Chapter 6

FIRST- AND SECOND-ORDER TRANSIENT CIRCUITS

Learning goals

- By the end of this chapter, the students should be able to:
- Calculate the initial values for inductor currents and capacitor voltages in transient circuits.
- Determine the voltages and currents in first-order transient circuits.
- Determine the voltages and currents in second-order transient circuits.

First-Order Circuits

GENERAL FORM OF THE RESPONSE EQUATIONS

- In first-order transient circuits , the solution of these circuits (i.e., finding a voltage or current) requires solving a first-order differential equation of the form.

$$\frac{dx(t)}{dt} + ax(t) = f(t)$$

If $x(t) = x_p(t)$ is the solution to the above DE and $x(t) = x_c(t)$ is any solution to the homogeneous DE.

$$\frac{dx(t)}{dt} + ax(t) = 0$$

Then $x(t) = x_p(t) + x_c(t)$

GENERAL FORM OF THE RESPONSE EQUATIONS

$x_p(t)$ is called *the particular integral solution*, or forced response, and $x_c(t)$ is called the *complementary solution*, or natural response.

Let $f(t) = A\ (constant)$

$$\frac{dx(t)}{dt} + ax(t) = A$$

The Integrating factor, is given by, $IF = e^{\int adt} = e^{at}$

$$e^{at}\frac{dx(t)}{dt} + ax(t)e^{at} = Ae^{at}$$

Thus $e^{at}x(t) = \frac{A}{a}e^{at} + c$

$$x(t) = K_1 + K_2e^{-t/\tau}$$

GENERAL FORM OF THE RESPONSE EQUATIONS

- K_1 is referred to as the *steady-state solution*: the value of the variable $x(t)$ as $t \rightarrow \infty$ when the second term becomes negligible.
- The constant τ is called the *time constant* of the circuit. The second term in the above equation is a decaying exponential that has a value, if $\tau > 0$, of K_2 for t=0 and a value of 0 for $t = \infty$.

