Chapter 1

Signals, Systems and Signal Processing

Chapter Outline

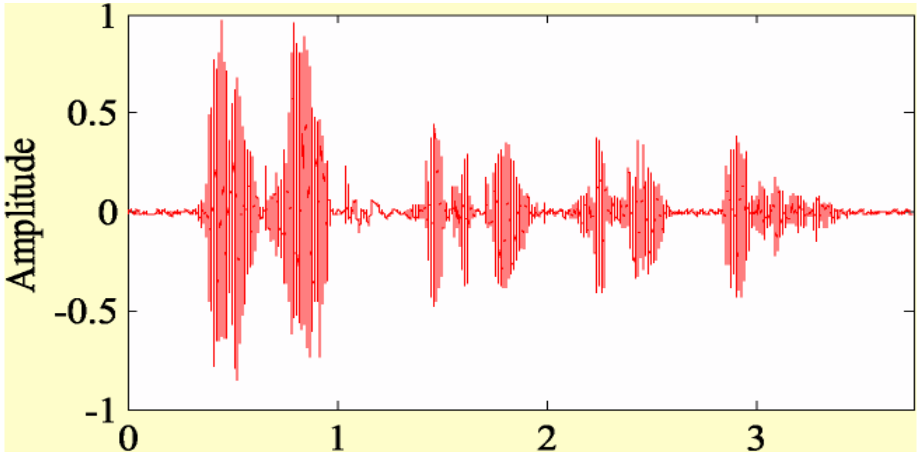
- Introduction
- Examples of signals
- Characteristics and classification of signals
- The Concept of Frequency in Continuous-Time and Discrete-Time Signals
- Analog-to-Digital and Digital-to-Analog Conversion

Introduction

- A *signal* is defined as any physical quantity that varies with time, space, or any other independent variable or variables. Mathematically , we describe a signal as a function of one or more independent variables. For example, the functions
- $s_1(t) = 5t \quad and \quad s_2(t) = 20t^2$
- Describe two signals, one that varies linearly with independent variable *t* (time) and a second that varies quadratically with *t.*
- $s(x, y) = 3x + 2xy + 10y^2$
- Is a signal of two independent variables x and y that could represent the two spatial coordinates in a plane.

Examples of signals

- **Speech and music signals:** Represent air pressure as a function of time at a point in space.
- Waveform of the speech signal *"I like digital signal processing"* is shown in figure below.



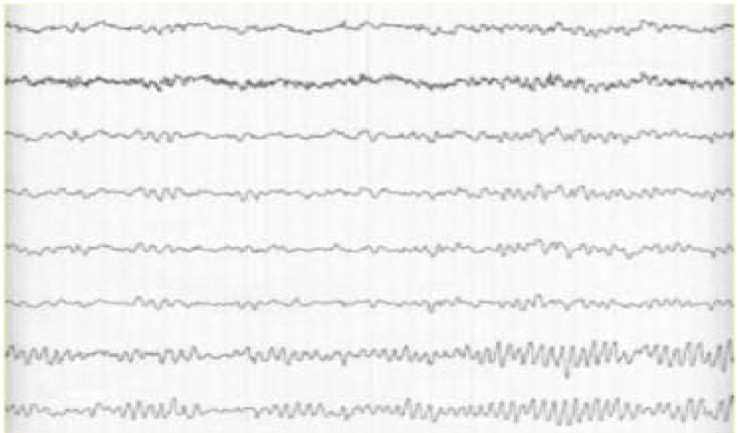
Examples of signals

- **Electrocardiography (ECG) Signal:** Represents the electrical activity of the heart.
- A typical ECG signal is shown in figure below.

- The ECG trace is a periodic waveform. One period of the waveform represents one cycle of the blood transfer process from the heart to the arteries.

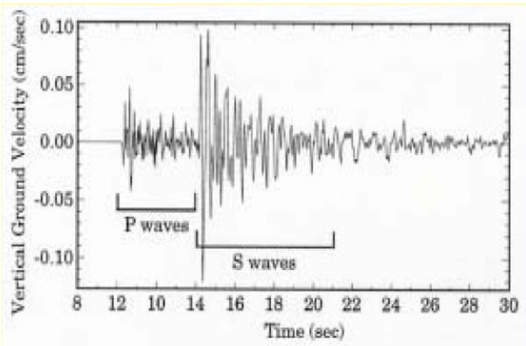
Examples of signals

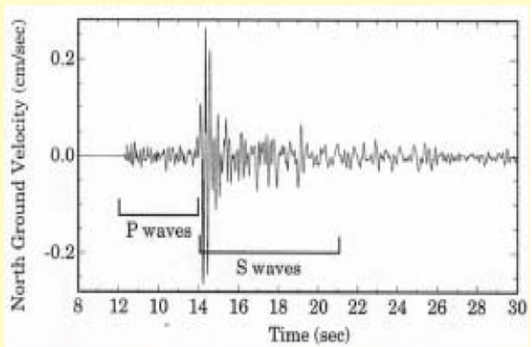
- **Electroencephalogram (EEG) Signals:** Represent the electrical activity caused by the random firings of billions of neurons in the brain. The typical EEG signal is shown in the figure below.

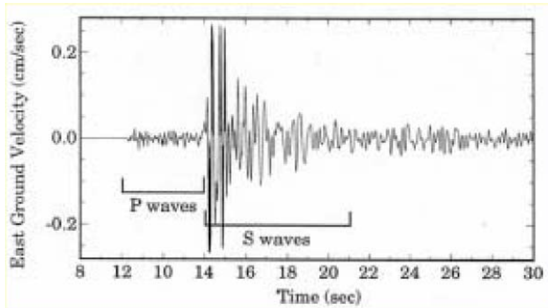


Examples of signals

- **Seismic Signals:** Caused by the movement of rocks resulting from an earthquake, a volcanic eruption, or an underground explosion.
- The ground movement generates 3 types of elastic waves that propagate through the body of the earth in all directions from the source of movement.
- Typical seismograph record is shown below.







Examples of signals

- **Black-and-white picture**: Represents light intensity as a function of two spatial coordinates.

- **Video signals:** Consists of a sequence of images, called frames, and is a function of 3 variables: 2 spatial coordinates and time.

Systems

- A system may also be defined as a physical device that performs an operation on a signal.
- For example, a filter used to reduce the noise and interference corrupting a desired information-bearing signal is called a system. In this case, the filter performs some operation(s) on the signal, which has the effect of reducing (filtering) the noise and intereference from the desired information-bearing signal.
- This filtering is an example of signal processing.

Systems

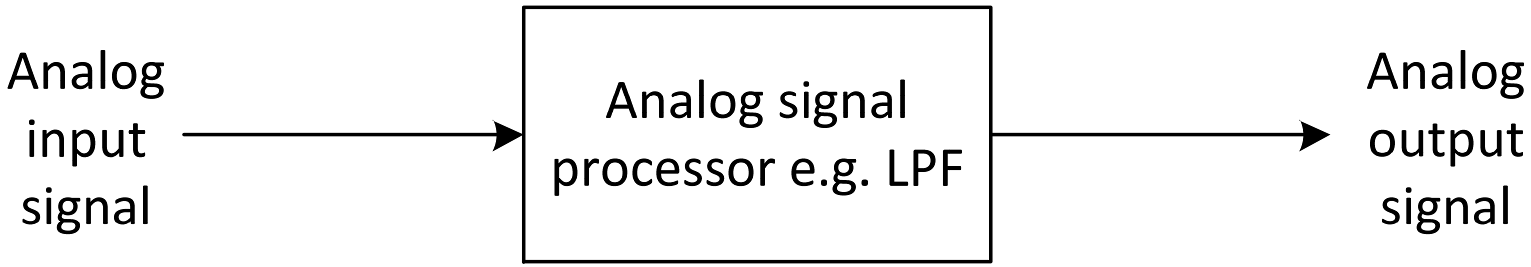
- The system does not only include the physical devices, but also software realizations of operations on a signal.
- In digital processing of signals on a digital computer, the operations performed on a signal consist of a number of mathematical operations as specified by a software program.
- In this case, the program represents an implementation of the system in software.
- Thus we have a system that is realized on a digital computer by means of a sequence of mathematical operations; that is, we have a digital signal processing system realized in software.

Systems

- For example, a digital computer can be programmed to perform digital filtering.
- Alternatively, the digital processing on the signal may be performed by digital hardware (logic circuits) configured to perform the desired specified operations.
- In such a realization, we have a physical device that performs the specified operations. In a broader sense, a digital system can be implemented as a combination of digital hardware and software, each of which performs its own set of specified operations.

Basic Elements of a DSP System

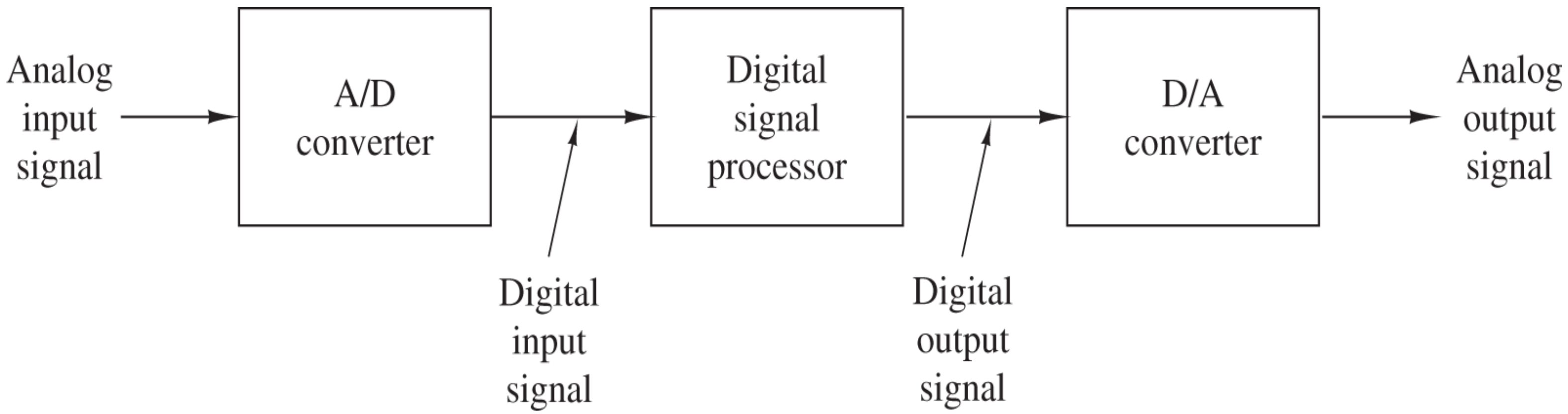
- **Analog Signal processing system**



Most of the signals encountered in science and engineering are analog in nature and can be processed by the analog system shown above where the input and output are analog.

