RANK OF MATRIX

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RANK OF A MATRIX

Rank of Matrix

The number r is called the **rank** of a matrix A if

(i) Every minor of order $(r + 1)$ of A is zero and *(ii)* There exists at least one minor of order r which is non-zero.

Note. (1) The rank of matrix A is denoted by ρ (A),

(2) Rank of a null matrix is zero.

(3) If $A = (a_{ij})_{m \times n}$ then $\rho(A) \leq \text{Min.}\ (m, n)$

(4) $\rho(I_n) = n$ (5) $\rho(O) = 0$

Example. (1) Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 5 & 7 \end{bmatrix}$

Then $$|A| = \begin{vmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 5 & 7 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 2 & 3 & 4 \end{vmatrix} = 0$$

A minor of order $2 = \begin{vmatrix} 1 & 2 \\ 2 & 3 \end{vmatrix} = 3 - 4 \neq 0 \therefore \rho(A) = 2$

Theorem. *The rank of the transpose of a matrix is equal to the rank of the matrix.*

Proof : Let $\rho(A) = r$ = There exists a square submatrix B of A such that det B $\neq 0$ and determinant of every square submatrix of A of order $> r$ is 0. Since A^t is obtained by changing rows into columns and columns into rows of A. B^t would be a square submatrix of A^t of order r and det (B^t) = det B $\neq 0$. Therefore, $\rho(A^t) \geq r$. We shall prove that $\rho(A) \not> r$. Suppose that there is a square submatrix C of A^t of order $> r$ such that det C $\neq 0$, then C^t would be square submatrix of A of order $> r$ such that det (C^t) = det C $\neq 0 \Rightarrow \rho(A) > r$, a contradiction.

Hence, $$\rho(A^t) = r = \rho(A).$$

Elementary Row Transformations

The following three types of operations on the rows of a given matrix are known as **elementary row transformations.** (*i.e.* E-row transformation)

(i) Interchanging any two rows of the given matrix.

(ii) Multiplying every element of any row of the given matrix by a non-zero number.

(iii) Adding a non-zero scalar multiple of the elements of any row to the corresponding elements of another row.

Note 1. *(i)* is denoted by R_{ij} *(ii)* is denoted by $R_{i(K)}$ *(iii)* $R_i + R_{j(K)}$ or $R_{ij(K)}$

Note 2. Similarly we can have **elementary column transformations.**

Note 3. Since the order of the largest non-singular square sub matrix of a given matrix is not affected by any of the elementary row transformations.

$\therefore$ *rank of a matrix remains unchanged by the application of any of the elementary row transformation on it.*

Elementary Operations. The above six operations *(i)* three elementary row operations *(ii)* three elementary column operations, are called **elementary operations.**

Def. Row Equivalent. A matrix A is said to be **row equivalent** to a matrix B if B can be obtained from A by applying in succession a finite number of elementary row operations on A and we write

$$A \overset{R}{\sim} B$$

Def. Column Equivalent

A matrix A is said to be **column equivalent** to a matrix B if B can be obtained from A by applying in succession a finite number of elementary column operations on A and we write $A \overset{C}{\sim} B$.

Note. *Column–equivalent matrices have the same order and the same rank.*

Echelon Form of a Matrix

A matrix is said to be in **echelon form** *if :*

(i) all the zero rows occur below non-zero rows

(ii) the number of zero before the first non–zero element in a row is less than the number of such zeros in the next row

(iii) the first non-zeros element in every non-zero row is 1.

For example, the matrix

$$\begin{bmatrix} 1 & 2 & 7 & 2 & 4 \\ 0 & 0 & 1 & 0 & 5 \\ 0 & 0 & 0 & 2 & 8 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & -2 & -3 & 5 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix} \text{ and } \begin{bmatrix} 1 & 3 & 0 & 4 & 9 \\ 0 & 0 & 1 & 3 & 7 \\ 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

are matrices in row–echelon form.

Note. From the above example, it is clear that if a matrix is in row–echelon form, then any column that contains a leading entry 1 must contain zeros below the leading entry 1. However the entries above the leading entry 1, are arbitrary.

Def. Row–reduced echelon form

A matrix in row–echelon form satisfying the following additional condition is said to be in **row–reduced echelon form condition.** *Each column which contains a leading entry 1 of a row has all the other entries zero.*

For example, the matrix

$$\begin{bmatrix} 1 & 4 & 0 & 0 & 9 \\ 0 & 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & p & 0 & q & 0 \\ 0 & 0 & 1 & r & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$ are in row–reduced echelon form.

Note 1. *A zero matrix and an identity matrix are always in row reduced echelon form.*

Note 2. *Row reduced echelon form is generally preferred to a row–echelon form in the solution of a system of linear equations.*

(*a*) Method to find Rank of a Matrix

Reduce the given matrix to echelon form by using elementary row transformations. Then, the number of non-zero rows in the echelon form, is the rank of the given matrix.