GENERAL FORM OF THE RESPONSE EQUATIONS

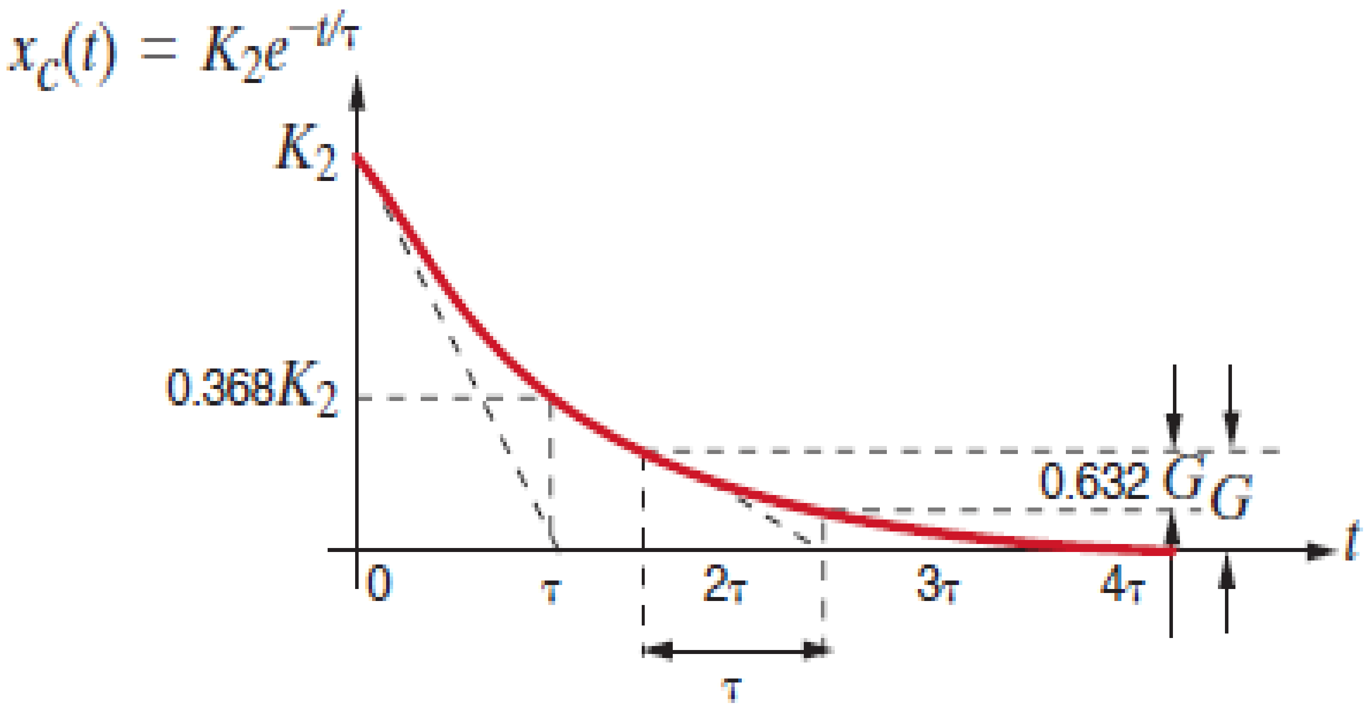


Figure 6.2: Time-constant illustration

Analysis of First-Order Circuits using Differential Equations

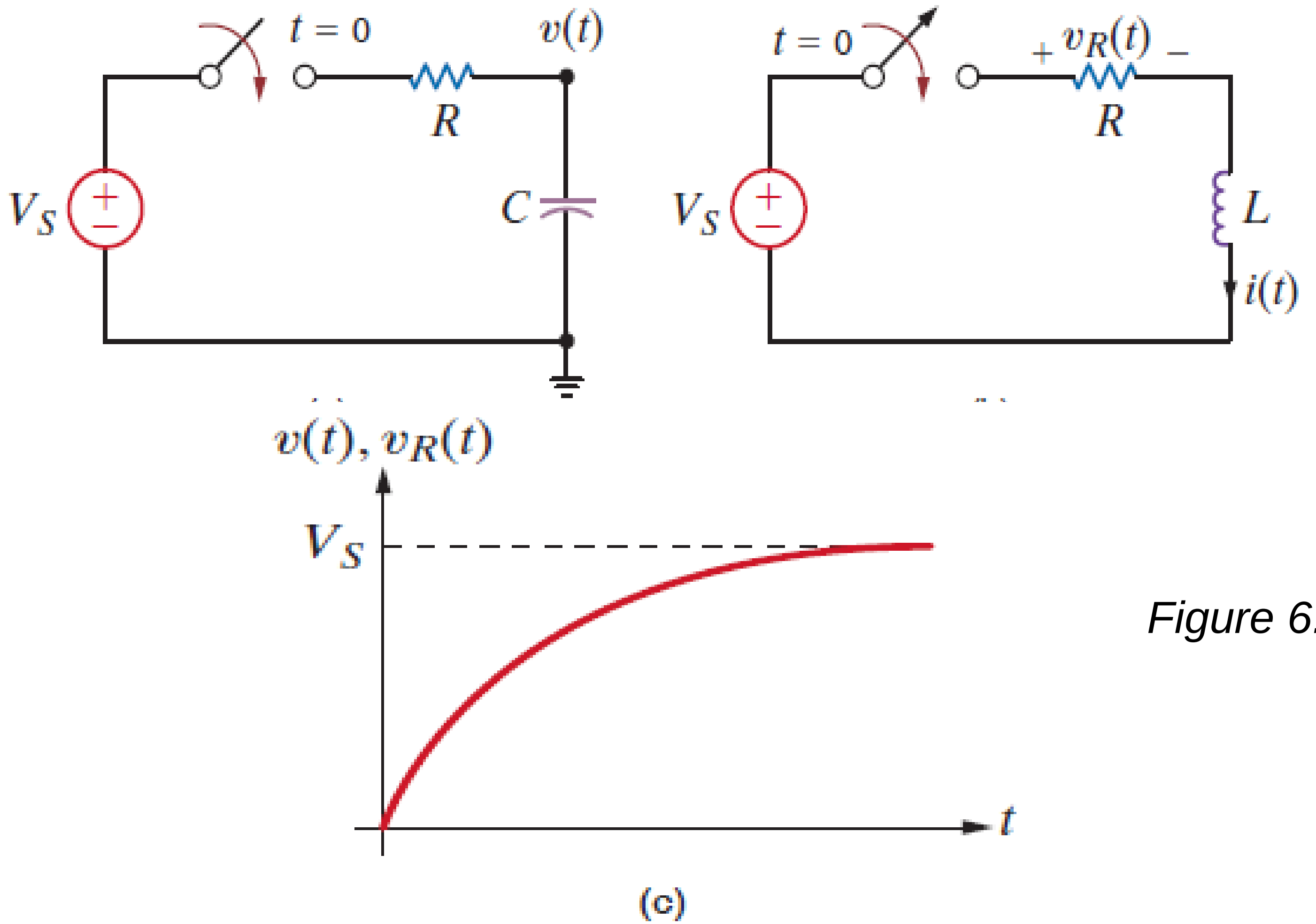


Figure 6.2

Analysis of First-Order Circuits using Differential Equations

Consider the circuit shown in Fig. 6.2a. At time *t=0* the switch closes. The KCL equation that describes the capacitor voltage for time *t>0* is

$$C\frac{dv(t)}{dt}+\frac{v(t)-V_s}{R}=0$$

$$\frac{dv(t)}{dt}+\frac{v(t)}{RC}=\frac{V_s}{RC}$$

The general solution is given by:

$$v(t)=K_1+K_2e^{-t/\tau}$$

Substituting this general solution into the DE and equating constant and exponential terms gives

Analysis of First-Order Circuits using Differential Equations

$K_1 = V_s \quad and \;\; \tau = RC$

$v(t) = V_s + K_2 e^{-t/RC}$

where V_s, is the steady-state value and *RC* is the network's time constant. K_2 is determined by the initial condition of the capacitor. For example, if the capacitor is initially uncharged (that is, the voltage across the capacitor is zero at *t=0*), then

$0 = V_s + K_2 \; or \; or \; K_2 = -V_s$

Hence, the complete solution for the voltage *v(t)* is

$v(t) = V_s - V_s e^{-t/RC}$

Analysis of First-Order Circuits using Differential Equations

- The circuit in Fig. 6.2b can be examined in a similar manner. The KVL equation that describes the inductor current for *t>0* is
- $L\frac{di(t)}{dt} + Ri(t) = V_s$
- Solving this gives
- $i(t) = \frac{V_s}{R} + K_2 e^{-\left(\frac{R}{L}\right)t}$
- Where $\frac{V_s}{R}$ is the steady-state value and L/R is the circuit's time constant. If there is no initial current in the inductor at, then at *t=0*

Analysis of First-Order Circuits using Differential Equations

$$0 = \frac{V_s}{R} + K_2 \; or \; K_2 = -\frac{V_s}{R}$$

Hence $i(t) = \frac{V_s}{R} - \frac{V_s}{R} e^{-\left(\frac{R}{L}\right)t}$ is the complete solution.

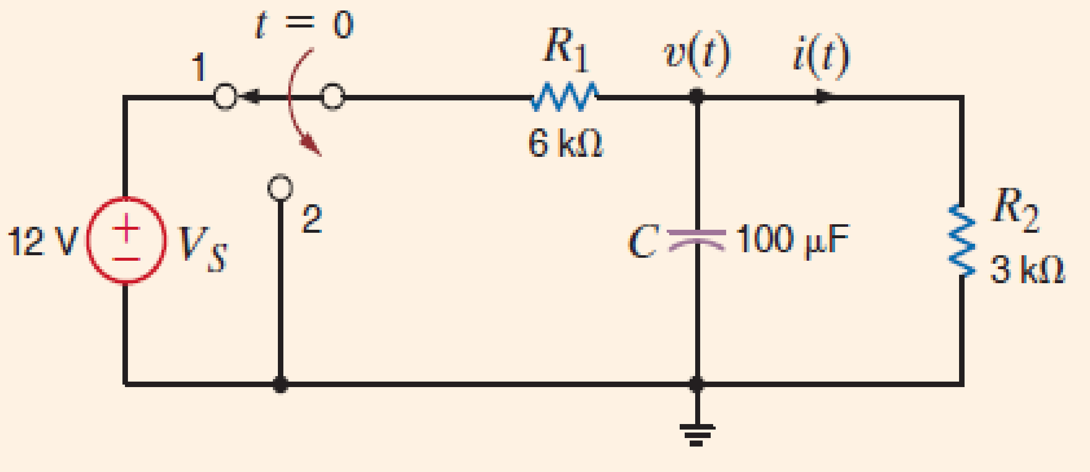
Note that if we wish to calculate the voltage across the resistor, then

$$V_R(t) = Ri(t) = V_s\left(1 - e^{-\left(\frac{R}{L}\right)t}\right)$$

Therefore, we find that the voltage across the capacitor in the *RC* circuit and the voltage across the resistor in the *RL* circuit have the same general form. A plot of these functions is shown in Fig. 6.2c.

Analysis of First-Order Circuits using Differential Equations

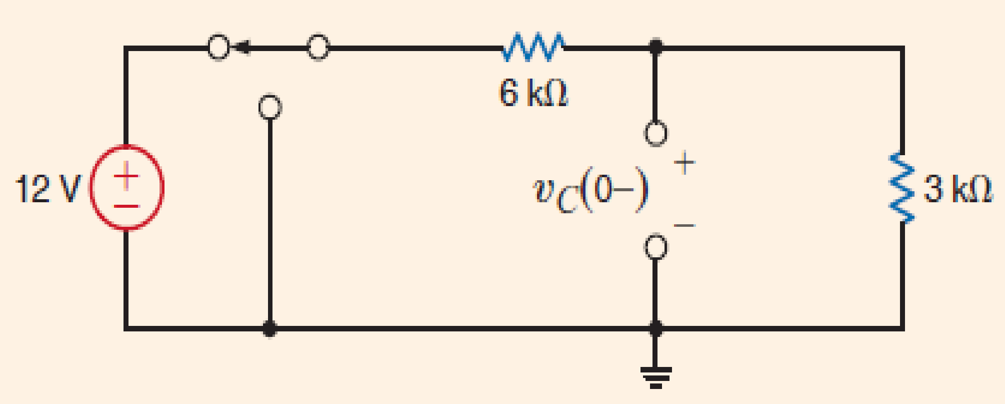
- **Example I**
- Consider the circuit. Assuming that the switch has been in position 1 for a long time, at time *t=0* the switch is moved to position 2. We wish to calculate the current *i(t)* for *t>0.*



(a)

Analysis of First-Order Circuits using Differential Equations

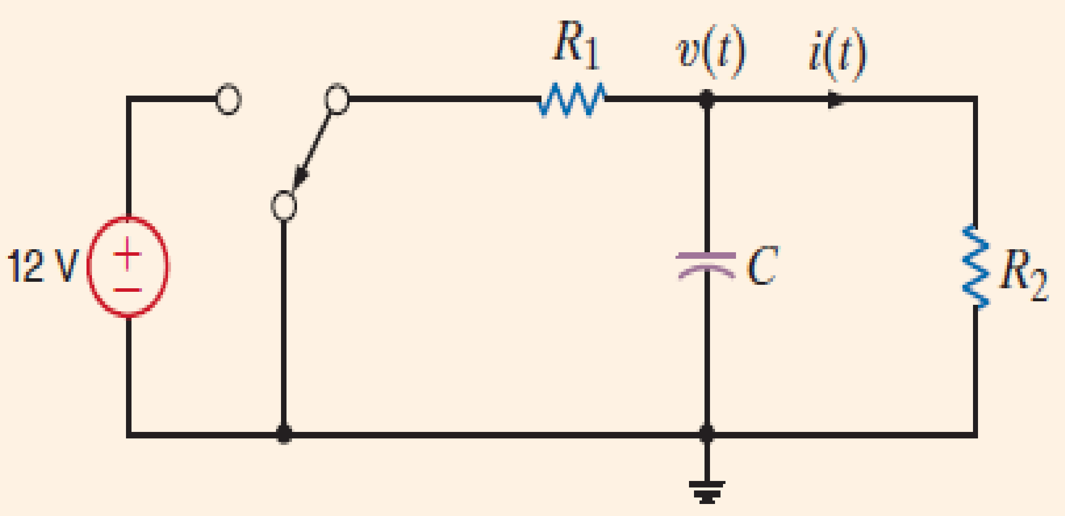
- At *t=0-* the capacitor is fully charged and conducts no current since the capacitor acts like an open circuit to dc. The initial voltage across the capacitor can be found using voltage division. As shown in the figure.
- $V_c(0-) = 12\left(\frac{3k}{6k+3k}\right) = 4V$



