MAKERERE UNIVERSITY

COLLEGE OF ENGINEERING, DESIGN, ART AND TECHNOLOGY

SCHOOL OF ENGINEERING

DEPARTMENT OF ELECTRICAL AND COMPUTER ENGINEERING

EMT 1101: ENGINEERING MATHEMATICS I LECTURE NOTES 2018/2019

CHAPTER ONE: CONCEPT OF A FUNCTION

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Course objectives

- To provide a general introductory treatment of mathematical concepts fundamental to Engineering.
- To help students develop the analytical and critical thinking abilities fundamental to problem solving in Engineering.

Course outline

1. Concept of a function
2. Differential calculus
3. Integral calculus
4. Infinite Series
5. Ordinary Differential Equations

Reference text books

1. Engineering mathematics by Stroud
2. Calculus and analytical geometry by Edwards
3. Introduction to University mathematics by Smyrl
4. Advanced Engineering mathematics by Wylie
5. Advanced engineering mathematics by Kreyszig
6. Advanced Engineering mathematics by Bajpai

Course assessment

Three course work assignments with an assignment at the end of every pair of chapters and two tests with a test at the end of three chapters.

- Course work and take home tests: 15%
- Continuous assessment tests: 25%
- Final examination: 60%

CONCEPT OF A FUNCTION

The key to the mathematical analysis of a geometric or scientific analysis is the identification of the relationship between variables that describe the situation. Such a relationship may be a formula that expresses one variable as a function of another e.g. the area, A, of the circle with radius r is given by $A = \pi r^2$, the volume of V of a sphere of radius r is given by $V = (4/3)\pi r^3$. These are examples of real valued functions of a real variable.

Definitions

- A *variable* is a symbol identifying elements of a given set.
- A *real – valued function, f,* defined on a set *D,* of real numbers is a rule that assigns to each number, *x,* in *D,* exactly one real number *f(x). f(x),* is called the value of the function, *f,* at the number or point, *x.*
- The set *D* of all those numbers *x* for which *f(x)* is defined is called the *domain* of the function.
- The set of all values *y* = *f(x)* is called the *range* of the function f.
  i.e. $\{y: y = f(x) for\ some\ x\ in\ D\}$
- When we describe the function *f* by writing a formula *y* = *f(x)*, then *x* is referred to as the *independent variable* and y is called the *dependent variable*.

Examples

1. Find the domain of the function given by; $g(x) = \frac{1}{\sqrt{2x+4}}$

   Solution

   $\sqrt{2x+4}$ is defined only if $2x + 4 \geq 0 \Rightarrow x \geq -2$

   In order that *g(x)* can be defined, then the denominator, $\sqrt{2x+4} \neq 0 \Rightarrow x \neq -2$

   Hence the domain of g is the interval; $D = (-2, +\infty) = -2 < x < +\infty$

Note

The curl brackets (or) are used in case the limit of the interval is not inclusive in the domain; this is called an *open interval*. Square brackets $[or]$ are used in case the limit at that point is inclusive in the domain range; this is called a *closed interval*.

2. Find the domain and range of the following functions:

(i) $f(x) = \frac{1}{x^2+x-6}$

For the function to be defined then $x^2 + x - 6 \neq 0$

$\therefore (x+3)(x-2) \neq 0$

This inequality is solved as follows:

	$x < -3$	$-3 < x < 2$	$x > 2$
$(x+3)$	-	+	+
$(x-2)$	-	-	+
$(x+3)(x-2)$	+	-	+

Therefore;

Domain, $x: x\ (-\infty, -3) \cup (-3,2) \cup (2, +\infty)$

Range, $y: y = f(x) = (-\infty, +\infty)$

(ii) $f(x) = (x^2 - x - 2)^{3/2}$

This can be re – written as; $f(x) = \left[(x^2 - x - 2)^{\frac{1}{2}}\right]^3$ therefore, f(x) is defined if,

$(x^2 - x - 2) \geq 0 \Rightarrow (x-2)(x+1) \geq 0$. This inequality can be solved according to the table below:

	$x < -1$	$-1 < x < 2$	$x > 2$
$(x-2)$	-	-	+
$(x+1)$	-	+	+
$(x-2)(x+1)$	+	-	+

Therefore,

$x: x\ (-\infty, -1] \cup [2, +\infty)$

$y: y = f(x) = (-\infty, +\infty)$

(iii) $f(x) = \frac{1}{\sqrt{x^2+x-6}}$

f(x) is only defined if $x^2 + x - 6 \geq 0\ and x^2 + x - 6 \neq 0$

$$\therefore x^2 + x - 6 > 0 \Rightarrow (x+3)(x-2) > 0$$

	$x < -3$	$-3 < x < 2$	$x > 2$
$(x+3)$	-	+	+
$(x-2)$	-	-	+
$(x+3)(x-2)$	+	-	+

$x: x\ (-\infty, -3) \cup (2, +\infty)$

$y: y = f(x) = (-\infty, +\infty)$

(iv) $f(x) = \sqrt{\frac{x}{x+1}}$

$\frac{x}{x+1} \geq 0\ \ and\ x + 1 \neq 0\ \ \Rightarrow x \neq -1$

	x < -1	*-1 < x < 0*	*x >0*
X	-	-	+
x +1	-	+	+
$\dfrac{x}{x+1}$	+	-	+

Therefore, $x: x\ (-\infty, -1)\ U\ [0, +\infty)$

$y: y = f(x)\ (-\infty, +\infty)$

Problems

Find the domain and range of the following functions:

(i) $f(x) = \sqrt{3x - 5}$

(ii) $f(t) = \sqrt{1 - 2t}$

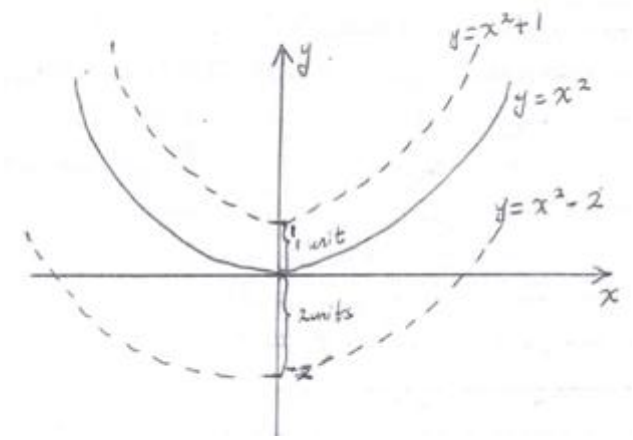
(iii) $f(x) = \frac{2}{3-x}$

(iv) $f(x) = \sqrt{x^2 - 9}$

(v) $f(x) = \sqrt{4 - \sqrt{x}}$

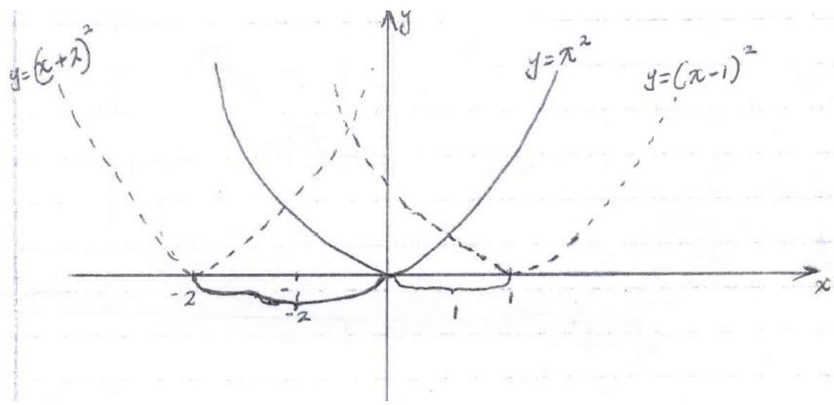
CURVE SHIFTING

Vertical shifting



- For y = f(x) + C, move the graph of y = f(x), C units up.
- For y = f(x) – C, move the graph of y = f(x), C units down.

Horizontal shifting



- For y = f(x – C), move the graph for y = f(x), C units to the right
- For y = f(x + C), move the graph of y = f(x), C units to the left.