Basic Elements of a DSP System

- Digital signal processing provides an alternative method for processing the analog signal as illustrated in the block diagram below.



Basic Elements of a DSP System

- The A/D converter converts the analogue input signal into a digital form.
- The D/A converter converts the processed signal back into analogue form.
- The heart of the system is the digital signal processor which may be based on a DSP chip such as Texas instruments TMS 320C60.
- The digital signal processor may implement one of the several DSP algorithms, for example digital filtering (low-pass filter) mapping the input *x[n]* into the output *s[n]*.
- Digital signal processor implies that the input signal must be in a digital form before it can be processed.

Advantages of Digital over Analog Signal Processing

There are many advantages to using DSP techniques for variety of applications, these include:

- High reliability and reproducibility
- Much better control of accuracy requirements.
- Flexibility and programmability
- The absence of component drift problem
- Compressed storage facility

Characterisation and classification of signals

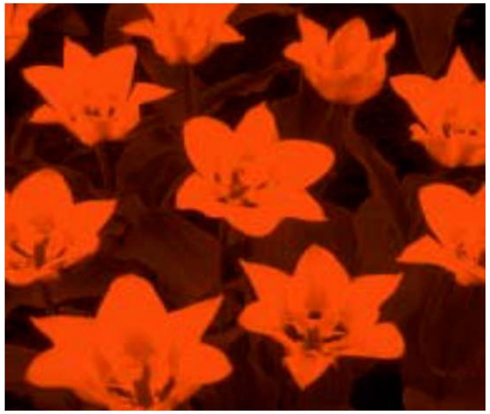
- Depending on the nature of the independent variables and the value of the function defining the signal, various types of signals can be defined.
- For example, the independent variables can be continuous or discrete. Likewise, the signal can be a continuous or discrete function of the independent variables.
- Moreover, the signal can be either a real-valued function or a complex-valued function.
- A signal generated by a single source is called a scalar signal

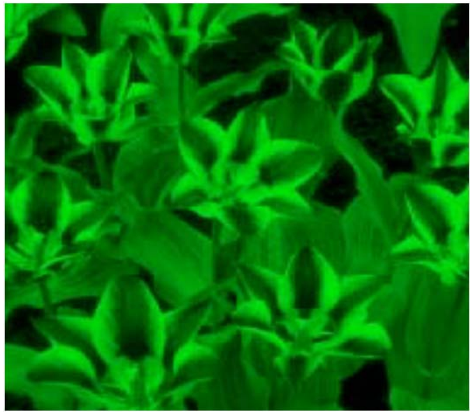
Characterisation and classification of signals

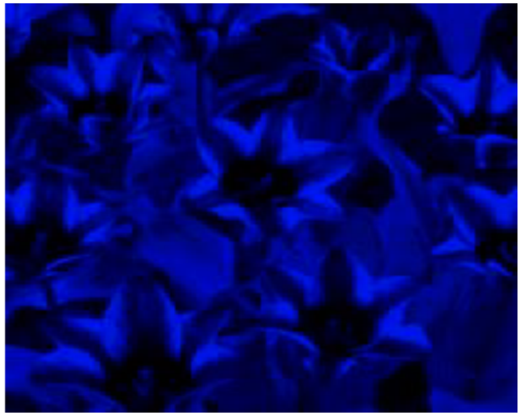
- A signal generated by multiple sources is called a vector signal or a multichannel signal.
- A one-dimensional (1 -D) signal is a function of a single independent variable.
- A multidimensional (M-D) signal is a function of more than one independent variables.
- The speech signal is an example of a 1 –D signal where the independent variable is time.
- An image signal, such as a photograph, is an example of a 2-D signal where the 2 independent variables are the 2 spatial variables.

Characterisation and classification of signals

- A color image signal is composed of three 2-D signals representing the three primary colors: red, green and blue (RGB).
- The 3 color components of a color image are shown below

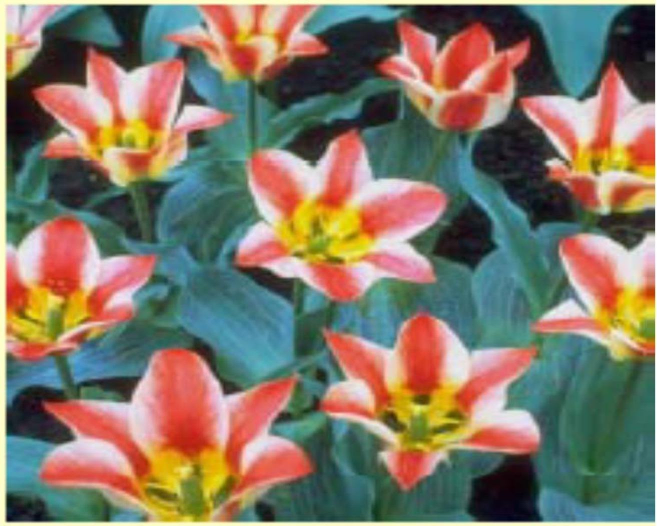






Characterisation and classification of signals

- The full color image obtained by displaying the previous 3 color components is shown below



Characterisation and classification of signals

- Each frame of a black-and-white digital video signal is a 2-D image signal that is a function of 2 discrete spatial variables, with each frame occurring at discrete instants of time.
- Hence, black-and-white digital video signal can be considered as an example of a 3-D signal where the 3 independent variables are the 2 spatial variables and time.
- A color video signal is a 3-channel signal c composed of three 3 -D signals representing the three primary colors: red, green and blue (RGB)

Characterisation and classification of signals

- For transmission purposes, the RGB television signal is transformed into another type of 3-channel signal composed of a luminance component and 2 chrominance components.
- For a 1 -D signal, the independent variable is usually labeled as time. If the independent variable is continuous, the signal is called a continuous-time signal.
- If the independent variable is discrete, the signal is called a discrete-time signal.

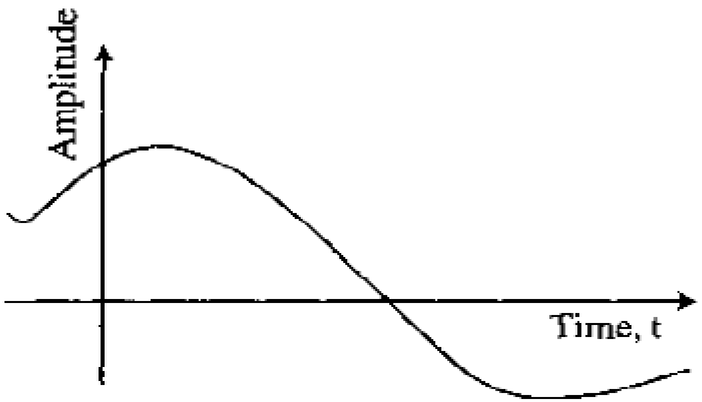
Characterisation and classification of signals

- A continuous-time signal is defined at every instant of time
- A discrete-time signal is defined at discrete instants of time, and hence, it is a sequence of numbers
- A continuous-time signal with a continuous amplitude is usually called an analog signal
- A speech signal is an example of an analog signal.
- A discrete-time signal with discrete-valued amplitudes represented by a finite number of digits is referred to as the digital signal.

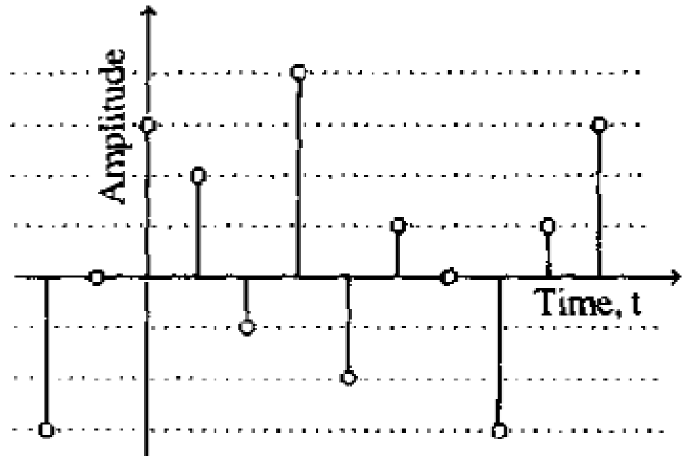
Characterisation and classification of signals

- An example of a digital signal is the digitized music signal stored in a CD-ROM disk.
- A discrete-time signal with continuous-valued amplitudes is called a sampled data signal.
- A digital signal is thus a quantized sampled-data signal
- A continuous-time signal with discrete-value amplitudes is usually called a quantized boxcar signal
- The figure below illustrates the 4 types of signals

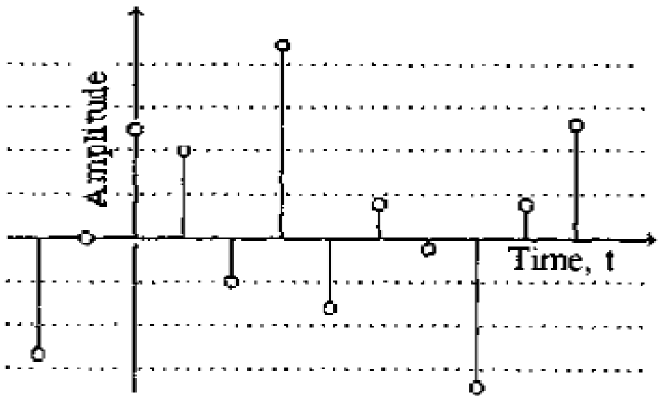
Characterisation and classification of signals