(b) $t = 0-$

Analysis of First-Order Circuits using Differential Equations

- The network for $t>0$ is shown below. The KCL equation for the voltage across the capacitor is
- $\frac{v(t)}{R_1} + C\frac{dv(t)}{dt} + \frac{v(t)}{R_2} = 0$



(c)

Analysis of First-Order Circuits using Differential Equations

- Using the component values, the equation becomes
- $\frac{dv(t)}{dt} + 5v(t) = 0$
- $v(t) = K_2 e^{-t/\tau}$
- Substituting into the DE gives $\tau = 0.2\ s$, Thus
- $v(t) = K_2 e^{-t/0.2}\ V$
- Using the initial condition $V_c(0-) = V_c(0+) = 4V$ we find that the complete solution is
- $v(t) = 4e^{-t/0.2}\ V$
- Then $i(t) = \frac{v(t)}{R} = \frac{4}{3} e^{-t/0.2}\ mA$

Example 2

- The switch in the network in Fig. 6.4 opens at *t=0.* Find the output voltage $v_0(t)$

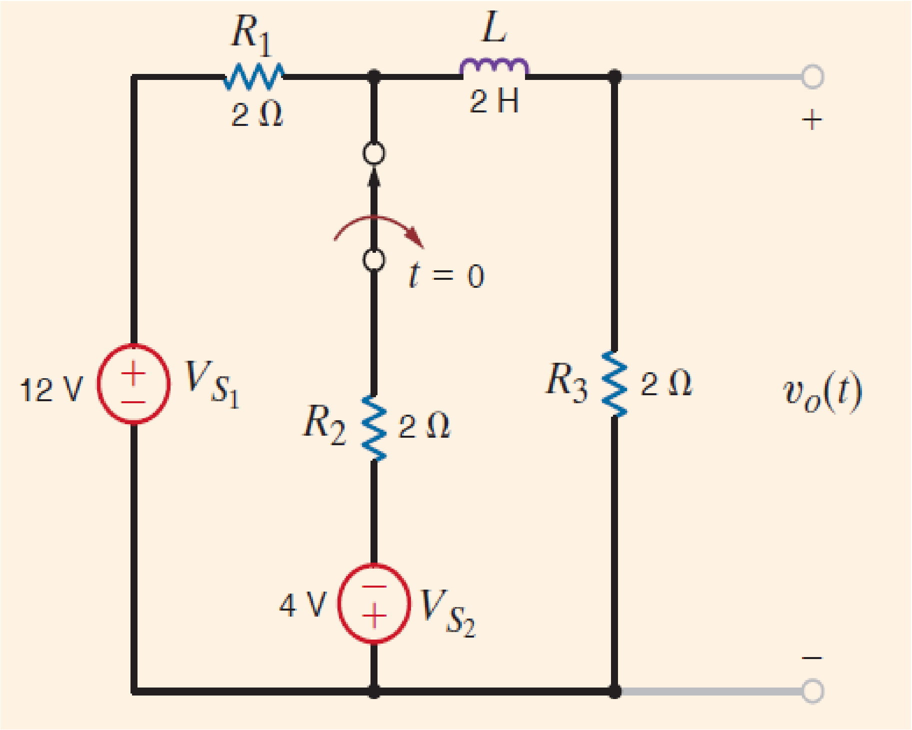


Figure 6.4

PULSE RESPONSE

- Thus far we have examined networks in which a voltage or current source is suddenly applied. As a result of this sudden application of a source, voltages or currents in the circuit are forced to change abruptly.
- A forcing function whose value changes in a discontinuous manner or has a discontinuous derivative is called a *singular function*. Two such singular functions that are very important in circuit analysis are the unit impulse function and the unit step function.

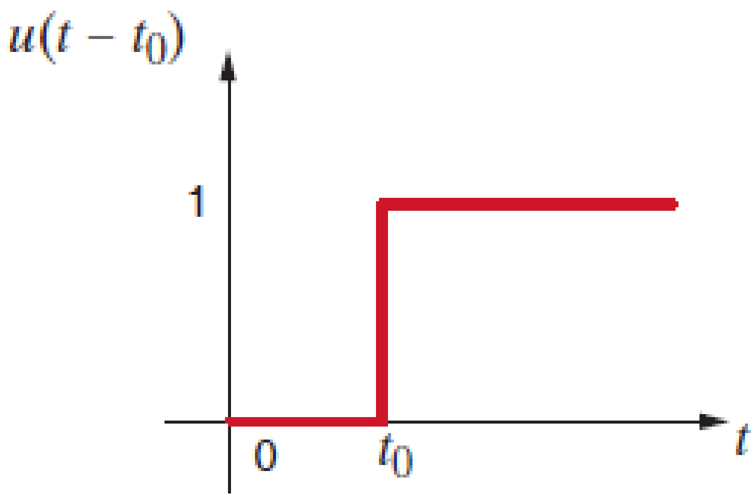
Unit Step Function

- The *unit step function* is defined by the following mathematical relationship:
- $u(t) = \begin{cases} 0 & t < 0 \\ 1 & t > 0 \end{cases}$

The unit step is dimensionless, and therefore a voltage step of V_0 volts or a current step of I_0 amperes is written as $V_0 u(t)$ and $I_0 u(t)$ respectively.

Unit step function

- If we use the definition of the unit step, it is easy to generalize this function by replacing the argument *t* by $t - t_0$. In this case
- $u(t - t_0) = \begin{cases} 0 & t < t_0 \\ 1 & t > t_0 \end{cases}$



Example

- Consider the circuit shown in Fig. 6.6a. The input function is the voltage pulse shown in Fig. 6.6b. Since the source is zero for all negative time, the initial conditions for the network are zero $[v_C(0-)=0]$.
- The response $v_0(t)$ for *0<t<0.3* is due to the application of the constant source at *t=0* and is not influenced by any source changes that will occur later. At *t=0.3s*, the forcing function becomes zero, and therefore $v_0(t)$ for *t>0.3s* is the source-free or natural response of the network. Let us determine the expression for the voltage $v_0(t)$.