Examples

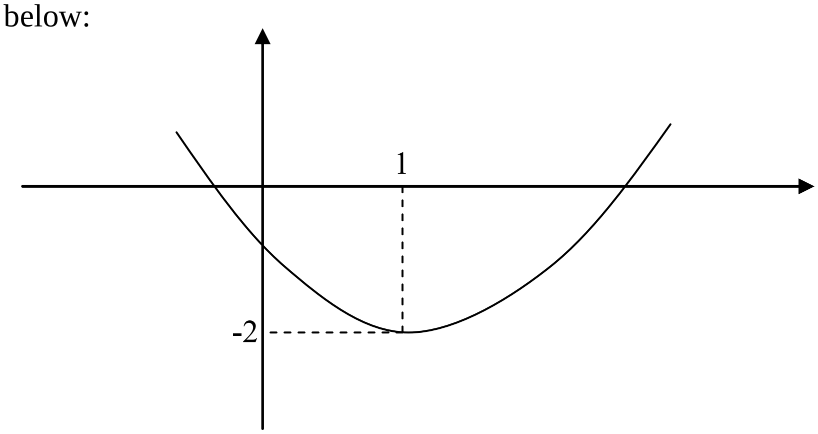
By completing squares, use translation and change of scale to sketch:

(i) $y = x^2 - 2x - 1$

Solution

$y = x^2 - 2x - 1 = x^2 - 2x + 1 - 1 - 1 = (x - 1)^2 - 2$

This can be sketched by translating the curve $y = x^2$, 2 units down and 1 unit to the right. See figure below:

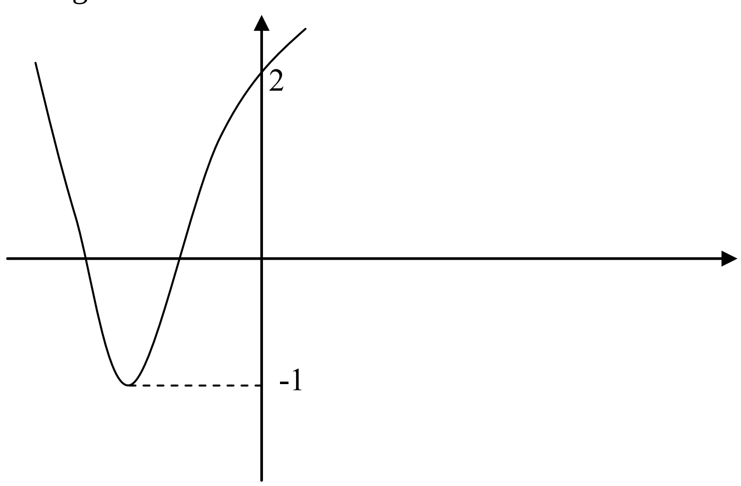


(ii) $y = 3x^2 + 6x + 2$

This can be written as; $y = 3(x^2 + 2x) + 2 = 3(x^2 + 2x + 1 - 1) + 2$

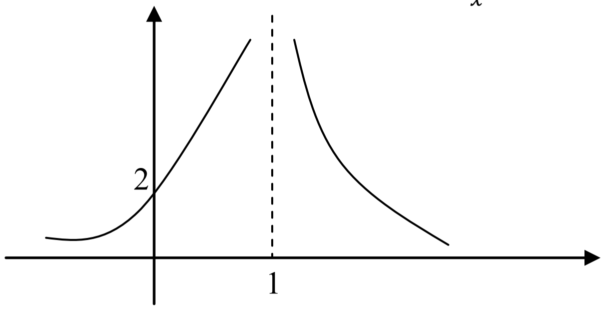
Therefore, $y = 3(x^2 + 2x + 1) + 2 - 3 = 3(x + 1)^2 - 1$

Hence this curve can be sketched by translating the curve $y = 3x^2$, 1 unit down and 1unit to the left as shown in the figure below:



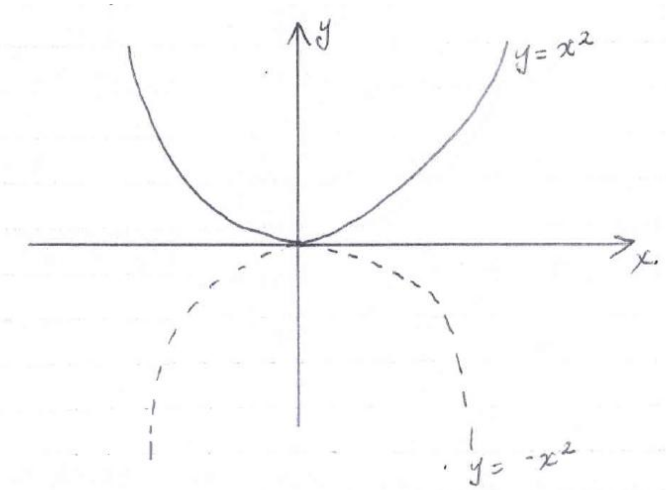
(iii) $y = \frac{2}{(x-1)^2}$

This function is sketched by translating the curve of $y = \frac{2}{x^2}$, 1 unit to the right.



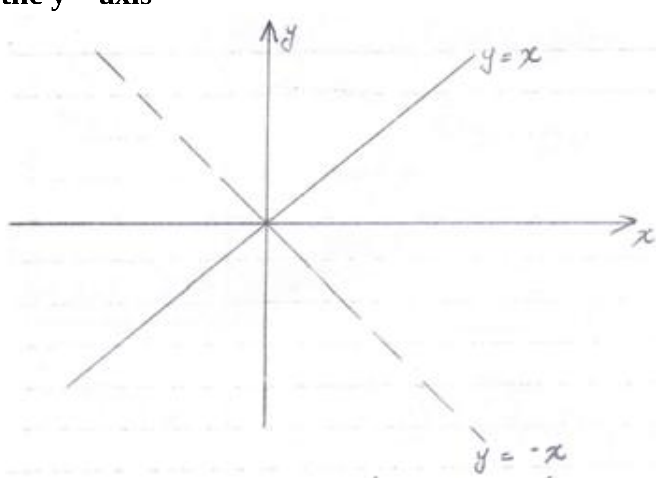
REFLECTION

(i) In the x – axis



y = f(x), on reflection in the x – axis becomes *y = - f(x).*

(i) In the y – axis



y = f(x) on reflection in the y – axis becomes y = f(-x)

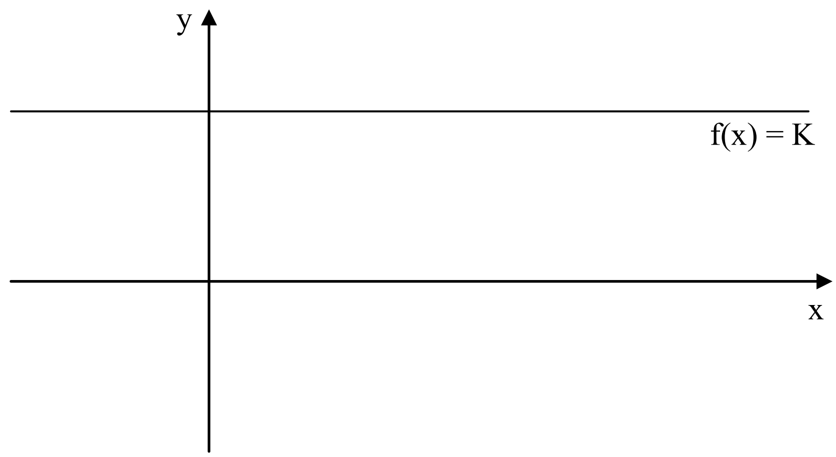
Question

Given the curve y = $3x^3 + 5x^2 - 4x + 10$, find its reflection in the y and x – axes.

CLASSIFICATION OF FUNCTIONS

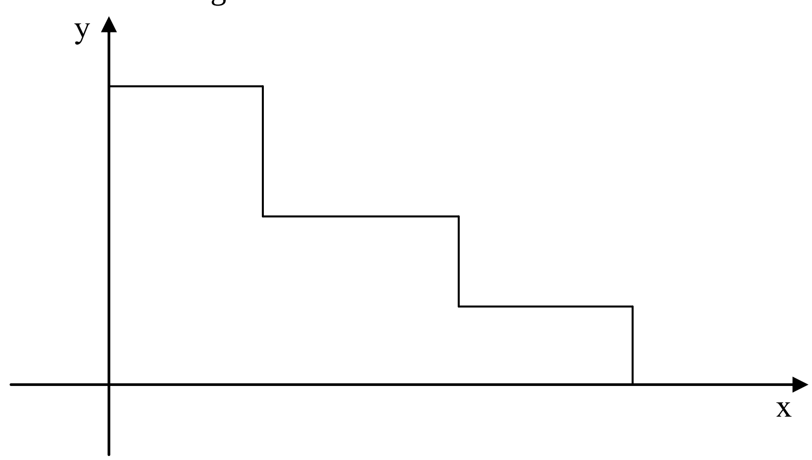
1. Constant function

This is a function that has a constant value over a given interval i.e. f(x) = K.



2. Step function

This is a function that is defined through some interval I and is constant over each one of a set of non – intersecting intervals.



