MAKERERE UNIVERSITY

COLLEGE OF ENGINEERING, DESIGN, ART AND TECHNOLOGY

SCHOOL OF ENGINEERING

DEPARTMENT OF ELECTRICAL AND COMPUTER ENGINEERING

EMT 1101: ENGINEERING MATHEMATICS I LECTURE NOTES 2018/2019

CHAPTER TWO: DIFFERENTIAL CALCULUS

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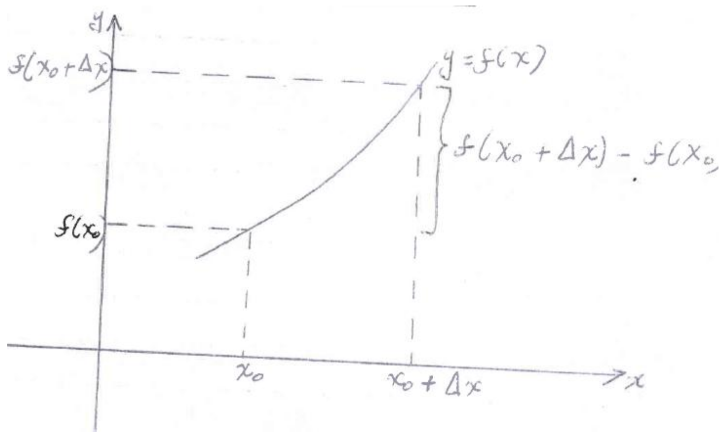
DIFFERENTIAL CALCULUS

The Derivative



Given a curve $y = f(x)$ we can always find a deferential quotient as;

$$f^1(x_0) = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x}$$

Which is the differential of *f(x)* at the point $x = x_0$.

If the limit exists at point $x = x_0$, we say *f* is ***differentiable*** at $x = x_0$. If the limit does not exist at $x = x_0$, then we say *f* is ***non – differentiable*** at $x = x_0$. It follows that the domain of f^1 consists of all points in the domain of *f* where *f* is differentiable.

The derivative function

The derivative of a function f(x) on the interval I is a function denoted $f^1(x)$ with values $f^1(x) = \lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x}$, provided that this exists for all values in I. This function is called the ***slope function*** of *f(x)*.

Examples

Find the derivatives of the following functions, from first principles:

1. $f(x) = \sqrt{x}$

$$f^1(x) = \lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\sqrt{x + \Delta x} - \sqrt{x}}{\Delta x} * \frac{\sqrt{x + \Delta x} + \sqrt{x}}{\sqrt{x + \Delta x} + \sqrt{x}}$$

$$= \lim_{\Delta x \to 0} \frac{x + \Delta x + \sqrt{x}\sqrt{x + \Delta} - \sqrt{x}\sqrt{x + \Delta x} - \left(\sqrt{x}\right)^2}{\Delta x\sqrt{x + \Delta x} + \Delta x\sqrt{x}}$$

$$= \lim_{\Delta x \to 0} \frac{\Delta x}{\Delta x\sqrt{x + \Delta x} + \Delta x\sqrt{x}} = \lim_{\Delta x \to 0} \frac{1}{\sqrt{x + \Delta x} + \sqrt{x}} = \frac{1}{2\sqrt{x}}$$

2. $f(x) = 3x^2 + 1$

$$f^1(x) = \lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x} = \lim_{\Delta x \to 0} \frac{3(x + \Delta x)^2 + 1 - (3x^2 + 1)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{3\left(x^2 + 2x\Delta x + (\Delta x)^2\right) + 1 - (3x^2 + 1)}{\Delta x} = \lim_{\Delta x \to 0} \frac{3x^2 + 6x\Delta x + 3(\Delta x)^2 + 1 - 3x^2 - 1}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{6x\Delta x + 3(\Delta x)^2}{\Delta x} = 6x$$

3. $f(x) = x^3 - 1$ (Ans: $3x^2$)

Differentiation

This is the process of finding $f^1(x)$ from $f(x)$ or more explicitly, known as differentiation with respect to x.

$$\frac{d}{dx} f(x) = f^1(x)$$

Or $Df = f^1$, symbol D being called an operator.

$\frac{d}{dx}$ - the operation of differentiation w.r.t.x.

Fundamental results

1. $\frac{d}{dx}\left(x^n\right) = nx^{n-1}$ for any rational number.

Examples

(i) $\frac{d}{dx}\left(x^{67}\right) = 67x^{66}$

(ii) $\frac{d}{dx}\left(\frac{1}{x^{29}}\right) = \frac{d}{dx}\left(x^{-29}\right) = -29x^{-30} = \frac{-29}{x^{30}}$

(iii) $\frac{d}{dx}\left(\sqrt{x}\right) = \frac{d}{dx}\left(x^{\frac{1}{2}}\right) = \frac{1}{2}x^{-1/2} = \frac{1}{2x^{1/2}} = \frac{1}{2\sqrt{x}}$ as seen earlier.

2. $\frac{d}{dx}(\sin x) = \cos x$

Proof from first principles

$$f^1(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{\sin(x + \Delta x) - \sin x}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{2\cos\frac{x + \Delta x + x}{2}\sin\frac{x + \Delta x - x}{2}}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{2\cos\left(x + \frac{\Delta x}{2}\right)\sin\frac{\Delta x}{2}}{\Delta x} = \lim_{\Delta x \to 0} \frac{\cos\left(x + \frac{\Delta x}{2}\right)\sin\frac{\Delta x}{2}}{\frac{\Delta x}{2}} = \lim_{\Delta x \to 0} \cos\left(x + \frac{\Delta x}{2}\right) \lim_{\Delta x \to 0} \frac{\sin\frac{\Delta x}{2}}{\frac{\Delta x}{2}}$$

$$= \cos x$$

Recall: $\lim_{x \to 0} \frac{\sin x}{x} = 0$

3. $\frac{d}{dx}(\cos x) = -\sin x$

Proof

$$f^1(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{\cos(x + \Delta x) - \cos x}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{-2\sin\frac{x + \Delta x + x}{2}\sin\frac{x + \Delta x - x}{2}}{\Delta x} = \lim_{\Delta x \to 0} \frac{-2\sin\left(x + \frac{\Delta x}{2}\right)\sin\frac{\Delta x}{2}}{\Delta x}$$

$$= \lim_{\Delta x \to 0} -\sin\left(x + \frac{\Delta x}{2}\right)\frac{\sin\frac{\Delta x}{2}}{\frac{\Delta x}{2}} = -\sin x$$

Rules of differentiation

We shall express these rules in terms of the functions *f*, *g* and also in terms of variables *u*, *v* where $u = f(x)$ and $v = g(x)$.

1. **Addition rules**

$$\frac{d}{dx}\left[f(x) \pm g(x)\right] = f^1(x) \pm g^1(x)$$

i.e. if $y = u \pm v$, then $\frac{dy}{dx} = \frac{du}{dx} \pm \frac{dv}{dx}$

Example

$$\frac{d}{dx}(\sin x + \cos x) = \cos x - \sin x$$

$$\frac{d}{dx}\left(x^{67} + \frac{1}{x^{29}}\right) = 67x^{66} - \frac{29}{x^{30}}$$

2. **Chain rule**

$$\frac{d}{dx} f(g(x)) = f^1(g(x)).g^1(x)$$

i.e. If *y* is a function of *v* and *v* is a function of *x*, then *y* is also a function of *x* and;

$$\frac{dy}{dx} = \frac{dy}{dv}.\frac{dv}{dx}$$

Example

Find $\frac{dy}{dx}$ for the following:

(i) $y=\left(2x^2+3\right)^2$

Let $v=2x^2+3$ then; $y=v^2 \Rightarrow \frac{dy}{dv}=2v$ and $\frac{dv}{dx}=4x$

Therefore, $\frac{dy}{dx}=\frac{dy}{dv}.\frac{dv}{dx}=2v.4x=2\left(2x^2+3\right)4x=8x\left(2x^2+3\right)$

(ii) $y=\left(3x^5+5x^2-1\right)^{27}$ $\left(Ans:135x\left(3x^3+2\right)\left(3x^5+5x^2-1\right)^{26}\right)$

(iii) $y=\sin^{27}x$

Let $v=\sin x$, then; $y=v^{27} \Rightarrow \frac{dy}{dv}=27v^{26}$ and $\frac{dv}{dx}=\cos x$

Therefore, $\frac{dy}{dx}=\frac{dy}{dv}.\frac{dv}{dx}=27v^{26}.\cos x=27\sin^{26}x\cos x$

Questions

Find $\frac{dy}{dx}$ of the following using the chain rule:

(i) $y=\frac{1}{\sqrt{x^2+1}}$

(ii) $y=\sin\left(x^{27}\right)$

(iii) $y=\sqrt{\cos x}$

(iv) $y=\cos\left(\sqrt{x}\right)$

3. Extended chain rule

We can easily extend the chain rule to show that if y is a function of v_1, v_1 a function of v_2, and v_2 is a function of x, then;

$$\frac{dy}{dx}=\frac{dy}{dv_1}\cdot\frac{dv_1}{dv_2}\cdot\frac{dv_2}{dx}$$

And more generally, if y is a function of $v_1, v_2, \ldots\ldots v_n$ a function of x, then;

$$\frac{dy}{dx}=\frac{dy}{dv_1}\cdot\frac{dv_1}{dv_2}\cdots\cdots\cdots\cdots\frac{dv_n}{dx}$$

Example

Use extended chain rule to find $\frac{dy}{dx}$ of the following:

(i) $y=\sin\sqrt{x^2+1}$

Let $v_1=\sqrt{x^2+1}$ $and\, v_2=x^2+1$

Then; $y=\sin v_1$ $and\, v_1=v_2^{\,1/2}$

Therefore, $\frac{dv_1}{dv_2}=\frac{1}{2}v_2^{\,-1/2}$, $\frac{dy}{dv_1}=\cos v_1$ $and\, \frac{dv_2}{dx}=2x$

$$\frac{dy}{dx}=\frac{dy}{dv_1}.\frac{dv_1}{dv_2}.\frac{dv_2}{dx}=\frac{1}{2}.\frac{1}{\sqrt{x^2+1}}.\cos\sqrt{x^2+1}.2x$$

$$=\frac{x\cos\sqrt{x^2+1}}{\sqrt{x^2+1}}$$

(ii) $y=\sqrt{\sin\left(x^2+1\right)}$ $\left(Ans: \frac{dy}{dx}=\frac{x\cos\left(x^2+1\right)}{\sqrt{\sin\left(x^2+1\right)}}\right)$

(iii) $y=\frac{1}{\sin^2(2x+3)}$ $\left(Ans: -\frac{4\cos(2x+3)}{\sin^3(2x+3)}\right)$

4. Product rule

$$\frac{d}{dx}\left[f(x).g(x)\right]=f(x).g^1(x)+f^1(x).g(x)$$

i.e. if $y=uv$, then;

$$\frac{dy}{dx}=u\frac{dv}{dx}+v\frac{du}{dx}$$

Examples

(i) If $y=x^2\sin x$, then;

$$\frac{dy}{dx}=x^2\cos x+2x\sin x$$

(ii) If $y=\sqrt{x}\cos x$, then;

$$\frac{dy}{dx}=-\sqrt{x}\sin x+\frac{1}{2\sqrt{x}}\cos x$$

5. Quotient rule

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right]=\frac{g(x).f^1(x)-f(x).g^1(x)}{\left[g(x)\right]^2}$$

i.e. if $y=\frac{u}{v}$, then;

$$\frac{dy}{dx}=\frac{v\frac{du}{dx}-u\frac{dv}{dx}}{v^2}$$

Examples

(i) If $y=\frac{x^2+1}{x^2-1}$, then;

$$\frac{dy}{dx}=\frac{\left(x^2-1\right)2x-\left(x^2+1\right)}{\left(x^2-1\right)^2}=\frac{-4x}{\left(x^2-1\right)^2}$$

(ii) If $y=\frac{\sin x}{1+\cos x}$, then;

$$\frac{dy}{dx} = \frac{(1+\cos x)\cos x - \sin x(-\sin x)}{(1+\cos x)^2}$$

$$= \frac{1}{1+\cos x}$$

The quotient rule may be used to derive another fundamental result:

$$\frac{d}{dx}(\tan x) = \frac{d}{dx}\left(\frac{\sin x}{\cos x}\right)$$

$$= \frac{\cos x \cos x - \sin x(-\sin x)}{\cos^2 x} = \frac{1}{\cos^2 x} = \sec^2 x$$

And similarly; $\frac{d}{dx}(\cot x) = -\cos ec^2 x$

6. Inverse rule

If f has an inverse f^{-1}, so that $y = f(x)$ and $x = f^{-1}(y)$, then;

$$\frac{d}{dx} f^{-1}(y) = \frac{1}{f^{-1}(x)}$$

i.e. $\frac{dx}{dy} = \frac{1}{\frac{dy}{dx}}$

Examples

(i) Find $\frac{d}{dx}\left(\sin^{-1} x\right)$

Let $y = \sin^{-1} x \Rightarrow x = \sin y$

$\frac{dx}{dy} = \cos y$; Hence, $\frac{dy}{dx} = \frac{1}{\cos y} = \frac{1}{\pm\sqrt{1-x^2}}$

For the inequalities, $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$ we see that only the positive value of the root should be taken. Hence;

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}$$

Note: $\frac{dy}{dx}$ is not defined at $x = \pm 1$, i.e. $sin^{-1}x$ is not differentiable at $x = \pm 1$

Question

Prove the following using the inverse rule:

(i) $\frac{d}{dx}\left(\cos^{-1} x\right) = -\frac{1}{\sqrt{1-x^2}}$

(ii) $\frac{d}{dx}\left(\tan^{-1} x\right) = \frac{1}{1+x^2}$

Differentiation of functions involving exponential, logarithmic and hyperbolic functions

The natural logarithm

$\ln x = \int_1^x \frac{1}{x} dx$ for all positive values of *x*. It follows that if $x > 1$, *lnx* is the area under the curve $y = \frac{1}{x}$ from point 1 to point *x* on the x – axis. If $0 < x < 1$, *lnx* is minus the area under this curve from *x* to the point 1, and *ln1 = 0*.

It follows immediately from the above definition that $\ln x$ is differentiable for all positive values of *x* and that;

$$\frac{d}{dx} \ln x = \frac{1}{x}$$

Generally this leads to the relationship; $\frac{d}{dx} f(x) = \frac{f^1(x)}{f(x)}$

Examples

Differentiate w.r.t.x the following:

1. $\ln(x^2 + 5)$

$$\frac{d}{dx} \ln(x^2 + 5) = \frac{2x}{x^2 + 5}$$

2. $\ln(3 - 2x)$

$$\frac{d}{dx} \ln(3 - 2x) = \frac{-2}{3 - 2x}$$

3. $x^2 \ln\left(1 + \frac{1}{x}\right)$

Using the product rule;

$$\frac{d}{dx}\left[x^2 \ln\left(1 + \frac{1}{x}\right)\right] = x^2\left(\frac{-x^{-2}}{1 + 1/x}\right) + 2x \ln\left(1 + \frac{1}{x}\right)$$

$$= 2x \ln\left(1 + \frac{1}{x}\right) - \left(\frac{x}{x+1}\right)$$

If $x > 0$ and b is a positive constant, then; $\frac{d}{dx} \ln(bx) = \frac{b}{bx} = \frac{1}{x}$

Therefore, $\frac{d}{dx} \ln(bx) = \frac{d}{dx} (\ln x)$

More examples

Differentiate the following w.r.t.x:

1. $\ln\left(\frac{2x-1}{2x+1}\right)$

Let $y=\ln\left(\frac{2x-1}{2x+1}\right)=\ln(2x-1)-\ln(2x+1)$

Then; $\frac{dy}{dx}=\frac{2}{2x-1}-\frac{2}{2x+1}=\frac{4}{4x^2-1}$

2. $\ln\sqrt{x^2+x+1}$

Let $y=\ln\sqrt{x^2+x+1}=\frac{1}{2}\ln\left(x^2+x+1\right)$

Then; $\frac{dy}{dx}=\frac{1}{2}\frac{2x+1}{x^2+x+1}$

3. $\ln\left[\frac{(x-1)^7}{\sqrt{2x+1}}\right]$ $\left(Ans:\frac{7}{x-1}-\frac{1}{2x+1}\right)$

4. $\frac{x\left(1+x^2\right)^3}{\sqrt[3]{1+x^3}}$

If $y=\frac{x\left(1+x^2\right)^3}{\sqrt[3]{1+x^3}}$; then $\ln y=\ln\frac{x\left(1+x^2\right)^3}{\sqrt[3]{1+x^3}}=\ln x+3\ln\left(1+x^2\right)-\frac{1}{3}\ln\left(1+x^3\right)$