Example

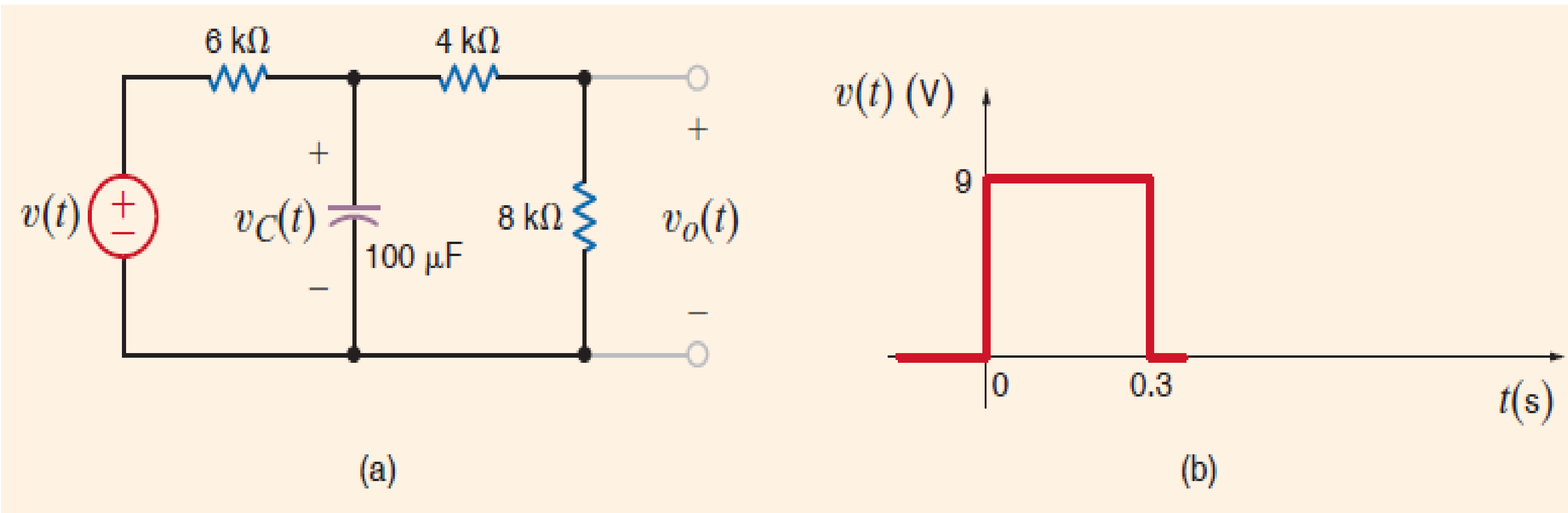


Figure 6.6

Example

- $For\ the\ period\ 0 < t < 0.3$
- $i_1 = i_c + i_0$
- $\frac{v(t)-v_C(t)}{6k} = 100 \times 10^{-6} \frac{dv_C(t)}{dt} + \frac{v_C(t)}{12k}$
- $\frac{dv_C(t)}{dt} + 2.5v_C(t) = 15$

The general solution is $v_C(t) = K_1 + K_2e^{-t/\tau}$
Substituting into the DE and recognising that $v_C(\infty) = 0$ gives $\tau = 0.4s$, $K_1 = 6\ \ and\ \ K_2 = -6$
Thus is $v_C(t) = 6 - 6e^{-2.5t}\ V$

Example

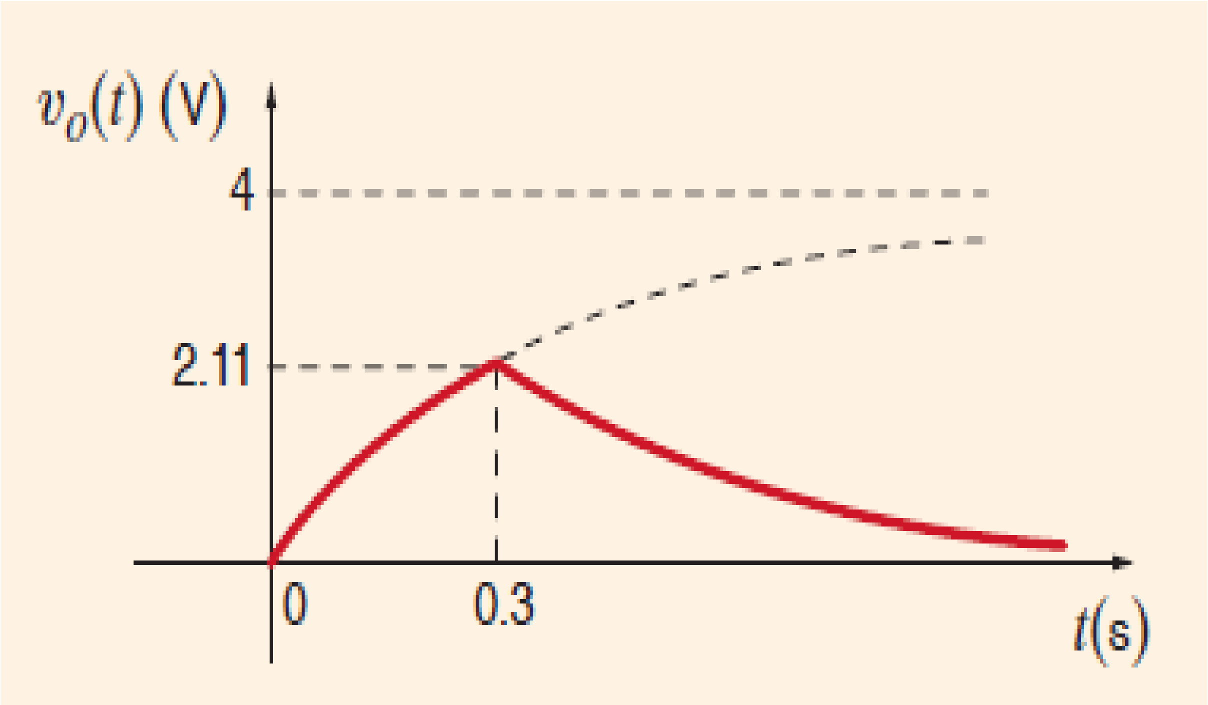
- $i_0(t) = \frac{v_C(t)}{12k} = \frac{1}{2} - \frac{1}{2}e^{-2.5t}\ mA$
- $v_0(t) = i_0(t) \times 8k = 4 - 4e^{-2.5t}\ V$. This is the response for $the\ period\ 0 < t < 0.3s$

$$v_0(0.3+) = 4 - 4e^{-2.5\times0.3} = 2.11V$$

- Since the source is zero for *t>0.3s* the final value for $v_0(t)$ as $t \to \infty$ is zero. Therefore, the expression for$v_0(t)$ for *t>0.3s* is
- $v_0(t) = 2.11e^{-2.5(t-0.3)}\ V \qquad for\ t > 0.3s$
- The term $e^{-2.5(t-0.3)}$ indicates that the exponential decay starts at *t=0.3s.*

Example

- $$v_0(t) = \begin{cases} 0 & t < 0 \\ 4(1 - e^{-2.5t})V & 0 < t < 0.3s \\ 2.11e^{-2.5(t-0.3)}\ V & t > 0.3s \end{cases}$$



Second-Order Circuits

- Consider the two basic RLC circuits shown in Fig. 6.8. We assume that energy may be initially stored in both the inductor and capacitor.

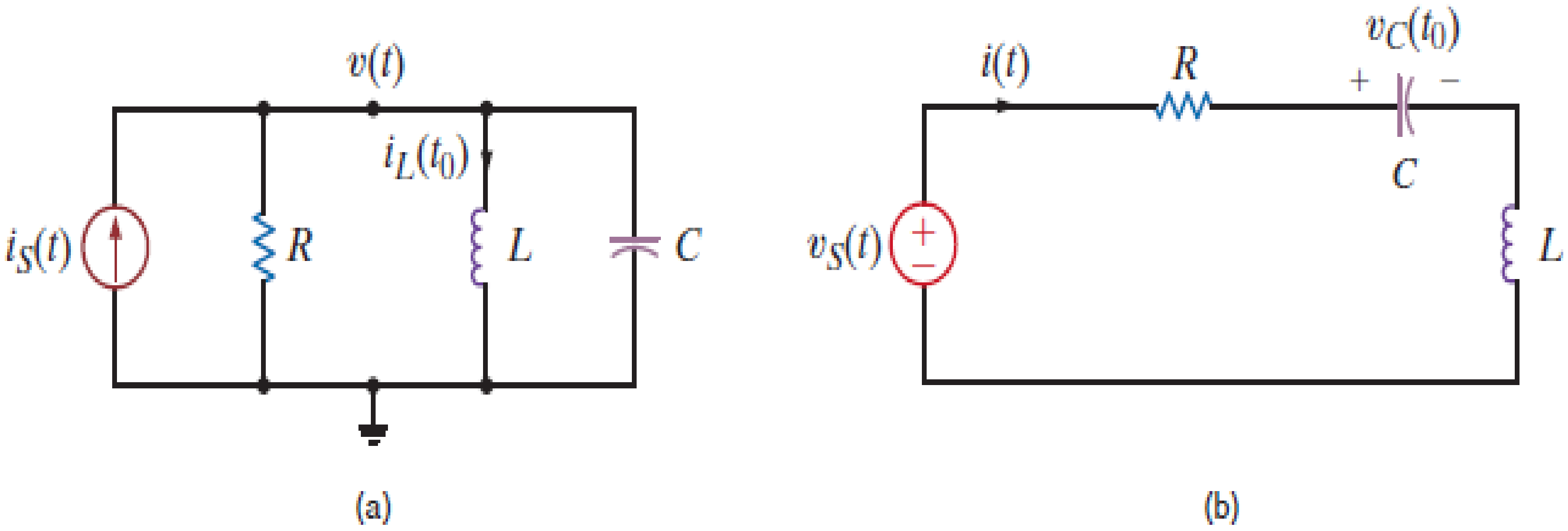


Figure 6.8 Parallel and series RLC circuits.