3. Multiple value function

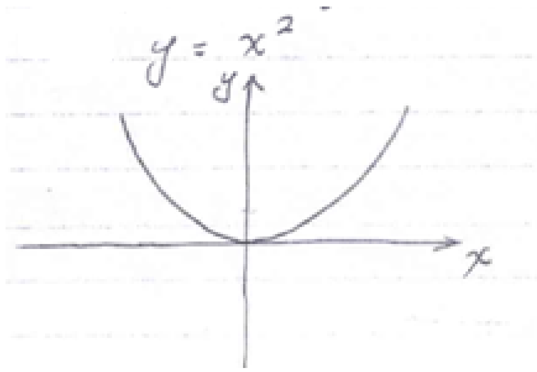
This is a function which has more than one value for the independent variable e.g $f(x) = \sqrt{x}$.

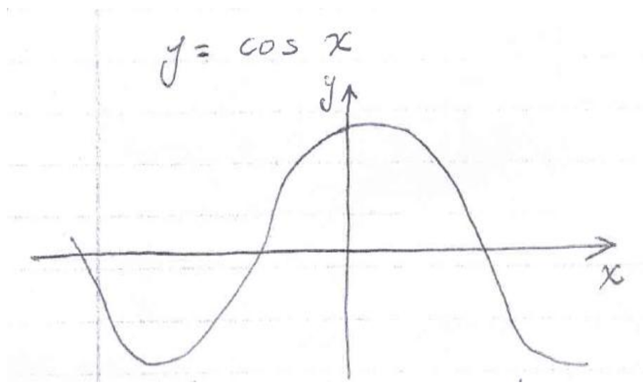
4. Absolute value function

This is a function that only considers magnitude e.g. $f(x) = |x - 2|$

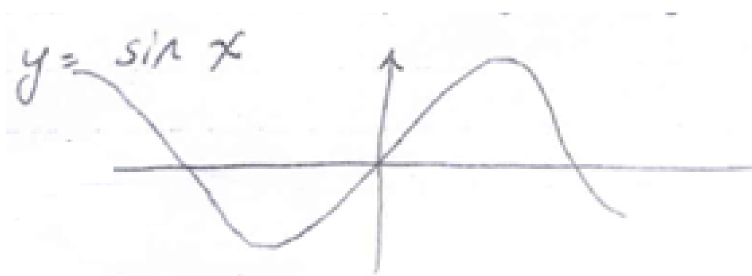
5. Odd and even functions

A function y =(x) is said to be an *even function* if $f(x) = f(-x)$. Examples are $y = x^2$ and $y = cosx$





A function y = f(x) is said to be *odd* if $-f(x) = f(-x)$, e.g. $y = sinx$



Examples

Identify each of the following as even, odd or neither:

(i) $y(x) = \frac{x^3+3x}{1-x^4}$

$$y(-x) = \frac{(-x)^3 + 3(-x)}{1-(-x)^4} = \frac{-x^3 - 3x}{1 - x^4} = -y(x)$$

Since y(-x) = -y(x), then the function is an odd function.

(ii) $f(x) = sin^2x$

$f(-x) = (\sin(-x))^2 = sin^2x = f(x) \Rightarrow Evenfunction$

(iii) $g(x) = \frac{tanx}{1+x^2}$

$g(-x) = \frac{\tan(-x)}{1+(-x)^2} = \frac{-tanx}{1+x^2} = -g(x)$; hence an odd function.

(iv) $f(x) = (1+x)^4$

$f(-x) = (1-x)^4 \neq \pm(1+x)^4$; hence function is neither odd nor even.

6. **Polynomial function**
   If the relationship is of the form $y = f(x) = A_n x^n + A_{n-1}x^{n-1} + \cdots \ldots \ldots . . A_1 x + A_0$ where n is a non – negative integer and A is a constant, then the given function is called a polynomial function. E.g. $y = 5x^4 + 7x^3 + x^2 - 3x + 8$
7. **Rational function**
   This is of the form $y = \frac{P(x)}{Q(x)}$ where $Q(x) \neq 0$ and $P(x) and\, Q(x)$ are both polynomial functions.
8. **Functions of several variables**
   If the domain, D, is a plane, then the function y = f(x, y) is a relationship that associates with each point (x , y) of the plane. E.g. $f(x, y) = xy + x^2y + 3y$
9. **Rounded function**
   It is a function whose numerical value is less than or equal to a certain given constant e.g. $Z = f(x, y)$ is rounded if there exists a constant K such that $\{K: f(x, y) \leq K\}$
10. **Inverse functions**
    If a function f(x) has a one to one mapping or correspondence in the open interval (a, b) and if each value f(x) of the interval corresponds to one value of x, then we can find the relationship g(x) such that g(x) is an inverse of f(x) and vice versa.

    Note

    The graphs of f(x) and g(x) are reflections of each other in the line y = x.

    Examples

    Find the inverse functions of the following:

    1. $f(x) = x^2$

       Letting $y = f(x) = x^2$

       Interchanging variables, gives $x = y^2$

       $\therefore y = \sqrt{x} \Rightarrow Inverse\ function, g(x) = \sqrt{x}$
    2. $f(x) = x^2 + 1$

       $y = x^2 + 1$

       $x = y^2 + 1 \Rightarrow y^2 = x - 1$

       $\therefore y = \pm\sqrt{x-1}$

       $Inverse\ function, g(x) = \sqrt{x-1}$
    3. $f(x) = \frac{1}{x^2+5}$

$$y = \frac{1}{x^2 + 5}$$

$$x = \frac{1}{y^2 + 5} \Rightarrow y^2 + 5 = \frac{1}{x}$$

$$y = \sqrt{\frac{1 - 5x}{x}}$$

$$\therefore g(x) = \sqrt{\frac{1 - 5x}{x}}$$

4. $f(x) = \frac{x-1}{2x+3}$ Ans: $g(x) = \frac{3x+1}{1-2x}$
5. $f(x) = x^2 + 2x$; Ans: g(x) = $\sqrt{x+1} - 1$

11. Inverse trigonometric functions

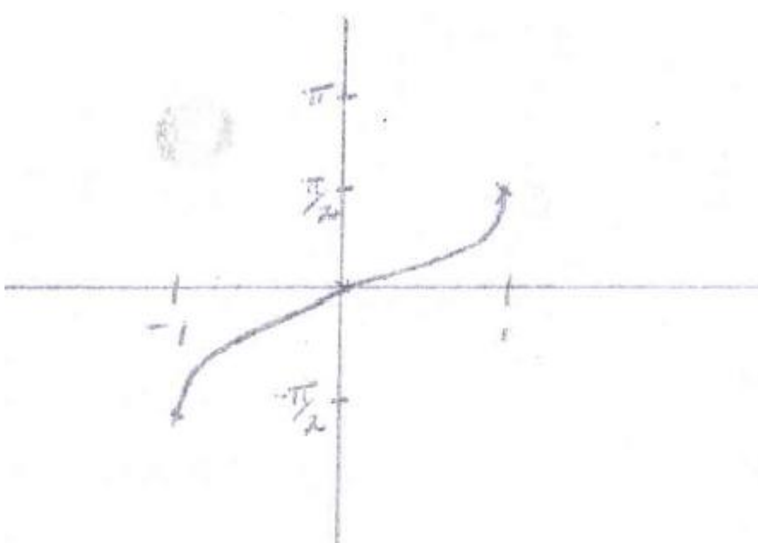
The inverse function of the trigonometric function;

$y = f(x) = sinx$

$g(x) = arcsinx \ or \ sin^{-1}x$

The inverse sine function is defined as y = $\sin^{-1}$x if and only if siny = x where $-1 \leq x \leq 1$

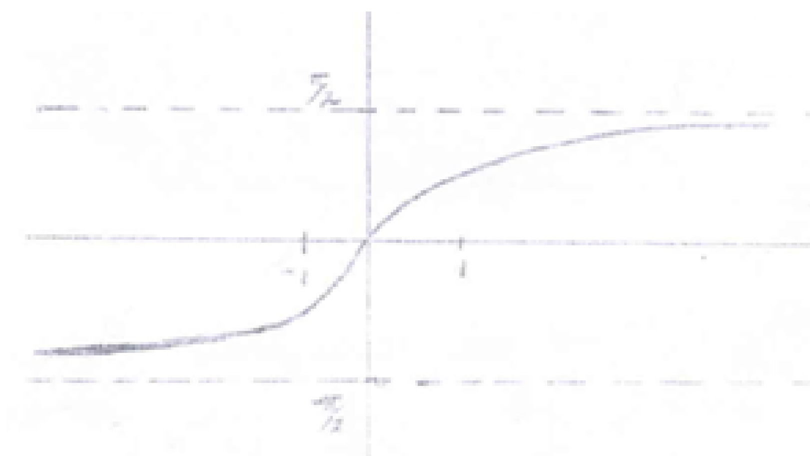
Graph of $y = sin^{-1}x$



Graph of $y = cos^{-1}x$

The inverse cosine function is defined as y = $\cos^{-1}$x if and only if cosy = x and $-1 \leq x \leq 1$

Graph of $y = tan^{-1}x$



The inverse tangent function is defined as y = $\tan^{-1}$x.