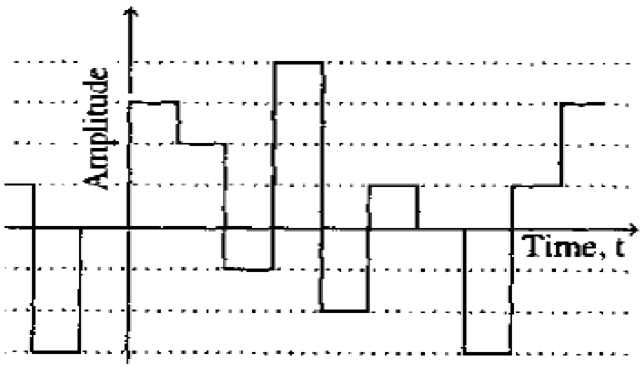
(a) A continuous time signal



(b) A Digital Signal



(c) A sampled data signal



(d) A quantized boxcar signal

Characterisation and classification of signals

- The functional dependence of a signal in its mathematical representation is often explicitly shown.
- For a continuous-time 1-D signal, the continuous independent variable is usually denoted by *t*
- For example, $u(t)$ represents a continuous time 1-D signal.
- For a discrete-time 1-D signal, the discrete independent variable is usually denoted by *n*
- For example, $\{v[n]\}$ represents a discrete time 1-D signal
- Each member, $v[n]$, of a discrete-time signal is called a sample

Characterisation and classification of signals

- In many applications, a discrete-time signal is generated by sampling a parent continuous-time signal at uniform intervals of time
- If the discrete instants of time at which a discrete-time signal is defined are uniformly spaced, the independent discrete variable n can be normalized to assume integer values.
- In the case of a continuous-time 2-D signal, the 2 independent variables are the spatial coordinates, usually denoted by x and y

Characterisation and classification of signals

- For example, the intensity of a black-and white image at location (x,y) can be expressed as $u(x,y)$.
- On the other hand, a digitized image is a 2-D discrete-time signal, and its 2 independent variables are discretized spatial variables, often denoted by m and n
- Thus, a digitized image can be represented as $v[m,n]$
- •A black-and-white video signal is a 3-D signal and can be represented as $u(x,y,t)$
- A color video signal is a vector signal composed of 3 signals representing the 3 primary colors: red, green, and blue.

Characterisation and classification of signals

- $u(x,y,t) = \begin{bmatrix} r(x,y,t) \\ g(x,y,t) \\ b(x,y,t) \end{bmatrix}$
- A signal that can be uniquely determined by a well-defined process, such as a mathematical expression or rule, or table look-up, is called a deterministic signal.
- A signal that is generated in a random fashion and cannot be predicted ahead of time is called a random signal.

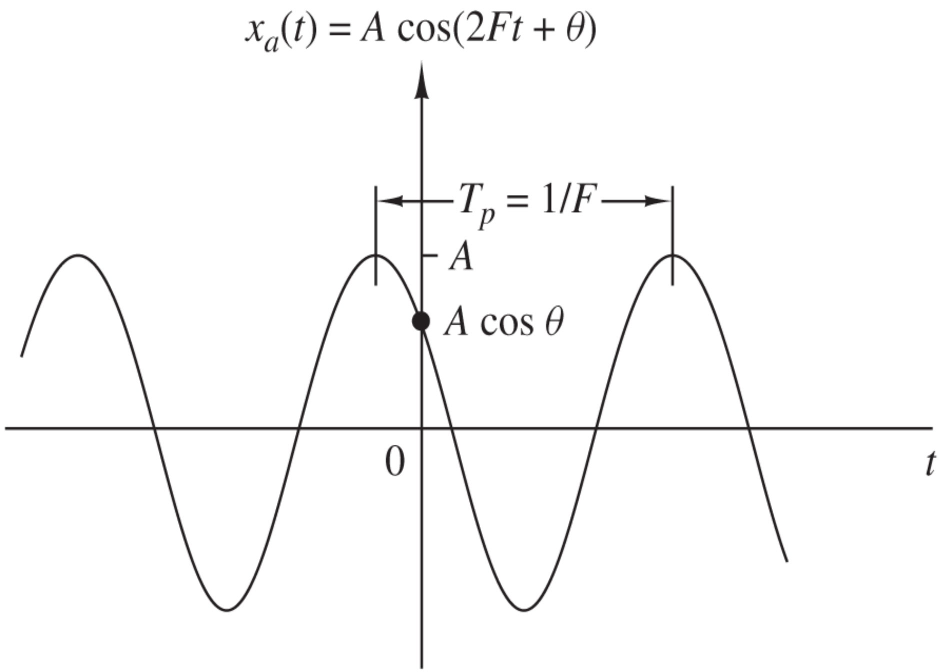
The Concept of Frequency in Continuous-Time and Discrete-Time Signals

Continuous-Time Sinusoidal Signals

- A simple harmonic oscillation is mathematically described by the following continuous- time sinusoidal signal:
- $x_a(t) = A\cos(\Omega t + \theta) \quad -\infty < t < \infty$
- This is an analog signal and is completely characterized by amplitude, A, frequency, Ω in radians per second (rad/s) and θ is the phase in radians.
- $\Omega = 2\pi F$ and thus
- $x_a(t) = A\cos(2\pi F t + \theta) \quad -\infty < t < \infty$

The Concept of Frequency in Continuous-Time and Discrete-Time Signals

- F is the frequency i.e. the number of cycles per second measured in Hertz (Hz).



Example of an analog sinusoidal signal.

The Concept of Frequency in Continuous-Time and Discrete-Time Signals

- The analog sinusoidal signal above is characterized by the following properties:
- For every fixed value of the frequency F, $x_a(t)$ is periodic. Indeed, it can easily be shown, using elementary trigonometry, that

$$x_a(t+T_p) = x_a(t)$$

- Where $T_p = 1/F$ is the fundamental period of the sinusoidal signal.
- Continuous-time sinusoidal signals with distinct (different) frequencies are themselves distinct.

The Concept of Frequency in Continuous-Time and Discrete-Time Signals

- Increasing the frequency F results in an increase in the rate of oscillation of the signal, in the sense that more periods are included in a given time interval.
- We observe that for F = 0, the value $T_p = \infty$ is consistent with the fundamental relation, $F = 1/\,T_p$.
- Due to continuity of the time variable t, we can increase the frequency F, without limit, with a corresponding increase in the rate of oscillation.
- The relationships we have described for sinusoidal signals carry over to the class of complex exponential signals $x_a(t) = Ae^{j(\Omega t+\theta)}$

The Concept of Frequency in Continuous-Time and Discrete-Time Signals

- This can easily be seen by expressing these signals in terms of sinusoids using the Euler identity
- $e^{\pm j\emptyset} = \cos\emptyset \pm j\sin\emptyset$
- By definition, frequency is an inherently positive physical quantity. This is obvious if we interpret frequency as the number of cycles per unit time in a periodic signal.
- However, in many cases, only for mathematical convenience, we need to introduce negative frequencies.

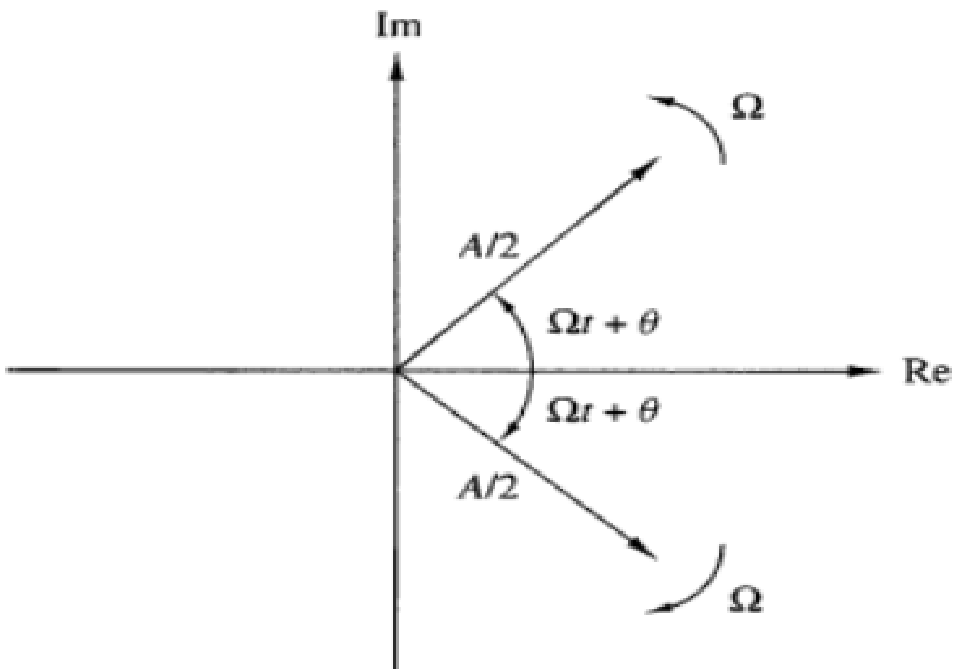
The Concept of Frequency in Continuous-Time and Discrete-Time Signals

- To see this, the sinusoidal is expressed as:
- $x_a(t) = A\cos(\Omega t + \theta) = \frac{A}{2}e^{j\Omega+\theta)} + \frac{A}{2}e^{-j(\Omega t+\theta)}$ using Euler identity.
- Note that a sinusoidal signal can be obtained by adding two equal-amplitude complex-conjugate exponential signals, sometimes called phasors, illustrated in Fig. next.
- As time progresses the phasors rotate in opposite directions with angular frequencies $\pm\Omega$ radians per second.

The Concept of Frequency in Continuous-Time and Discrete-Time Signals

- Since a positive frequency corresponds to counterclockwise uniform angular motion, a negative frequency simply corresponds to clockwise angular motion.
- For mathematical convenience, both negative and positive frequencies are used. Hence the frequency range for analog sinusoids is $-\infty < F < \infty$.

The Concept of Frequency in Continuous-Time and Discrete-Time Signals



Representation of a cosine function by a pair of complex-conjugate exponentials (phasors).