Second-Order Circuits

- The node equation for the parallel RLC circuit is
- $\frac{v}{R} + \frac{1}{L}\int_{t_0}^{t} v(x)dx + i_L\,(t_0) + C\,\frac{dv}{dt} = i_s(t)$
- Similarly, the loop equation for the series *RLC* circuit is
- $Ri + \frac{1}{C}\int_{t_0}^{t} i(x)dx + v_C\,(t_0) + L\,\frac{di}{dt} = v_s(t)$
- If the two equations are differentiated with respect to *t*.
- $C\,\frac{d^2v}{dt^2} + \frac{1}{R}\frac{dv}{dt} + \frac{v}{L} = \frac{di_s}{dt}$
- $L\,\frac{d^2i}{dt^2} + R\,\frac{di}{dt} + \frac{i}{C} = \frac{dv_s}{dt}$
- These are second order DEs with constant coefficients.

THE RESPONSE EQUATIONS

- The Des are of the form
- $\frac{d^2x(t)}{dt^2} + a_1\frac{dx(t)}{dt} + a_2x(t) = f(t)$
- $x(t) = x_p(t) + x_c(t)$
- If $f(t) = A$, $\boldsymbol{x(t) = \frac{A}{a_2} + x_c(t)}$
- The Homogeneous equation is
- $\frac{d^2x(t)}{dt^2} + a_1\frac{dx(t)}{dt} + a_2x(t) = 0$

THE RESPONSE EQUATIONS

- For our analysis, this equation can be written as
- $\frac{d^2x(t)}{dt^2} + 2\xi\omega_0\frac{dx(t)}{dt} + {\omega_0}^2x(t) = 0$
- Where $a_1 = 2\xi\omega_0 \quad and \quad a_2 = {\omega_0}^2$
- The characteristic equation is
- $s^2 + 2\xi\omega_0 s + {\omega_0}^2 = 0$
- ξ is the exponential damping ratio.
- ω_0 is the undamped natural frequency.

THE RESPONSE EQUATIONS

- Solving the characteristic equation gives.
- $s = \frac{-2\xi\omega_0 \pm \sqrt{4\xi^2{\omega_0}^2 - 4{\omega_0}^2}}{2} = -\xi\omega_0 \pm \omega_0\sqrt{\xi^2 - 1}$
- Therefore the two values of s_1 and s_2 are:
- $s_1 = -\xi\omega_0 + \omega_0\sqrt{\xi^2 - 1}$
- $s_2 = -\xi\omega_0 - \omega_0\sqrt{\xi^2 - 1}$
- In general the complementary solution of the DE is:
- $x_c(t) = K_1 e^{s_1 t} + K_2 e^{s_2 t}$

THE RESPONSE EQUATIONS

- $K_1 \ and \ K_2$ are constants that can be evaluated via the initial conditions, $x(0) \ \ and \ dx(0)/dt$
- Since
- $x(t) = K_1 e^{s_1 t} + K_2 e^{s_2 t}$
- Then $x(0) = K_1 + K_2$
- And $\frac{dx(t)}{dt}|_{t=0} = \frac{dx(0)}{dt} = s_1 K_1 + s_2 K_2$
- $x(0) \ and \ dx(0)/dt$ are two simultaneous equations to be solved for $K_1 \ and \ K_2$

Case 1: $\xi > 1$. Overdamped.

- The natural frequencies s_1 and s_2 are real and unequal; therefore, the natural response of the network described by the second-order differential equation is of the form
- $x_c(t) = K_1 e^{-(\xi\omega_0 - \omega_0\sqrt{\xi^2-1})t} + K_2 e^{-(\xi\omega_0 + \omega_0\sqrt{\xi^2-1})t}$
- This indicates that the natural response is the sum of two decaying exponentials.

Case 2: $\xi < 1$. *Underdamped.*

- The roots of the characteristic equation are complex, given by:
- $s_1 = -\xi\omega_0 + j\omega_0\sqrt{1-\xi^2} = -\sigma + j\omega_d$
- $s_2 = -\xi\omega_0 - j\omega_0\sqrt{1-\xi^2} = -\sigma - j\omega_d$
- Where $\sigma = \xi\omega_0$ and $\omega_d = \omega_0\sqrt{1-\xi^2}$ .Thus the natural frequencies are complex numbers. The natural response is then of the form
- $x_c(t) = e^{-\xi\omega_0 t}(A_1 \cos\omega_0\sqrt{1-\xi^2}t + A_2 \sin\omega_0\sqrt{1-\xi^2}t$
- Where A_1 and A_2 are constants evaluated from the initial conditions. This illustrates that the natural response is an exponentially damped oscillatory response.

Case 3: $\xi = 1$ Critically damped.

- Therefore $s_1 = s_2 = -\xi\omega_0$.
- In the case where the characteristic equation has repeated roots, the general solution is of the form:
- $x_c(t) = B_1 e^{-\xi\omega_0 t} + B_2 t e^{-\xi\omega_0 t}$
- Where B_1 and B_2 are constants derived from the initial conditions.
- The three cases are compared in Figure 6.9.

overdamped, critically damped, and Underdamped

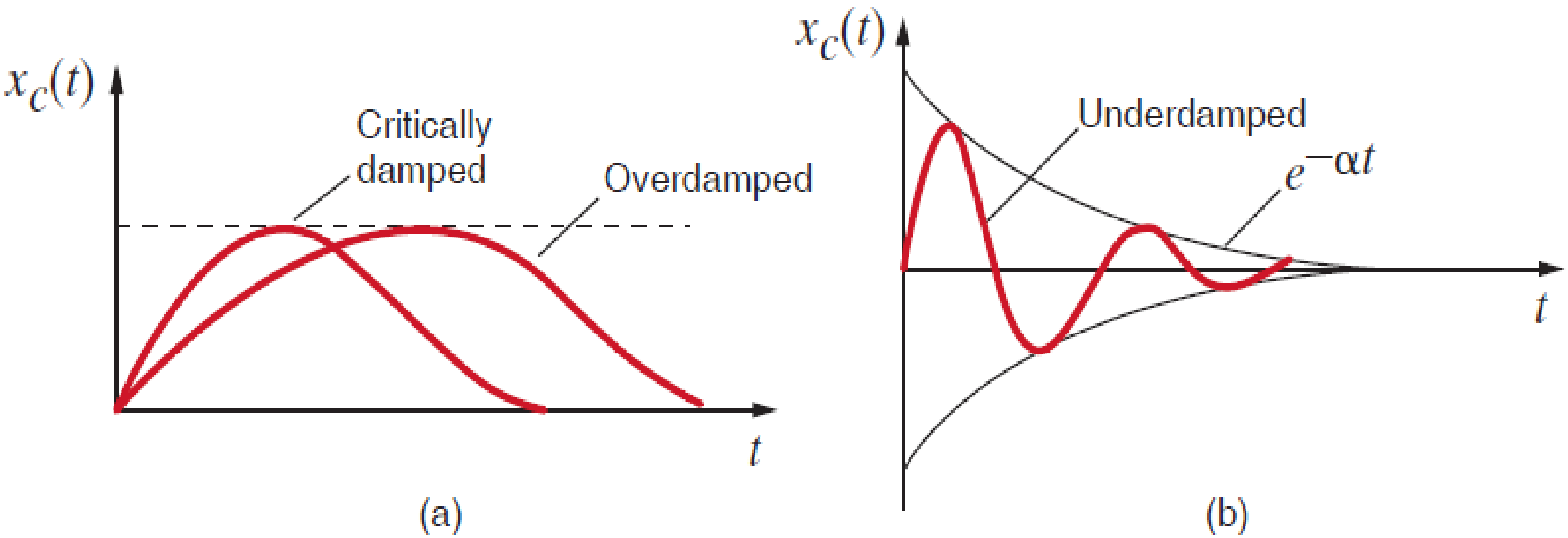


Figure 6.9: Comparison of overdamped, critically damped, and Underdamped responses.

