



## Scalar and Vector Fields

**Scalar:** A geometrical or physical quantity that can completely be characterized by a single number.

- For example: length of a bar, mass of an object, electrical resistivity of a metal, viscosity of a fluid, temperature of an object, pressure at a point, etc.

**Vector:** A physical quantity that requires for its complete characterization not only the specification of a **magnitude** but also the specification of a **direction** and a **sense**.

- For example: velocity of a moving particle, force acting on a body, angular momentum of rotating rigid body.

If a scalar is considered to have different values at different points, say  $P$ , in three dimensional space, then it is referred to as a **scalar function**, designated as  **$f(P)$** .



### Scalar Field:

A single-valued, real, scalar function  $f(P)$  which is defined at each point in a domain  $D$ .

Note that the values of this function depend only on the points  $P$  in  $D$ , but not on the particular choice of coordinate system used.

The domain  $D$  called as the **domain of definition** of  $f(P)$  can be a curve, a surface, or a three dimensional region in space.

### Example:

The Euclidian distance of any point  $P(x,y,z)$  in space from a given fixed point  $P_o(x_o,y_o,z_o)$  is a scalar field defined by the following scalar function in Cartesian coordinates

$$f(P) = f(x,y,z) = \sqrt{(x-x_o)^2 + (y-y_o)^2 + (z-z_o)^2}$$

whose domain of definition is the whole space.

A scalar function may also be function of time, that is  $f(P,t)$ .



### **Example:**

The temperature distribution  $T(x,y,z)$  at every point within a room is a scalar function.

- It defines a scalar field whose domain of definition  $D$  is given by the three dimensional interior region of the room.
- Even though the temperature of a point outside  $D$ , say a point in the corridor, may be defined, it is not considered to be part of the scalar field defined as the temperature distribution within the room of interest.

Similar to the scalar field, the **field** concept can be extended to vectors.



### Vector Field:

A single-valued, real, vector function  $\vec{v}(P)$  which is defined at each point  $P$  in a domain of definition  $D$ .

- Just as in scalar fields, the values of this vector function depend only on the points  $P$  in  $D$  but not on the particular choice of coordinate system used.
- $D$  can be a curve, a surface, or a three dimensional region in space, and a vector function may also be function of time, that is  $(P, t)$ .
- If the Cartesian coordinates are used, then a vector function may be represented as

$$\vec{v}(P) = \vec{v}(x, y, z) = v_1(x, y, z)\vec{i} + v_2(x, y, z)\vec{j} + v_3(x, y, z)\vec{k}$$

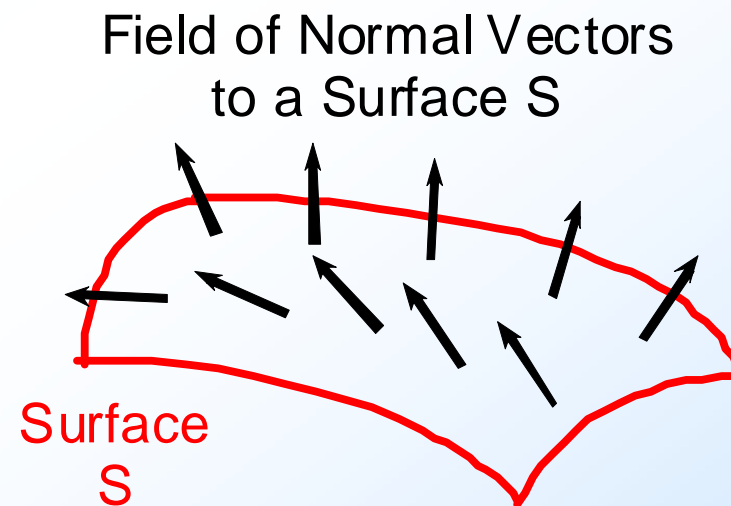
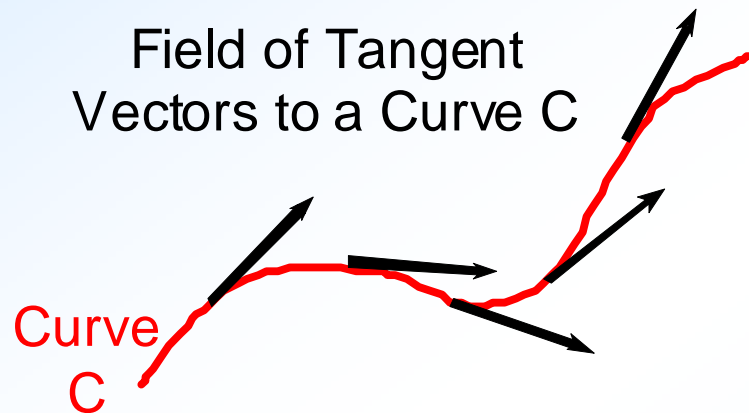
where  $\vec{i}$ ,  $\vec{j}$ ,  $\vec{k}$  are the unit vectors along  $x$ ,  $y$ ,  $z$  directions, respectively.