LIMITS AND CONTINUITY

Investigation of limits of functions

1. Consider the function $y = x^2 + 3$; as we get closer to $x = 2$;

X	Y
2.0500	7.2025
2.0100	7.0401
2.0010	7.0040
2.0000	7.0000
1.9990	6.9960
1.9900	6.9601
1.9000	6.6100

The closer we get to 2, the closer *y* gets to 7.0

2. Let us now investigate $y = f(x) = \frac{x^2-1}{x-1}$ as x tends to 1

x	y
1.0100	2.0100
1.0001	2.0001
1.0000	?
0.9990	1.9990
0.9900	1.9900

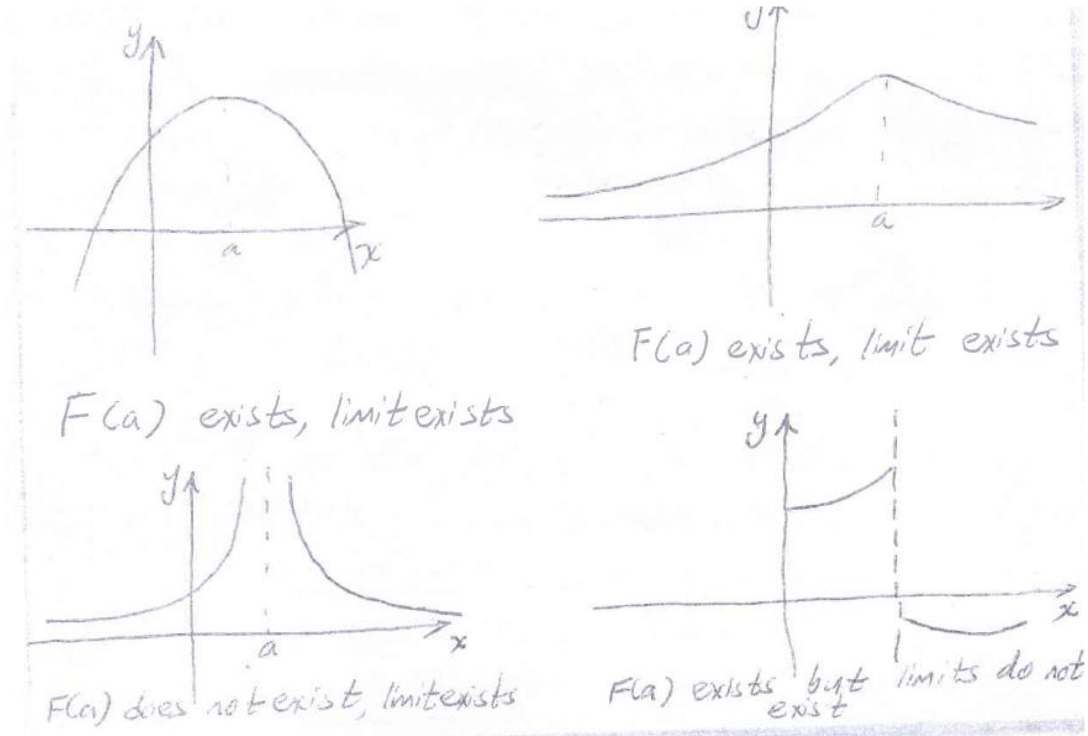
Hence the function tends to 2 as x tends to 1.0; however the function does not exist at x = 1

Definition

Let L be a real number and let f(x) be defined in an open interval containing a constant a, but not necessarily, then; $\lim_{x \to a} f(x) = L$.

Condition for the limit to exist

To discuss the limit of *f* at *a* does not require that *f(x)* be defined at *a*. Neither its value there nor the possibility that it has no value at *a*, is of importance. All we need is that f(x) is defined in some deleted neighbourhood of *a* i.e. it is necessary that f(x) be defined on both sides of a point, *a*, for the limit to exist. If f(x) is undefined on one side or if f(x) is discontinuous at $x = a$, the limit does not exist.



Finding limits

1. Given a function f(x) defined at a point, a, then $\lim_{x\to a} f(x) = f(a)$

   Example; if $fx) = \frac{x+4}{2x+1}$, then $\lim_{x\to 3} f(x) = \lim_{x\to 3}\frac{x+4}{2x+1} = f(3) = \frac{7}{7} = 1$

2. If $f(x) = \frac{(x-a)^{\propto}g(x)}{(x-a)^{\propto}h(x)}$; then *f(a)* is undefined, but $f(x) = \frac{g(x)}{h(x)}$ for all $x \neq a$

   Therefore, $\lim_{x\to a} f(x) = \lim_{x\to a}\frac{g(x)}{h(x)}$. This means that limits are concerned with values of x close to, *a*, but different from, *a*.

   If the expression is a quotient of two functions, any common factors should be eliminated from the numerator and denominator.

Examples

Find the following limits:

1. $\lim_{x\to 3}\frac{x^2-9}{x^2-x-6}$

   f(3) is not defined but the limit exists; (*Ans: 6/5*)

2. $\lim_{x\to 0}\frac{1+\frac{1}{x}}{1-\frac{1}{x}}$

   $$\lim_{x\to 0}\frac{1+\frac{1}{x}}{1-\frac{1}{x}}=\lim_{x\to 0}\frac{(x+1)\cancel{/x}}{(x-1)\cancel{/x}}=\lim_{x\to 0}\frac{x+1}{x-1}=-1$$

Limits laws

1. **Constant law**

   $\lim_{x\to a}C=C$, (C being a constant)

Note: in the following three laws, assume that, $\lim_{x\to a}f(x)=L$ and $\lim_{x\to a}g(x)=M$

2. **Additional law**

   $\lim_{x\to a}\left[f(x)\pm g(x)\right]=L\pm M$

   The limit of the sum is the sum of the limits and the limit of the difference is the difference of the limits.

3. **Product law**

   $\lim_{x\to a}\left[f(x)g(x)\right]=LM$

   The limit of the product of two or more functions is the product of their limits.

4. **Quotient law**

   $\lim_{x\to a}\frac{f(x)}{g(x)}=\frac{L}{M}$ if $M\neq 0$

   The limit of the quotient of two functions is the quotient of the limits, provided that the limit of the denominator is not zero.

5. **Root law**

$\lim\limits_{x \to a} \sqrt[n]{x} = \sqrt[n]{a}$ n is a positive integer and a > 0 if n is even.

(i) $\lim\limits_{x \to a} x^n = a^n$

(ii) $\lim\limits_{x \to a} [f(x)]^n = \left[\lim\limits_{x \to a} f(x)\right]^n = L^n$

Examples

Find the following limits

1. $\lim\limits_{x \to 3} (x^2 + 2x + 4)$

$$\lim_{x \to 3} (x^2 + 2x + 4) = \lim_{x \to 3} x^2 + 2\lim_{x \to 3} x + \lim_{x \to 3} 4$$
$$= (3)^2 + 2(3) + 4$$
$$= 19$$

2. $\lim\limits_{x \to 3} \dfrac{2x+5}{x^2+2x+4}$

$$\lim_{x \to 3} \frac{2x+5}{x^2+2x+4} = \frac{2(\lim\limits_{x \to 3} x) + \lim\limits_{x \to 3} 5}{\lim\limits_{x \to 3}(x^2+2x+4)}$$
$$= \frac{6+5}{19}$$
$$= \frac{11}{19}$$

3. $\lim\limits_{x \to 4} \dfrac{x^2+3x}{3x+2}$; *Ans: 2*

4. $\lim\limits_{x \to 2} \dfrac{x^2-4}{x^2+x-6}$; Ans 4/5

6. Substitution law

Suppose that $\lim\limits_{x \to a} g(x) = L$ and $\lim\limits_{x \to L} f(x) = f(L)$, then $\lim\limits_{x \to a} f(g(x)) = f\left(\lim\limits_{x \to a} g(x)\right) = f(L)$.

The condition under which the law holds is that the limit of the outer function *f* does not only exist at x = L but it is also equal to the value of the function, *f(L)* at x = L.

Special cases of the substitution law

(i) With $f(x) = x^{\frac{1}{n}}$ where n is a positive integer, then $\lim_{x \to a} \sqrt[n]{g(x)} = \sqrt[n]{\lim_{x \to a} g(x)}$. This holds under the assumption that the limit of g(x) exists as *x tends to a* (and is positive if n is even).

(ii) With $g(x) = x^m$, where m is a positive integer, the equation above becomes $\lim_{x \to a} x^{m/n} = a^{m/n}$ assuming a > 0. These equations may be regarded as generalized root laws.

