Chapter 7

AC STEADY STATE ANALYSIS

Learning goals

- By the end of this chapter, the students should be able to:
- Describe the basic characteristics of sinusoidal functions.
- Perform phasor and inverse phasor transformations.
- Draw phasor diagrams.
- Calculate impedance and admittance for basic circuit elements: R, L, C.
- Determine the equivalent impedance of basic circuit elements connected in series and parallel.
- Determine the equivalent admittance of basic circuit elements connected in series and parallel.
- Redraw a circuit in the frequency domain given a circuit with a sinusoidal source.
- Apply our circuit analysis techniques to frequency domain circuits.

Sinusoids

- Considering the sine wave $x(t) = X_M \sin \omega t$ where $x(t)$ could represent *v(t)* or *i(t). X_M* is the *amplitude*, *maximum value,* or peak value; ω is the *radian* or *angular frequency*; and ωt is the *argument* of the sine function. The function repeats itself every 2π radians. This condition is described mathematically as
- $x(\omega t + 2\pi) = x(\omega t)$, or in general for period T, as
- $x[\omega(t + T)] = x(\omega t)$
- meaning that the function has the same value at time *t+T* as it does at time t.

Sinusoids

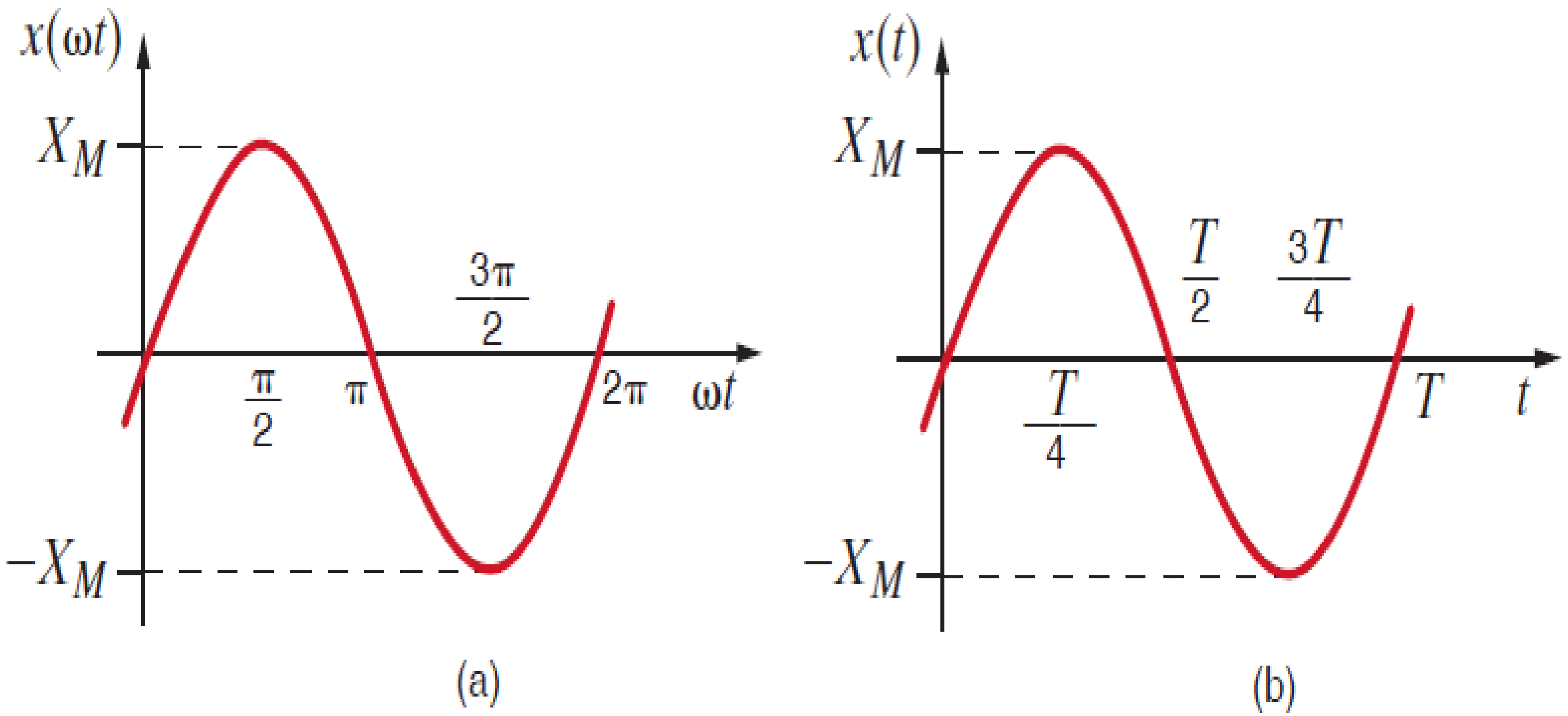


Figure 7.1: *Plots of a sine wave as a function of both ωt and t.*

Sinusoids

- The number of cycles per second, called Hertz, is the frequency f, where
- $f = \frac{1}{T}$
- Since *ωT=2π(Fig. 7.1a)*
- $\omega = \frac{2\pi}{T} = 2\pi f$
- let us consider the following general expression for a sinusoidal function:
- $x(t) = X_M \sin(\omega t + \theta)$
- θ is called the phase angle

Sinusoids

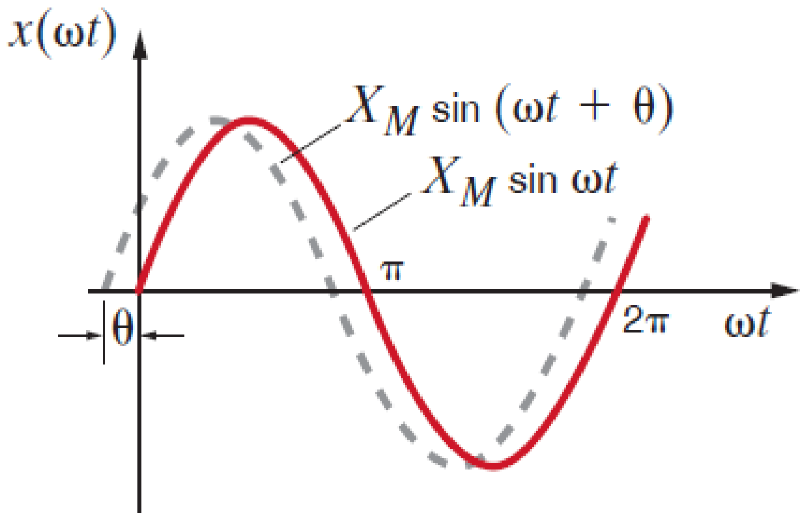


Figure 7.2: *Graphical illustration of $X_M \sin(\omega t + \theta)$ leading $X_M \sin \omega t$ by θ radians.*

- Because of the presence of the phase angle, any point on the waveform $X_M \sin(\omega t + \theta)$ occurs θ radians earlier in time than the corresponding point on the waveform $X_M \sin \omega t$. Therefore, we say that $X_M \sin \omega t$ *lags* $X_M \sin(\omega t + \theta)$ by θ radians.

Sinusoids

- Adding to the argument integer multiples of either 2π radians or 360˚ does not change the original function
- The cosine function could be easily used as well, since the two waveforms differ only by a phase angle; that is,
- $\cos\omega t = \sin\left(\omega t + \frac{\pi}{2}\right)$ and $\sin\omega t = \cos\left(\omega t - \frac{\pi}{2}\right)$
- Note:
- $-\cos(\omega t) = \cos(\omega t \pm 180^0)\ \ and$
- $-\sin(\omega t) = \sin(\omega t \pm 180^0)$

Examples

- Determine the frequency and the phase angle between the two voltages $v_1(t) = 12\sin(1000t + 60^0)\,V \;\; and\; v_2(t) = -6\cos(1000t + 30^0)\,V$

Solution

- $f = \frac{\omega}{2\pi} = \frac{1000}{2\pi} = 159.2\ Hz$
- $v_2(t) = -6\cos(\omega t + 30^0)\,V = 6\cos(\omega t + 210^0)V = 6\sin(\omega t + 300^0) = 6\sin(\omega t - 60^0)$
- Therefore, the phase angle between $v_1(t)\; and\; v_2(t) = 60 - -60 = 120^0$, i.e $v_1(t)$ leads $v_2(t)$ by 120^0.