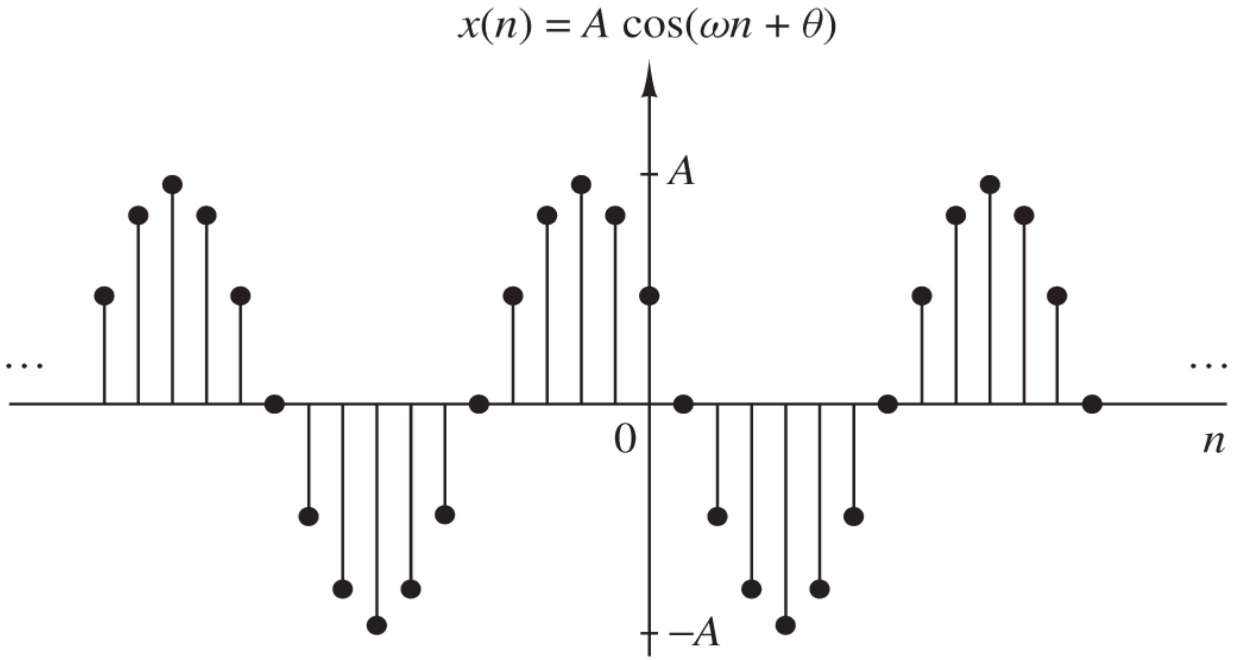
The Concept of Frequency in Continuous-Time and Discrete-Time Signals

Discrete-Time Sinusoidal Signals

- A discrete-time sinusoidal signal may be expressed as
- $x(n) = A\cos(\omega n + \theta), -\infty < n < \infty$
- where *n* is an integer variable, called the sample number, *A* is the amplitude of the sinusoid, ω is the frequency in radians per sample, and θ is the phase in radians.
- $\omega = 2\pi f$
- $x(n) = A\cos(2\pi f n + \theta), -\infty < n < \infty$
- f has dimensions of cycles per sample.

The Concept of Frequency in Continuous-Time and Discrete-Time Signals

- Figure below shows a sinusoid with frequency $\omega = \pi/6$ radians per sample $\left(f = \frac{1}{12}\ cyles\ per\ sample\right)$ and phase $\theta = \pi/3$



The Concept of Frequency in Continuous-Time and Discrete-Time Signals

- The discrete-time sinusoids are characterized by the following properties:
- A discrete-time sinusoid is periodic only if its frequency is a rational number.
- By definition, a discrete-time signal x(n) is periodic with period N (N > 0) if and only if
- $x(n+N) = x(n) \quad for\ all\ n$
- The smallest value of N for which equation above is true is called the fundamental period.

The Concept of Frequency in Continuous-Time and Discrete-Time Signals

Proof of the periodicity property.

- For a sinusoid with frequency f_0 to be periodic, we should have
- $\cos[2\pi f_0(N+n)+\theta] = \cos(2\pi f_0 n+\theta)$
- This relation is true if and only if there exists an integer k such that
- $2\pi f_0 N = 2k\pi$ or $f_0 = \frac{k}{N}$
- Thus a discrete-time sinusoidal signal is periodic only if its frequency f_0 can be expressed as the ratio of two integers (i.e. f_0 is rational).

The Concept of Frequency in Continuous-Time and Discrete-Time Signals

- To determine the fundamental period N of a periodic sinusoid, we express its frequency f_0 as $f_0 = \frac{k}{N}$ and cancel common factors so that k and N are relatively prime. Then the fundamental period of the sinusoid is equal to N.
- Observe that a small change in frequency can result in a large change in the period. For example, note that $f_1 = 31/60$ implies that $N_1 = 60$, whereas $f_2 = 30/60$ implies that $N_2 = 2$

The Concept of Frequency in Continuous-Time and Discrete-Time Signals

- Discrete-time sinusoids whose frequencies are separated by an integer multiple of 2π are identical.
- To prove this assertion, let us consider the sinusoid $\cos(\omega_0 n + \theta)$. It easily follows that
- $\cos[(\omega_0 + 2\pi)n + \theta] = \cos(\omega_0 n + 2\pi n + \theta) = \cos(\omega_0 n + \theta)$
- As a result, all sinusoidal sequences
- $x_k(n) = A\cos(\omega_k n + \theta), \qquad k = 0,1,2, \dots.$
- Where $\omega_k = \omega_0 + 2k\pi, \quad -\pi \le \omega_0 \le \pi$ are indistinguishable (i.e., identical).

The Concept of Frequency in Continuous-Time and Discrete-Time Signals

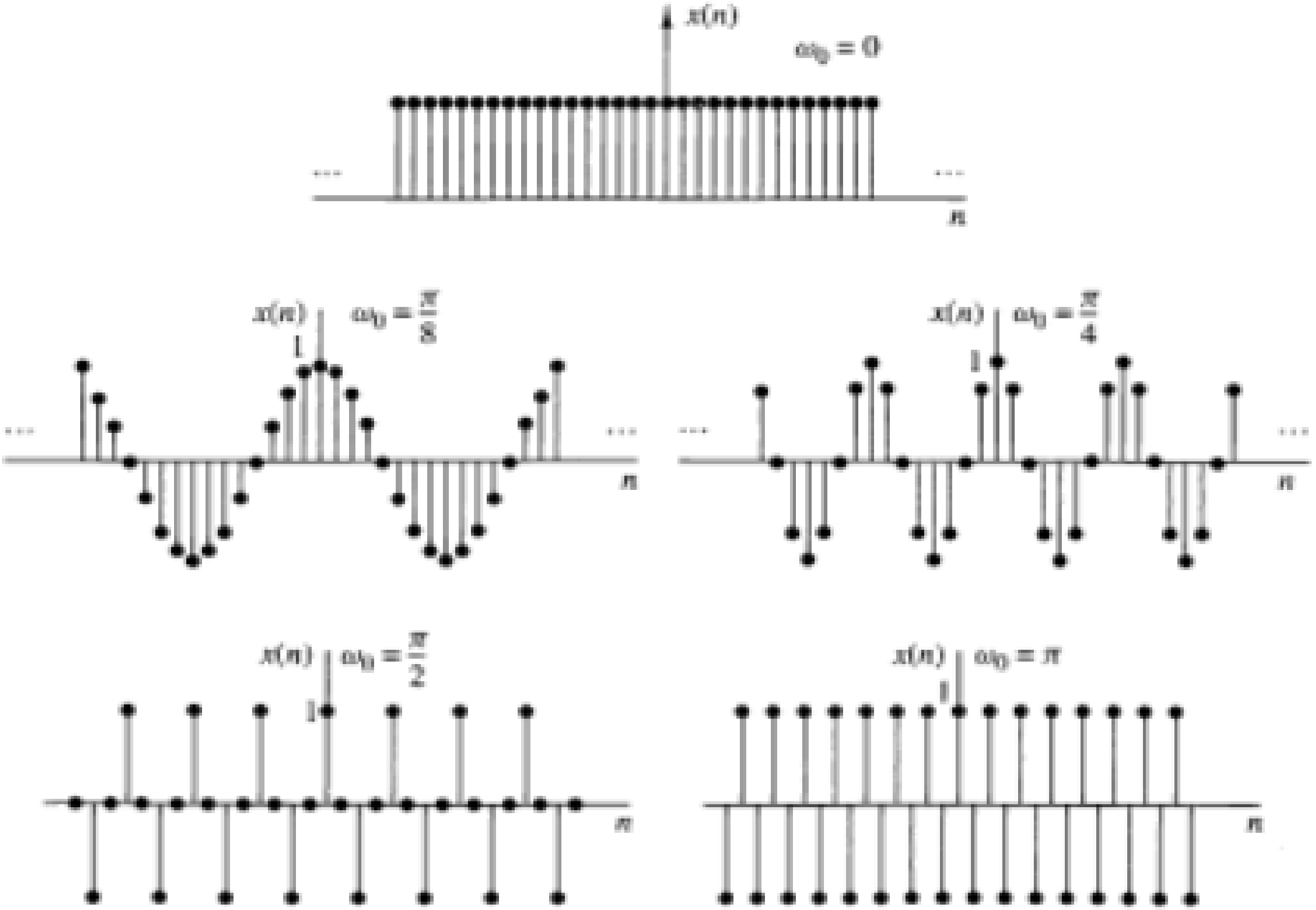
- Any sequence resulting from a sinusoid with frequency $|\omega| > \pi \;\; of \; |f| > \frac{1}{2}$, is identical to a sequence obtained from a sinuoisal signal with frequency $|\omega| < \pi$.
- Because of this similarity, we call the sinusoid having the frequency $|\omega| > \pi$ an alias of a corresponding sinusoid with frequency $|\omega| < \pi$.
- Thus we regard frequencies in the range $-\pi \leq \omega \leq \pi, or -\frac{1}{2} \leq f \leq \frac{1}{2},$ as unique. and all frequencies $|\omega| > \pi$ or $|f| > \frac{1}{2}$ as aliases.

The Concept of Frequency in Continuous-Time and Discrete-Time Signals

- The highest rate of oscillation in a discrete-time sinusoid is attained when $\omega = \pi \; (or \; \omega = -\pi\,)$ or equivalently $f = \frac{1}{2} \; \left(or \; f = -\frac{1}{2}\right)$
- To illustrate this property, let us investigate the characteristics of the sinusoidal signal sequence

$x(n) = \cos \omega_0 \, n$ when the frequency varies from 0 to π.