## Some Examples to Vector Fields

### Example:

The figures below show two geometrically motivated examples for the concept of vector field.

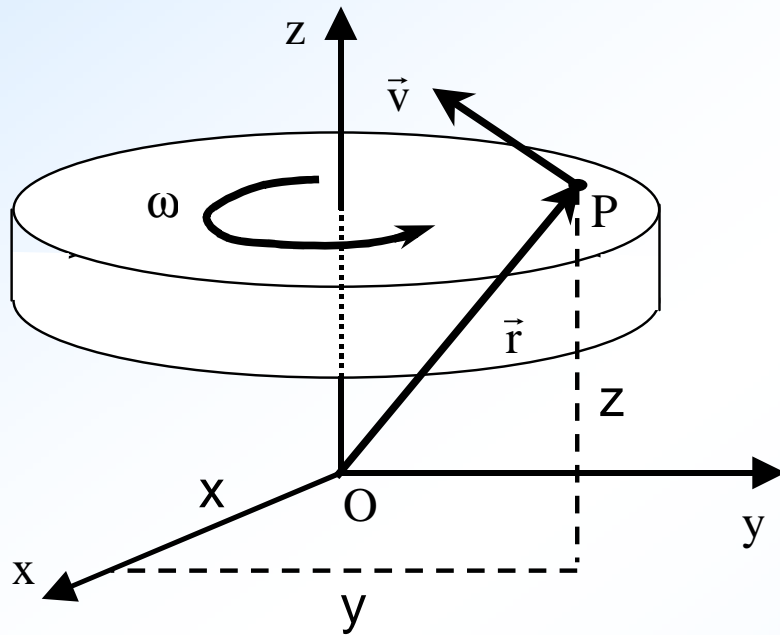


Note that the domains of definitions  $D$  is the curve  $C$  in the field of tangent vectors and the surface  $S$  in the field of normal vectors.



### **Example:** (Velocity field of a Rotating Body)

Consider a disk-shaped rigid body, which is rotating about its axis of symmetry at an angular speed of  $\omega$  as shown in the figure.



The velocity of a point P located on (or within) this disk can be written as

$$\vec{v}(P) = \vec{v}(x, y, z) = \vec{\omega} \times \vec{r}$$

where  $\vec{\omega}$  is the angular velocity of the rotating rigid body and can be expressed as

$$\vec{\omega} = \omega \vec{k}$$

and  $\vec{r}$  is the position vector of the point P and is expressed as

$$\vec{r}(P) = \vec{r}(x, y, z) = x\vec{i} + y\vec{j} + z\vec{k}$$



Hence, the equation for the velocity field  $\vec{v}(P)$  of points of the disk becomes

$$\vec{v}(P) = \vec{v}(x,y,z) = \omega \vec{k} \times (x \vec{i} + y \vec{j} + z \vec{k}) = -\omega y \vec{i} + \omega x \vec{j}$$

- Note that the vector function defined by the above equation can be evaluated mathematically for any point  $P(x,y,z)$  in the whole space.
- However, the resulting velocity vectors are physically meaningful only for those points belonging to the rotating disk. Therefore the **domain of the definition  $D$**  of this velocity field is given by the rotating disk itself.
- Note also that the position vector expression given by the equation constitutes also a vector field.