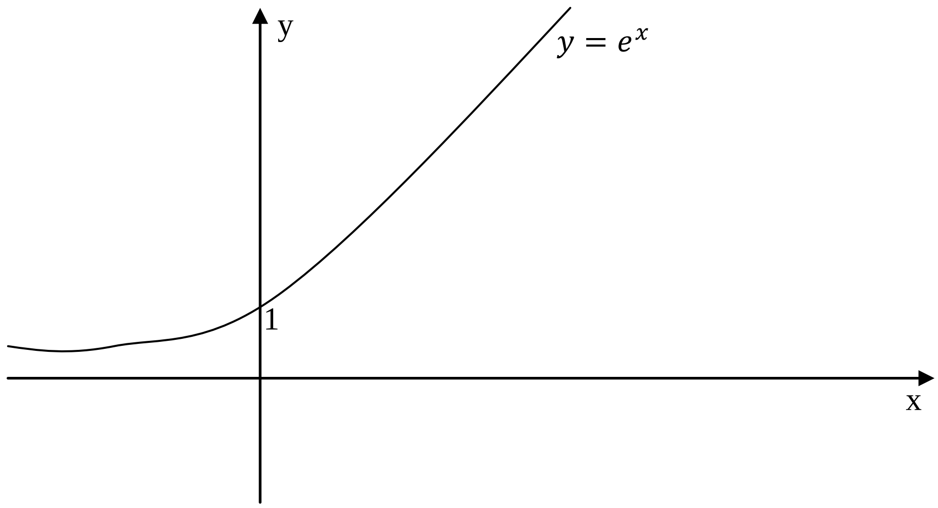
$$\frac{1}{y}\frac{dy}{dx}=\frac{1}{x}+\frac{6x}{1+x^2}-\frac{x^2}{1+x^3}=\frac{\left(1+x^2\right)\left(1+x^3\right)+6x(x)\left(1+x^3\right)-x^2\left(1+x^2\right)x}{x\left(1+x^2\right)\left(1+x^3\right)}$$

$$=\frac{1+7x^2+6x^5}{x\left(1+x^2\right)\left(1+x^3\right)}$$

Hence, $\frac{dy}{dx}=\frac{\left(1+7x^2+6x^5\right)\left(1+x^2\right)}{\left(1+x^3\right)^{4/3}}$

The exponential function

From the graph of *lnx* we can see that the natural logarithm has an inverse function, the graph of which is shown below:



This inverse function is called the exponential function. Thus: $y = e^x \Rightarrow x = \ln y$

If *x* is rational then;

$x = \ln y \Rightarrow y = e^x$ and so $\frac{dx}{dy} = \frac{1}{y}$ that is $\frac{dy}{dx} = y = e^x$

Hence, $\frac{d}{dx}\left(e^x\right) = e^x$

This suggests the general expression; $\frac{d}{dx}\left(e^{f(x)}\right) = f^1(x)e^{f(x)}$

Examples

Differentiate the following *w.r.t.x*:

1. $e^{\left(x^2+x+1\right)}$

   Let $y = e^{\left(x^2+x+1\right)}$, then; $\frac{dy}{dx} = (2x+1)e^{\left(x^2+x+1\right)}$

2. $y = e^{\frac{1}{x}}$

   $\frac{dy}{dx} = -x^{-2}e^{1/x} = -\frac{1}{x^2}e^{1/x}$

3. $y = e^{\sin x}$

   $\frac{dy}{dx} = \cos x e^{\sin x}$

Differentiation of b^x

If b is any positive number, we have $b = e^{\ln b}$

Therefore, $b^x = \left(e^{\ln b}\right)^x = e^{x\ln b}$. Since $e^{x\ln b}$ is defined for all *x*, this result suggests an obvious way of extending the definition of b^x; we simply redefine b^x by saying;

$b^x = e^{x\ln b}$ for all *x*. Using this we see that;

$$\frac{d}{dx}\left(b^x\right) = (\ln b)e^{x\ln b} = b^x \ln b$$

Examples

Differentiate *w.r.t.x* the following:

1. 5^x

   Let $y = 5^x$, then; $\ln y = \ln 5^x = x\ln 5$

   Hence; $\frac{1}{y}\frac{dy}{dx} = \ln 5 \Rightarrow \frac{dy}{dx} = 5^x \ln 5$

2. $3^{\left(x^2+x+1\right)}$

   Let $y = 3^{\left(x^2+x+1\right)} \Rightarrow \ln y = \ln 3^{\left(x^2+x+1\right)} = \left(x^2 + x + 1\right)\ln 3$

   $\frac{1}{y}\frac{dy}{dx} = (2x+1)\ln 3$.Therefore; $\frac{dy}{dx} = (2x+1)\left(3^{x^2+x+1}\right)\ln 3$

3. $y = x^x$

   $\ln y = \ln x^x = x\ln x$

   Applying the product rule; $\frac{1}{y}\frac{dy}{dx} = x.\frac{1}{x} + \ln x$. Hence $\frac{dy}{dx} = x^x(1+\ln x)$

4. $\left(x^2+1\right)^{\sin x}$ $\left(Ans: \left(x^2+1\right)^{\sin x}\left[\frac{2x\sin x}{x^2+1} + \cos x \ln\left(x^2+1\right)\right]\right)$

Hyperbolic functions

We define hyperbolic sine of *x* (written as *sinhx*) and the hyperbolic cosine of *x* (written as *coshx*) by the relations;

$$\sinh x = \frac{1}{2}\left(e^x - e^{-x}\right) \text{ and } \cosh x = \frac{1}{2}\left(e^x + e^{-x}\right)$$

Note Recall all identities involving hyperbolic sines and cosines.

Derivatives of hyperbolic functions

- $\frac{d}{dx}(\sinh x) = \cosh x$
- $\frac{d}{dx}(\cosh x) = \sinh x$

From these results we can deduce that:

- $\frac{d}{dx}(\tanh x) = \sec h^2 x$
- $\frac{d}{dx}(\coth x) = -\cos ech^2 x$
- $\frac{d}{dx}(\sec hx) = -\sec hx \tanh x$
- $\frac{d}{dx}(\cos echx) = -\cos echx \coth x$

Derivatives of inverse hyperbolic functions

These can be derived using the inverse rule of differentiation; $\frac{dx}{dy} = \frac{1}{dy/dx}$

Or alternatively by using, the relations:

$\sinh^{-1} x = \ln\left[x + \sqrt{x^2 + 1}\right]$,

$\cosh^{-1} x = \ln\left[x + \sqrt{x^2 - 1}\right]$ and;

$\tan nh^{-1} x = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)$

The results are:

- $\frac{d}{dx}(\sinh^{-1} x) = \frac{1}{\sqrt{1+x^2}}$
- $\frac{d}{dx}(\cosh^{-1} x) = \frac{1}{\sqrt{x^2 - 1}}$
- $\frac{d}{dx}(\tanh^{-1} x) = \frac{1}{1-x^2}$

Implicit differentiation

From the chain rule it follows that; $\frac{d}{dx}(\sinh^{-1} x) = \frac{1}{\sqrt{1+x^2}}$

Examples

1. $\frac{d}{dx}(y^2) = 2y\frac{dy}{dx}$
2. $\frac{d}{dx}(\sin y) = \cos y \frac{dy}{dx}$

From an implicit equation we can thus find $\frac{dy}{dx}$ by differentiating both sides of the equation with respect to *x*.

Examples

1. If $x^2 + y^2 = 1$, then;

   $$2x + 2y\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = -\frac{x}{y}$$

2. If $\sqrt{x} + \sin\ y = x^2$, then

   $$\frac{1}{2\sqrt{x}} + \cos y\frac{dy}{dx} = 2x \Rightarrow \frac{dy}{dx} = \frac{2x - \frac{1}{2\sqrt{x}}}{\cos y}$$

Special care is needed with terms that contain both *x* and *y*, e.g

1. $\frac{d}{dx}(xy) = x\frac{dy}{dx} + y$
2. $\frac{d}{dx}\left(x^2y^3\right) = x^2.3y^2\frac{dy}{dx} + 2xy^3 = 3x^2y^2\frac{dy}{dx} + 2xy^3$
3. $\frac{d}{dx}\left(\frac{x^2}{\sin y}\right) = \frac{\sin y.2x - \cos y\frac{dy}{dx}x^2}{\sin^2 y} = \frac{2x\sin y - x^2\cos y\frac{dy}{dx}}{\sin^2 y}$
4. $\frac{d}{dx}[\sin(xy)]$

   Let $u = xy \Rightarrow \frac{du}{dx} = x\frac{dy}{dx} + y$

   $$\frac{d}{dx}[\sin u] = \cos u\frac{du}{dx}$$

   $$\frac{d}{dx}[\sin(xy)] = \cos(xy)\left(x\frac{dy}{dx} + y\right)$$

More examples

Find $\frac{dy}{dx}$ given that;

1. $x^2 + xy + y^2$

   $$2x + x\frac{dy}{dx} + 2y\frac{dy}{dx} \Rightarrow \frac{dy}{dx} = -\frac{(2x+y)}{x+2y}$$

2. $\sin\left(\frac{x}{y}\right) = xy$

   Let $u = \frac{x}{y} \Rightarrow \frac{du}{dx} = \frac{y - x\frac{dy}{dx}}{y^2}$

   $$\sin u = xy \Rightarrow \cos u\left(\frac{du}{dx}\right) = y + x\frac{dy}{dx}$$

$$\therefore \cos\left(\frac{x}{y}\right)\left(\frac{y - x\frac{dy}{dx}}{y^2}\right) = x\frac{dy}{dx} + y \Rightarrow \frac{dy}{dx} = \frac{y\left[\cos\left(\frac{x}{y}\right) - y^2\right]}{x\left[\cos\left(\frac{x}{y}\right) + y^2\right]}$$

Parametric differentiation

A relation between x and y is sometimes given by expressing both x and y in terms of another variable, say, t called a parameter.