Example. (*i*) The matrix $A = \begin{bmatrix} 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ is in echelon form and it has 2 non–zero rows.

$\therefore \qquad \rho(A) = 2$

Theorem. *If A and B are two matrices conformable for multiplication, then*

$$\rho(AB) \le \rho(A), \;\; \rho(AB) \le \rho(B)$$

OR

rank (AB) $\le$ min (rank A, rank B)

Theorem. *The elementary operations do not alter the rank of a matrix.*

Example *Find the rank of the matrix*

$$A = \begin{bmatrix} 4 & 5 & -6 & 1 \\ 7 & -3 & 0 & 8 \end{bmatrix}$$

Sol. Operate $R_2 - 2R_1$ $\quad A \sim \begin{bmatrix} 1 & 5 & -6 & 1 \\ -1 & -13 & 12 & 6 \end{bmatrix}$

Operate R_{12} $\quad A \sim \begin{bmatrix} -1 & -13 & 12 & 6 \\ 4 & 5 & -6 & 1 \end{bmatrix}$

Operate $R_{1(-1)}$ $\quad A \sim \begin{bmatrix} 1 & 13 & -12 & -6 \\ 4 & 5 & -6 & 1 \end{bmatrix}$

Operate $R_2 - 4R_1$ $\quad \sim \begin{bmatrix} 1 & 13 & -12 & -6 \\ 0 & -47 & 42 & 25 \end{bmatrix}$

Operate $R_{2\left(-\frac{1}{47}\right)}$ $\quad \sim \begin{bmatrix} 1 & 13 & -12 & -6 \\ 0 & 1 & \frac{-42}{47} & \frac{-25}{47} \end{bmatrix}$

Since it contains two non zero rows. Hence $\rho(A) = 2$

Row Rank and Column Rank of a Matrix

Def. Row Rank. The number of non–zero rows in the row–echelon form of a matrix A, is called the **row rank of A** and is denoted by $\rho_R(A)$.

Def. Column Rank. The number of non-zero rows in the column echelon form of matrix A, is called the column rank of A and is denoted by $\rho_C(A)$.

Note 1. Since all row–echelon form of a square matrix (if there are more than one form) have the same number of non–zero rows $\therefore$ row rank of a matrix is independent of the row–echelon form of the matrix.

In general, we can say that

"Every matrix is row equivalent to a row echelon matrix".

Similarly every matrix is column equivalent to a column–echelon matrix.

Note 2. If A is in row echelon form, then its transpose A′ is in the column echelon form.

Example *Find the row rank of*

$$A = \begin{bmatrix} 1 & 2 & -1 & 0 \\ 3 & 1 & 4 & 2 \\ 1 & -3 & 6 & 2 \end{bmatrix}$$

Sol. Operate $R_{31(-1)}$, $R_{21(-3)}$

$$A \sim \begin{bmatrix} 1 & 2 & -1 & 0 \\ 0 & -5 & 7 & 2 \\ 0 & -5 & 7 & 2 \end{bmatrix}$$

Operate $R_{2\left(-\frac{1}{5}\right)}$ $\quad \sim \begin{bmatrix} 1 & 2 & -1 & 0 \\ 0 & 1 & -\frac{7}{5} & -\frac{2}{5} \\ 0 & -5 & 7 & 2 \end{bmatrix}$

Operate $R_{32(5)}$, $R_{12(-2)}$ $\quad \sim \begin{bmatrix} 1 & 2 & -1 & -1 \\ 0 & 1 & -\frac{7}{5} & -\frac{2}{5} \\ 0 & 0 & 0 & 0 \end{bmatrix}$

which is in row-reduced echelon form. Clearly $\rho_R(A) = 2$ [$\because$ there are 2 non-zero rows in the echelon form]

Normal Form of a Matrix

We know that successive application of elementary row operations on a matrix reduce it to the row echelon form. Now if we apply both elementary row and column operations, we shall get a still simple matrix called **normal or canonical form** of the matrix.

If we apply successively elementary operations, we can reduce any matrix A of rank $r > 0$ in one of the forms.

$$I_r, \begin{bmatrix} I_r & O \\ O & O \end{bmatrix}, [I_r \quad O], \begin{bmatrix} I_r \\ O \end{bmatrix}$$

(where I_r is the identity matrix of order r and O is some row matrix).

All these are known as **normal form** of A.

If $r = 0$, then A is in the normal form iff $A = O$.

$$\text{If } A = I_r \text{ or } \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} \text{ or } [I_r \quad 0], \begin{bmatrix} I_r \\ 0 \end{bmatrix} \text{ then } (\rho(A) = r.$$

If A is an n–square matrix in the normal form, then A is non–singular iff $A = I_n$.