Examples

Find the following limits;

(i) $\lim_{x \to -4} \sqrt{x^2 + 9}$

This can be written as $\lim_{x \to -4} f(g(x))$, with $f(x) = \sqrt{x}$ and $g(x) = x^2 + 9$

$\lim_{x \to -4} (x^2 + 9) = 25$

Hence, $\lim_{x \to -4} \sqrt{x^2 + 9} = \lim_{x \to 25} \sqrt{x} = 5$

Alternatively, this can be easily evaluated as below;

$\lim_{x \to -4} \sqrt{x^2 + 9} = \sqrt{\lim_{x \to -4} (x^2 + 9)} = \sqrt{(16 + 9)} = 5$

(ii) $\lim_{x\to 4}\left(3x^{3/2}+20\sqrt{x}\right)^{1/3}$

$$=\left[\lim_{x\to 4}\left(3x^{3/2}+20\sqrt{x}\right)\right]^{1/3}=\left[3\lim_{x\to 4}x^{3/2}+20\lim_{x\to 4}\sqrt{x}\right]^{1/3}$$

$$=\left[3(4)^{3/2}+20\sqrt{4}\right]^{1/3}=\sqrt[3]{64}=4$$

7. Squeeze law

Suppose that $f(x) \leq g(x) \leq h(x)$ in some deleted neighborhood of a and also that, $\lim_{x\to a} f(x) = L = \lim_{x\to a} h(x)$, then $\lim_{x\to a} g(x) = L$ as well.

Examples

1. Apply the squeeze law to prove that $\lim_{x\to 0}\frac{x^2}{1+(1+x^4)^{5/2}}=0$

   Since $1+(1+x^4)^{5/2} \geq 1$ for all x, we see that,

   $0 \leq \frac{x^2}{1+(1+x^4)^{5/2}} \leq x^2$, for all values of x

   We can now apply the squeeze law with $f(x) \equiv 0$ and$h(x) \equiv x^2$. Because $\lim_{x\to 0} f(x) = \lim_{x\to 0} h(x) = 0$, it follows that the limit as x tends to zero of the trapped function,

   $$\lim_{x\to 0}\frac{x^2}{1+(1+x^4)^{5/2}}=0$$

2. Using the squeeze law show that, $\lim_{x\to 0}\frac{|x|}{1+x^2}=0$

   We can take, $f(x)=\frac{|x|}{(1+x^2)^2}$ and $h(x)=|x|$

   $$\lim_{x\to 0} f(x)=\frac{|x|}{\left(1+x^2\right)^2}=\frac{|0|}{1^2}=0 \text{ and } \lim_{x\to 0} h(x)=\lim_{x\to 0}|x|=0$$

   From the squeeze law, since $f(x) \leq g(x) \leq h(x)$ and $\lim_{x\to 0} f(x) = \lim_{x\to 0} h(x) = 0$

   It implies that, $\lim_{x\to 0} g(x) = \lim_{x\to 0}\frac{|x|}{1+x^2}=0$

3. Find $\lim_{x\to 0}\frac{x^2}{(1+x^4)^{1/2}}$ using the squeeze law.

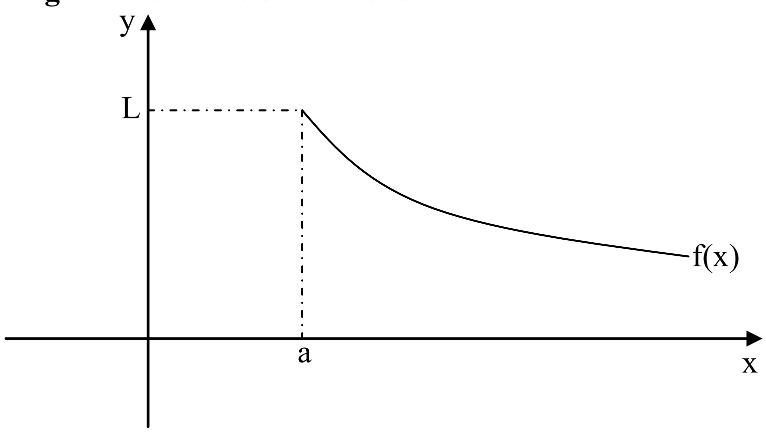
   We can take, $f(x)=\frac{x^2}{1+x^4}$ and $h(x)=x^2$

$$\lim_{x\to 0} f(x) = \lim_{x\to 0}\frac{x^2}{(1+x^4)} = 0 \text{ and } \lim_{x\to 0} h(x) = \lim_{x\to 0} x^2 = 0$$

From the squeeze law, since $f(x) \le g(x) \le h(x)$ and $\lim_{x\to 0} f(x) = \lim_{x\to 0} h(x) = 0$, it follows that, $\lim_{x\to 0} g(x) = \lim_{x\to 0}\frac{x^2}{\left(1+x^4\right)^{1/2}} = 0$

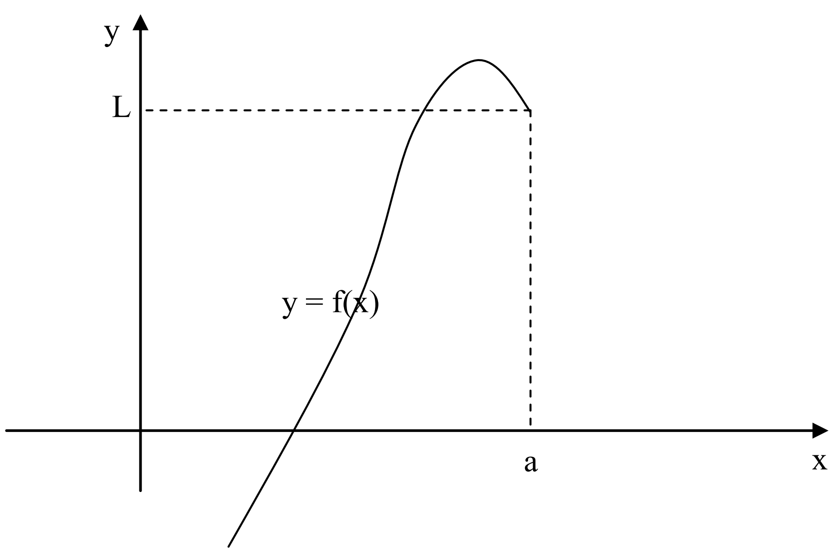
One sided limits

1. **Right hand limits of a function**



The right hand limit of *f(x)* is L suppose that *f* is defined on an open interval (a, c). Then we say that the number L is the right – hand limit of *f(x)* as x approaches, *a*, from the right and we write; $\lim_{x\to a^+} f(x) = L$, provided that the number *f(x)* can be made as close to L as one approached merely by choosing the point, *x*, in (a, c) sufficiently near the number, *a*.

2. **The left hand limit of a function**



Suppose that *f* is defined on an open interval (c, a). Then we say that the number, *L,* is the left – hand limit of *f(x)* as x approaches a, and we write this as; $\lim_{x\to a^-} f(x) = L$, provided that the number *f(x)* can be made as close to *L* as one pleases merely by choosing the point, *x,* in (c, a) sufficiently near the number, *a*.

Theorem (Conditional for existence of the limit as x approaches a)

Suppose that the function *f* is defined on the deleted neighbourhood of the point *a*. then the limit $\lim_{x\to a} f(x)$ exists and is equal to the number L if and only if the one – sided limits $\lim_{x\to a^+} f(x)$ and $\lim_{x\to a^-} f(x)$ both exist and both equal the number L.

Examples

1. Discuss the existence of the limit $\lim_{x\to 0} f(x)$, where;

   $$f(x) = \begin{cases} x^2+1 & \text{If } \ x<0 \\ x^2-1 & \text{If } \ x>0 \end{cases}$$

   Left hand limit

   $$\lim_{x\to 0^-} f(x) = \lim_{x\to 0}\left(x^2+1\right) = 1$$

   Right hand limit

   $$\lim_{x\to 0^+} f(x) = \lim_{x\to 0}\left(x^2-1\right) = -1$$

   Since $\lim_{x\to 0^-} f(x) \neq \lim_{x\to 0^+} f(x)$, the limit does not exist.

2. Find the following one sided limits;

   (i) $\lim_{x\to 0^+}\left(4+3x^{3/2}\right)$; (Ans: 4)

   (ii) $\lim_{x\to 0^+}\sqrt{\dfrac{4x}{x-4}}$ (Ans: Limit does not exist)

Limits at infinity and infinite limits

1. **Limits at infinity**

   If *f(x)* is defined for large values of x and if $f(x) \to L$ as *x* increases without bound, then $\lim_{x\to\infty} f(x) = L$

   If the limit is taken to infinity all terms are divided by the largest power of *x,* the independent variable in the expression. This will leave at least one constant. Any quantity divided by a power of *x* vanishes as x tends to infinity.