If x = f(t), y = g(t), then by the chain rule, we have;

$\frac{dy}{dx} = g^1(x)\frac{dt}{dx}$ and since $\frac{dt}{dx} = \frac{1}{\left(dx/dt\right)}$ by the inverse rule, it follows that:

$$\frac{dy}{dx} = \frac{g^1(t)}{f^1(t)} \text{ i.e. } \frac{dy}{dx} = \frac{dy/dt}{dx/dt}$$

Examples

1. If $x = \cos t,\ y = \sin t$, then;

$$\frac{dy}{dx} = \frac{\cos t}{-\sin t} = -\cot t$$

2. If $x = t + \frac{1}{t},\ y = 3t^{1/2} + t^{3/2}$, then;

$$\frac{dy}{dx} = \frac{\frac{3}{2}t^{-1/2} + \frac{3}{2}t^{1/2}}{1 - \frac{1}{t^2}} = \frac{3t^{3/2}}{2(t-1)}$$

Significance of sign of derivative

We say that f(x) is increasing in the interval $a \le x \le b$ if f(x) increases as x increases from a to b inclusive. We say that f(x) is increasing at the point x = x_0 if there is some neighbourhood of x_0 (i.e some interval with point x_0 inside it) in which f(x) is increasing.

We say that f(x) is decreasing in the interval $a \le x \le b$ if f(x) is decreasing as x increases from a to b inclusive, and f(x) is decreasing at point x_0 if there is some neighbouhood of x_0 in which f(x) is decreasing.

The slope of the curve y = f(x) at any point equals to the value of $f^1(x)$ at that point. It follows that f(x) is increasing at a point where $f^1(x) > 0$ and f(x) is decreasing at a point where $f^1(x) < 0$. A point where $f^1(x) = 0$ is called a *stationary point* and the tangent to the curve is parallel to the x – axis.

Summary

- $\frac{dy}{dx} > 0 \Rightarrow$ y is increasing
- $\frac{dy}{dx} < 0 \Rightarrow$ y is decreasing
- $\frac{dy}{dx} = 0 \Rightarrow$ y is stationary.

Stationary point

If $f^1(x)$ changes sign from positive on the left of the stationary point to negative on its right, such a point is reffered to as a *maximum turning point.*

If the sign of $f^1(x)$ changes from negative on the left of a stationary point to positive to its right, then the point is a minimum turning point.

No change of sign from to right of a stationary point normally means a *point of inflexion* at that point.

Higher derivatives

The derivative of the function f^1 is denoted f^{11},the derivative of f^{11} is denoted f^{11}, and so on.

i.e if $y = f(x)$

$$\frac{dy}{dx} = f^1(x)$$

$$\frac{d^2y}{dx^2} = f^{11}(x)$$

$\frac{d^3y}{dx^3} = f^{111}(x)$ and so on.

Example

Find $\frac{d^2y}{dx^2}$ given that;

$x^2 + y^2 = 1$

$$2x + 2y\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = -\frac{x}{y}$$

$$\frac{d^2y}{dx^2} = \frac{y(-1) - -x\frac{dy}{dx}}{y^2} = \frac{-y + x\left(-\frac{x}{y}\right)}{y^2} = -\frac{x^2+y^2}{y^3} = -\frac{1}{y^3}$$

Note: Although $\frac{dx}{dy} = \frac{1}{dy/dx}$, in general $\frac{d^2x}{dy^2} \neq \frac{1}{d^2y/dx^2}$

Example 2

Find $\frac{d^2y}{dx^2}$ when $x = sint$ and $y = cost$

By chain rule; $\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$

$$\frac{dy}{dt} = -\sin t \quad and \quad \frac{dx}{dt} = \cos t \Rightarrow \frac{dy}{dx} = \frac{-\sin t}{\cos t} = -\tan t$$

By chain rule again;

$$\frac{d^2y}{dx^2} = \frac{d}{dt}\left(\frac{dy}{dx}\right)\frac{dt}{dx} = -sec^2t.\frac{1}{cost} = -sec^3t$$

Theorem of Leibnitz

This theorem is an extension of the product rule to higher derivatives. We use the notation:

$u_r = \frac{d^ru}{dx^r}$ and $v_r = \frac{d^rv}{dx^r}$

Theorem

If $y = uv$, then;

$$\frac{d^ny}{dx^n} = u_nv + \overset{n}{\underset{1}{C}}u_{n-1}v_1 + \overset{n}{\underset{2}{C}}u_{n-2}v_2 + \dots\dots + \overset{n}{\underset{r}{C}}u_{n-r}v_r + \dots\dots + uv_n$$

Where the coefficients $\overset{n}{\underset{1}{C}}, \overset{n}{\underset{2}{C}}, \dots\dots\dots\dots$ are those which occur in the Binomial expansion of $(1 + x)^n$.

This theorem can be easily verified for n = 1, 2, 3, ………. Using the product rule. For $y = uv$, then;

$$\frac{dy}{dx} = u_1v + uv_1$$

$$\frac{d^2y}{dx^2} = u_2v + 2u_1v_1 + uv_2$$

$$\frac{d^3y}{dx^3} = u_3v + 3u_2v_1 + 3u_1v_2 + uv_3$$

Question

Find $\frac{d^6y}{dx^6}$, when $y = x^3 \sin x$ $\quad \left(Ans: 6\left(3x^2 - 20\right)\cos x - x\left(x^2 - 90\right)\sin x\right)$

Significance of the sign of the second derivative

If:

- $\frac{d^2y}{dx^2} > 0 \Rightarrow$ the curve is concave upwards
- $\frac{d^2y}{dx^2} < 0 \Rightarrow$ the curve is convex upwards or concave downwards.
- $\frac{d^2y}{dx^2} = 0 \Rightarrow$the curve has a point of inflexion, provided it changes sign.

Examples

Find the points of inflexion on the following curves:

1. $y = x^3 + 6x^2 + 7x + 1$

$$\frac{dy}{dx} = 3x^2 + 12x + 7 \ and \ \frac{d^2y}{dx^2} = 6x + 12$$

For point of inflexion;

$$\frac{d^2y}{dx^2} = 6x + 12 = 0 \Rightarrow x = -2$$

Clearly $\frac{d^2y}{dx^2} = 0$ and it changes sign at $x = -2$ i.e. the point $(-2,3)$ is a point of inflexion.

2. $y = x^4 + 4x^3 + 6x^2 - 2$

$$\frac{dy}{dx} = 4x^3 + 12x^2 + 12x \quad \Rightarrow \frac{d^2y}{dx^2} = 12x^2 + 24x + 12 = 12(x+1)^2$$

The only point where we have $\frac{d^2y}{dx^2} = 0$ is at $x = -1$, but $\frac{d^2y}{dx^2}$ does not change sign there, hence there are no points of inflexion on this curve.

Criteria for maximum and minimum turning points

The curve $y = f(x)$ has a maximum turning point at $x = a$ if $f^1(a) = 0$ and if $f^1(x)$ changes sign from positive to negative as x increases through the value a. Alternatively, there is a maximum turning point at $x = a$ if $f^1(a) = 0$ and $f^{11}(a) < 0$

The curve $y = f(x)$ has a minimum turning point at $x = a$ if $f^1(a) = 0$ and $f^1(a)$ changes sign from negative to positive as x increases through the value of a. Alternatively, there is a minimum turning point at $x = a$ if $f^1(a) = 0$ and $f^{11}(a) > 0$.

The alternative form does not enable us decide the nature of the turning point if $f^{11}(a) = 0$, it may also be inconvenient if $f^{11}(x)$ is difficult to find.

Examples

Find the turning points on the following curves and determine their nature.

1. $y = 2x^3 - 15x^2 + 36x - 28$

   $\frac{dy}{dx} = 6x^2 - 30x + 36 = 6(x-2)(x-3)$

   $\frac{d^2y}{dx^2} = 12x - 30$

   Hence, $\frac{dy}{dx} = 0$ when $x = 2$ and $x = 3$

   When $x = 2$, $\frac{d^2y}{dx^2} = -6 < 0$ so that this point is a maximum turning point.

   When x =3, $\frac{d^2y}{dx^2} = 6 > 0$ so that this point is a minimum turning point

   Therefore, the turning points are:
   (2, -72), maximum and (3, -109); minimum

2. $y = (x-1)^4$

   $\frac{dy}{dx} = 4(x-1)^3$

   $\frac{d^2y}{dx^2} = 12(x-1)^2$

   Hence $\frac{dy}{dx} = 0$ only when x =1

   When x =1, $\frac{d^2y}{dx^2} = 0$, so the test fails to determine the nature of the point. However, it is easy to see that $\frac{dy}{dx}$ changes sign from – to + as x increases through the value $x = 1$, so that the point is a minimum turning point.