Exercise

1. A parallel *RLC* circuit has the following circuit parameters: R=1Ω, L=2 H, and C=2 F. Compute the damping ratio and the undamped natural frequency of this network.
2. A series *RLC* circuit consists of R=2 Ω , L=1 H, and a capacitor. Determine the type of response exhibited by the network if (a) C = 1/2 F, (b) C=1 F, and (c) C=2 F.

Example 1

- Consider the parallel *RLC* circuit shown in Fig. 6.10.

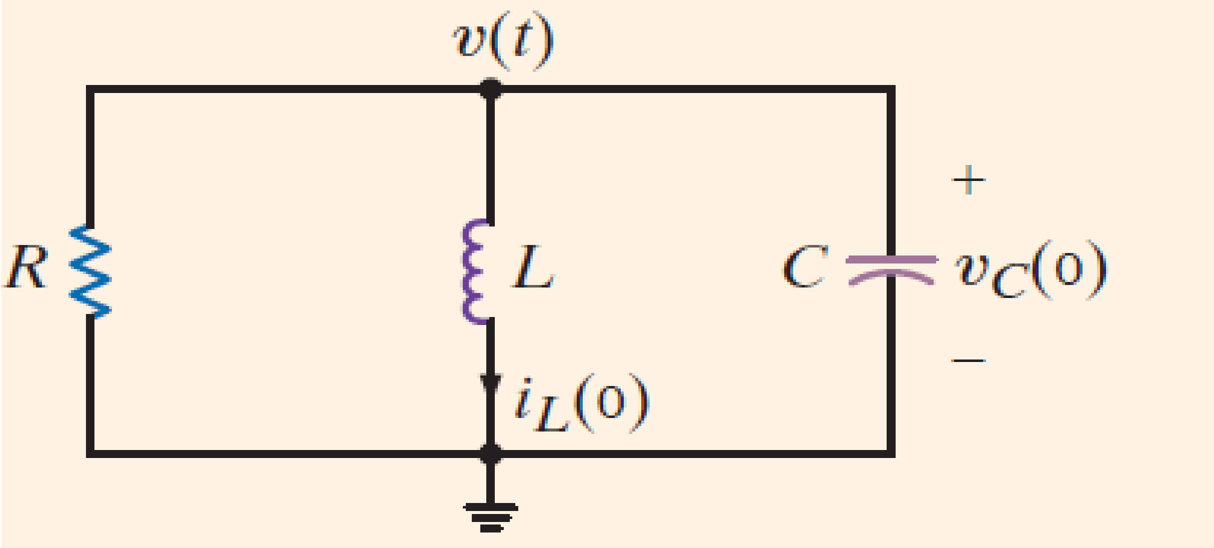


Figure 6.10: Parallel RLC circuit

Example 1

- The second-order differential equation that describes the voltage *v(t)* is.
- $\frac{d^2v}{dt^2} + \frac{1}{RC}\frac{dv}{dt} + \frac{v}{LC} = 0$
- Comparing with the previous equations, the damping term is $1/2RC$ and the undamped natural frequency is $1/\sqrt{LC}$. If $R = 2\Omega, C = 1/5\ F, and\ L = 5H$. the equation becomes
- $\frac{d^2v}{dt^2} + 2.5\frac{dv}{dt} + v = 0$

Example 1

- Let us assume that the initial conditions on the storage elements are $i_L(0) = -1A \; and \; v_c(0) = 4V$
  Let us find the node voltage *v(t)* and the inductor current.
- The characteristic equation is
- $s^2 + 2.5s + 1 = 0$
- $s_1 = -2 \; and \; s_2 = -0.5$
- Since the roots are real and unequal, the circuit is overdamped, and *v(t)* is of the form
- $v(t) = K_1e^{-2t} + K_2e^{-0.5t}$

Example 1

- The initial conditions are now employed to determine the constants K_1 and K_2. Since $v(t) = v_c(t)$
- $v_c(0) = 4 = K_1 + K_2$
- The second equation needed to determine K_1 and K_2 is normally obtained from the expression
- $\frac{dv(t)}{dt} = -2K_1e^{-2t} - 0.5K_2e^{-0.5t}$.
- The node equation for the circuit can be written as
- $C\frac{dv(t)}{dt} + \frac{v(t)}{R} + i_L(t) = 0$ or

- $\frac{dv(t)}{dt} = -\frac{1}{RC}v(t) - \frac{i_L(t)}{C}$
- At $t = 0$, $\frac{dv(0)}{dt} = -\frac{1}{RC}v(0) - \frac{i_L(0)}{C} = -2.5(4) - 5(-1) = -5$
- Thus $-5 = -2K_1 - 0.5K_2$
- Solving the two simultaneous equations gives $K_1 = 2 \; and \; K_2 = 2.$
- Therefore $v(t) = 2e^{-2t} + 2e^{-0.5t} \; V$
- The response curve is shown in the Figure 6.11