Exercise

1. Given that $v(t) = 120\cos(314t + \pi/4)\ V$, determine the frequency of the voltage in Hertz and the phase angle in degrees.
2. Three branch currents in a network are known to be

$$i_1(t) = 2\sin(377t + 45^0)A\,, \quad i_2\cos(377t + 10^0(t) = 0.5)A\,, \ \ and\ i_3(t) = -0.25\sin(377t + 60^0)A\,,$$

Determine the phase angles by which $i_1(t)$ leads $i_2(t)$ and $i_1(t)$ leads $i_3(t)$.

Sinusoidal and Complex Forcing Functions

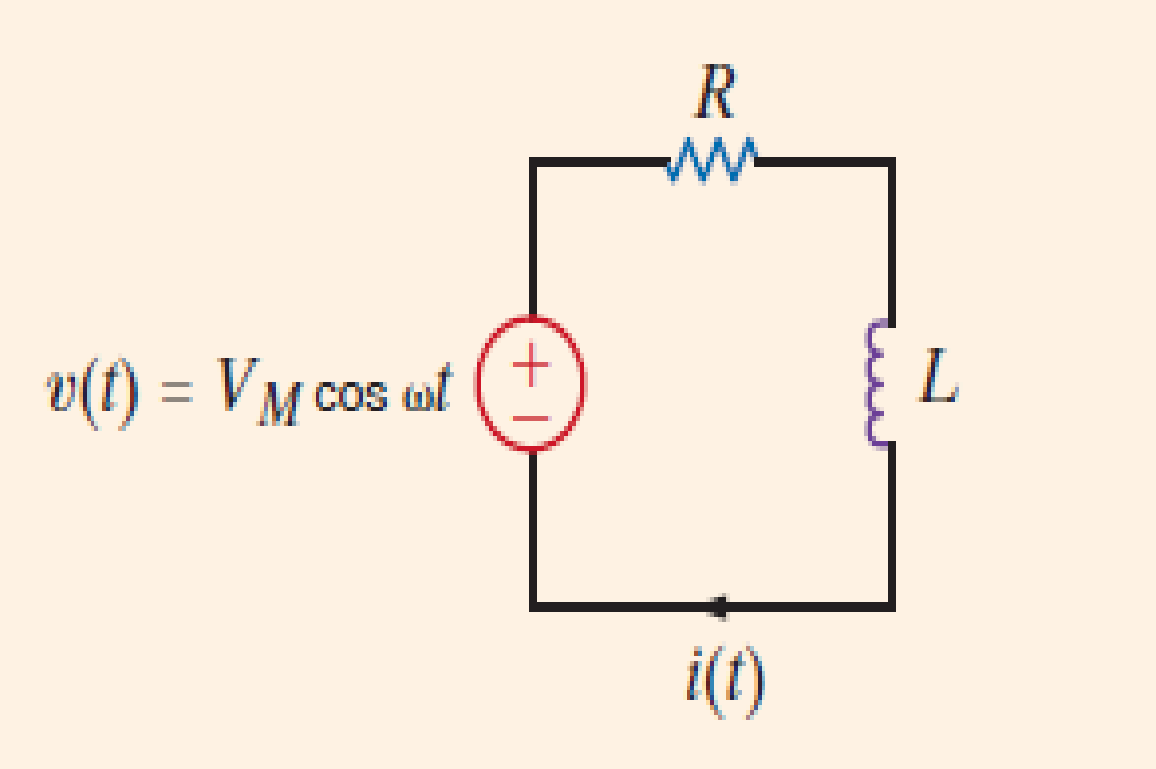


Figure 7.3: *A simple RL circuit*

Sinusoidal and Complex Forcing Functions

- KVL equation for the circuit is
- $L\frac{di(t)}{dt} + Ri(t) = V_M \cos\omega t$
- Since the input forcing function is $V_M \cos\omega t$, we assume that the forced response component of the current *i(t)* is of the form
- $i(t) = A\cos(\omega t + \emptyset)$
- $i(t) = A\cos(\omega t + \emptyset) =$
  $A\cos\emptyset \cos\omega t - A sin\emptyset \sin\omega t = A_1 \cos\omega t + A_2 \sin\omega t$
- Substituting into the DE

Sinusoidal and Complex Forcing Functions

- $L\frac{d}{dt}(A_1\cos\omega t + A_2\sin\omega t) + R(A_1\cos\omega t + A_2\sin\omega t) = V_M\cos\omega t$

Evaluating and equating the coefficients

- $-A_1\omega L + A_2 R = 0$
- $A_1 R + A_2\omega L = V_M$

Solving the above equations simultaneously and substituting in $i(t)$ gives

- $i(t) = \frac{RV_M}{R^2+\omega^2L^2}\cos\omega t + \frac{\omega L V_M}{R^2+\omega^2L^2}\sin\omega t$

Sinusoidal and Complex Forcing Functions

Which can be written as

- $i(t) = A\cos(\omega t + \emptyset)$
- Where $A\cos\emptyset = \frac{RV_M}{R^2+\omega^2L^2}$ and $A\sin\emptyset = -\frac{\omega LV_M}{R^2+\omega^2L^2}$
- Thus $\tan\ \emptyset = -\frac{\omega L}{R}$
- $(A\cos\emptyset)^2 + (A\sin\emptyset)^2 = A^2 = \frac{V_M^2}{R^2+\omega^2L^2}$
- $A = \frac{V_M}{\sqrt{R^2+\omega^2L^2}}$
- Hence the final expression is
- $i(t) = \frac{V_M}{\sqrt{R^2+\omega^2L^2}}\cos\left(\omega t - tan^{-1}\frac{\omega L}{R}\right)$

Sinusoidal and Complex Forcing Functions

- The preceding analysis indicates that ∅ is zero if *L=0* and hence *i(t)* is in phase with *v(t).* If *R=0*, $\emptyset = -90^0$, and the current lags the voltage by 90°. If *L* and *R* are both present, the current lags the voltage by some angle between 0° and 90°.
- This example illustrates an important point: solving even a simple one-loop circuit containing one resistor and one inductor is very complicated compared to the solution of a singleloop circuit containing only two resistors. It would be more complicated to solve a more complicated circuit using this procedure.

Sinusoidal and Complex Forcing Functions

- Let us determine the current in the RL circuit examined in Figure 7.3.
- We will apply $V_M e^{j\omega t}$
- The forced response will be
- $i(t) = I_M e^{j(\omega t + \emptyset)}$
- Substituting this into the DE, and differentiating, then dividing by the common factor will give.
- $RI_M e^{j\emptyset} + j\omega L I_M e^{j\emptyset} = V_M$
- which is an algebraic equation with complex coefficients.

Sinusoidal and Complex Forcing Functions

- This equation can be written as
- $I_M e^{j\emptyset} = \frac{V_M}{R+j\omega L}$
- Converting the right-hand side of the equation to exponential or polar form produces the equation
- $I_M e^{j\emptyset} = \frac{V_M}{\sqrt{R^2+\omega^2 L^2}} e^{j\left(-tan^{-1}\frac{\omega L}{R}\right)}$
- This shows that
- $I_M = \frac{V_M}{\sqrt{R^2+\omega^2 L^2}}$ $\quad and \quad$ $\emptyset = -tan^{-1}\frac{\omega L}{R}$

Sinusoidal and Complex Forcing Functions

- However, since our actual forcing function was $V_M \cos \omega t$ rather than $V_M e^{j\omega t}$ our actual response is the real part of the complex response:
- $i(t) = A\cos(\omega t + \emptyset) = \frac{V_M}{\sqrt{R^2+\omega^2 L^2}} \cos\left(\omega t - tan^{-1}\frac{\omega L}{R}\right)$
- Similar to the one obtained previously.