3. $y = \frac{x}{\sqrt{x-1}}$

   $$\frac{dy}{dx} = \frac{\sqrt{x-1} - \frac{x}{2\sqrt{x-1}}}{x-1} = \frac{x-2}{2(x-1)^{3/2}}$$

   Hence, $\frac{dy}{dx} = 0$ when $x = 2$. In this case it is more laborious to find $\frac{d^2y}{dx^2}$ than to check that $\frac{dy}{dx}$ changes sign from – to + as x increases through value 2.

Applications of differential calculus

Introduction

If $y = f(x)$, then the value of $\frac{dy}{dx}$ at the point $x = x_0$, i.e. $f^1(x_0)$ measures:

a) The rate of change of y with respect to x at the point $x = x_0$, or, in other words,
b) The slope of the curve $y = f(x)$ at the point $x = x_0$
Recall: y is increasing, decreasing or stationary as x increases through the value x_0 accordingly as $f^1(x_0)$ is positive, negative or zero.

Related rates

If y is a function of x and both x and y are changing with respect to time, then there must be some connection between their two rates of change. This connection is given by the chain rule, since we must have:

$$\frac{dy}{dt} = \frac{dy}{dx}.\frac{dx}{dt}$$

Recommended procedure for solving related rates problems

(i) If the problem can be interpreted geometrically, draw a sketch of the problem.

(ii) Label the important quantities on your sketch, such as variables and constants according to the statement of the problem.

(iii) From the sketch, together with known relationships; either given in the problem or known from geometry or trigonometry, write down an equation relating the relevant variables.

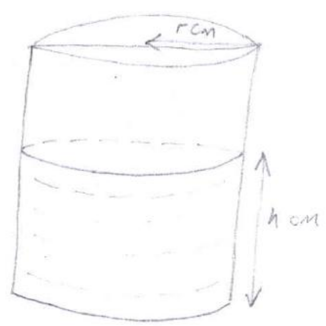
(iv) Differentiate both sides of the equation with respect to time, t, using the chain rule where necessary to obtain an equation relating the rates.

(v) Solve for the desired rate, and after doing so, substitute the given data or information into this expression to obtain the solution.

(vi) Express the solution in terms of the original problem.

Examples

1. Water pours into a right circular cylindrical tank at the rate of 4 cm^3/s. At what rate is the level rising in the tank?
Solution
The volume ($V\ cm^3$) of the water in the tank at any time t is given by; $V = \pi r^2 h$
Where r cm is the radius of the tank and h cm is the depth of the water in the tank.



Hence;

$$\frac{dV}{dt}=\frac{dV}{dh}.\frac{dh}{dt}$$

$$\frac{dV}{dh}=\pi r^2$$

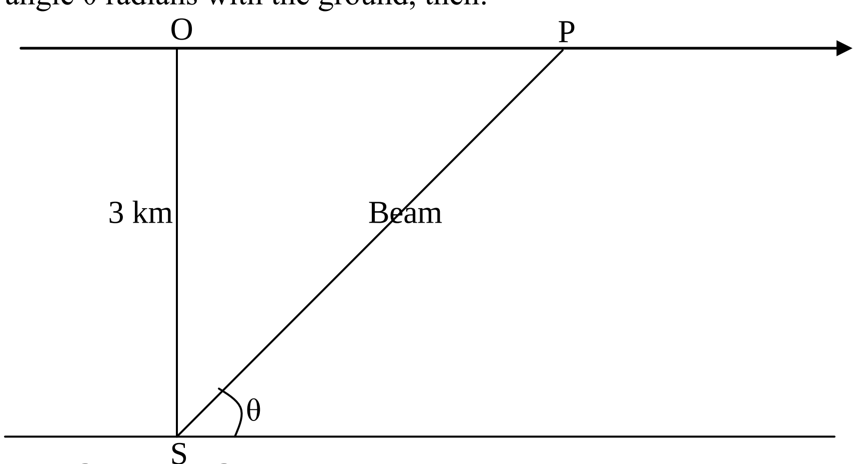
$$\frac{dV}{dt}=\pi r^2.\frac{dh}{dt} \text{ but } \frac{dV}{dt}=4 \text{ and so; } \frac{dh}{dt}=\frac{4}{\pi r^2}$$

Hence the level is rising at the rate of $\frac{4}{\pi r^2}cm/s$

2. Solve problem 1. Above for the case in which the tank is conical with vertex angle 60^0. (Ans: $\frac{12}{\pi r^2}$)
3. An aero plane flying in a straight path at a constant height of 3 km passes directly over a search light on the ground and is subsequently caught in the beam of the search light with the beam at an angle of 60^0 to the level ground. In order to keep the aeroplane in the beam it is found necessary to start lowering the beam at 4^0/s. What is the speed of the aeroplane (in km/h) at that moment?

Solution

Let S be the position of the search light, O the point of the flight path 3 km above S. Let P be the position of the aero plane at time, t, and suppose the distance OP is x km while SP makes an angle θ radians with the ground, then:



$$\tan\theta=\frac{3}{x}\Rightarrow x=\frac{3}{\tan\theta};\ x=3\cot\theta$$

$$\frac{dx}{dt}=\frac{dx}{d\theta}.\frac{d\theta}{dt}$$

$$\frac{dx}{d\theta}=-3\cos ec^2\theta$$

$$\therefore\frac{dx}{d\theta}=-3\cos ec^2\theta.\frac{d\theta}{dt}$$

Given the line SP rotates at 4 degrees/s, then θ is changing at $\frac{4\pi}{180}$ rad/s. since SP is rotating clockwise, θ is decreasing so, that $\frac{d\theta}{dt}$ is negative. Hence; $\frac{d\theta}{dt} = -\frac{4\pi}{180}$.

Given also $\theta = 60^0 = \frac{\pi}{3}$, then;

$$\frac{dx}{dt} = -3\cos ec^2\left(\frac{\pi}{3}\right).\left(-\frac{4\pi}{180}\right) = \frac{16\pi}{180}$$

It follows that x is increasing at a rate of $\frac{16\pi}{180}$km/s i.e 320π km/h, and therefore, the plane is flying at a rate of 320π km/h.

4. A man is standing 30 m from a straight – road track. A train is approaching moving along the track at a rate of 90 km/h. How fast is the distance between the train and the man decreasing when the train is 50 m away from the man? (Ans: 72 km/h)

5. A ladder 8 m long leans against a wall 4 m high. The lower end of the ladder is pushed away from the wall at a rate of 2 m/s. How fast is the angle between the top of the ladder and the wall changing when the angle between the ladder and the wall is 60^0? (Ans: 0.5 rad s^{-1})

Problems concerning maximum and minimum values

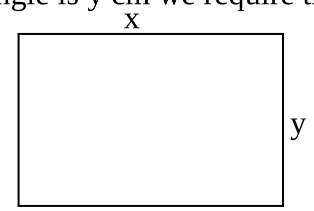
Many problems reduce to the fundamental problem of finding the maximum or minimum value of *f(x)* in same closed interval, where the function *f* is continuous on that interval and differentiable at all points of the interval except (possibly) the end points.

Procedure generally applied

(a) Assign letter names to all variables and constants in the problem and if possible draw a sketch or geometrical implication of the problem.
(b) Identify the variable for which the extreme is being sought.
(c) Find an equation of this variable in terms of other constants and variables. This is called the *principle equation*.
(d) Find any other equations involving the given variables and constants. These are called the *auxiliary equations*.
(e) Use the auxiliary equations to substitute for variables in the principle equation until a function in terms of a single independent variable of interest is obtained.
(f) Determine the closed interval $[a, b]$ over which the function is defined.
(g) Find the maximum or minimum value of the function over the interval (a, b) using the known techniques of differentiation.
(h) Describe the solution found in (g) in the language of the original problem.

Examples

1. How should a wire of length 48 cm be bent to produce a rectangle of maximum area?
   Suppose the wire is bent at a point to form one side of length *x*. If the other side of the rectangle is y cm we require that,



$$2x + 2y = 48$$

$$y = 24 - x$$

The area of the rectangle is therefore, given by;

$$A = xy = x(24 - x)$$

$$A = 24x - x^2$$

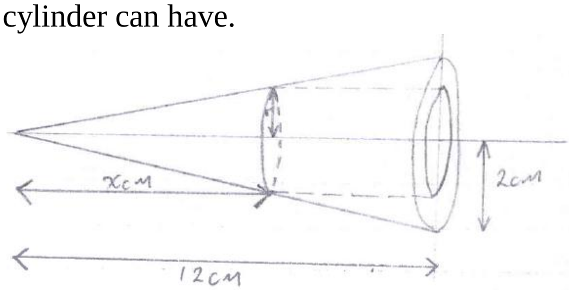
It is clear that A lies between 0 and 24, therefore, we can regard the problem as that of finding the maximum value of $24x - x^2$ for values of x in the interval; $0 \le x \le 24$.

A is continuous and differentiable on this interval and;

$$\frac{dA}{dx} = 24 - 2x$$

A is maximum, when $\frac{dA}{dx} = 0$ therefore; $x = 12$. This means the wire should be bent to form a square of side *12 cm*.