## Vector Calculus

The basic concepts of calculus such as **convergence**, **continuity**, and **differentiability** can be introduced to vector analysis.

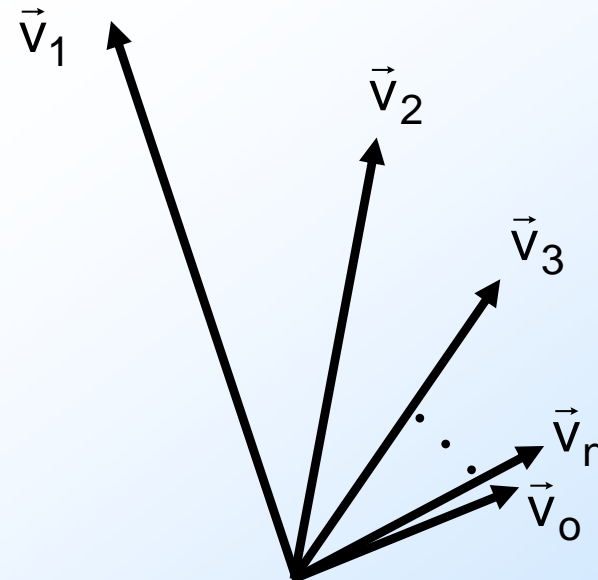
### Convergence:

An infinite sequence of vectors,  $\vec{v}_n$   $n = 1, 2, \dots$ , is said to **converge** to a **limit vector**  $\vec{v}_0$  if

$$\lim_{n \rightarrow \infty} |\vec{v}_n - \vec{v}_0| = 0$$

Then we write

$$\lim_{n \rightarrow \infty} \vec{v}_n = \vec{v}_0$$







### Limit:

A vector function  $\vec{v}(t)$  of a real variable  $t$  is said to have the **limit**  $\vec{v}_0$  as  $t$  approaches  $t_0$ , if  $\vec{v}(t)$  is defined in some neighborhood (that is, in some interval on the  $t$ -axis containing  $t_0$  as an interior point) of  $t_0$  (possibly except at  $t_0$ ) and if

$$\lim_{t \rightarrow t_0} |\vec{v}(t) - \vec{v}_0| = 0$$

Then we write

$$\lim_{t \rightarrow t_0} \vec{v}(t) = \vec{v}_0$$

### Continuity:

A vector function  $\vec{v}(t)$  is said to be **continuous** at  $t = t_0$ , if it has a limit at  $t = t_0$  and if this limit is equal to  $\vec{v}(t_0)$



## Differentiability and Derivative of a Vector Function:

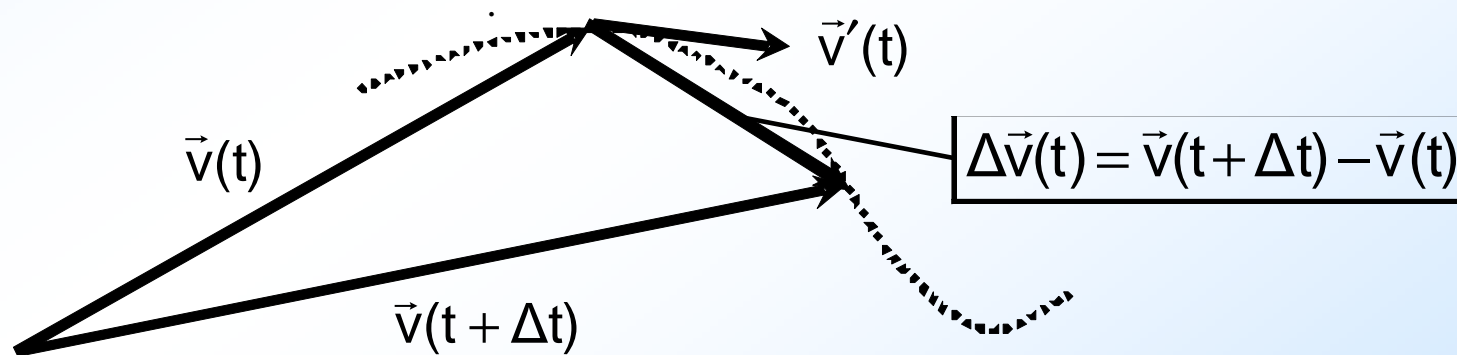
A vector function  $\vec{v}(t)$  is said to be **differentiable** at a point  $t$ , if the limit

