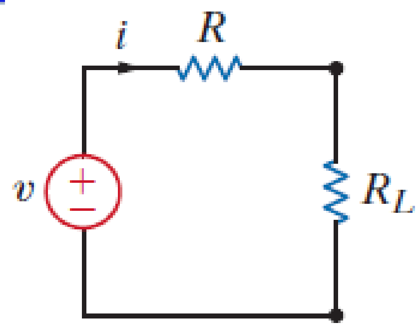


(-1.94mA)

Maximum Power Transfer

- In circuit analysis we are sometimes interested in determining the maximum power that can be delivered to a load. By employing Thévenin's theorem, we can determine the maximum power that a circuit can supply and the manner in which to adjust the load to effect maximum power transfer.
- Suppose that we are given the circuit shown in Fig. below.
- The power that is delivered to the load is given by the following expression

Maximum Power Transfer



Equivalent circuit for examining maximum power transfer

$$P_{load} = i^2 R_L = \left(\frac{v}{R + R_L}\right)^2 R_L$$

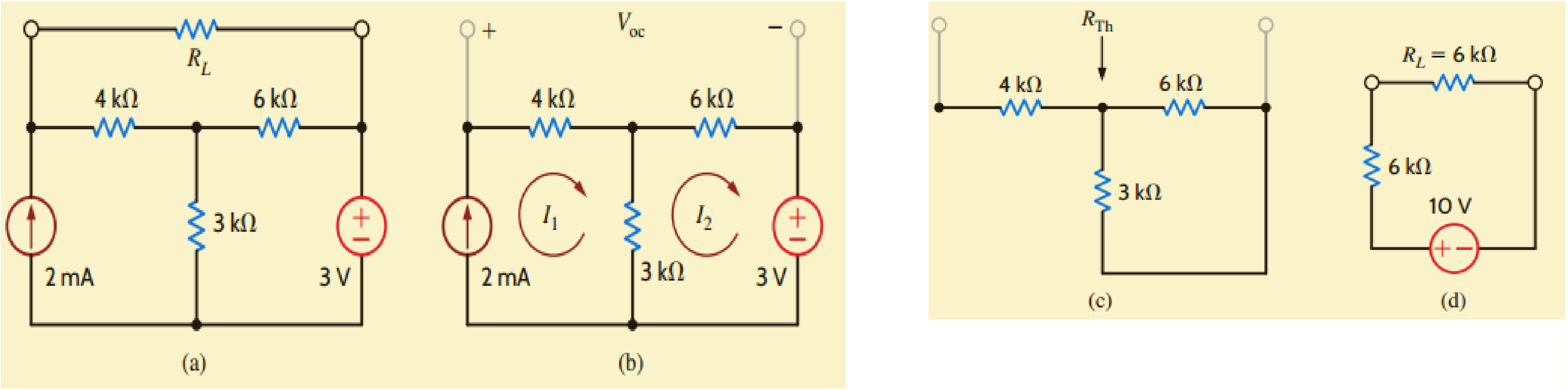
We want to determine the value of R_L that maximizes this quantity. Hence, we differentiate this expression with respect to R_L and equate the derivative to zero:

Maximum Power Transfer

- $\frac{dP_{load}}{dR_L} = \frac{(R+R_L)^2 v^2 - 2v^2 R_L (R+R_L)}{(R+R_L)^4} = 0$
- Which yields
- $R_L = R$
- In other words, maximum power transfer takes place when $R_L = R$
- Although this is a very important result, we have derived it using the simple network in Fig. above
- However, we should recall that v and R in Fig. above could represent the Thévenin equivalent circuit for any linear network.

Maximum Power Transfer

- **Example**
- Let us find the value of R_L for maximum power transfer in the network in Fig. a and the maximum power that can be transferred to this load.



Maximum Power Transfer

- To begin, we derive the Thévenin equivalent circuit for the network exclusive of the load.
- V_{oc} can be calculated from the circuit in Fig. b. The mesh equations for the network are
- $I_1 = 2 \times 10^{-3}$
- $3k(I_2 - I_1) + 6kI_2 + 3 = 0$
- Solving the two, gives $I_2 = \frac{1}{3} mA, hence\ V_{oc} = 4kI_1 + 6kI_2 = 10V$
- $R_{th}\ is\ 6k\Omega; Thus\ R_L = R_{Th} = 6k\Omega$ for maximum power transfer.
- The maximum power transferred to the load in Fig. d is
- $P_L = \left(\frac{10}{12k}\right)^2 (6k) = \frac{25}{6} mW$

Maximum Power Transfer

- **Exercise**
- Find R_L for maximum transfer and the maximum power transferred to R_L in the figure below. (24/13kΩ, 27/26mW)