Phasors

- Again, we consider the RL circuit in Figure 7.3. Let us use phasors to determine the expression for the current.
- The DE is $L\frac{di(t)}{dt} + Ri(t) = V_M \cos \omega t$
- The forcing function can be replaced by a complex forcing function that is written as $\boldsymbol{V}e^{j\omega t}$ with phasor $\boldsymbol{V} = V_M \angle 0^0$. Similarly, the forced response component of the current *i(t)* can be replaced by a complex function $\boldsymbol{I}e^{j\omega t}$that is written as with phasor $\boldsymbol{I} = I_M \angle \emptyset$

Phasors

- Using the complex forcing function, we find that the differential equation becomes
- $L\frac{d}{dt}\left(\boldsymbol{I}e^{j\omega t}\right) + R\boldsymbol{I}e^{j\omega t} = \boldsymbol{V}e^{j\omega t}$
- The common factor can be eliminated, leaving the phasors; that is,
- $j\omega L\boldsymbol{I} + R\boldsymbol{I} = \boldsymbol{V}$
- Thus $\boldsymbol{I} = \frac{V}{R+j\omega L} = I_M\angle\emptyset = \frac{V_M}{\sqrt{R^2+\omega^2L^2}}\angle - tan^{-1}\frac{\omega L}{R}$
- Therefore, $i(t) = \frac{V_M}{\sqrt{R^2+\omega^2L^2}}\cos\left(\omega t - tan^{-1}\frac{\omega L}{R}\right)$

Phasors

- We define relations between phasors after the $e^{j\omega t}$ term has been eliminated as "phasor, or frequency domain, analysis."
- The phasors are then simply transformed back to the time domain to yield the solution of the original set of differential equations.
- In addition, we note that the solution of sinusoidal steady-state circuits would be relatively simple if we could write the phasor equation directly from the circuit description.

Phasors

Time Domain	Frequency Domain
$A\cos(\omega t \pm \theta)$	$A\angle \pm \theta$
$A\sin(\omega t \pm \theta)$	$A\angle \pm \theta - 90^0$

Table 8.1: Phasor Representation

Exercise

1. Convert the following voltage functions to phasors

- $v_1(t) = 12\cos(377t - 425^0)\,V$
- $v_2(t) = 18\sin(2513t + 4.2^0)\,V$

2. Convert the following Phasors to the time domain if the frequency is 400 Hz.

- $V_1 = 10\angle 20^0$
- $V_2 = 12\angle -60^0$

Phasor Relationships for Circuit Elements

- **Resistor**

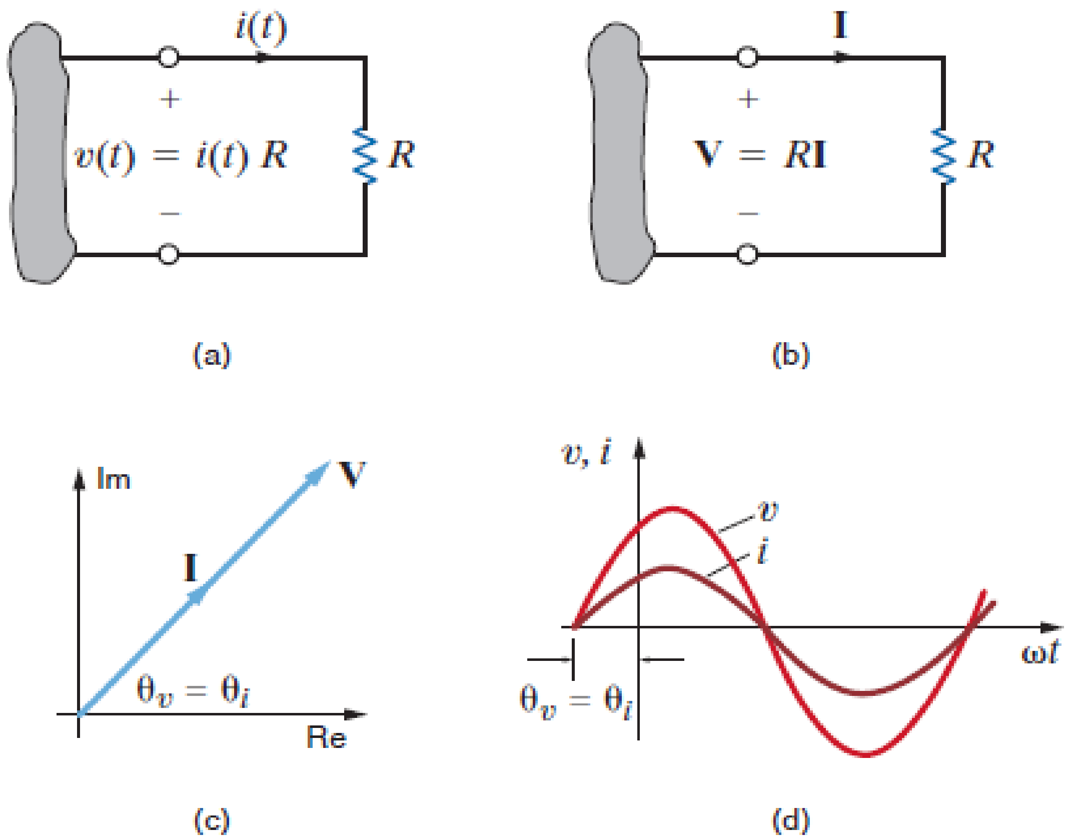


Figure 7.4: *Voltage–current relationships for a resistor.*

Resistor

- $v(t) = Ri(t)$

Applying the complex voltage $V_M e^{j(\omega t+\theta_v)}$ results in the complex current $I_M e^{j(\omega t+\theta_i)}$ and therefore

- $V_M e^{j(\omega t+\theta_v)} = RI_M e^{j(\omega t+\theta_i)}$
- $V_M e^{j\theta_v} = RI_M e^{j\theta_i}$

In phasor form

- $\boldsymbol{V} = R\boldsymbol{I}$
- Where $\boldsymbol{V} = V_M e^{j\theta_v} = V_M\angle\theta_v$ $\quad and \quad$ $\boldsymbol{I} = I_M e^{j\theta_i} = I_M\angle\theta_i$.
- Where $\theta_v = \theta_i$. Thus current and voltage for this circuit are in phase.

Resistor

- **Exercise**

The current in a 4Ω resistor is known to be $\boldsymbol{I} = 12\angle 60^0 A$.

Express the voltage across the resistor as a time function if the frequency of the current is 4 kHz.