Example

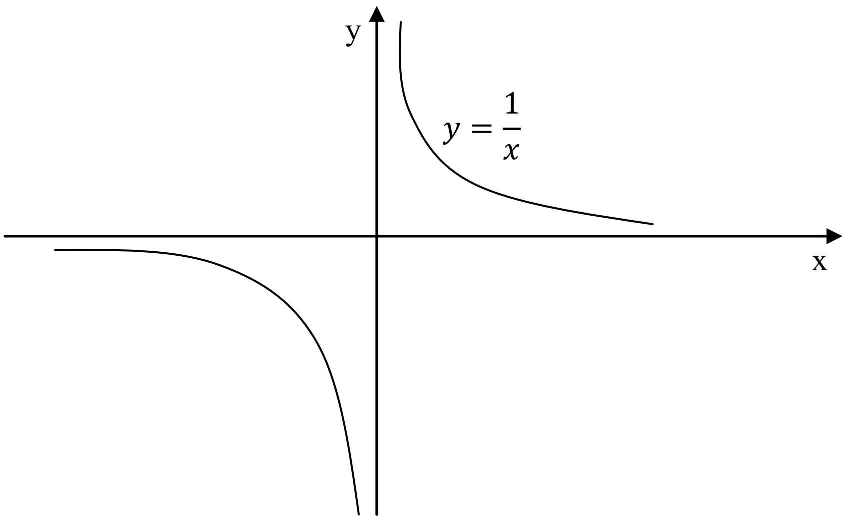
Find $\lim_{x\to\infty}\frac{2x^2-2x+1}{x^2+4x}$

$$\lim_{x\to\infty}\frac{2x^2-2x+1}{x^2+4x}=\lim_{x\to\infty}\frac{\frac{2x^2}{x^2}-\frac{2x}{x^2}+\frac{1}{x^2}}{\frac{x^2}{x^2}+\frac{4x}{x^2}}=\lim_{x\to\infty}\frac{2-\frac{2}{x}+\frac{1}{x^2}}{1+4/x}$$

$$=\lim_{x\to\infty}2=2$$

2. Infinite limits

Consider the function $f(x) = 1/x$

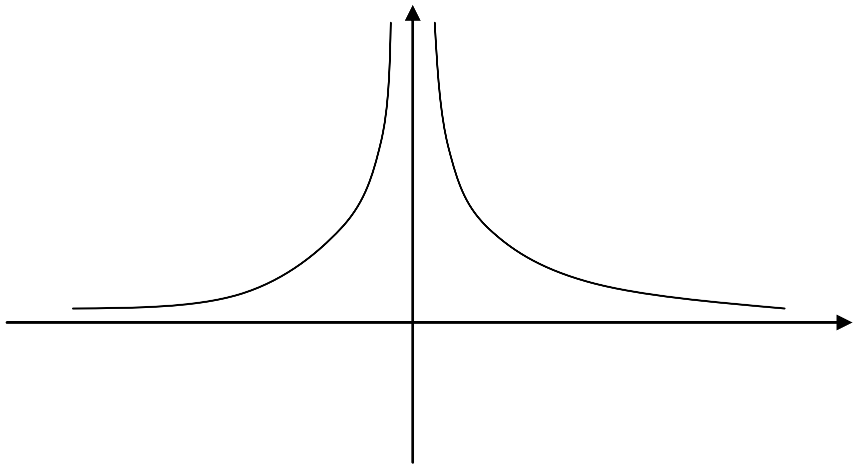


Note that 1/x increases without bound as x approaches zero from the right but decreases without bound as approaches zero from the left, so;

$$\lim_{x\to 0^-} f(x)=\lim_{x\to 0^-}\frac{1}{x}=-\infty \text{ (Does not exist)}$$

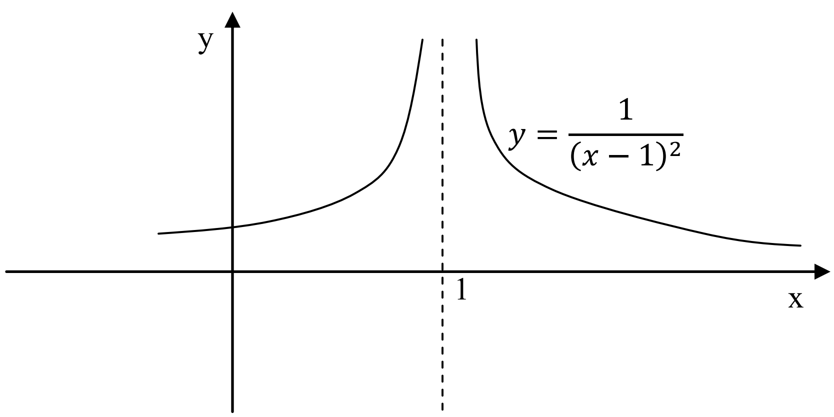
$$\lim_{x\to 0^+} f(x)=\lim_{x\to 0^+}\frac{1}{x}=+\infty \text{ (Does not exist)}$$

Let us now consider the function, $f(x) = \frac{1}{x^2}$



$\lim_{x \to 0^+} \frac{1}{x^2} = +\infty$ and $\lim_{x \to 0^-} \frac{1}{x^2} = +\infty$, the limit does not exist.

And even for, $f(x) = \frac{1}{(x-1)^2}$



$\lim_{x \to 1^+} \frac{1}{(x-1)^2} = +\infty$ and $\lim_{x \to 1^-} \frac{1}{(x-1)^2} = +\infty$

L'Hopital's Rule

This is used when the numerator and denominator of the expression both approach zero (0) or infinity as x tends to a value a.

If $f(x) = \frac{h(x)}{g(x)}$ and $\lim_{x \to a} \frac{h(x)}{g(x)} = \frac{0}{0} \, or \, \frac{\infty}{\infty}$, then;

$$\lim_{x \to a} f(x) = \lim_{x \to a} \frac{h(x)}{g(x)} = \lim_{x \to a} \frac{h^1(x)}{g^1(x)} = \lim_{x \to a} \frac{h^{11}(x)}{g^{11}(x)} = \ldots\ldots = \lim_{x \to a} \frac{h^n(x)}{g^n(x)}$$

Examples

Find the following limits:

(i) $\lim_{x\to 0}\left(\frac{e^x+e^{-x}-2}{2cos2x-2}\right)$

Since $\lim_{x\to 0}\frac{e^x+e^{-x}-2}{2\cos 2x-2}=\frac{0}{0}$, then by L'Hopital's rule;

$\lim_{x\to 0}\frac{e^x+e^{-x}-2}{2\cos 2x-2}=\lim_{x\to 0}\frac{e^x-e^{-x}}{-4\sin 2x}=\frac{0}{0}$. Applying L'Hopital's rule again;

$$\lim_{x\to 0}\frac{e^x+e^{-x}-2}{2\cos 2x-2}=\lim_{x\to 0}\frac{e^x+e^{-x}}{-8\cos 2x}=\frac{2}{-8}=-\frac{1}{4}$$

(ii) $\lim_{x\to 0}\frac{a-\sqrt{a^2-x^2}}{x^2}$

Since, $\lim_{x\to 0}\frac{a-\sqrt{a^2-x^2}}{x^2}=\lim_{x\to 0}\frac{a-\sqrt{a^2-0}}{0}=\frac{0}{0}$, then by L'Hopital's rule;

$$\lim_{x\to 0}\frac{a-\sqrt{a^2-x^2}}{x^2}=\lim_{x\to 0}\frac{\frac{1}{2}\left(a^2-x^2\right)^{-1/2}(-2x)}{2x}=\lim_{x\to 0}\frac{1}{2\sqrt{a^2-x^2}}=\frac{1}{2\sqrt{a^2}}=\frac{1}{2a}$$

Continuity of the function at a point

Definition

Suppose that the function, *f*, is defined in a neighbourhood of, *a*. we say that, *f*, is continuous at, *a*, provided that $\lim_{x\to a} f(x)$ exists and moreover, that the value of this limit is *f (a)*.

Therefore, the conditions for the function to be continuous at a point $x = a$ are:

(i) *f(x)* must be defined at $x = a$ i.e. *f(a)* must exist.
(ii) The limit of *f(x)* as *x* approaches *a* must exist i.e. $\lim_{x\to a} f(x)$ must exist
(iii) The limit $\lim_{x\to a} f(x)$ must be equal to *f(a)*

If any of these conditions is not satisfied, then the function *f(x)* is not continuous at, *a*, i.e. *f(x)* is discontinuous at *a*.

Examples

1. Discuss the continuity of f(x), given;

$$f(x)=\begin{cases} x^2+4; & x<2 \\ x^3; & x\geq 2 \end{cases}$$

Solution

<u>For *x < 2*</u>

$f(2) = 2^2 + 4 = 8$

$\lim_{x \to 2^-} f(x) = \lim_{x \to 2}(x^2 + 4) = 8$, hence, $\lim_{x \to 2^-} f(x) = f(2) = 8$

For $x \geq 2$

$f(2) = 2^3 = 8$

$\lim_{x \to 2^+} f(x) = \lim_{x \to 2} x^3 = 8$, hence, $\lim_{x \to 2^+} f(x) = f(2) = 8$

Therefore, the function *f(x)* is continuous everywhere.