2. A circular cylinder is cut from a cone as shown in the figure below. Given that the radius of the base of the cone is 2 cm and its height is 12 cm, find the maximum volume that the cylinder can have.



Solution

Suppose the top of the cylinder is x cm from the vertex of the cone. Then the height of the cylinder is (12 – x) cm. Let the radius of the cylinder be r cm. From similar triangles, we have:

$$\frac{r}{x} = \frac{2}{12} \Rightarrow r = \frac{1}{6}x$$

Hence the volume of the cylinder (V cm^3) is given by:

$$V = \pi r^2 h = \pi r^2 h = \pi \left(\frac{1}{6} x \right)^2 (12 - x)$$

$$V = \frac{\pi}{36}\left(12x^2 - x^3\right)$$

We wish to find the maximum value of V on the interval $0 \le x \le 12$. Clearly V is continuous and differentiable on this interval, and;

$$\frac{dV}{dx} = \frac{\pi}{36}\left(24x - 3x^2\right) = \frac{\pi}{12} x(8 - x)$$

Therefore, the stationary values of V occur when x = 0 and x = 8 and are V = 0 and $V = \frac{64\pi}{9}$ respectively.

Hence the maximum value of the volume in the interval is $V_{\max} = \frac{64\pi}{9}$

3. A water trough is to be constructed from 3 metal sheets of length 6 m and width 1 m. It is to be constructed with end panels in shape of trapezoids. The end panels are made from other materials. Find the angle at which the metal sheets should be joined so as to provide a trough of maximum volume. (Ans: 120^0)

Approximations and small errors

From the derivative $f^1(x)$ it follows immediately that:

$$\frac{f(x + \Delta x) - f(x)}{\Delta x} \approx f^1(x) \text{ for small } \Delta x.$$

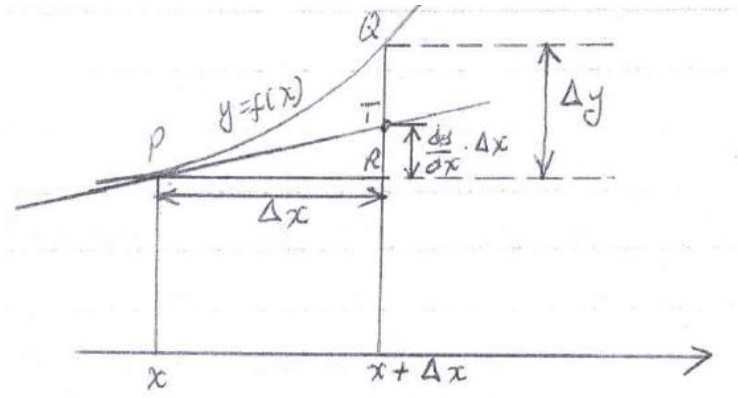
Hence;

$$f(x + \Delta x) - f(x) \approx f^1(x)\Delta x$$

Therefore, for small Δx;

$$\Delta y \approx \left(\frac{dy}{dx} \right) . \Delta x$$

This approximation is illustrated in the figure below:



P and Q are two points on the curve *y = f(x)*, PT is the tangent to the curve at P, and we are saying that the length RQ is approximately equal to the length RT provided that Q is close to P.

Consequently, if the values of f(x) and $f^1(x)$ are both known at one point, the value of f(x) anywhere in the neighbourhood of that point can be calculated approximately.

Example

Using the concept of differentiation, calculate $\sqrt{3.987}$ approximately.

$y = \sqrt{x}\ when\, x = 4, y = 2$

$\frac{dy}{dx} = \frac{1}{2\sqrt{x}} = \frac{1}{4}$ when $x = 4$

$3.987 = 4 + \Delta x$

$\Delta x = -0.013$

Then

$\Delta y = \left(\frac{dy}{dx}\right).\Delta x$

$\therefore \Delta y = \frac{1}{4}.(-0.013) = -0.00327$

Hence; $\sqrt{3.987} \approx 2 - 0.00327 = 1.99675$

Curve sketching

To sufficiently sketch the curve of a rational function, the following points must be noted:

(a) Domain and range of the function.
(b) *x* and *y* intercepts.
(c) Establish the interval over which the function f is either increasing or decreasing.
(d) Look for maxima and minima points of the function.
(e) Find the interval over which the function f is concave upwards or concave downwards.
(f) Look for points of inflexion.

(g) Look for asymptotes to the curve. An asymptote to a curve is a straight line to which the shape of the curve approximates at a great distance from the origin i.e. the distance from a point P on the curve to the line tends to zero as the distance OP from the origin to P tends to infinity. We distinguish between three types of asymptotes:

1. **Horizontal asymptotes (asymptotes parallel to the x – axis)**
   The line $y = b$ is an asymptote to the curve y = f(x), if $f(x) \to b$ as $x \to \infty$ or if $f(x) \to b$ as $x \to -\infty$ i.e. $\lim_{x\to+\infty} f(x) = b$ or $\lim_{x\to-\infty} f(x) = b$
2. **Vertical asymptotes (Asymptotes parallel to the y – axis)**
   The line $x = a$ is an asymptote to the curve y = f(x) if $f(x) \to \pm\infty$ as $x \to a$ or as $x \to a-$
3. **Slant asymptotes**
   The line $y = mx + C$ is an asymptote to the curve *y = f(x)* if $f(x) - mx - C \to 0$ as $x \to \infty$ or as $x \to -\infty$

Examples

Sketch the following curves:

1. $f(x) = \dfrac{x+1}{x+2}$

<u>Solution</u>

Domain and range

$x + 2 \neq 0 \Rightarrow x \neq -2$

Domain: $(-\infty,-2)U(-2,+\infty)$

Range: $(-\infty, +\infty)$

y – intercept

when $x = 0$, $y = \dfrac{1}{2}$

$$f^1(x) = \frac{(x+2).1-(x+1)}{(x+1)^2} = \frac{1}{(x+2)^2}$$

For the interval we use the domain:

Interval	**f^1(x)**	**Increasing/decreasing**
$(-\infty, -2)$	+	Increasing
$(-2, \infty)$	+	Increasing

Maxima or minima

$f^1(x) = \dfrac{1}{(x+2)^2} = 0$; Undefined hence there are no maxima or minima points.

Concavity

$$f^{11}(x) = -2(x+2)^{-3} = \frac{-2}{(x+2)^3}$$

Interval	$\mathbf{f^{11}(x)}$	**Concavity**
$(-\infty,-2)$	+	Concave upwards
$(-2,\infty)$	-	Concave downwards

Points of inflexion

$f^{11}(x) = 0$; $f^{11}(x) = \dfrac{-2}{(x+2)^3} = 0$; undefined, hence no points of inflexion.

Horizontal asymptotes

$$\lim_{x\to\pm\infty} f(x) = b = \lim_{x\to\pm\infty}\frac{x+1}{x+2} = \lim_{x\to\pm\infty}\frac{1+1/x}{1+2/x} = 1$$

Therefore, $y = 1$ is a horizontal asymptote.

Vertical asymptotes

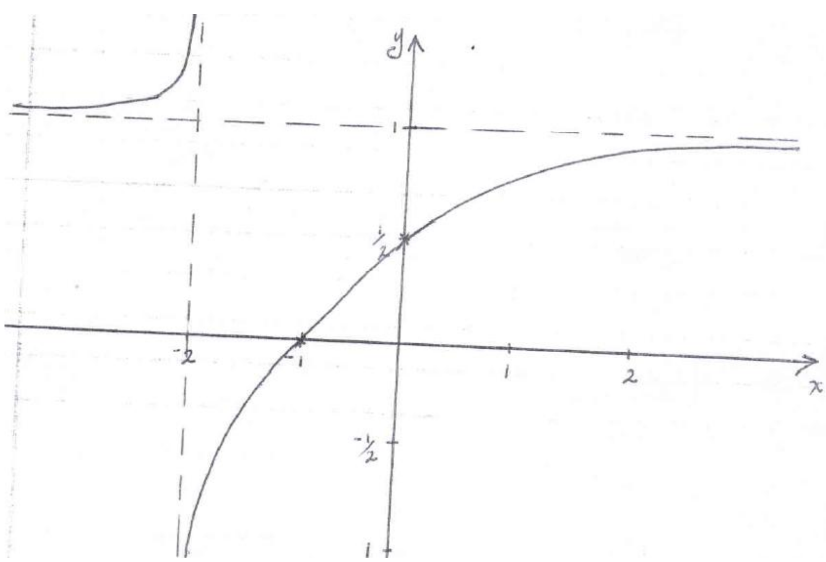
f(x) is not defined at $x = -2$,

$$\lim_{x\to-2+}\frac{x+1}{x+2} = \infty$$

$$\lim_{x\to-2-}\frac{x+1}{x+2} = -\infty$$

hence x = -2 is a vertical asymptote.