Inductor

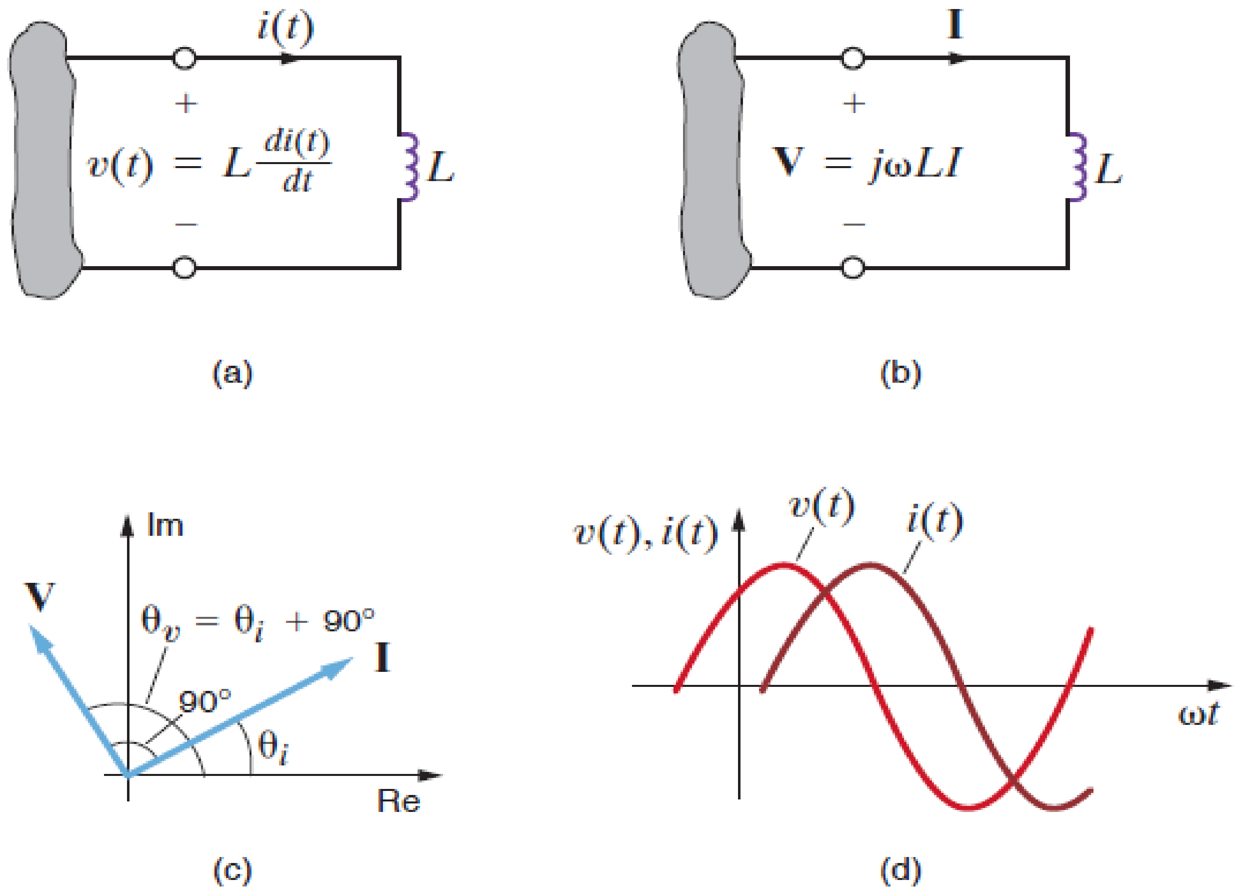


Figure 7.5: *Voltage–current relationships for an inductor.*

Inductor

- $v(t) = L\frac{di(t)}{dt}$
- $V_M e^{j(\omega t+\theta_v)} = L\frac{d}{dt} I_M e^{j(\omega t+\theta_i)}$
- $V_M e^{j\theta_v} = j\omega L I_M e^{j\theta_i}$
- In phasor notation
- $\boldsymbol{V} = j\omega L\boldsymbol{I}$
- Since the imaginary operator $j = 1e^{j90^0} = 1\angle 90^0 = \sqrt{-1}$
- $V_M e^{j\theta_v} = \omega L I_M e^{j\left(\theta_i+90^0\right)}$

Inductor

- Therefore, the voltage and current are *90° out of phase*, and in particular the voltage leads the current by 90° or the current lags the voltage by 90°. The phasor diagram and the sinusoidal waveforms for the inductor circuit are shown in Figs. 8.5c and d, respectively.

Example

- The voltage $v(t) = 12\cos(377t + 20^0)V$ is applied to a 20-mH inductor as shown in Fig. 7.7a. Find the resultant current.

Capacitor

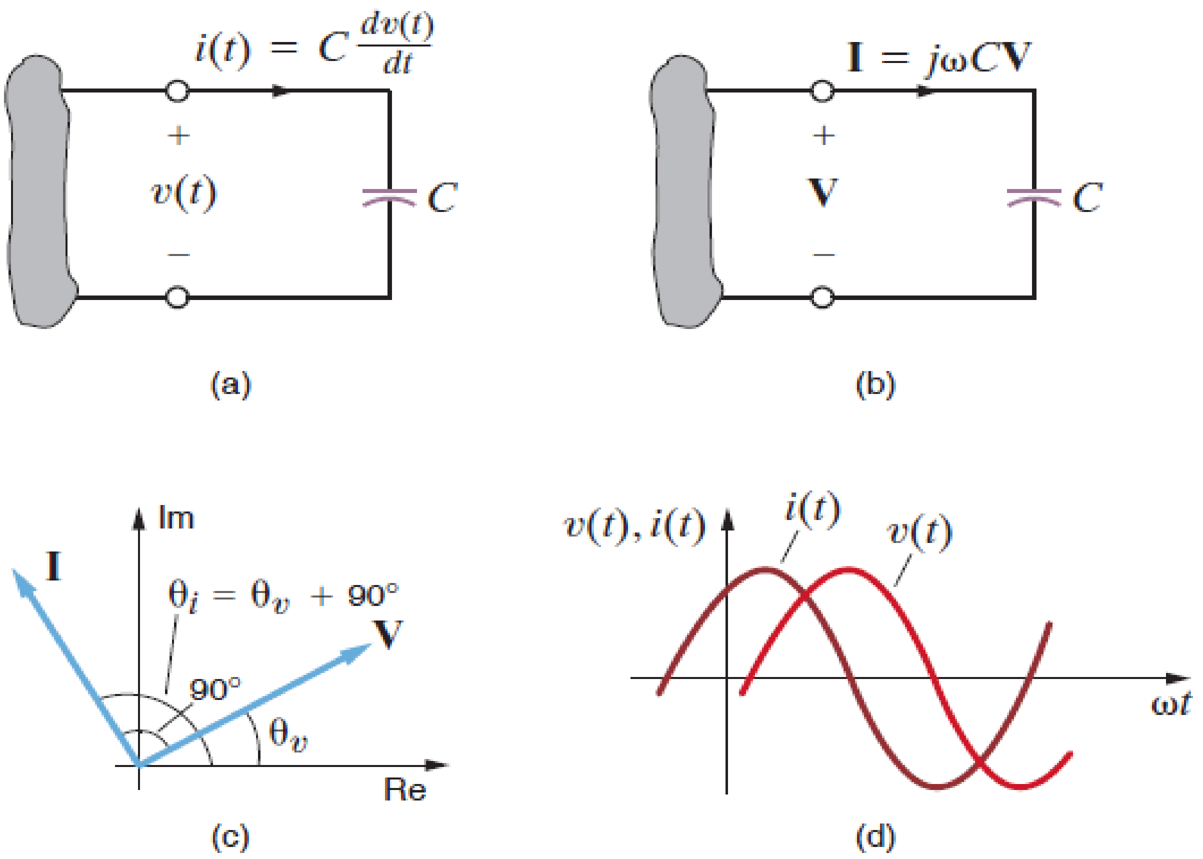


Figure 7.6: *Voltage–current relationships for a capacitor.*

Capacitor

- $i(t) = C\frac{dv(t)}{dt}$
- $I_M e^{j(\omega t+\theta_i)} = C\frac{d}{dt}V_M e^{j(\omega t+\theta_v)}$
- Which reduces to
- $I_M e^{j\theta_i} = j\omega C V_M e^{j\theta_v}$
- In phasor notation this becomes
- $I = j\omega CV$
- **And** $I_M e^{j\theta_i} = \omega C V_M e^{j(\theta_v+90^0)}$
- Note that the voltage and current are *90° out of phase*. In particular, the current leads the voltage by 90°

Exercise

1. The voltage $v(t) = 100\cos(314t + 15^0)V$ is applied to a 100 µF capacitor as shown in Fig. 7.6a. Find the current.
2. The current in a 150- µF capacitor is $I = 3.6\angle -145^0\ A$. If the frequency of the current is 60 Hz, determine the voltage across the capacitor.