The Concept of Frequency in Continuous-Time and Discrete-Time Signals

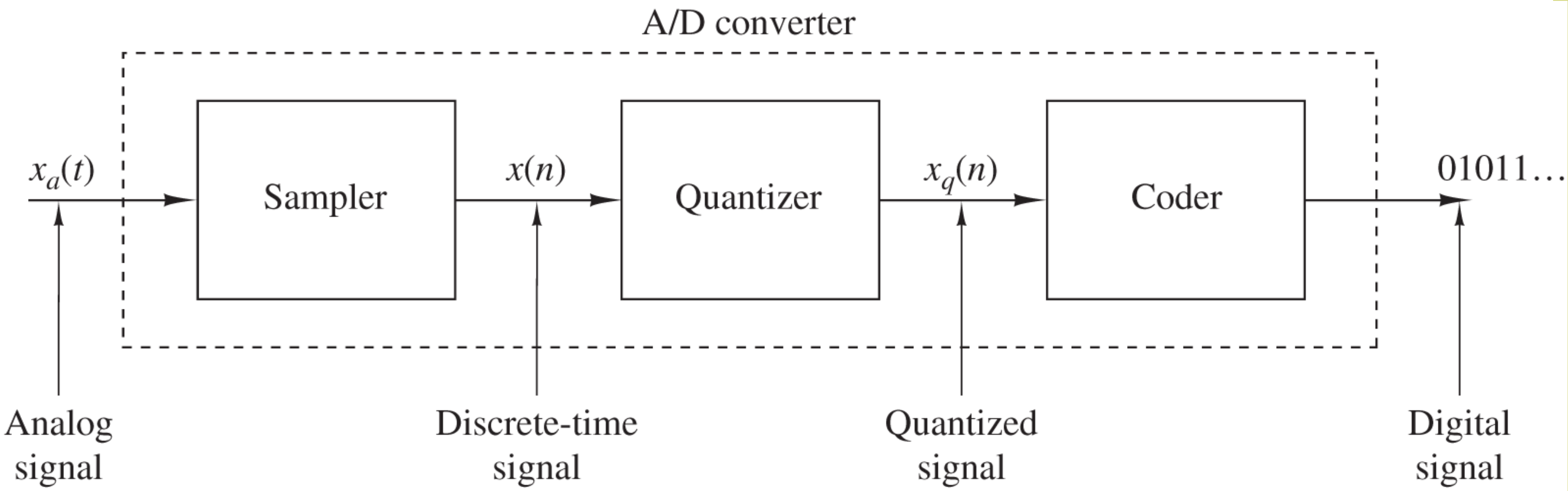


$$x(n) = \cos \omega_0 n$$

Analog-to-Digital and Digital-to-Analog Conversion

- Most signals of practical interest, such as speech, biological signals, seismic signals, radar signals, sonar signals, audio and video signals are analog.
- To process analog signals by digital means, it is first necessary to convert them into digital form.
- This procedure is called analog-to-digital (A/D) conversion, and the corresponding devices are called A/D converters (ADCs).
- The A/D conversion is conceptually a three-step process illustrated next.

Analog-to-Digital and Digital-to-Analog Conversion



Basic parts of an A/D converter

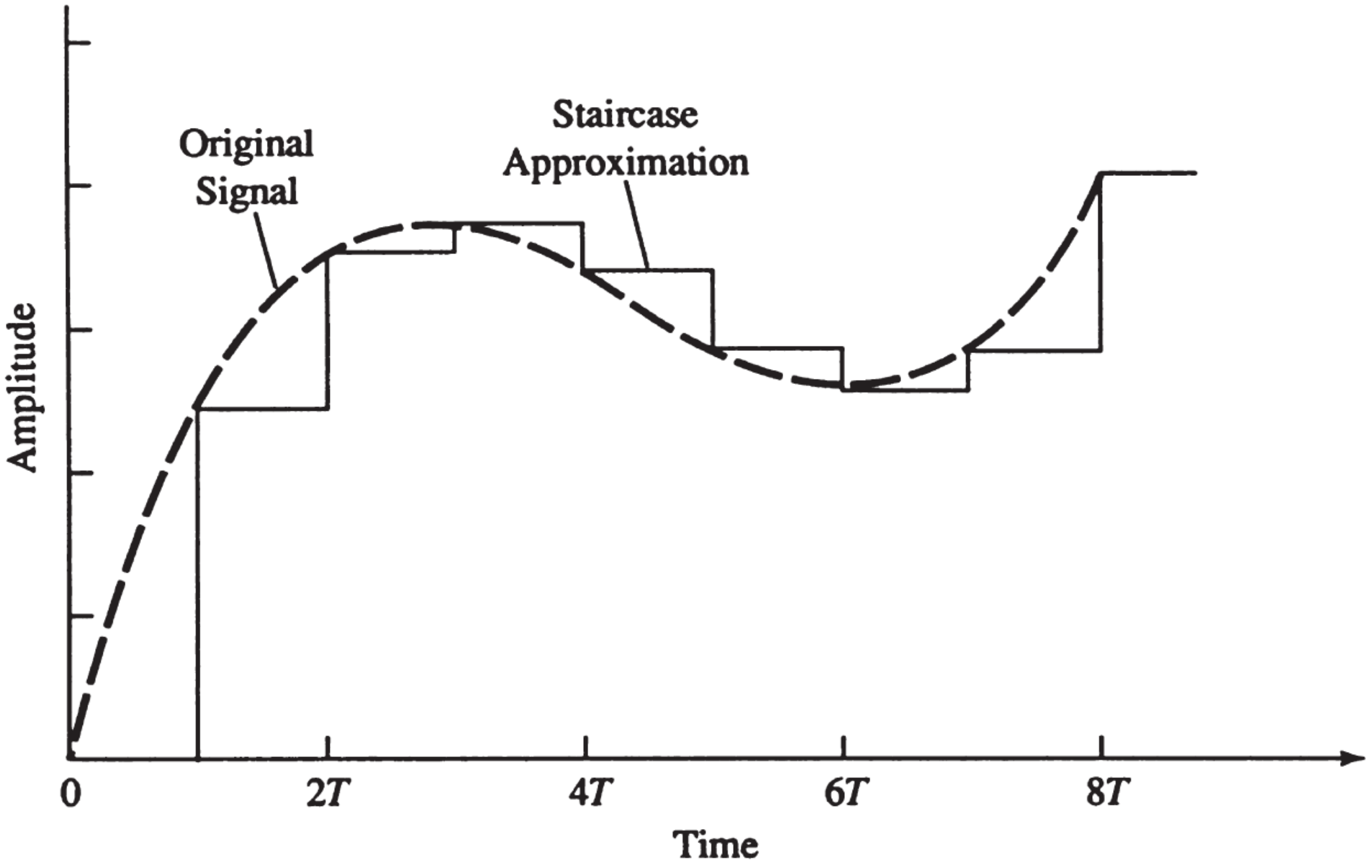
Analog-to-Digital and Digital-to-Analog Conversion

- **Sampling:** This is the conversion of a continuous-time signal into a discrete-time signal obtained by taking 'samples' of the continuous-time signal at discrete–time instants. Thus, if $x_a(t)$ is the input to the sampler, the output is $x_a(nT) \equiv x(n)$. Where T is called the sampling interval.
- **Quantization:** This is the conversion of a discrete-time continuous-valued signal into a discrete-time, discrete-valued (digital) signal.
- The value of each signal sample is represented by a value selected from a finite set of possible values.
- The difference between the unquantized sample $x(n)$ and the quantized output $x_q(n)$ is called the quantization error.

Analog-to-Digital and Digital-to-Analog Conversion

- **Coding:** In the coding process, each discrete value $x_q(n)$ is represented by a *b*-bit binary sequence.
- In many cases of practical interest (e.g., speech processing) it is desirable to convert the processed digital signals into analog form. (Obviously, we cannot listen to the sequence of samples representing a speech signal or see the numbers corresponding to a TV signal.) The process of converting a digital signal into an analog signal is known as digital-to-analog (D/A) conversion. All D/A converters "connect the dots" in a digital signal by performing some kind of interpolation, whose accuracy depends on the quality of the D/A conversion process.

Analog-to-Digital and Digital-to-Analog Conversion



Zero-order hold digital-to-analog (D/A) conversion

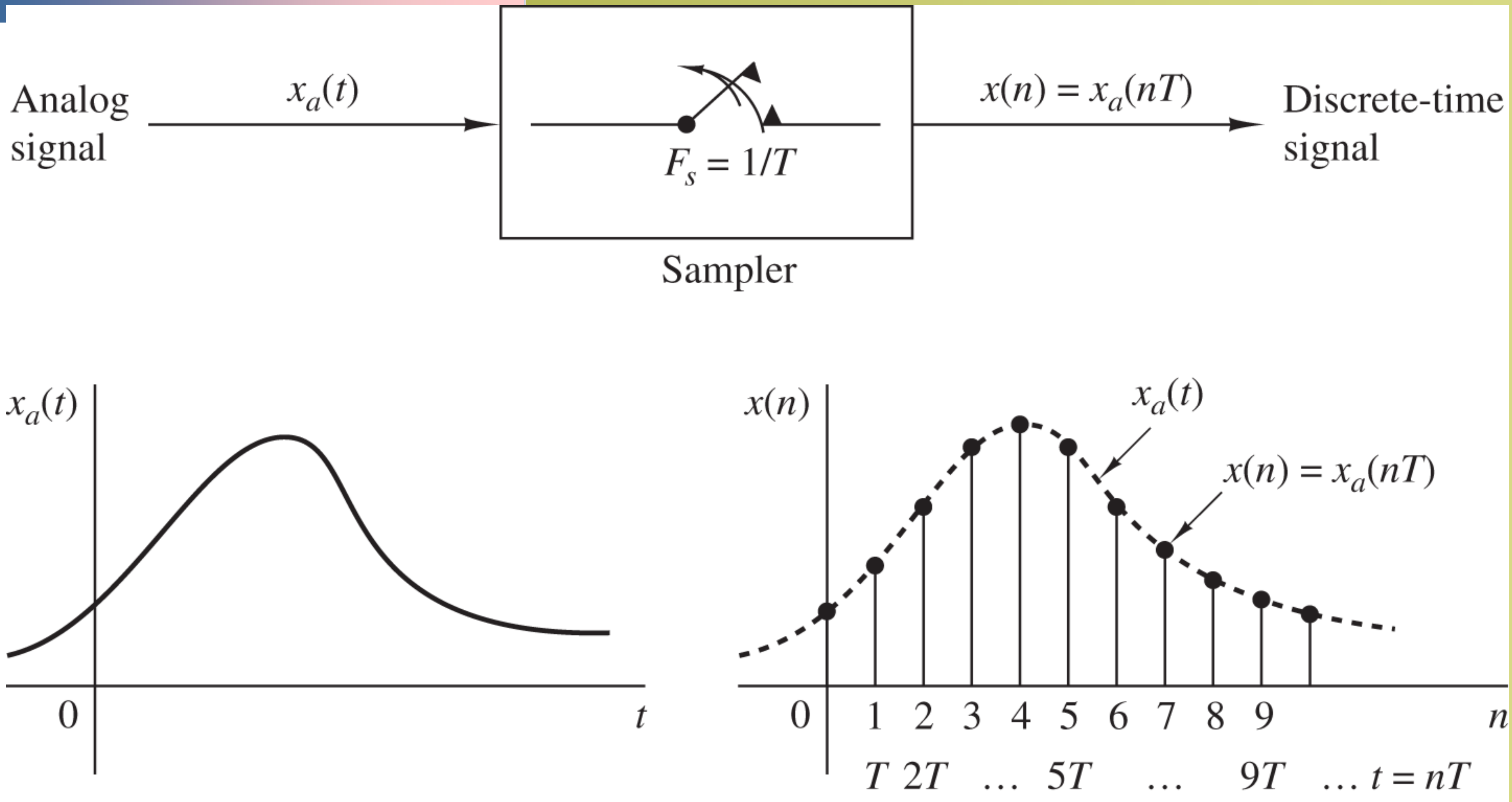
Analog-to-Digital and Digital-to-Analog Conversion