$$\lim_{\Delta t \rightarrow 0} \frac{\vec{v}(t + \Delta t) - \vec{v}(t)}{\Delta t} = \lim_{\Delta t \rightarrow 0} \frac{\Delta \vec{v}(t)}{\Delta t}$$

exists.

Then the vector

$$\vec{v}'(t) = \lim_{\Delta t \rightarrow 0} \frac{\vec{v}(t + \Delta t) - \vec{v}(t)}{\Delta t} = \lim_{\Delta t \rightarrow 0} \frac{\Delta \vec{v}(t)}{\Delta t}$$



[or  $\dot{\vec{v}}(t)$  if  $t$  is time] is called the **derivative** of  $\vec{v}(t)$



Note that the derivative of a vector function can be obtained by taking the derivative of its components.

Given

$$\vec{v}(t) = v_1(t)\vec{i} + v_2(t)\vec{j} + v_3(t)\vec{k}$$

its derivative can be written as

$$\vec{v}'(t) = v'_1(t)\vec{i} + v'_2(t)\vec{j} + v'_3(t)\vec{k}$$

### **Example:**

Let a vector function be defined as  $\vec{u}(t) = (\cos^2 t)\vec{i} + (2\sin t)\vec{j} + (t^2)\vec{k}$

Then, its first and second derivatives can be obtained as

$$\vec{u}'(t) = (-2\cos t \sin t)\vec{i} + (2\cos t)\vec{j} + (2t)\vec{k} = (-\sin 2t)\vec{i} + (2\cos t)\vec{j} + (2t)\vec{k}$$

$$\vec{u}''(t) = (-2\cos 2t)\vec{i} + (-2\sin t)\vec{j} + (2)\vec{k}$$



The familiar rules of differentiation yield corresponding rules for vector differentiation:

$$(c\vec{v})' = c\vec{v}' \quad \text{where } c \text{ is a constant}$$

$$(\vec{u} + \vec{v})' = \vec{u}' + \vec{v}'$$

$$(\vec{u} \cdot \vec{v})' = \vec{u}' \cdot \vec{v} + \vec{u} \cdot \vec{v}'$$

$$(\vec{u} \times \vec{v})' = \vec{u}' \times \vec{v} + \vec{u} \times \vec{v}'$$

$$(\vec{u} \cdot \vec{v} \times \vec{w})' = \vec{u}' \cdot \vec{v} \times \vec{w} + \vec{u} \cdot \vec{v}' \times \vec{w} + \vec{u} \cdot \vec{v} \times \vec{w}'$$

**Example:** *(Derivative of a vector with constant magnitude)*

Consider a vector function such that its magnitude is constant; that is,  $|\vec{u}(t)| = c$ .

Then,

$$|\vec{u}(t)|^2 = \vec{u}(t) \cdot \vec{u}(t) = c^2 \quad \longrightarrow \quad \frac{d}{dt}(\vec{u} \cdot \vec{u}) = \frac{d\vec{u}}{dt} \cdot \vec{u} + \vec{u} \cdot \frac{d\vec{u}}{dt} = 2\vec{u} \cdot \frac{d\vec{u}}{dt} = 0$$

Hence, the derivative of a vector function  $\vec{u}(t)$  whose magnitude is constant, is either the null (zero) vector or is orthogonal to  $\vec{u}(t)$ .



END OF WEEK 5