2. Discuss the continuity of the function; $f(x) = \begin{cases} \dfrac{1}{(x-3)(x+2)}; & x < 0 \\ \dfrac{1}{(x-2)(x+3)}; & x \geq 0 \end{cases}$

For x < 0

$$f(x) = \frac{1}{(x-3)(x+2)}$$

Therefore, $f(0) = \frac{1}{(-3)(2)} = -\frac{1}{6}$

$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^+} \frac{1}{(x-3)(x+2)} = -\frac{1}{6}$$

Therefore, $\lim_{x \to 0^-} f(x) = f(0) = -\dfrac{1}{6}$

For $x \geq 0$

$$f(x) = \frac{1}{(x-2)(x+3)}$$

Hence, $f(0) = \frac{1}{(-2)(3)} = -\frac{1}{6}$

$$\lim_{x \to 0^+} f(x) = \lim_{x \to 0} \frac{1}{(x-2)(x+3)} = -\frac{1}{6}$$

Therefore, $\lim_{x \to 0^+} f(x) = f(0) = -\dfrac{1}{6}$

Thus the function *f(x)* is continuous everywhere including the transition point $x = 0$, but it is discontinuous at $x = -2$ and $x = 2$

Theorems of continuity

Theorem 1: Continuity of compositions

The composition of two continuous functions is itself continuous. More precisely, if *g* is itself continuous at *a* and *f* is continuous at *g(a)*, then *f(g(x))* is continuous at *a*.

Examples

Find the following limits

1. $\lim_{x\to 0}\sin(x^2+\pi)$

$g(x) = x^2 + \pi$ and $f(x) = sinx$

$$\lim_{x\to 0}\sin(x^2+\pi) = \sin\left(\lim_{x\to 0}(x^2+\pi)\right) = \sin\pi = 0$$

2. $\lim_{x\to 3}\left(\frac{x^2-9}{x-3}\right)^3$ (Ans: 216)

Theorem 2: Continuity of inverse functions

Let *f* be a continuous function with domain in the open interval I. suppose that *f* has an inverse function *g* with domain in the open interval J. then *g* is continuous at each point of J.

For example consider $f(x) = x^n$ (n is a positive integer). If *n* is odd, I is the whole real line$(-\infty, +\infty)$. If n is even, I is the positive half – line$(0, +\infty)$. Then *f(x)* has its inverse function, $(x) = \sqrt[n]{x}$. The domain of *g* is in the interval J, where $J = (-\infty, +\infty)$ if n is odd and $J = (0, +\infty)$ if n is even.

Since $f(x) = x^n$ is continuous on I, theorem 2 implies that *g(x)* is continuous on J. Thus $\lim_{x\to a}\sqrt[n]{x} = \sqrt[n]{a}$

With the stipulation that $a > 0$ if *n* is even, this proves the root law.

Continuous functions on closed intervals

The function *f(x)* defined on the closed interval $[a, b]$ is said to be continuous on the interval provided that it is continuous at each point of the open interval (a, b) and that $\lim_{x\to a^+} f(x) = f(a)$ and that $\lim_{x\to a^-} f(x) = f(b)$.

The last two conditions mean that at each end point, the value of the function is equal to its limit from within the interval.

Continuous functions defined on closed intervals are very special. Such functions have maximum value property and intermediate value property.

Definition: maximum and minimum value

If c is in the interval$[a, b]$, then *f(c*) is called the maximum value of f(x) on$[a, b]$ if $f(c) \geq f(x)$ for all x in$[a, b]$. Similarly for the value d in$[a, b]$, *f(d)* is called minimum value of *f(x*) if $f(d) \leq f(x)$ for all x in $[a, b]$.

Theorem 3: Maximum value property

If *f(x)* is continuous on the closed interval$[a, b]$, then there exists a number c in $[a, b]$ such that *f(c)* is the maximum value of f on$[a, b]$.

Theorem 4: Intermediate value property

Suppose that the function *f* is continuous on the closed interval$[a, b]$. Then *f(x)* assumes every intermediate value between *f(a)* and *f(b)*. Thus, if γ is any number between *f(a)* and *f(b)*, then there exists at least one point c in $[a, b]$ such that *f(c) = γ*.

Piece – wise continuous

A function is said to be *piece – wise continuous* in an interval $[a, b]$ if that interval can be sub – divided into finite number of smaller intervals in each of which the function is continuous and has finite left and right values.

Trigonometric functions and their limit

$$\lim_{\theta\to 0}\cos\theta = 1$$

$$\lim_{\theta\to 0}\sin\theta = 0$$

$$\lim_{\theta\to 0}\tan\theta = 0$$

Theorem 1: Continuity of sine and cosine

The functions *sinx* and *cosx* are continuous functions of x on the whole real line.

Theorem 2: The basic trigonometric limit

$$\lim_{x\to 0}\frac{\sin\theta}{\theta} = 1$$

Examples

1. Show that $\lim_{x\to 0}\frac{1-\cos\theta}{\theta} = 0$

   Solution

$$\lim_{x\to 0}\frac{1-\cos\theta}{\theta} = \lim_{x\to 0}\frac{1-\cos\theta}{\theta}.\frac{1+\cos\theta}{1+\cos\theta}$$

$$= \lim_{\theta\to 0}\frac{\sin^2\theta}{\theta(1+\cos\theta)} = \lim_{\theta\to 0}\frac{\sin\theta}{\theta}\left(\lim_{\theta\to 0}\frac{\sin\theta}{1+\cos\theta}\right)$$

$$= \lim_{\theta\to 0}(1)\left(\frac{0}{1+1}\right) = 0$$

2. Evaluate $\lim_{x\to 0}\frac{\tan 3x}{x}$

$$\lim_{x\to 0}\frac{\tan 3x}{x} = \lim_{x\to 0}\frac{3\tan 3x}{3x} = 3\lim_{x\to 0}\frac{\tan 3x}{x}$$

   Letting $3x = \theta$

$$\therefore \lim_{x\to 0}\frac{\tan 3x}{x} = 3\lim_{\theta\to 0}\frac{\tan\theta}{\theta} = 3\left(\lim_{\theta\to 0}\frac{\sin\theta}{\theta}\right)\left(\lim_{\theta\to 0}\frac{1}{\cos\theta}\right)$$

$$= (3)(1)\left(\frac{1}{1}\right) = 3$$

SEQUENCES

A sequence is the one that can be written in the form, $s_n = f(n)$ where n is a positive integer.

e.g. $\{s_n\} = s_1, s_2, s_3, \dots \dots \dots \dots \dots . . s_n$, this is a finite sequence.

$\{s_n\} = s_1, s_2, s_3, \dots \dots . s_n, \dots \dots \dots \dots$, this is an infinite sequence.

Examples

1. $a_n = \frac{1}{2}n$

   Therefore, $\{a_n\} = \left\{\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \dots \dots \dots \dots \dots \dots .\right\}$

2. $a_n = 2n + 1$

   Then $\{a_n\} = \{3,5,7,9,11, \dots \dots \dots \dots \dots . \dots . .\}$

Since a sequence is a function, it can be represented as a graph, but a smooth or continuous curve cannot be drawn since a sequence is only defined for integers and not for values in between i.e. it is defined for discrete values.

Limits of sequences

If a sequence S_n has a finite limit S for a considerable large value of n, then we say; $\lim_{n\to\infty} S_n = S$, then the sequence is said to be *convergent* to S as n tends to infinity.

A sequence is said to be *divergent* if it has an infinite limit, i.e. when $\lim_{n\to\infty} S_n = \infty$

Examples

1. $\lim_{n\to\infty} 2n+1 = \infty$ - divergent sequence

2. $\lim_{n\to\infty} \frac{1}{n} = 0$ - convergent sequence

Properties of limits of sequences

1. Suppose the limit $\lim_{x\to\infty} f(x) = L$ and if it is also defined for a positive integer n, i.e. $S_n = f(n)$

   $\lim_{n\to\infty} S_n = \lim_{x\to\infty} f(x) = L$

Example

1. If $a_n = \frac{3n^2+5}{n^2} \Rightarrow f(x) = \frac{3x^2+5}{x^2}$

$$\lim_{n\to\infty} S_n = \lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{3x^2+5}{x^2} = \lim_{x\to\infty} \frac{3+\frac{5}{x^2}}{1} = 3$$

$$\therefore \lim_{n\to\infty} S_n = \lim_{x\to\infty} \frac{3n^2+5}{n^2} = 3$$

2. Let r be a real number, then

$$\therefore \lim_{n\to\infty} r^n = \begin{cases} 0 & |r| < 1 \\ \infty & |r| > 1 \end{cases}$$

3. If $\lim_{n\to\infty} a_n = L$ and $\lim_{n\to\infty} b_n = M$, then;

   (i) $\therefore \lim_{n\to\infty} (a_n \pm b_n) = L \pm M$

   (ii) $\lim_{n\to\infty} Ca_n = C.L$

   (iii) $\lim_{n\to\infty} a_n b_n = L.M$

   (iv) $\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{L}{M}$

4. If $\lim_{n\to\infty} a_n = A$ and f(x) is continuous at f(x) = A; then $\lim_{n\to\infty} f(a_n) = f(A)$

Example

Find $\lim_{n\to\infty} \tan\left[\frac{\pi n^2+1}{3-4n^2}\right]$

$$\lim_{n\to\infty} \tan\left[\frac{\pi n^2+1}{3-4n^2}\right] = \tan \lim_{n\to\infty} \left[\frac{\pi n^2+1}{3-4n^2}\right] = \tan \lim_{n\to\infty} \left[\frac{\pi + \frac{1}{n^2}}{\frac{3}{n^2} - 4}\right]$$

$$= \tan\left(-\frac{\pi}{4}\right) = -1$$

5. **Squeeze law for sequences**

   If a_n, b_n, and c_n are general terms of sequences for all n and $a_n < b_n < c_n$; and $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$, then $\lim_{n\to\infty} b_n = L$ also.