Impedance and Admittance

- Impedance is defined as the ratio of the phasor voltage **V** to the phasor current **I**:
- $\boldsymbol{Z} = \frac{\boldsymbol{V}}{\boldsymbol{I}}$

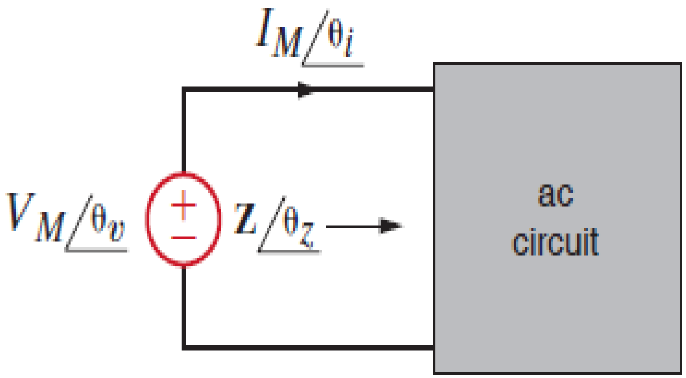


Figure 7.7: *General impedance relationship.*

Impedance and Admittance

- $Z = \frac{V_M \angle \theta_v}{I_M \angle \theta_i} = \frac{V_M}{I_M} \angle \theta_v - \theta_i = Z \angle \theta_z$
- Since **Z** is the ratio of **V** to **I**, the units of **Z** are ohms. Thus, impedance in an ac circuit is analogous to resistance in a dc circuit.
- In rectangular form, impedance is expressed as
- $Z(\omega) = R(\omega) + jX(\omega)$
- Where $R(\omega)$ is the real, or resistive, component and $X(\omega)$ is the imaginary, or reactive, component. In general, we simply refer to R as the resistance and X as the reactance.
- The above equations indicate that $Z\langle\theta_z = R + jX$

Impedance and Admittance

- Thus $Z = \sqrt{R^2 + X^2} \quad and \quad \theta_z = \tan^{-1}\frac{X}{R}$
- Where $R = Zcos\theta_z \quad and\ X = Zsin\theta_z$.
- For the individual passive elements the impedance is as shown in Table 8.2.

Impedance and Admittance

Passive Element	Impedance
R	$Z = R$
L	$Z = j\omega L = jX_L = \omega L\angle 90^0, \quad X_L = \omega L$
C	$Z = \frac{1}{j\omega C} = jX_C = -\frac{1}{\omega C}\angle 90^0, X_C = -\frac{1}{\omega C}$

Table 8.2: Passive Element Impedance

Impedance and Admittance

- KCL and KVL are both valid in the frequency domain.
- Impedances can be combined using the same rules that we established for resistor combinations.
- $Z_s = Z_1 + Z_2 + \cdots Z_n \quad and \quad \frac{1}{Z_p} = \frac{1}{Z_1} + \frac{1}{Z_2} + \cdots \frac{1}{Z_n}$

Impedance and Admittance

- Determine the equivalent impedance of the network shown in Fig. 7.8 if the frequency is f=60 Hz. Then compute the current *i(t)* if the voltage source is $v(t) = 50\cos(\omega t + 30^0)V$
- Finally, calculate the equivalent impedance if the frequency is *f=400 Hz.*

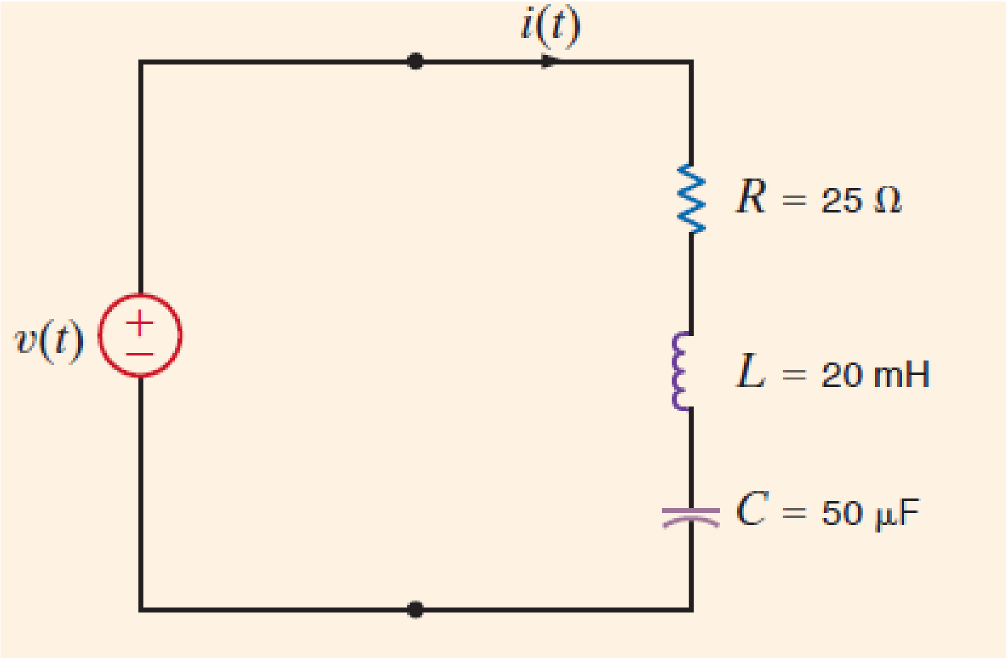
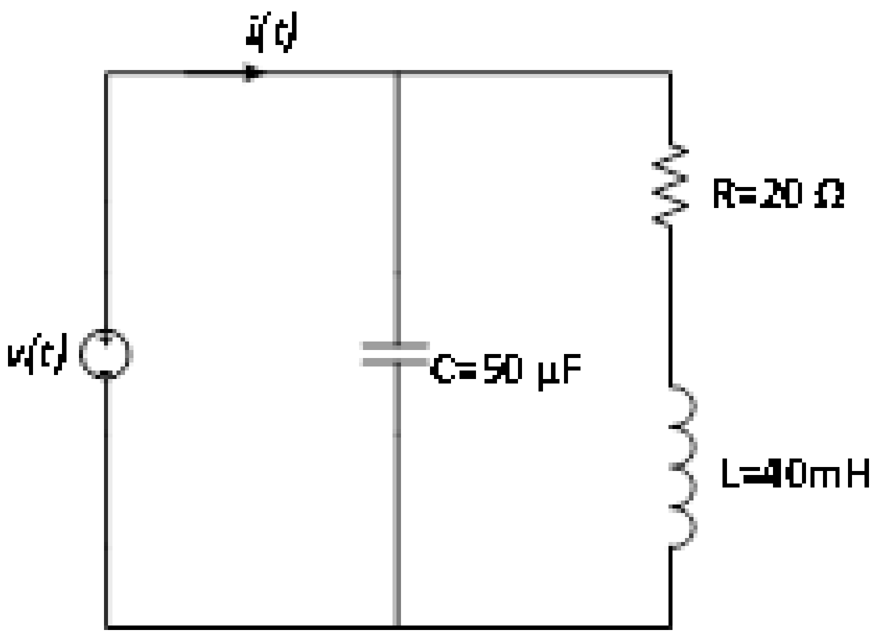


Figure 7.8: *Series ac circuit.*

Impedance and Admittance

- Find the current *i(t)* in the network if
  $v(t) = 120\sin(377t + 60^0)V$



Impedance and Admittance

- Another quantity that is very useful in the analysis of ac circuits is the two-terminal input *admittance*, which is the reciprocal of impedance; that is,
- $\boldsymbol{Y} = \frac{1}{\boldsymbol{Z}} = \frac{\boldsymbol{I}}{\boldsymbol{V}}$
- The units of **Y** are siemens, and this quantity is analogous to conductance in resistive dc circuits. Since **Z** is a complex number, **Y** is also a complex number.
- $\boldsymbol{Y} = Y_M \langle \boldsymbol{\theta_y}$
- which is written in rectangular form as
- $Y = G + jB$

Impedance and Admittance

- From the expression
- $G = \frac{R}{R^2+X^2} \quad and \quad B = -\frac{X}{R^2+X^2}$
- and in a similar manner, we can show that
- $R = \frac{G}{G^2+B^2}, \quad X = \frac{-B}{G^2+B^2}$
- The admittance of the individual passive elements are
- $Y_R = \frac{1}{R} = G \;,\; Y_L = \frac{1}{j\omega L} = -\frac{1}{\omega L}\angle 90^0 \;,\; Y_C = j\omega C = \omega C\angle 90^0$

Impedance and Admittance

- The rules for combining admittances are the same as those for combining conductances;
- $Y_p = Y_1 + Y_2 + \cdots Y_n \qquad and \qquad \frac{1}{Y_s} = \frac{1}{Y_1} + \frac{1}{Y_2} + \cdots \frac{1}{Y_n}$

Impedance and Admittance

- Calculate the equivalent admittance Y_p for the network in Fig. 7.9 and use it to determine the current **I** if $V_s = 60\langle 45^0\ V$.