- Figure above illustrates a simple form of D/A conversion, called a zero-order hold or a staircase approximation.
- Other approximations are possible, such as linearly connecting a pair of successive samples (linear interpolation), fitting a quadratic through three successive samples (quadratic interpolation), and so on. Is there an optimum (ideal) interpolator? For signals having a limited frequency content (finite bandwidth), the sampling theorem introduced in the following section specifies the optimum form of interpolation.

Sampling of Analog Signals

- Periodic sampling or uniform sampling is the sampling most often used in practice and is described by the relation
- $x(n) = x_a(nT), \qquad -\infty < n < \infty$
- where $x(n)$ is the discrete-time signal obtained by "taking samples" of the analog signal $x_a(t)$ every T seconds.
- This procedure is illustrated next. The time interval *T* between successive samples is called the sampling period and its reciprocal $1/T = F_s$ is called the sampling rate or the sampling frequency (hertz).

Sampling of Analog Signals



Sampling

Sampling of Analog Signals

- $t = nT = \frac{n}{F_s}$
- Thus there exists there exists a relationship between the frequency variable *F* (or Ω) for analog signals and the frequency variable *f* (or ω) for discrete- time signals. To establish this relationship, consider an analog sinusoidal signal of the form
- $x_a(t) = A\cos(2\pi Ft + \theta)$
- $x_a(nT) \equiv x(n) = A\cos(2\pi FnT + \theta) = A\cos\left(\frac{2\pi nF}{F_s} + \theta\right)$

Sampling of Analog Signals

- we note that the frequency variables *F* and *f* are linearly related as
- $f = \frac{F}{F_s}$
- or, equivalently, as
- $\omega = \Omega T$
- Where $F_s = sampling\ frequency$
- $F = frequencyof\ the\ analog\ signal$
- $f = frequency\ of\ the\ digital\ signal$
- $\omega = relative\ or\ normalised\ frequency$

Sampling of Analog Signals

Continuous-time signals		Discrete-time signals
$\Omega = 2\pi F$		$\omega = 2\pi f$
$\frac{\text{radians}}{\text{sec}}$ Hz		$\frac{\text{radians}}{\text{sample}}$ $\frac{\text{cycles}}{\text{sample}}$
	$\xrightarrow{\omega = \Omega T,\, f = F/F_s}$	$-\pi \leq \omega \leq \pi$
	$\xleftarrow{\Omega = \omega/T,\, F = f \cdot F_s}$	$-\frac{1}{2} \leq f \leq \frac{1}{2}$
$-\infty < \Omega < \infty$		$-\pi/T \leq \Omega \leq \pi/T$
$-\infty < F < \infty$		$-F_s/2 \leq F \leq F_s/2$

Sampling of Analog Signals

- Since the highest frequency in a discrete-time signal is $\omega = \pi \; or \; f = \frac{1}{2}$. It follows that, with a sampling rate F_s, the corresponding highest values of F and Ω are
- $F_{max} = \frac{F_s}{2} = \frac{1}{2T}$
- $\Omega_{max} = \pi F_s = \frac{\pi}{T}$
- Therefore, sampling introduces an ambiguity, since the highest frequency in a continuous-time signal that can be uniquely distinguished when such a signal is sampled at a rate $F_s = 1/T$ is $F_{max} = F_s/2$, or $\Omega_{max} = \pi F_s$. To see what happens to frequencies above $F_s/2$, let us consider the following example.

Sampling of Analog Signals

Example

- The implications of these frequency relations can be fully appreciated by considering the two analog sinusoidal signals
- $x_1(t) = \cos 2\pi(10)t$
- $x_2(t) = \cos 2\pi(50)t$
- which are sampled at a rate *F_s = 40 Hz.* The corresponding discrete-time signals or sequences are
- $x_1(n) = \cos 2\pi\left(\frac{10}{40}\right)n = \cos\frac{\pi}{2}n$
- $x_2(n) = \cos 2\pi\left(\frac{50}{40}\right)n = \cos\frac{5\pi}{2}n$

Sampling of Analog Signals

- However $\cos\frac{5\pi}{2}n = \cos\left(2\pi n + \frac{\pi n}{2}\right) = \cos\frac{\pi}{2}n$. Thus $x_1(n) = x_2(n)$.
- Thus the sinusoidal signals are identical and, consequently, indistinguishable.
- Since $X_2(t)$ yields exactly the same values as $x_1(t)$ when the two are sampled at F_s = 40 samples per second, we say that the frequency F_2 = 50 Hz is an alias of the frequency F_1 = 10 Hz at the sampling rate of 40 samples per second.

Sampling of Analog Signals

- All of the sinusoids $\cos 2\pi(F_1 + 40k)t$, k=1,2,3,… sampled at 40 samples per second yield identical values. Consequently they are all aliases of F_1 = 10 Hz.
- In general, the sampling of a continuous-time signal
- $x_a(t) = A\cos(2\pi F_0 t + \theta)$
- With a sampling rate $F_s = 1/T$ results in a discrete-time signal
- $x(n) = A\cos(2\pi f_0 n + \theta)$
- Where $f_0 = F_0/F_s$ is the relative frequency of the sinusoid.

Sampling of Analog Signals

- If we assume that
- $-F_s/2 \leq F_0 \leq F_s/2$, the frequency f_0 of $x(n)$ is in the range $-1/2 \leq f_0 \leq 1/2$, which is the frequency range of the discrete-time signals and hence it is possible to identify (or reconstruct) the analog signal $x_a(t)$ from the samples $x(n)$.
- On the other hand, if the sinusoids
- $x_a(t) = A\cos(2\pi F_k t + \theta)$
- Where $F_k = F_0 + kF_s, \qquad k = \pm 1, \pm 2 \ldots.$
- Are sampled at a rate F_s, it is clear that the frequency F_k is outside the fundamental frequency range $-F_s/2 \leq F_0 \leq F_s/2$.

Sampling of Analog Signals

- Consequently, the sampled signal is
- $x(n) \equiv x_a(nT) = A\cos\left(2\pi \frac{F_0+kF_s}{F_s} n + \theta\right)$
- $= A\cos\left(2\pi n \frac{F_0}{F_s} + \theta + 2\pi k n\right)$
  $= A\cos(2\pi f_0 n + \theta)$
- Which is identical to the discrete-time signal obtained earlier. Thus an infinite number of continuous-time sinusoids is represented by sampling the same discrete-time signal (i.e. by the same set of samples). Consequently, if we are given $x(n)$, an ambiguity exists as to which continuous time-signal $x_a(t)$ these values represent.

Sampling of Analog Signals

- **Example**

Consider the analog signal $x_a(t) = 3\cos 100\pi t$

a) Determine the minimum sampling rate required to avoid aliasing.

b) Suppose that the signal is sampled at the rate $F_s = 200$ *Hz*. What is the discrete-time signal obtained after sampling?

c) Suppose that the signal is sampled at the rate $F_s = 75$ *Hz*. What is the discrete-time signal obtained after sampling?

d) What is the frequency $0 < F < F_s/2$ of a sinusoid that yields samples identical to those obtained in part (c)?

Sampling of Analog Signals

- **Solution.**

(a) The frequency of the analog signal is F = 50 Hz. Hence the minimum sampling rate required to avoid aliasing is F_s = 100 Hz.