<u>**Sketch**</u>



2. $f(x)=\dfrac{1-x^2}{x^3}$

Domain and range

$x \neq 0$

Domain: $(-\infty,0)U(0,+\infty)$

Range: $(-\infty,+\infty)$

x – intercept

When y = 0; $\dfrac{1-x^2}{x^3}=0 \Rightarrow 1-x^2=0$

$x=\pm 1$, hence the x – intercepts are; (1, 0) and (-1, 0).

y – intercept

When x = 0, $y=\dfrac{1-0}{0}$; undefined, hence no y – intercepts.

$$f^1(x)=\frac{x^3(-2x)-(1-x^2)(3x^2)}{x^6}=\frac{-2x^4-3x^2+3x^4}{x^6}=\frac{x^2-3}{x^6}$$

Maxima or minima

$f^1(x)=0 \Rightarrow \dfrac{x^2-3}{x^4}=0$; $x=\pm\sqrt{3}$

For the interval we use the domain, putting the turning points in consideration:

Interval	$\mathbf{f^1(x)}$	**Increasing/decreasing?**
$(-\infty,-\sqrt{3})$	+	Increasing
$(-\sqrt{3},0)$	-	Decreasing
$(0,\sqrt{3})$	-	Decreasing
$(\sqrt{3},\infty)$	+	Increasing

For $x=+\sqrt{3}$; $y=\dfrac{1-(\sqrt{3})^2}{(\sqrt{3})^3}=-0.385 \Rightarrow (\sqrt{3},-0.385)$ is a minima point.

For $x=-\sqrt{3}$; $y=\dfrac{1-(-\sqrt{3})^2}{(-\sqrt{3})^3}=0.385 \Rightarrow (\sqrt{3},0.385)$ is a maxima point.

Points of inflexion

$$f^{11}(x)=\frac{-2x^2+12}{x^5}$$

$f^{11}(x)=\dfrac{-2x^2+12}{x^5}=0 \Rightarrow 2x^2=12$; $x=\pm\sqrt{6}$

For $x = +\sqrt{6}$; $y = \dfrac{1-\left(\sqrt{6}\right)^2}{\left(\sqrt{6}\right)^3} = -0.34 \Rightarrow \left(\sqrt{6}, -0.34\right)$

For $x = -\sqrt{6}$; $y = \dfrac{1-\left(-\sqrt{6}\right)^2}{\left(-\sqrt{6}\right)^3} = 0.34 \Rightarrow \left(\sqrt{6}, 0.34\right)$

Horizontal asymptotes

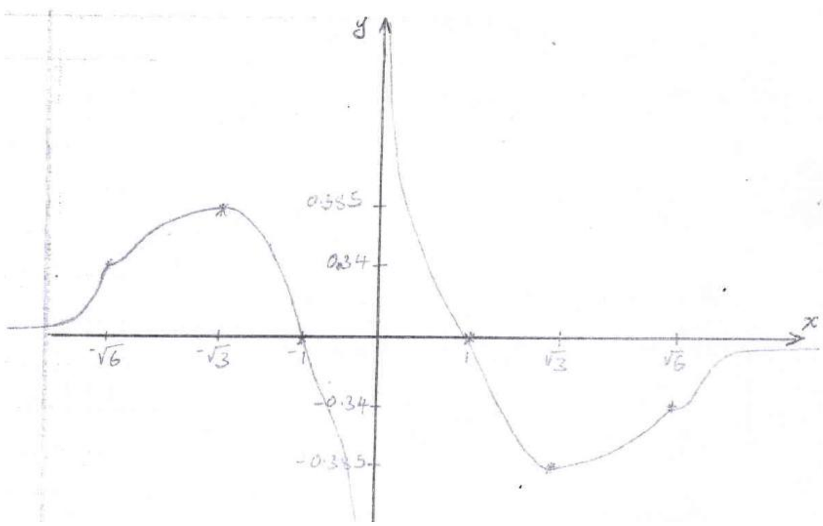
$\lim\limits_{x\to\pm\infty} \dfrac{1-x^2}{x^3} = 0 \Rightarrow y = 0$ is a horizontal asymptote.

Vertical asymptote

$\lim\limits_{x\to 0^+} \dfrac{1-x^2}{x^3} = -\infty$ and;

$\lim\limits_{x\to 0^-} \dfrac{1-x^2}{x^3} = +\infty$

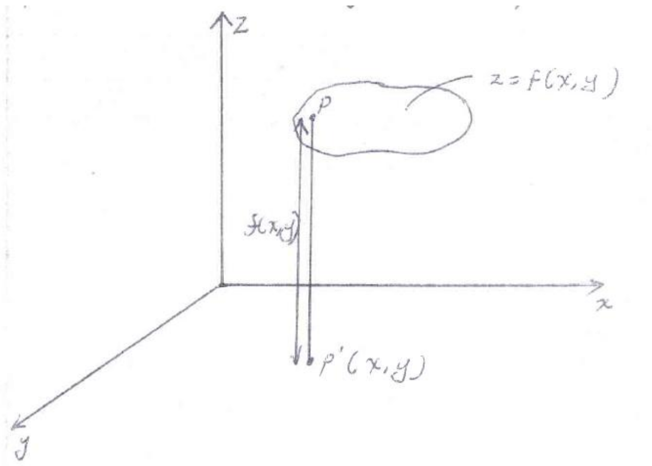
<u>Sketch</u>



Functions of two variables

A function of two variables defined on a domain D in a plane is a relationship that associates each point *(x, y)* in D with a real number *f(x, y)*. e.g. $f(x, y) = \dfrac{x^2 + y}{x^2 + y^2}$.

The graph of a function of two variables is frequently a height above or below the plane domain. It is a set of points (x, y, z) in space related by the equation $Z = f(x, y)$.



A function of two variables f(x, y) defined over a domain D has got a limit as (x, y) approaches (a, b).

$\lim\limits_{(x,y)\to(a,b)} f(x, y) = l$, provided that f(x, y) gets close to l as (x, y) approaches (a, b) along any curve.

Examples

1. Find $\lim\limits_{x\to(1,1)} 4 - x - y$

   $$\lim_{(x,y)\to(1,1)} 4 - x - y = \lim_{x\to 1} 3 - x = 2$$

2. $\lim\limits_{(x,y)\to(0,0)} \dfrac{y^2 - x^2}{y^2 + x^2}$ $(Ans: Limit\,does not exist)$

Continuity

If f is a function defined in the neighbourhood of (a, b) at point (x_0, y_0), f is continuous at (x_0, y_0) if the limit as $(x, y) \to (x_0, y_0)$ or if $f(x_0, y_0)$ exists i.e.

$$\lim_{(x,y)\to(x_0,y_0)} f(x, y) = f(x_0, y_0)$$

Partial derivatives

The partial derivative of f with respect to x is denoted by $\dfrac{\partial f}{\partial x}$ and is defined by the equation:

$\dfrac{\partial f}{\partial x} = \lim\limits_{\Delta x\to 0} \dfrac{f(x + \Delta x, y) - f(x, y)}{\Delta x}$ i.e. $\dfrac{\partial f}{\partial x}$ or $\dfrac{\partial z}{\partial x}$ is the rate of change of f(x, y) with respect to x when y is held fixed.

Similarly;

$\dfrac{\partial f}{\partial y} = \lim\limits_{\Delta y\to 0} \dfrac{f(x, y + \Delta y) - f(x, y)}{\Delta y}$ i.e. $\dfrac{\partial f}{\partial y}$ or $\dfrac{\partial z}{\partial y}$ is the change of f(x, y) or z with respect to y when x is held fixed.

The notation; $f_x = \frac{\partial f}{\partial x}$ and $f_y = \frac{\partial f}{\partial y}$ is commonly used.

To find $\frac{\partial f}{\partial x}$, we regard f(x, y) as a function of the single variable x, treating y as a constant, and we differentiate with respect to x in the usual way.

Similarly, we can find $\frac{\partial f}{\partial y}$ by ordinary differentiation with respect to y, treating x as a constant.