Example 1

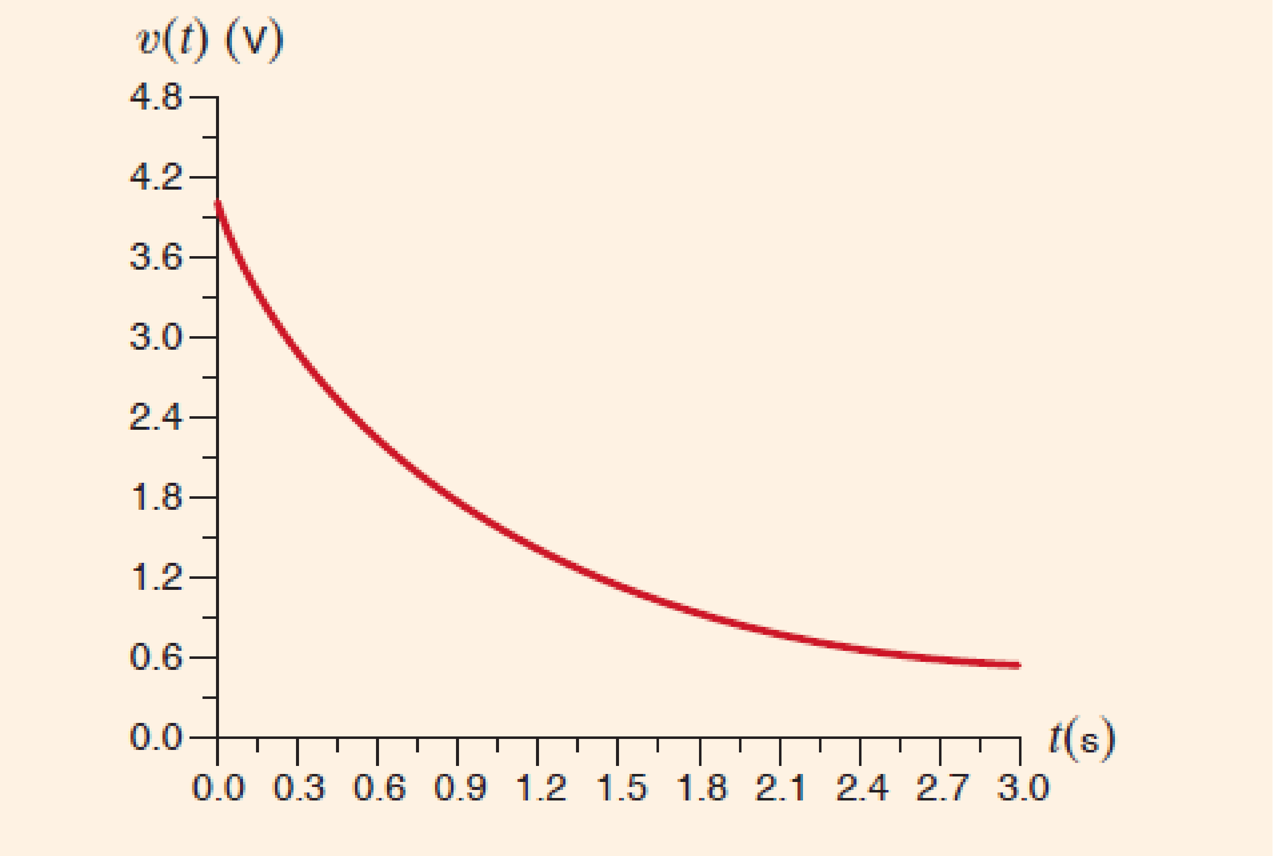


Figure 6.11: Overdamped response.

Example 1

- The inductor current is related to $v(t) by$
- $i_L(t) = \frac{1}{L}\int v(t)dt$
- $i_L(t) = -\frac{1}{5}e^{-2t} - \frac{4}{5}e^{-0.5t}A$
- In comparison with *RL* and *RC* circuits, the response of this *RLC* circuit is controlled by two time constants. The first term has a time constant of ½ s and the second term has a time constant of 2 s.

Example II

- The series *RLC* circuit shown in Fig. 6.12 has the following parameters: $C = 0.04\ F, L = 1H, R = 6\Omega, i_L(0) = 4A \quad and \ \ v_C(0) = -4V$

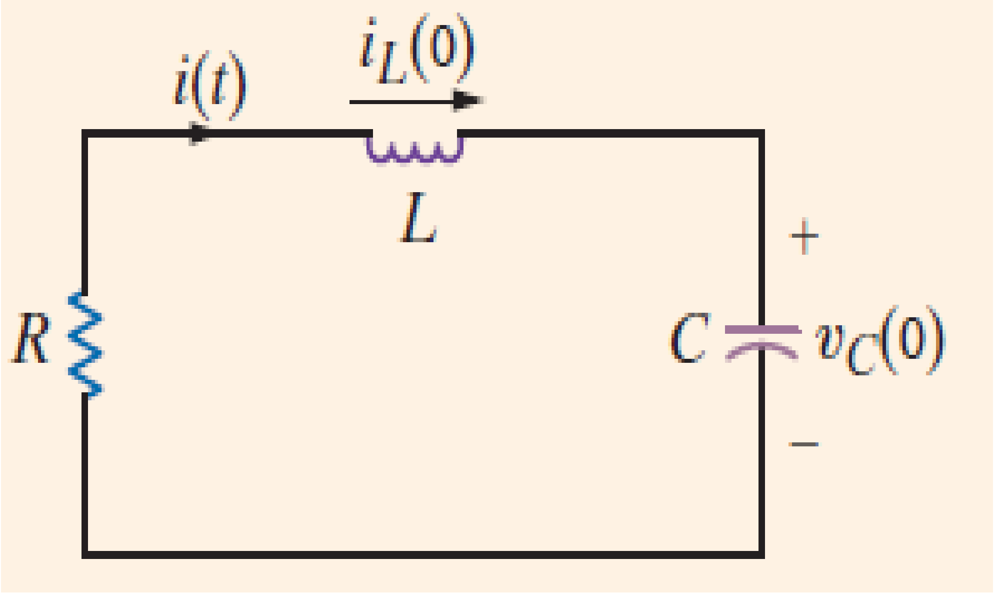


Figure 6.12: Series RLC circuit.

Example II

- The equation for the current in the circuit is given by the expression
- $\frac{d^2i}{dt^2} + \frac{R}{L}\frac{di}{dt} + \frac{i}{LC} = 0$
- The damping term is $R/2L$ and the undamped natural frequency is $1/\sqrt{LC}$.
- Substituting the circuit elements gives.
- $\frac{d^2i}{dt^2} + 6\frac{di}{dt} + 25i = 0$
- Characteristic equation is $s^2 + 6s + 25 = 0$
- $s_1 = -3 + j4 \;\; and \; s_2 = -3 - j4$

Example II

- Since the roots are complex, the circuit is Underdamped, and the expression for *i(t)* is
- $i(t) = K_1 e^{-3t} \cos 4t + K_2 e^{-3t} \sin 4t$
- Using the initial conditions, we find that
- $i(0) = 4 = K_1$
- And $\frac{di}{dt} = -4K_1 e^{-3t} \sin 4t - 3K_1 e^{-3t} \cos 4t + 4K_2 e^{-3t} \cos 4t - 3K_2 e^{-3t} \sin 4t$
- And thus $\frac{di(0)}{dt} = -3K_1 + 4K_2$
- $Ri(0) + L\frac{di(0)}{dt} + v_C(0) = 0$
- Or $\frac{di(0)}{dt} = -\frac{R}{L} i(0) - \frac{v_C(0)}{L} = -20$
- Thus $-3K_1 + 4K_2 = -20$

Example II

- This gives $K_2 = -2$. Thus
- $i(t) = 4e^{-3t}\cos 4t - 2e^{-3t}\sin 4t \;\; A$
- The voltage across the capacitor could be determined via KVL using this current:
- $Ri(t) + L\frac{di(t)}{dt} + v_C(t) = 0$
- $v_C(t) = -Ri(t) - L\frac{di(t)}{dt}$
- Substituting the preceding expression for *i(t)* into this equation yields
- $v_C(t) = -4e^{-3t}\cos 4t + 22e^{-3t}\sin 4t \;\; V$
- A plot of the function is shown in Fig. 6.13:

Example II

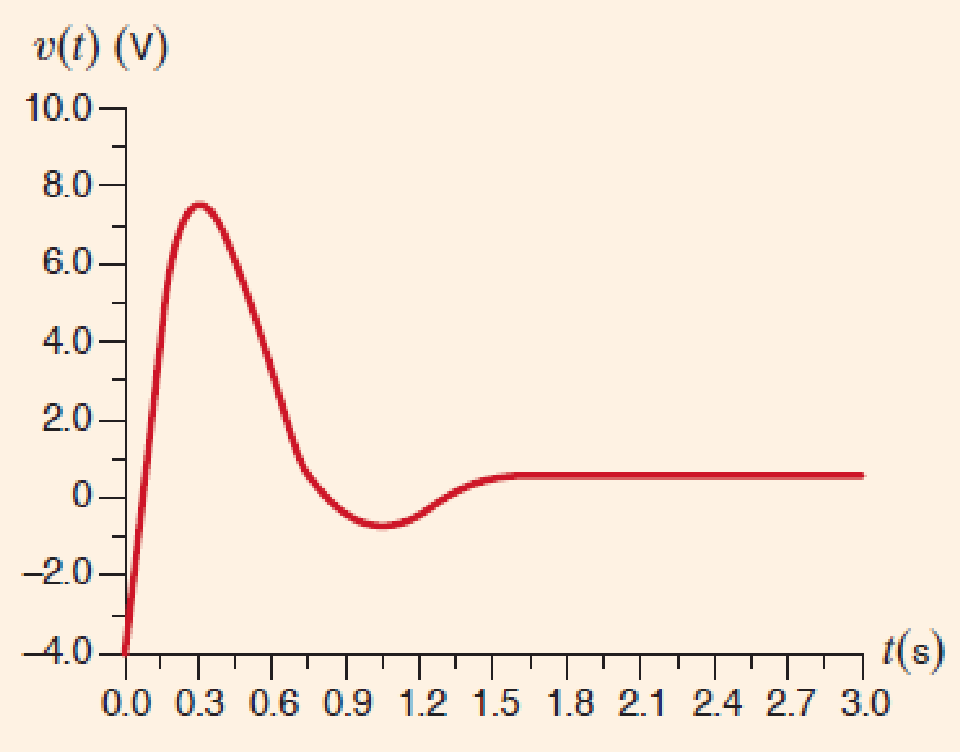


Figure 6.13: Underdamped response.