(b) If the signal is sampled at F_s = 200 Hz, the discrete-time signal is

$$x(n) = 3\cos\frac{100\pi}{200}n = 3\cos\frac{\pi}{2}n$$

(c) If the signal is sampled at F_s = 75 Hz, the discrete-time signal is

Sampling of Analog Signals

- $x(n) = 3\cos\frac{100\pi}{75}n =$
  $3\cos\frac{4\pi}{3}n = 3\cos\left(2\pi - \frac{2\pi}{3}\right)n = 3\cos\frac{2\pi}{3}n$

(d) For the sampling rate of F_s = 75 Hz, we have

$F = fF_s = 75f$

The frequency of the sinusoid in part (c) is $f = \frac{1}{3}$. Hence

F = 25 Hz

- Clearly, the sinusoidal signal
- $y_a(t) = 3\cos 2\pi Ft = 3\cos 50\pi t$ sampled at F_s=75 samples/s yields identical samples. Hence F = 50 Hz is an alias of F = 25 Hz for the sampling rate F_s = *75 Hz.*

Sampling of Analog Signals

The Sampling Theorem

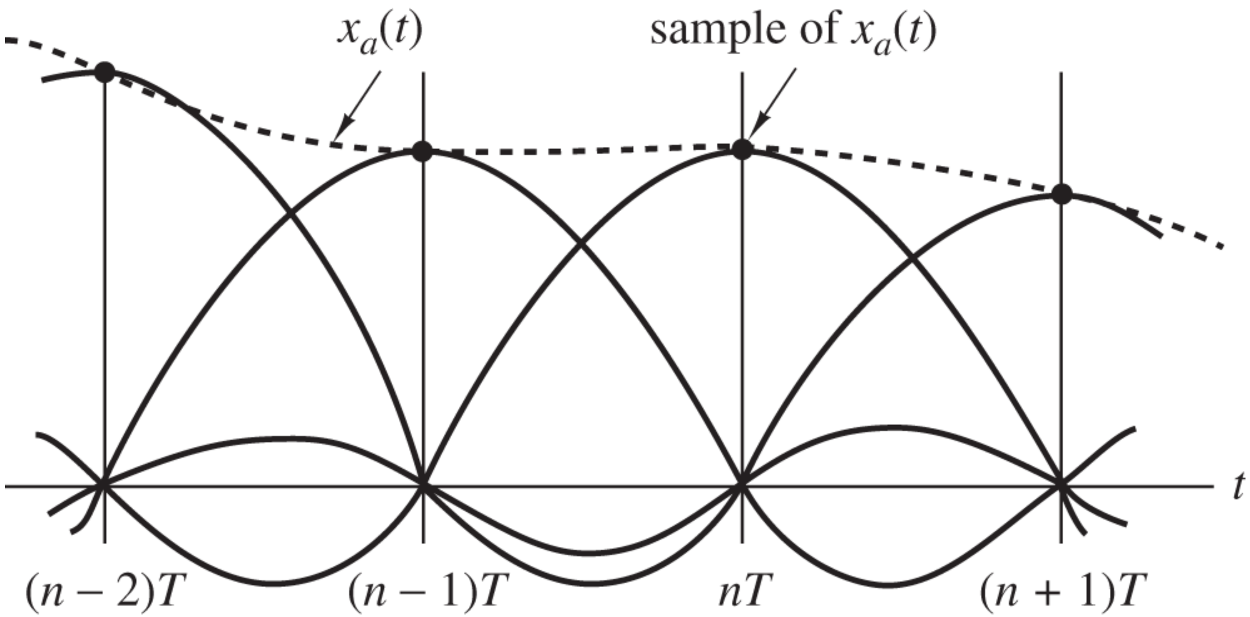
- If the highest frequency contained in an analog signal $x_a(t)$ is $F_{max} = B$ and the signal is sampled at a rate $F_s > 2F_{max} \equiv 2B$, then $x_a(t)$ can be exactly recovered from its sample values using the interpolation function
- $g(t) = \frac{\sin 2\pi Bt}{2\pi Bt}$
- Thus $x_a(t)$ may be expressed as
- $x_a(t) = \sum_{n=-\infty}^{\infty} x_a\left(\frac{n}{F_s}\right) g\left(t - \frac{n}{F_s}\right)$
- $x_a(n/F_s) = x_a(nT) \equiv x(n)$ are the samples of $x_a(t)$.

Sampling of Analog Signals

- When the sampling of $x_a(t)$ is performed at the minimum sampling rate $F_s = 2B$, the reconstruction formula becomes.
- $x_a(t) = \sum_{n=-\infty}^{\infty} x_a\left(\frac{n}{2B}\right) \frac{\sin 2\pi B(t-n/2B)}{2\pi B(t-n/2B)}$
- The sampling rate $F_N = 2B = 2F_{max}$ is called the Nyquist rate.
- The reconstruction of $x_a(t)$ from the sequence x(n) is a complicated process, involving the weighted sum of g(t) and its time shifted versions $g(t - nT)$ for $-\infty < n < \infty$ where the weighting factors are samples of x(n)

Sampling of Analog Signals

- Figure below illustrates the ideal D/A conversion process using the interpolation function



Ideal D/A conversion (interpolation).

Sampling of Analog Signals

Example

Consider the analog signal

$x_a(t) = 3\cos 50\pi t + 10\sin 300\pi t - \cos 100\pi t$

What is the Nyquist rate for this signal?

Solution. The frequencies present in the signal above are *F_1 = 25 Hz, F_2 = 150 Hz, F_3 = 50 Hz*

Thus F_{max} = 150 Hz

$F_s > 2F_{max}$ = 300 Hz

- The Nyquist rate is $F_N = 2F_{max}$=300 Hz

Sampling of Analog Signals

- **Discussion:** It should be observed that the signal component $10\sin 300\pi t$, sampled at the Nyquist rate F_N = 300, results in the samples $10\sin n\pi$ which are identically zero. In other words, we are sampling the analog sinusoid at its zero-crossing points, and hence we miss this signal component completely.
- This situation does not occur if the sinusoid is offset in phase by some amount θ. In such a case we have $10\sin(300\pi t+\theta)$ sampled at the Nyquist rate

Sampling of Analog Signals

- F_N = 300 samples per second, which yields the samples
- $10\sin(n\pi+\theta) = 10(\sin\pi n\cos\theta + \cos\pi n\sin\theta) = (-1)^n 10\sin\theta$
- Thus if $\theta \neq 0\ or\ \pi$, the samples of the sinusoid taken at the Nyquist rate are not all zero.
- However, we still cannot obtain the correct amplitude from the samples when the phase θ is unknown.
- A simple remedy that avoids this potentially troublesome situation is to sample the analog signal at a rate higher than the Nyquist rate.

Sampling of Analog Signals

Example

Consider the analog signal

$$x_a(t) = 3\cos 2000\pi t + 5\sin 6000\pi t + 10\cos 12000\pi t$$

a) What is the Nyquist rate for this signal?

b) Assume now that we sample this signal using a sampling rate F_s = *5000* samples/s. What is the discrete-time signal obtained after sampling?

c) What is the analog signal $y_a(t)$ that we can reconstruct from the samples if we use ideal interpolation?

Sampling of Analog Signals

- **Solution**

(a) The frequencies existing in the analog signal are

- $F_1 = 1\ kHz,\ F_2 = 3\ kHz,\ F_3 = 6\ kHz$
- Thus $F_{max} = 6\ kHz,$ and according to the sampling theorem,
- $F_s > 2F_{max} = 12\ kHz$
- The Nyquist rate is
- $F_N = 12\ kHz$

(b) Since we have chosen $F_s = 5\ kHz,$ the folding frequency is $F_s/2 = 2.5$ kHz

- and this is the maximum frequency that can be represented uniquely by the sampled signal.

Sampling of Analog Signals

- $x(n) = x_a(nT) = x_a\left(\frac{n}{F_s}\right) = 13\cos 2\pi\left(\frac{1}{5}\right)n - 5\sin 2\pi\left(\frac{2}{5}\right)n$
- (c) Since the frequency components at only 1 kHz and 2 kHz are present in the sampled signal, the analog signal we can recover is
- $y_a(t) = 13\cos 2000\pi t - 5\sin 4000\pi t$
- which is obviously different from the original signal $x_a(t)$. This distortion of the original analog signal was caused by the aliasing effect, due to the low sampling rate used.

Quantization of Continuous-Amplitude Signals

Quantization of Continuous-Amplitude Signals

- The process of converting a discrete-time continuous-amplitude signal into a digital signal by expressing each sample value as a finite (instead of an infinite) number of digits is called quantization.
- The error introduced in representing the continuous-valued signal by a finite set of discrete value levels is called quantization error or quantization noise.
- Let the quantizer operation on the samples $x(n)$ be $Q[x(n)]$ and let $x_q(n)$ denote the sequence of quantized samples at the output of the quantizer.

Quantization of Continuous-Amplitude Signals

- Hence
- $x_q(n) = Q[x(n)]$
- Then the quantization error is a sequence $e_q(n)$ defined as the difference between the quantized value and the actual sample value. Thus
- $e_q(n) = x_q(n) - x(n)$.

Quantization of Continuous-Amplitude Signals

- **Example**
- Consider the discrete- time signal
- $x(n) = \begin{cases} 0.9^n, & n \geq 0 \\ 0, & n < 0 \end{cases}$
- obtained by sampling the analog exponential signal $x_a(t) = 0.9^t$, $t > 0$ with a sampling frequency F_s = 1 Hz. Table below, which shows the values of the first 10 samples of x(n), reveals that the description of the sample value x(n) requires n significant digits. It is obvious that this signal cannot be processed by using a calculator or a digital computer since only the first few samples can be stored and manipulated. For example, most calculators process numbers with only eight significant digits.

Quantization of Continuous-Amplitude Signals

n	$x(n)$ Discrete-time signal	$x_q(n)$ (Truncation)	$x_q(n)$ (Rounding)	$e_q(n) = x_q(n) - x(n)$ (Rounding)
0	1	1.0	1.0	0.0
1	0.9	0.9	0.9	0.0
2	0.81	0.8	0.8	−0.01
3	0.729	0.7	0.7	−0.029
4	0.6561	0.6	0.7	0.0439
5	0.59049	0.5	0.6	0.00951
6	0.531441	0.5	0.5	−0.031441
7	0.4782969	0.4	0.5	0.0217031
8	0.43046721	0.4	0.4	−0.03046721
9	0.387420489	0.3	0.4	0.012579511

Numerical Illustration of Quantization with One Significant Digit Using Truncation or Rounding

Quantization of Continuous-Amplitude Signals

- However, let us assume that we want to use only one significant digit. To eliminate the excess digits, we can either simply discard them (truncation) or discard them by rounding the resulting number (rounding).
- The resulting quantized signals $x_q(n)$ are shown in Table.
- The rounding process is graphically illustrated in Fig. below. The values allowed in the digital signal are called the quantization levels, whereas the distance Δ between two successive quantization levels is called the quantization step size or resolution.