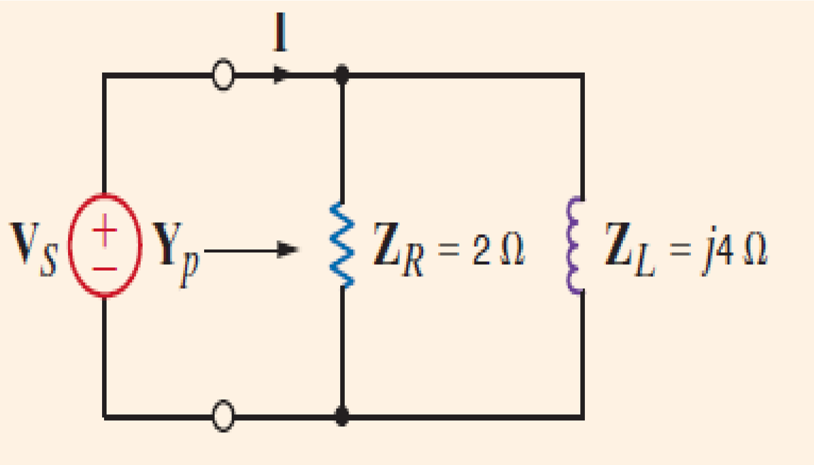


Figure 7.9: *Example parallel circuit*

$$Y_R = \frac{1}{Z_R} = \frac{1}{2} S$$

$$Y_L = \frac{1}{Z_L} = -\frac{j}{4} S$$

Thus $Y_p = \frac{1}{2} - j\frac{1}{4} S$

$$Hence\ I = Y_p V_s$$

$$= \left(\frac{1}{2} - j\frac{1}{4}\right)(60\angle 40^0)$$

$$= 33.5\langle 18.43^0\ A$$

Phasor Diagrams

- Impedance and admittance are functions of frequency, and therefore their values change as the frequency changes.
- These changes in **Z** and **Y** have a resultant effect on the current–voltage relationships in a network. This impact of changes in frequency on circuit parameters can be easily seen via a phasor diagram

Phasor Diagrams-Example

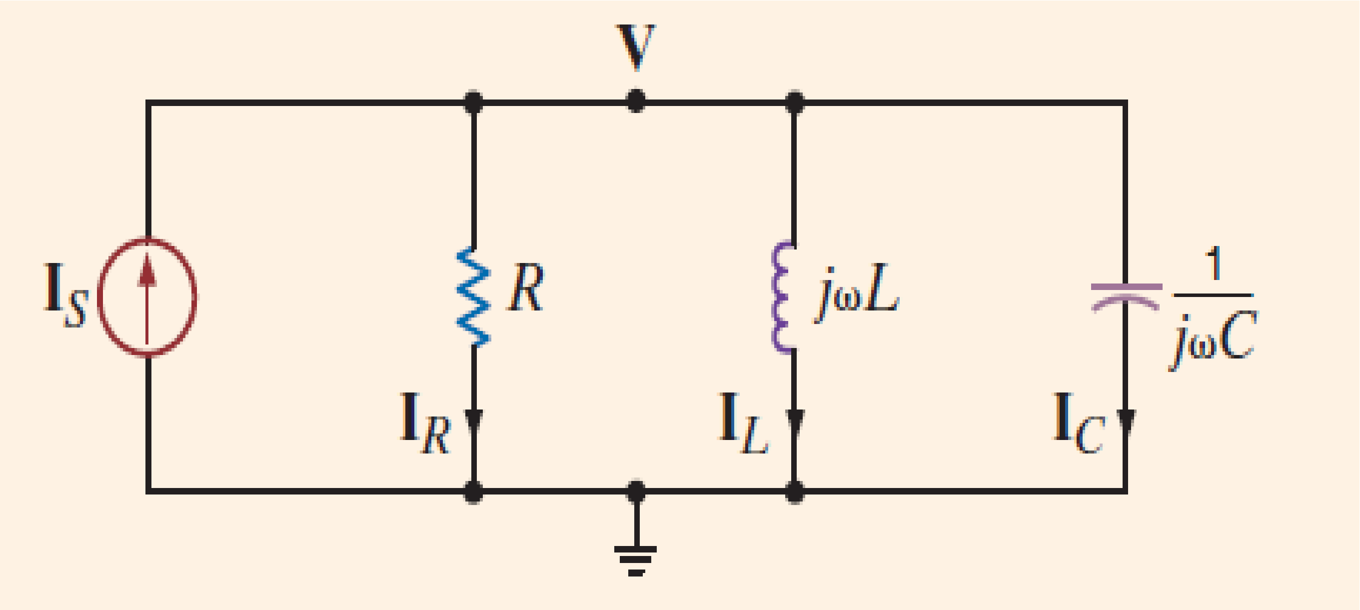


Figure 7.11: *Example parallel circuit*

Phasor diagrams-example

- At the upper node in the circuit KCL is
- $I_S = I_R + I_L + I_C = \frac{V}{R} + \frac{V}{j\omega L} + \frac{V}{1/j\omega L}$
- $Since\ V = V_M\angle 0^0, then \quad I_S = \frac{V_M\angle 0^0}{R} + \frac{V_M\angle -90^0}{\omega L} + V_M\omega C\angle 90^0$
- Note that I_S is in phase with **V** when **I**$_C$ = **I**$_L$ or, in other words, when $\omega L = \frac{1}{\omega C}$. Hence, the node voltage **V** is in phase with the current source when $\omega = \frac{1}{\sqrt{LC}}$
- This can also be seen from the KCL equation
- $I = \left[\frac{1}{R} + j\left(\omega C - \frac{1}{\omega L}\right)\right] V$