Theorem *If A is an $m \times n$ matrix of rank r, then A is equivalent to the matrix $\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$ in the normal form*

Theorem. *If A and B square matrices of order n, prove that $\rho(AB) \geq \rho(A) + \rho(B) - n$*

Theorem. *Show that $\rho(AA') = \rho(A)$, where A' is the transpose of A.*

Sol. We know that $\rho(AA') \leq \rho(A)$ ...(1)

Again $\quad A' = IA' = A^{-1}AA' \quad \therefore \quad \rho(A') = \rho(A^{-1}AA')$

But $\quad \rho(A) = \rho(A')$

$\therefore \quad \rho(A) \leq \rho(AA')$ ...(2)

From (1) and (2), we get $\rho(AA') = \rho(A)$

Relation between rank, row rank and column rank of a matrix

Theorem. *For any matrix*

$$\rho_R(A) = \rho(A) = \rho_C(A)$$

Example *Reduce to the normal form the matrix*

$$A = \begin{bmatrix} 0 & 2 & 3 & 4 \\ 2 & 3 & 5 & 4 \\ 4 & 8 & 13 & 12 \end{bmatrix}$$

Sol. $$A = \begin{bmatrix} 0 & 2 & 3 & 4 \\ 2 & 3 & 5 & 4 \\ 4 & 8 & 13 & 12 \end{bmatrix}$$

Operate R_{12} $$\sim \begin{bmatrix} 2 & 3 & 5 & 4 \\ 0 & 2 & 3 & 4 \\ 4 & 8 & 13 & 12 \end{bmatrix}$$

Operate $R_{1(1/2)}$ $$\sim \begin{bmatrix} 1 & \frac{3}{2} & \frac{5}{2} & 2 \\ 0 & 2 & 3 & 4 \\ 4 & 8 & 13 & 12 \end{bmatrix}$$

Operate $R_{31(-4)}$ $$\sim \begin{bmatrix} 1 & \frac{3}{2} & \frac{5}{2} & 2 \\ 0 & 2 & 3 & 4 \\ 0 & 2 & 3 & 4 \end{bmatrix}$$

Operate $R_{2(1/2)}$ $$\sim \begin{bmatrix} 1 & \frac{3}{2} & \frac{5}{2} & 2 \\ 0 & 1 & \frac{3}{2} & 2 \\ 0 & 2 & 3 & 4 \end{bmatrix}$$

Operate $R_{32(-2)}$ $$\sim \begin{bmatrix} 1 & \frac{3}{2} & \frac{5}{2} & 2 \\ 0 & 1 & \frac{3}{2} & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Operate $C_{21}\left(-\frac{3}{2}\right)$ $$\sim \begin{bmatrix} 1 & 0 & \frac{5}{2} & 2 \\ 0 & 1 & \frac{3}{2} & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Operate $C_{41(-2)}$, $C_{31(-5/2)}$ $$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & \frac{3}{2} & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Operate $C_{42(-2)}$, $C_{32(-3/2)}$ $$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} I_2 & O \\ O & O \end{bmatrix}$$ which is in the normal form.

Clearly $\rho(A) = 2$

Theorem. *If A is an $m \times n$ matrix of rank r, then $\exists$ non-singular matrices P and Q of* order m and n *respectively, such that* $PAQ = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$

Example *Find two non-singular matrices P and Q such that PAQ is in the normal form where* $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & -1 \\ 3 & 1 & 1 \end{bmatrix}$. *Also find the rank of the matrix A.*

Sol. Since A = IAI

$$\therefore \quad \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & -1 \\ 3 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

We want to reduce the matrix on the L.H.S. to the normal form by elementary row (column) operation. Since every E–row operations will also be applied on the prefactor I on the R.H.S. and every E–column operation to the host factor I on the R.H.S.

$\therefore$ operate

$R_{21(-1)}$, $R_{31(-3)}$, we get

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & -2 & -2 \\ 0 & -2 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Operate $C_{21(-1)}$, $C_{31(-1)}$, we get

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & -2 \\ 0 & -2 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Operate $R_{2(-1/2)}$, we get

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & -2 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{2} & -\frac{1}{2} & 0 \\ -3 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Operate $R_{32(2)}$, we get

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{2} & -\frac{1}{2} & 0 \\ -2 & -1 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Operate $C_{32(r)}$, we get

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{2} & -\frac{1}{2} & 0 \\ -2 & -1 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow \quad \begin{bmatrix} I_2 & 0 \\ 0 & 0 \end{bmatrix} = PAQ$$

where $P = \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{2} & -\frac{1}{2} & 0 \\ -2 & -1 & 1 \end{bmatrix}$, $Q = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$

Since $A \sim \begin{bmatrix} I_2 & O \\ O & O \end{bmatrix}$

$\therefore \quad \rho(A) = 2$