Quantization of Continuous-Amplitude Signals

- Quantisation error $e_q(n)$ in rounding is limited to the range of $-\frac{\Delta}{2} \leq e_q(n) \leq \frac{\Delta}{2}$.
- In other words, the instantaneous quantization error cannot exceed half of the quantization step

Quantization of Continuous-Amplitude Signals

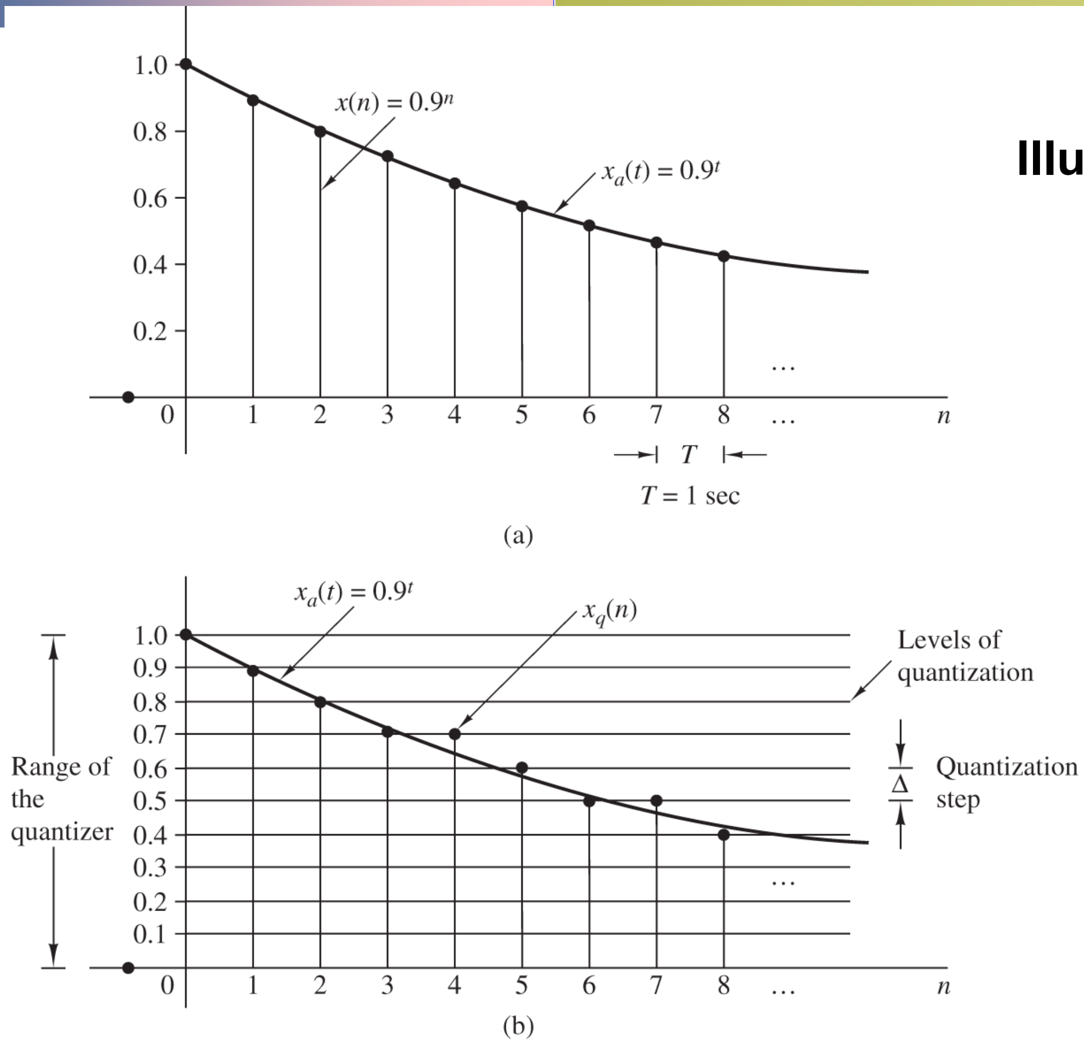
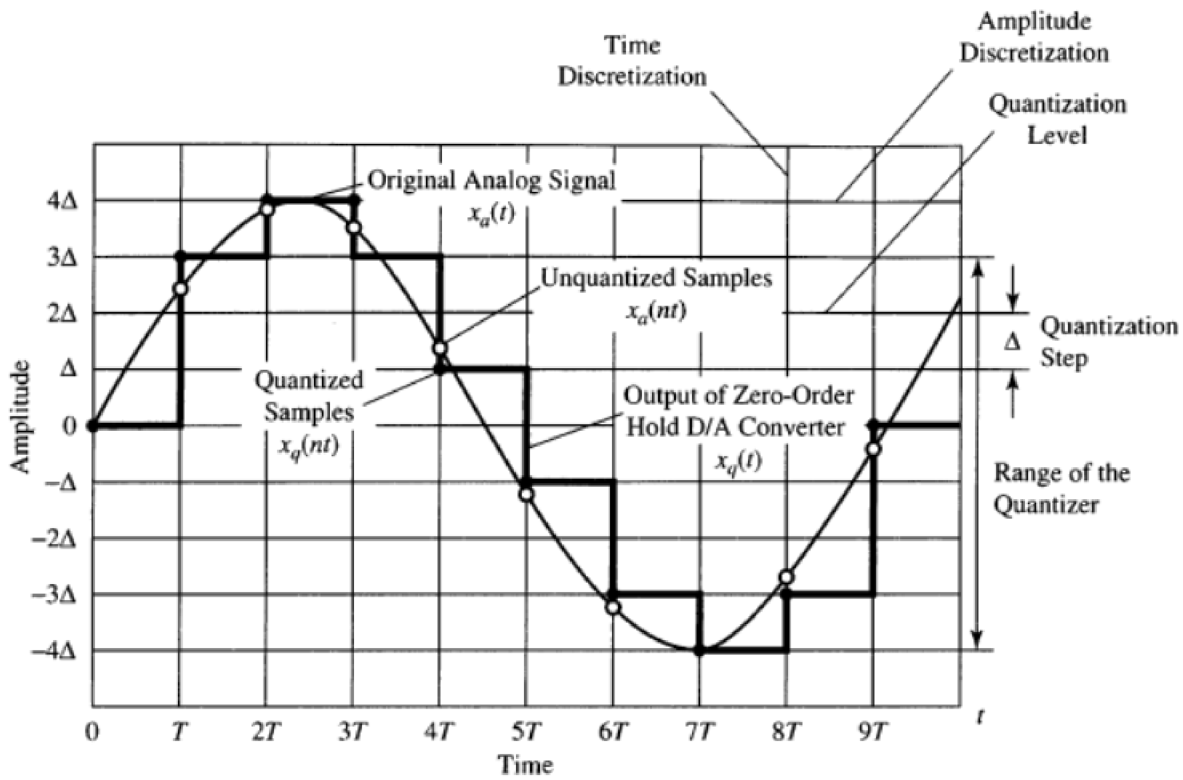


Illustration of quantization.

Quantization of Sinusoidal Signals



Sampling and quantization of a sinusoidal signal.

Quantization of Sinusoidal Signals

- Figure above illustrates the sampling and quantization of an analog sinusoidal signal
- $x_a(t) = A\cos\Omega_0 t$ using a rectangular grid. Horizontal lines within the range of the quantizer indicate the allowed levels of quantization. Vertical lines indicate the sampling times.
- Thus, from the original analog signal $x_a(t)$ we obtain a discrete-time signal $x(n) = x_a(nT)$ by sampling and a discrete-time, discrete-amplitude signal $x_q(nT)$ after quantization.

Quantization of Sinusoidal Signals

- In practice, the staircase signal $x_q(t)$ can be obtained by using a zero-order hold.
- This analysis is useful because sinusoids are used as test signals in A/D converters.
- If the sampling rate F_s satisfies the sampling theorem, quantization is the only error in the A/D conversion process.
- Thus we can evaluate the quantization error by quantizing the analog signal $x_a(t)$ instead of the discrete-time signal $x(n) = x_a(nT)$.
- The quantisation error is given by
- $e_q(n) = x_q(n) - x(n)$.

Quantization of Sinusoidal Signals

- τ denotes the time that $x_a(t)$ stays within the quantisation levels. The mean square error power P_q is.
- $P_q = \frac{1}{2\tau}\int_{-\tau}^{\tau} e_q^2(t)dt = \frac{1}{\tau}\int_{0}^{\tau} e_q^2(t)dt$
- Since $e_q(t) = (\Delta/2\tau)t, \quad -\tau \le t \le \tau$
- Then $P_q = \frac{1}{2\tau}\int_{-\tau}^{\tau} e_q^2(t)dt = \frac{1}{\tau}\int_{0}^{\tau} \left(\frac{\Delta}{2\tau}\right)^2 t^2 dt = \frac{\Delta^2}{12}$
- If the quantizer has b bits of accuracy and the quantizer covers the entire range 2A, the quantization step is $\Delta = 2A/2^b$
- Hence
- $P_q = \frac{A^2/3}{2^{2b}}$

Quantization of Sinusoidal Signals

- The average power of the signal $x_a(t)$ is
- $P_a = \frac{1}{T_p}\int_0^{\tau_p}(A\cos\Omega_0 t\,)^2 dt = \frac{A^2}{2}$
- The quality of the output of the A/D converter is usually measured by the signal-to- quantization noise ratio (SQNR), which provides the ratio of the signal power to the noise power:
- $SQNR = \frac{P_a}{P_q} = \frac{3}{2}.\,2^{2b}$
- $SQNR\ (dB) = 10\log SQNR = 1.76 + 6.02b$
- This implies that the SQNR increases approximately 6 dB for every bit added to the word length, that is, for each doubling of the quantization levels.

Coding of Quantized Samples

- The coding process in an A/D converter assigns a unique binary number to each quantization level. If we have L levels we need at least L different binary numbers. With a word length of b bits we can create 2^b different binary numbers. Hence we have $2^b \geq L$ or equivalently, $b \geq \log_2 L$
- Thus the number of bits required in the coder is the smallest integer greater than or equal to $\log_2 L$. In our example it can easily be seen that we need a coder with b = 4 bits.
- Commercially available A/D converters may be obtained with finite precision of b = 16 or less.
- Generally, the higher the sampling speed and the finer the quantization, the more expensive the device becomes.

Digital-to-Analog Conversion

- To convert a digital signal into an analog signal we can use a digital-to-analog (D/A) converter. As stated previously, the task of a D/A converter is to interpolate between samples.
- The sampling theorem specifies the optimum interpolation for a bandlimited signal. However, this type of interpolation is too complicated and, hence, impractical, as indicated previously.
- From a practical viewpoint, the simplest D/A converter is the zero-order hold shown discussed previously, which simply holds constant the value of one sample until the next one is received.