   Example

   Evaluate $\lim_{n\to\infty} \frac{\sin n}{n}$ using the squeeze law.

   We can take $a_n = \frac{sinn}{n^2}$ and $c_n = sinn$

   $\lim_{n\to\infty} \frac{\sin n}{n^2} = 0$ and $\lim_{n\to\infty} \sin n = 0$. By the squeeze law, since $\frac{sinn}{n^2} \leq \frac{sinn}{n} \leq sinn$, then $\lim_{n\to\infty} \frac{\sin n}{n} = 0$ also.

6. **L'Hopital's rule**

   If $a_n = f(n)$ and $b_n = g(n)$, and if $\lim_{n\to\infty} \frac{f(n)}{g(n)} = \frac{0}{0} or \frac{\infty}{\infty}$, then;

   $$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{f(n)}{g(n)} = \lim_{n\to\infty} \frac{f^1(n)}{g^1(n)} = \lim_{n\to\infty} \frac{f^{11}(n)}{g^{11}(n)} = \ldots\ldots\ldots\ldots \lim_{n\to\infty} \frac{f^n(n)}{g^n(n)}$$

Examples

1. Evaluate $\lim_{n\to\infty} \frac{3n^2}{e^{2n}}$

   $\lim_{n\to\infty} \frac{3n^2}{e^{2n}} = {}^{\infty}\!/_{\infty}$ Applying L'Hopital's rule,

   $\lim_{n\to\infty} \frac{3n^2}{e^{2n}} = \lim_{n\to\infty} \frac{6n}{2e^{2n}} = \frac{\infty}{\infty}$, Applying L'Hopital's rule again;

   $$= \lim_{n\to\infty} \frac{6}{4e^{2n}} = \frac{6}{\infty} = 0$$

2. Find the $\lim_{n\to\infty} \frac{e^n}{n^2}$ (Ans: divergent)

3. Prove that the limit of $\{a_n\} = \left\{\left(1 + \frac{1}{n}\right)^n\right\} = e$

   $$\{a_n\} = \left\{\left(1 + \frac{1}{n}\right)^n\right\} = \left(\frac{n+1}{n}\right)^n$$

Let $y = \left(\frac{n+1}{n}\right)^n \Rightarrow lny = nln\left(\frac{n+1}{n}\right) = \frac{ln\left(\frac{n+1}{n}\right)}{\frac{1}{n}}$

$$\therefore \lim_{n\to\infty}\ln y = \lim_{n\to\infty}\frac{\ln\left(\frac{n+1}{n}\right)}{1/n} = \frac{0}{0}$$

Considering the RHS; Applying L'Hopital's rule;

$$\lim_{n\to\infty}\frac{\ln\left(\frac{n+1}{n}\right)}{1/n} = \lim_{n\to\infty}\frac{\frac{1}{n+1}-\frac{1}{n}}{-\frac{1}{n^2}} = \lim_{n\to\infty}\frac{\frac{-1}{n(n+1)}}{-\frac{1}{n^2}} = \lim_{n\to\infty}\frac{n}{n+1}$$

$$= \lim_{n\to\infty}\frac{1}{1+\frac{1}{n}} = 1$$

$$\therefore \ln\left(\lim_{n\to\infty} y\right) = 1 \Rightarrow \left(\lim_{n\to\infty} y\right) = e$$

Thus, $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e$

4. Find $\lim_{n\to\infty}(1+n)^{\frac{1}{n}}$ (Ans: 1.0)

Exponential, hyperbolic and logarithmic functions of a real variable

Definition: The natural logarithm

The natural logarithm *lnx* of the positive number *x* is defined to be $lnx = \int_1^x \frac{1}{t}dt$

Note that *lnx* is not defined for $x \leq 0$. Geometrically, *lnx* is the area under the graph of $y = \frac{1}{t}$ from *t = 1* to *t = x*.

The exponential function

The natural logarithm function *lnx* is continuous and increasing for *x > 0*, and it attains large positive and negative values. It follows that *lnx* has an inverse function that is defined for all *x*. because *lnx* is an increasing function, there is only one such number *x* with *ln x = y*. This inverse function to *lnx* is called the natural *exponential function*.

Definition: The natural exponential function

The natural exponential function is defined for all x by; $e^x = y$ if and only if $lny = x$. Thus e^x is simply that positive number y whose natural logarithm is x. It is an immediate consequence of that $ln(e^x) = x$ for all x.

Laws of logarithms

If x and y are positive numbers and r is a rational number, then;

$$\ln xy = \ln x + \ln y$$

$$\ln\left(\frac{1}{x}\right) = -\ln x$$

$$\ln\left(\frac{x}{y}\right) = \ln x - \ln y$$

$$\ln\left(x^r\right) = r\ln x$$

Laws of exponentials

If x, x_1, and x_2 are real numbers and r is a rational, then;

$$e^{x_1}e^{x_2} = e^{x_1+x_2}$$

$$e^{-x} = \frac{1}{e^x}$$

$$(e^x)^r = e^{rx}$$

"Recall trigonometric functions"

Hyperbolic functions

The hyperbolic cosine and the hyperbolic sine of real number x are denoted by *coshx* and *sinhx* respectively and are defined as;

$$\cosh x = \frac{e^x + e^{-x}}{2} \text{ and } \sinh x = \frac{e^x - e^{-x}}{2}$$

The particular combinations of familiar exponentials are useful in certain applications of calculus and are also helpful in evaluation of certain integrals.

$$\tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

$$\coth x = \frac{\cosh x}{\sinh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}} \text{ (for } x \neq 0\text{)}$$

$$\sec hx = \frac{1}{\cosh x} = \frac{2}{e^x + e^{-x}}$$

$$\operatorname{cosech} x = \frac{1}{\sinh x} = \frac{2}{e^x - e^{-x}} \text{ (for } x \neq 0)$$

Recall the following identities

- $\cosh^2 x - \sinh^2 x = 1$
- $1 - \tanh^2 x = \sec h^2 x$
- $\coth^2 x - 1 = \operatorname{cosech}^2 x$
- $\sinh(x+y) = \sinh x \cosh y + \cosh x \sinh y$
- $\cosh(x+y) = \cosh x \cosh y + \cosh x \sinh y$
- $\sinh 2x = 2 \sinh x \cosh x$
- $\cosh 2x = \cosh^2 x + \sinh^2 x$

Example

Prove that $\cosh^2 x - \sinh^2 x = 1$

From, $\cosh x = \dfrac{e^x + e^{-x}}{2}$ and $\sinh x = \dfrac{e^x - e^{-x}}{2}$

$$\Rightarrow \cosh^2 x - \sinh^2 x = \frac{1}{4}\left(e^x + e^{-x}\right)^2 - \frac{1}{4}\left(e^x - e^{-x}\right)^2 = \frac{1}{4}\left(e^{2x} + 2 + e^{-2x}\right) - \frac{1}{4}\left(e^{2x} - 2 + e^{-2x}\right)$$

$$= \frac{e^{2x}}{4} + \frac{1}{2} + \frac{e^{-2x}}{4} - \frac{e^{2x}}{4} + \frac{1}{2} - \frac{e^{-2x}}{4} = 1$$

Note:

(i) The other identities can be derived similarly.

(ii) The identity $\cosh^2 \theta - \sinh^2 \theta = 1$ tells us that the point *(coshθ, sinhθ)*, lies on the hyperbola $x^2 - y^2 = 1$, and this is the reason for the name “hyperbolic functions”.

Question

Prove the following expressions:

(i) $sinh^{-1}x = ln\left(x + \sqrt{x^2 + 1}\right)$

(ii) $cosh^{-1}x = ln\left(x + \sqrt{x^2 - 1}\right)$

(iii) $tanh^{-1}x = \frac{1}{2} ln\left(\frac{1+x}{1-x}\right)$

END