Phasor diagrams-Example

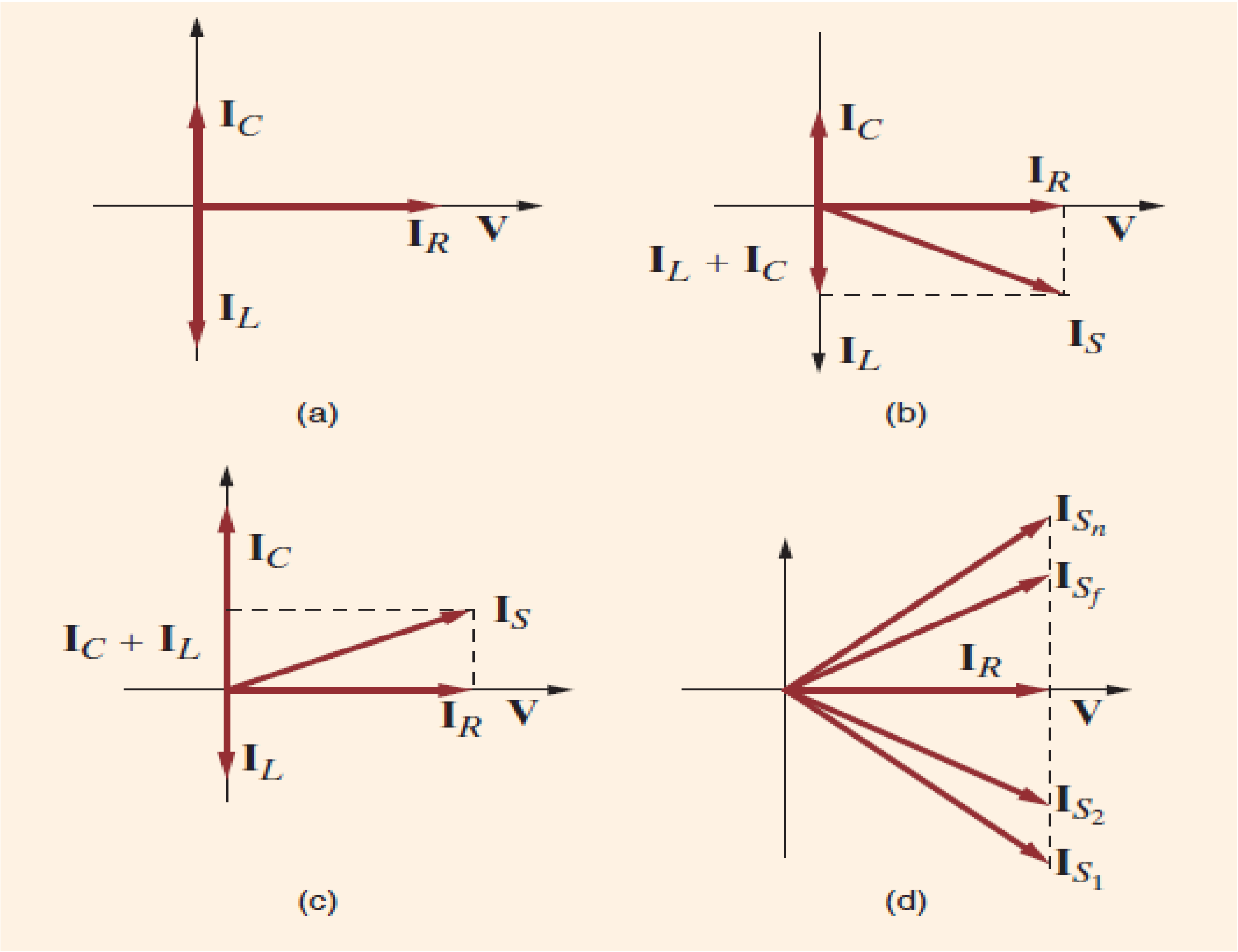


Figure 7.12: *Phasor diagrams for the circuit in Fig. 7.11.*

Basic Analysis Using Kirchhoff's Laws

- We wish to calculate all the voltages and currents in the circuit shown in Fig. 7.14a.

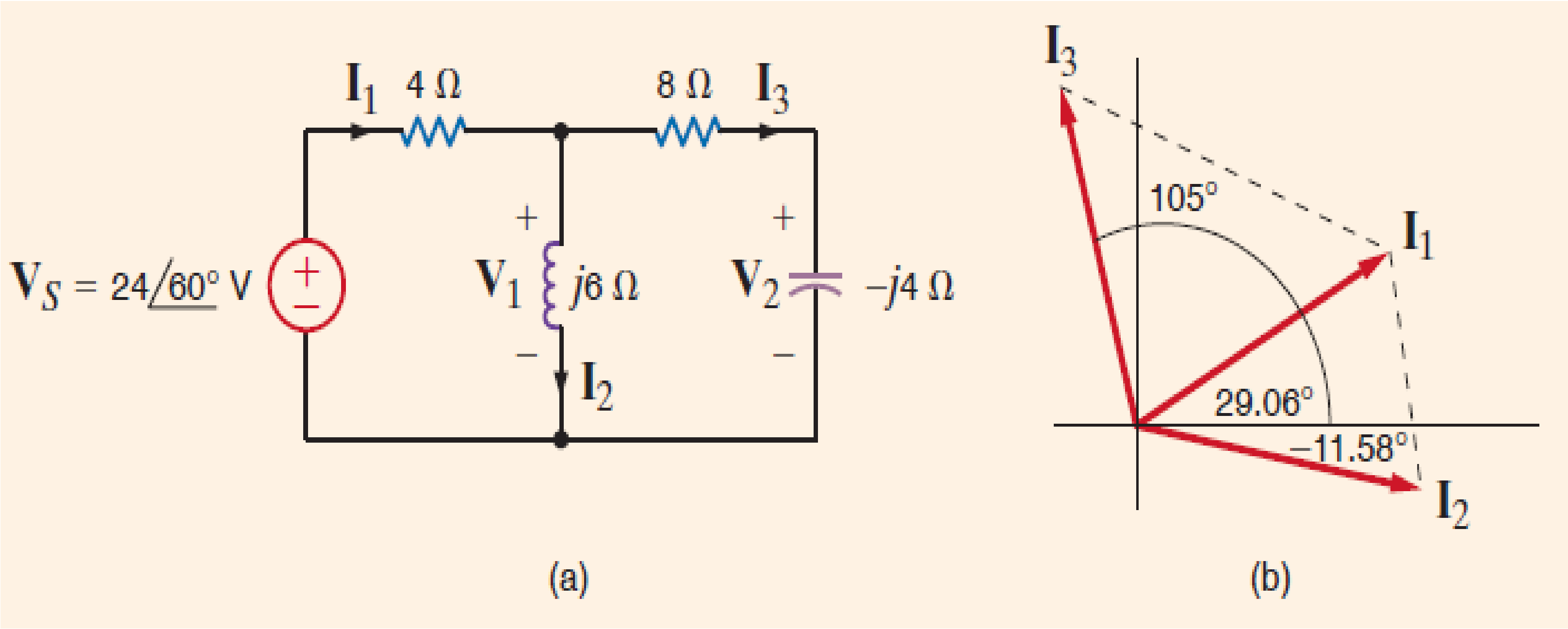


***Figure 7.14** (a) Example ac circuit, (b) phasor diagram for the currents (plots are not drawn to scale).*

Basic Analysis Using Kirchhoff's Laws

- **Solution**
- $Z_{eq} = 4 + \frac{(j6)(8-j4)}{j6+8-j4} = 8.24 + j4.94 = 9.61\angle 30.94^0 \Omega$
- $I_1 = \frac{V_S}{Z_{eq}} = \frac{24\langle 60^0}{9.61\langle 30.94^0} = 2.5\angle 29.06^0\ A$
- V_1 can be determined using KVL:
- $V_1 = V_S - 4I_1 = 16.26\angle 78.43^0 A$ (This can also be determined by voltage division)
- $I_2 = \frac{V_1}{j6} = 2.71\angle -11.58^0\ A$

Basic Analysis Using Kirchhoff's Laws

- $I_3 = \frac{V_1}{8-j4} = 1.82\angle 105^0\ A$
- $(I_2\ and\ I_3$ can also be determined by current division.)
- $V_2 = I_3(-j4) = 7.28\angle 15^0\ V$
- The phasor diagram for the currents $I_1, I_2\ and\ I_3$ and is shown in Fig. 7.14b and is an illustration of KCL.