Examples

1. If $z = x^2 + y^2 + 5xy + xy^2 + 2x^2y$, then;

   $\frac{\partial f}{\partial x} = 2x + 5y + y^2 + 4xy$ and

   $\frac{\partial f}{\partial y} = 2y + 5x + 2xy + 2x^2$

2. If $z = \frac{y}{x} + \frac{x}{y}$, then;

   $\frac{\partial f}{\partial x} = -\frac{y}{x^2} + \frac{1}{y}$ and

   $\frac{\partial f}{\partial y} = \frac{1}{x} - \frac{x}{y^2}$

3. If $z = e^{x^2+y^2}$

   $\frac{\partial f}{\partial x} = 2xe^{x^2+y^2}$ and

   $\frac{\partial f}{\partial x} = 2ye^{x^2+y^2}$

Chain rule

If $z = g(u)$ where $u = h(x, y)$, then z is a function of x and y, and;

$$\frac{\partial z}{\partial x} = \frac{dz}{du}.\frac{\partial u}{\partial x} \text{ and } \frac{\partial z}{\partial y} = \frac{dz}{du}.\frac{\partial u}{\partial y}$$

4. If $z = \tan^{-1}\left(\frac{y}{x}\right)$, then;

   We can let $u = \frac{y}{x}$ and hence $z = \tan^{-1} u$

   $$\frac{\partial z}{\partial x} = \frac{dz}{du}.\frac{\partial u}{\partial x} = \frac{1}{1+u^2}.\left(-\frac{y}{x^2}\right) = \frac{-y}{x^2+y^2}$$

   $$\frac{\partial z}{\partial y} = \frac{dz}{du}.\frac{\partial u}{\partial y} = \frac{1}{1+u^2}.\left(\frac{1}{x}\right) = \frac{x}{x^2+y^2}$$

5. If $z = \frac{1}{2x + y}$

Let $u = 2x + y$ therefore; $z = \frac{1}{u}$

$$\frac{\partial z}{\partial x} = \frac{dz}{du}.\frac{\partial u}{\partial x} = -\frac{1}{u^2}.2 = \frac{-2}{(2x+y)^2}$$

$$\frac{\partial z}{\partial y} = \frac{dz}{du}.\frac{\partial u}{\partial y} = -\frac{1}{u^2}.1 = \frac{-1}{(2x+y)^2}$$

6. If $z = y + x\ln\left(\frac{x}{y}\right)$

Using product rule;

$$\frac{\partial z}{\partial x} = x\frac{\partial}{\partial x}\left[\ln\left(\frac{x}{y}\right)\right] + \ln\left(\frac{x}{y}\right) = x.\frac{1}{(x/y)}.\frac{1}{y} + \ln\left(\frac{x}{y}\right) = 1 + \ln\left(\frac{x}{y}\right)$$

$$\frac{\partial z}{\partial y} = 1 + \frac{x}{x/y}\left(-\frac{x}{y^2}\right) = 1 - \frac{x}{y}$$

Note

The idea of partial differentiation extends easily to functions of more than two variables e.g. if z is function of three variables x, y, and t then z has three partial derivatives; $\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}\ and\ \frac{\partial z}{\partial t}$

To obtain $\frac{\partial z}{\partial x}$ we differentiate with respect to x treating both y and t constants, and similarly for $\frac{\partial z}{\partial y}$ and $\frac{\partial z}{\partial t}$.

Partial derivatives of higher order

If z = f(x, y), then $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ are themselves functions of x and y, and have partial derivatives with respect to x and y. we define the second order partial derivatives of z as:

- $\frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right)$ ……………………………………………………………………….(1)
- $\frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right)$ ……………………………………………………………………….(2)
- $\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right)$ ……………………………………………………………………….(3)
- $\frac{\partial^2 z}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right)$ ……………………………………………………………………….(4)

The second order derivatives (1) to (4) are also denoted by f_{xx}, f_{yy}, f_{xy} and f_{yx} respectively.

It can be shown that;

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 z}{\partial y \partial x}$$

This result is known as the *commutative property* of partial derivatives.

Partial derivatives of any order are defined in a similar manner. For example to find $\frac{\partial^5 z}{\partial x^3 \partial y^2}$

We differentiate with respect to x three times and with respect to y twice, and the result is independent of the order in which these differentiations are carried out.

Examples

1. Show that if $z = x^3 - 3xy^2$, then $\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 z}{\partial y \partial x}$ and $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0$

   Solution

$$\frac{\partial z}{\partial x} = 3x^2 - 3y^2$$

$$\frac{\partial^2 z}{\partial x^2} = 6x$$

$$\frac{\partial^2 z}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = -6y$$

$$\frac{\partial z}{\partial y} = -6xy$$

$$\frac{\partial^2 z}{\partial y^2} = -6x$$

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = -6y$$

$$\therefore \frac{\partial^2 z}{\partial y \partial x} = \frac{\partial^2 z}{\partial y \partial x} \text{ and, } \frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 6x + -6x = 0$$

2. Show that if $z = \left(x^2 + y^2\right)\tan^{-1}\left(\frac{y}{x}\right)$, then;

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 4\tan^{-1}\left(\frac{y}{x}\right) \text{ and } \frac{\partial^2 z}{\partial y \partial x} = \frac{x^2 - y^2}{x^2 + y^2}$$

Extended chain rule

Suppose that z =f(x, y) and that x, y are both function of two other variables u, v, then z may be expressed in terms of u and v and $\frac{\partial z}{\partial u}, \frac{\partial z}{\partial v}$ may be found by normal methods of partial differentiation. Alternatively, $\frac{\partial z}{\partial u}, \frac{\partial z}{\partial v}$ may be found by using a chain rule;

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}.\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}.\frac{\partial y}{\partial u}$$ and;

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}.\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}.\frac{\partial y}{\partial v}$$. These equations can be regarded as an extension of the chain rule.

Example

Given z = f(x, y) with $x = \frac{1}{2}\left(u^2 - v^2\right) and\ y = uv$, show that: $u\frac{\partial z}{\partial v} - v\frac{\partial z}{\partial u} = 2\left(x\frac{\partial z}{\partial y} - y\frac{\partial z}{\partial x}\right)$

Solution

$$\frac{\partial x}{\partial u} = u; \frac{\partial x}{\partial v} = -v; \frac{\partial y}{\partial u} = v; \frac{\partial y}{\partial v} = u$$

By extended chain rule;

$$\frac{\partial z}{\partial u} = u\frac{\partial z}{\partial x} + v\frac{\partial z}{\partial y}$$

$$\frac{\partial z}{\partial v} = -v\frac{\partial z}{\partial x} + u\frac{\partial z}{\partial y}$$. Which gives;

$$u\frac{\partial z}{\partial v} - v\frac{\partial z}{\partial u} = u\left(-v\frac{\partial z}{\partial x} + u\frac{\partial z}{\partial y}\right) - v\left(u\frac{\partial z}{\partial x} + v\frac{\partial z}{\partial y}\right)$$

$$= -uv\frac{\partial z}{\partial x} + u^2\frac{\partial z}{\partial y} - uv\frac{\partial z}{\partial x} - v^2\frac{\partial z}{\partial y} = \left(u^2 - v^2\right)\frac{\partial z}{\partial y} - 2uv\frac{\partial z}{\partial x}$$

Total derivative

A simple extension of the chain rule applies when z = f(x, y) and x and y are both functions of the single variable u. Putting derivatives with respect to v equal to zero in the extended chain rule we see that the equation reduces to the single equation:

$$\frac{dz}{du} = \frac{\partial z}{\partial x}.\frac{dx}{du} + \frac{\partial z}{\partial y}.\frac{dy}{du}$$

Note that in this case z is a function of the single variable *u*, and $\frac{dz}{du}$ (sometimes called the total derivative of z) is not a partial derivative.

Implicit functions

We have previously found $\frac{dz}{du}$ using the implicit equation of the form *f(x, y) = 0*. We can now give a general formula, differentiating both sides of the equation *f(x, y) = 0* with respect to x we get:

$$f_x + f_y\frac{dy}{dx} = 0$$

$$\therefore \frac{dy}{dx} = \frac{-f_x}{f_y}$$

We may extend this idea to the equation *f(x, y, z) = 0* which expresses z implicitly as a function of the two variables x and y. differentiating the equation *f(x, y, z) = 0*, first with respect to x and then with respect to y, we get;

$$f_x + f_z \frac{\partial z}{\partial x} = 0 \text{ and } f_y + f_z \frac{\partial z}{\partial y} = 0$$

Hence; $\frac{\partial z}{\partial x} = \frac{-f_x}{f_z}$ and $\frac{\partial z}{\partial y} = \frac{-f_y}{f_z}$

Examples

1. Find $\frac{dy}{dx}$ at any point of the curve $x^3 + y^3 - 3xy = a^3$

   Solution

   Here we have that;

   $f(x, y) = x^3 + y^3 - 3xy - a^3$

   $f_x = 3x^2 - 3y = 3(x^2 - y)$

   $f_y = 3y^2 - 3x = 3(y^2 - x)$

   Hence; $\frac{dy}{dx} = \frac{-f_x}{f_y} = \frac{-3(x^2 - y)}{3(y^2 - x)} = \frac{y - x^2}{y^2 - x}$

2. Find $\frac{\partial z}{\partial x}\, and \frac{\partial z}{\partial y}$ at any point on the surface; $x^3 + y^3 + z^3 - 3xyz = a^3$

   $f(x, y, z) = x^3 + y^3 + z^3 - 3xyz - a^3$

   $f_x = 3(x^2 - yz)$

   $f_y = 3(y^2 - xz)$

   $f_z = 3(z^2 - xy)$

   Therefore;

   $\frac{\partial z}{\partial x} = -\frac{f_x}{f_z} = \frac{-3(x^2 - yz)}{3(z^2 - xy)} = \frac{yz - x^2}{z^2 - xy}$ and

   $\frac{\partial z}{\partial y} = -\frac{f_y}{f_z} = \frac{-3(y^2 - xz)}{3(z^2 - xy)} = \frac{xz - y^2}{z^2 - xy}$

END