Example III

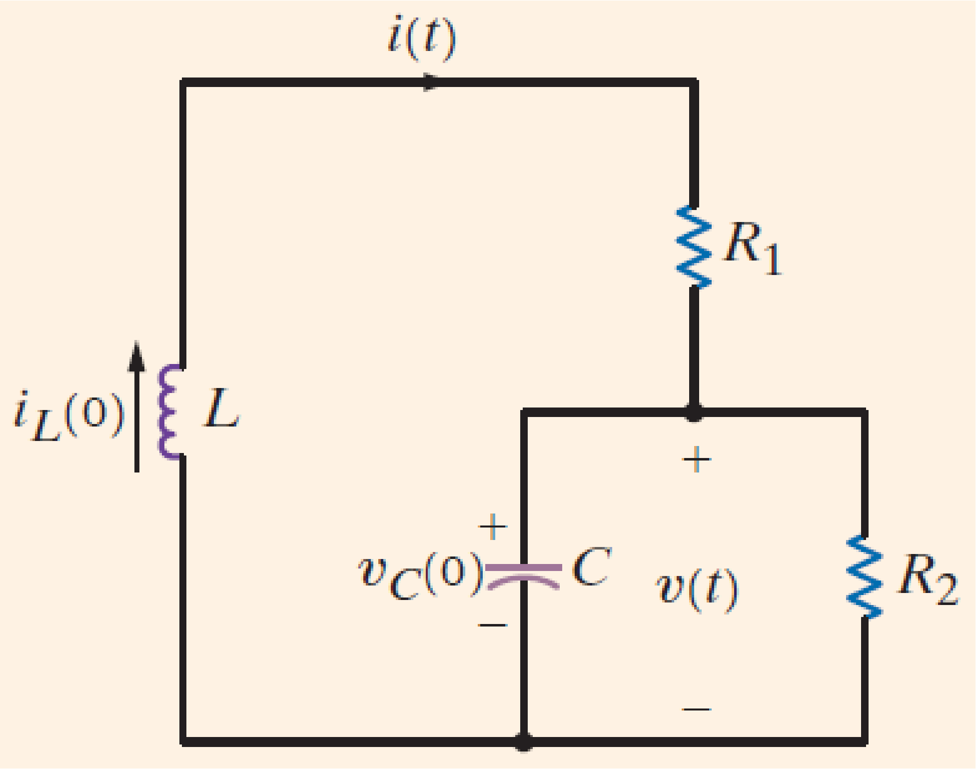


Figure 6.14: Series-parallel RLC circuit.

Example III

- The two equations that describe the network are
- $L\frac{di(t)}{dt} + R_1 i(t) + v(t) = 0$ and $i(t) = C\frac{dv(t)}{dt} + \frac{v(t)}{R_2}$
- $\frac{d^2v}{dt^2} + \left(\frac{1}{R_2C} + \frac{R_1}{L}\right)\frac{dv}{dt} + \frac{R_1+R_2}{R_2LC}v = 0$
- If the circuit parameters and initial conditions are
- $R_1 = 10\,\Omega, C = \frac{1}{8}F, R_2 = 8\Omega, L = 2\,H, v_C(0) = 1V\ and\ i_L(0) = \frac{1}{2}A$
- The DE becomes $\frac{d^2v}{dt^2} + 6\frac{dv}{dt} + 9v = 0$

Example III

- The characteristic equation is
- $s^2 + 6s + 9 = 0$
- Hence the roots are
- $s_1 = -3 \quad and\ s_2 = -3$
- Since the roots are real and equal, the circuit is critically damped. The term *v(t)* is then given by the expression
- $v(t) = K_1 e^{-3t} + K_2 t e^{-3t}$
- Since $v(t) = v_C(t)$
- $v(0) = v_C(0) = 1 = K_1$

Example III

- In addition,
- $\frac{dv(t)}{dt} = -3K_1e^{-3t} + K_2e^{-3t} - 3K_2e^{-3t}$
- However $\frac{dv(t)}{dt} = \frac{i(t)}{C} - \frac{v(t)}{R_2C}$
- Setting these two expressions equal to one another and evaluating the resultant equation at *t=0* yields
- $3 = -3K_1 + K_2$. Thus $K_2 = 6$
- $v(t) = e^{-3t} + 6te^{-3t}$

Example III

- The current *i(t)* can be determined from the nodal analysis equation at *v(t):*
- $i(t) = C\frac{dv(t)}{dt} + \frac{v(t)}{R_2}$
- $i(t) = \frac{1}{2}e^{-3t} - \frac{3}{2}te^{-3t}\ A$
- A plot of this critically damped function is shown in Fig. 6.15.

Example III

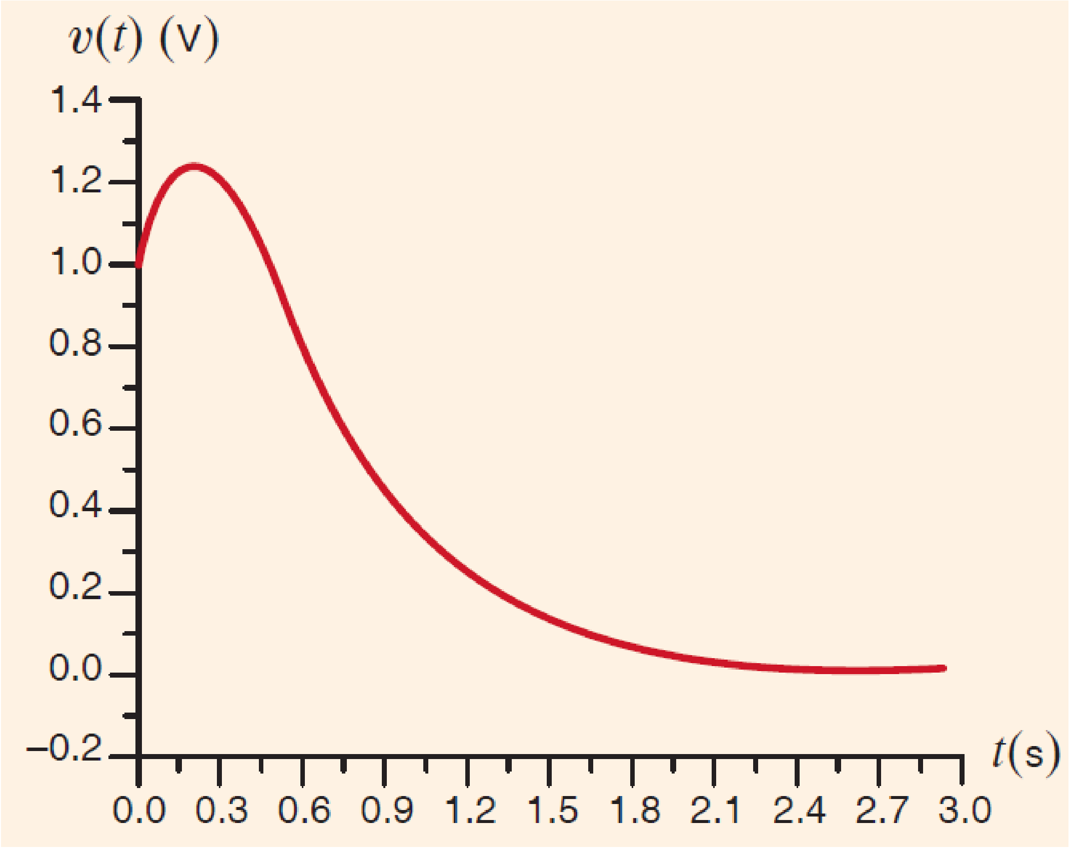


Figure 6.15: Critically damped response

Comments/Questions