Analysis Techniques-Nodal

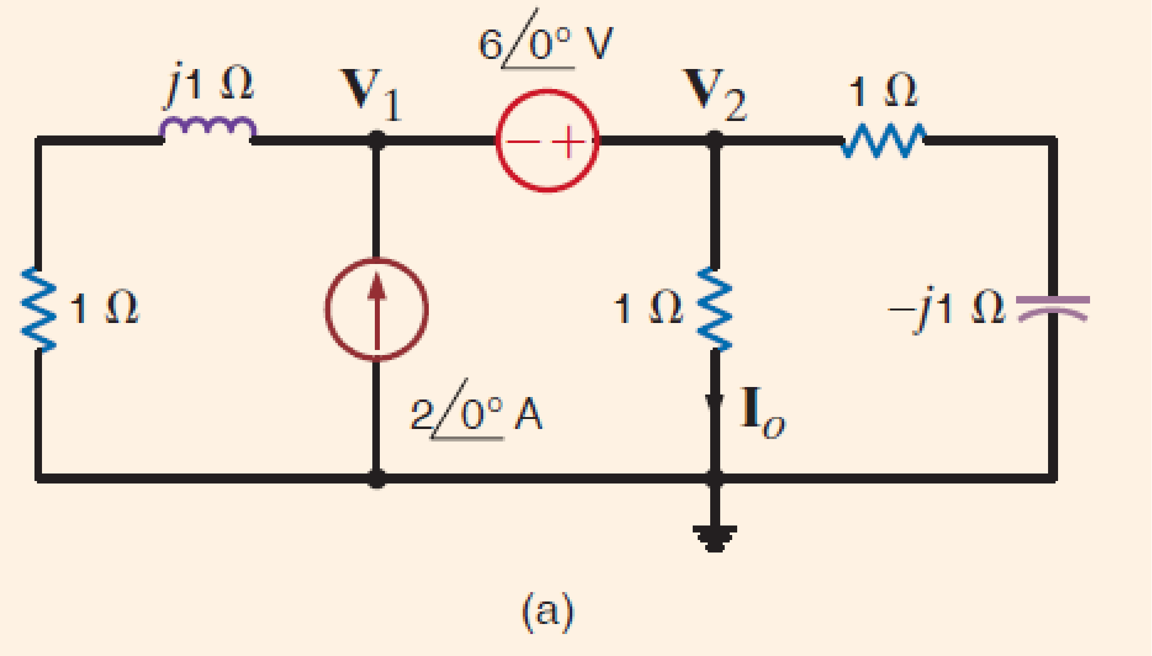
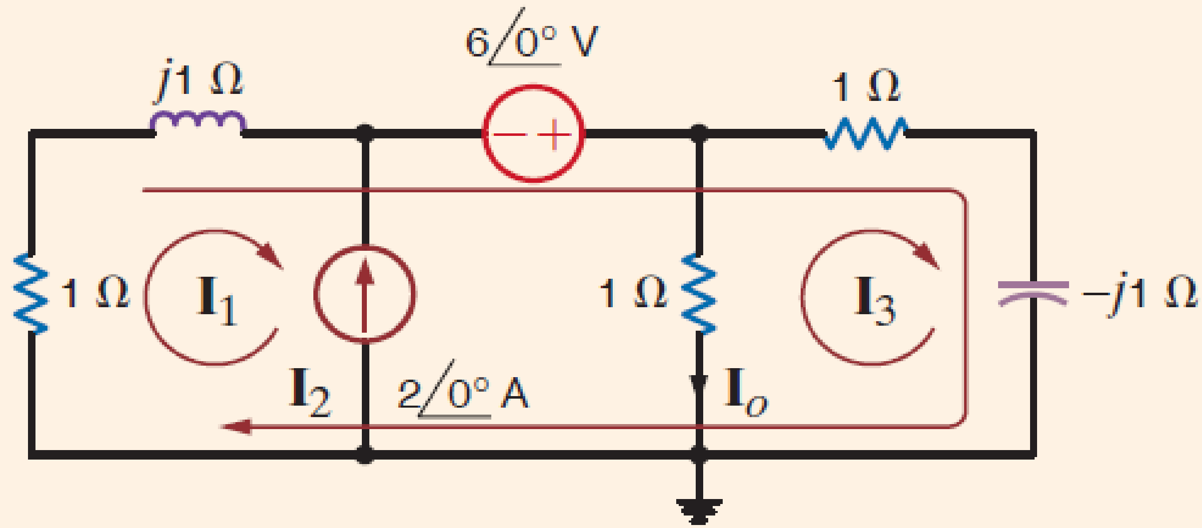


Figure 7.15 *Circuits for node analysis*

Analysis Techniques-Nodal

- The KCL equation for the supernode that includes the voltage source is.
- $\frac{V_1}{1+j} + \frac{V_2}{1} + \frac{V_2}{1-j} = 2\angle 0^0$
- and the associated KVL constraint equation is
- $V_2 - V_1 = 6\angle 0^0$
- $Therefore, V_1 = V_2 - 6$. Substituting this into the first equation and solving gives
- $V_2 = \left[\frac{5}{2} - j\,\frac{3}{2}\right] V$
- Therefore $I_0 = \left[\frac{5}{2} - j\,\frac{3}{2}\right] A$

Analysis Techniques-Loop

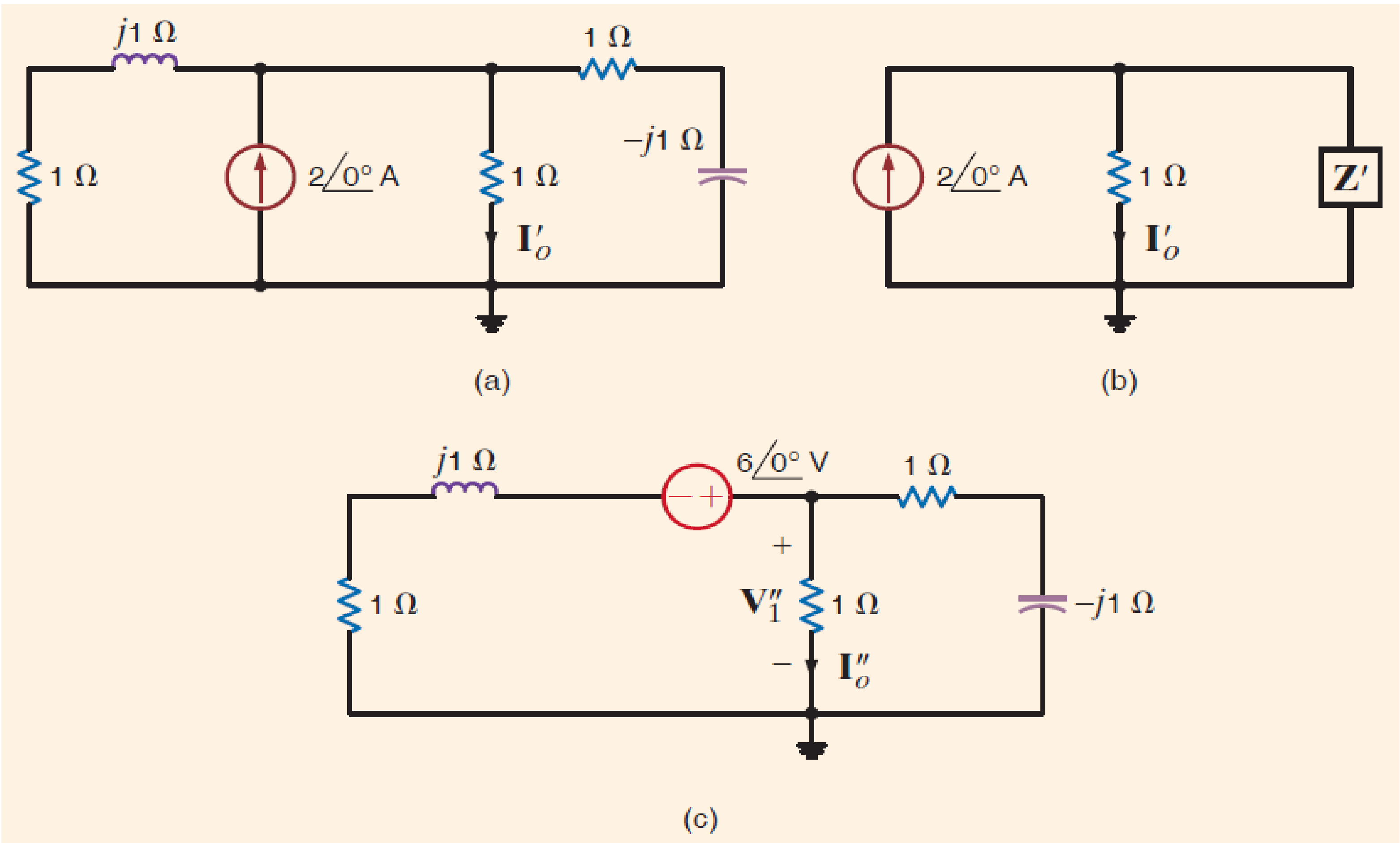


(b)

Analysis Techniques-Loop

- The three loop equations are
- $I_1 = -2\langle 0^0$
- $I_1(1+j) + I_2(2) + I_3(1-j) = 6\angle 0^0$
- $I_2(1-j) + I_3(2-j) = 0$
- Solving the above equations gives
- $I_3 = \left(-\frac{5}{2} + j\frac{3}{2}\right) A$
- And finally $I_0 = -I_3 = \left[\frac{5}{2} - j\frac{3}{2}\right] A$

Analysis Techniques-Superposition



Exercise

- Use Thevenin’s and Norton’s analysis techniques to obtain a similar result.