MAKERERE UNIVERSITY

COLLEGE OF ENGINEERING, DESIGN, ART AND TECHNOLOGY

SCHOOL OF ENGINEERING

DEPARTMENT OF ELECTRICAL AND COMPUTER ENGINEERING

EMT 1201: ENGINEERING MATHEMATICS II LECTURE NOTES 2015/2016

CHAPTER ONE: COMPLEX ALGEBRA

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COMPLEX NUMBERS

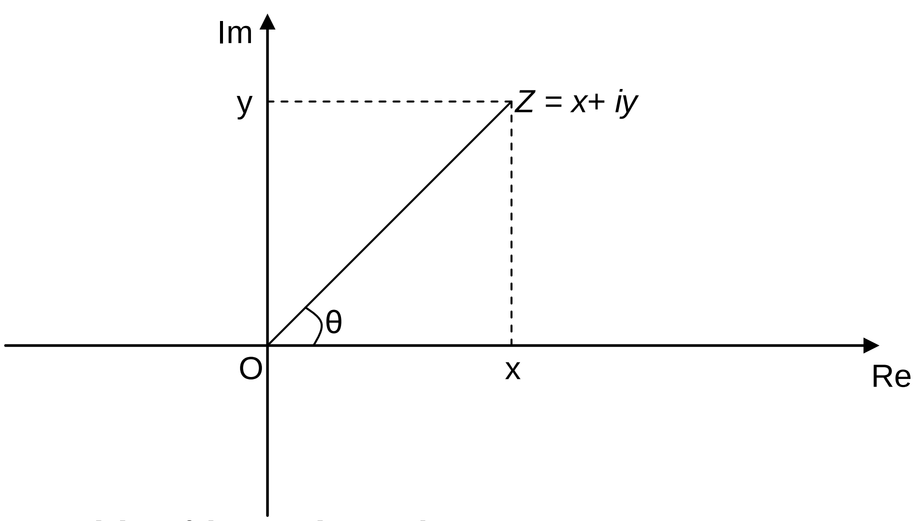
A complex number is a number described as $Z = (real\ part) + i(imaginary part)$; where $i = \sqrt{-1}$

A complex number can be written in various forms:

- $Z = x + iy$ - Cartesian form
- $Z = r(cos\theta + isin\theta)$ - polar form
- $Z = re^{i\theta}$ - Exponential form.

Graphical representation of complex numbers (The argand diagram)

We choose two perpendicular coordinate axes; the horizontal *x – axis*, called the *real axis*, and the vertical y – axis, called the imaginary axis. On both axes we choose the same unit length. This is called the *Cartesian coordinate system*. We now plot $Z = x + iy$ as the point P with coordinates *(x, y)*. The xy – plane in which the complex numbers are representend in this way is called the *complex plane* or the *Argand diagram*.



OZ – modulus of the complex number

θ – argument of the complex number. It is always measured in the positive – anticlockwise direction.

Arithmetic operations of complex numbers

Addition

The sum of $Z_1 = x_1 + iy_1$ and $Z_2 = x_2 + iy_2$ can be written as;

$Z_1 + Z_2 = (x_1 + x_2) + i(y_1 + y_2)$

Subtraction

Given $Z_1 = x_1 + iy_1$ and $Z_2 = x_2 + iy_2$, then;

$Z_1 - Z_2 = (x_1 - x_2) + i(y_1 - y_2)$

Multiplication

The product Z_1Z_2 can be determined as;

$$Z_1Z_2 = (x_1 + iy_1)(x_2 + iy_2)$$

$$= (x_1x_2 - y_1y_2) + i(x_1y_2 + x_2y_1)$$

This is obtained by the rules of arithmetic and using $i^2 = -1$

Complex conjugates

Let $Z = x + iy$ be any complex number. Then $x - iy$ is called the <u>*complex conjugate*</u> of Z and is denoted, $\bar{Z}$.

Thus, if $Z = x + iy$, then $\bar{Z} = x - iy$.

Conjugates are useful since, $Z\bar{Z} = x^2 + y^2$ is real, a property used in division of complex numbers.

Moreover, their addition and subtraction yields;

$Z + \bar{Z} = 2x$ and $Z - \bar{Z} = 2iy$, so that we can express the real part and imaginary part of Z by the important formulae;

$Re_Z = x = \frac{1}{2}(Z + \bar{Z})$ and

$$Im_Z = y = \frac{1}{2i}(Z - \bar{z})$$

Important properties of conjugates

- $\overline{(Z_1 + Z_2)} = \overline{Z_1} + \overline{Z_2}$
- $\overline{(Z_1 - Z_2)} = \overline{Z_1} - \overline{Z_2}$
- $\overline{(Z_1 Z_2)} = \overline{Z_1 Z_2}$
- $\overline{\left(\frac{Z_1}{Z_2}\right)} = \frac{\overline{Z_1}}{\overline{Z_2}}$

Division

Given $Z_1 = x_1 + iy_1$ and $Z_2 = x_2 + iy_2$, then;

$$Z = \frac{Z_1}{Z_2} = \frac{x_1 + iy_1}{x_2 + iy_2} * \frac{x_2 - iy_2}{x_2 - iy_2}$$

$= \frac{x_1 x_2 - (-1) y_1 y_2 + (x_2 y_1 - x_1 y_2) i}{x_2{}^2 - x_2 y_2 i + x_2 y_2 i - i^2 y_2{}^2}$, but $i^2 = -1$

Therefore, $\frac{Z_1}{Z_2} = \frac{x_1 x_2 - (-1) y_1 y_2 + (x_2 y_1 - x_1 y_2}{x_2{}^2 - (-1) y_2{}^2}$

$$= \frac{x_1 x_2 + y_1 y_2}{x_2{}^2 + y_2{}^2} + \left(\frac{x_2 y_1 - x_1 y_2}{x_2{}^2 + y_2{}^2}\right) i$$

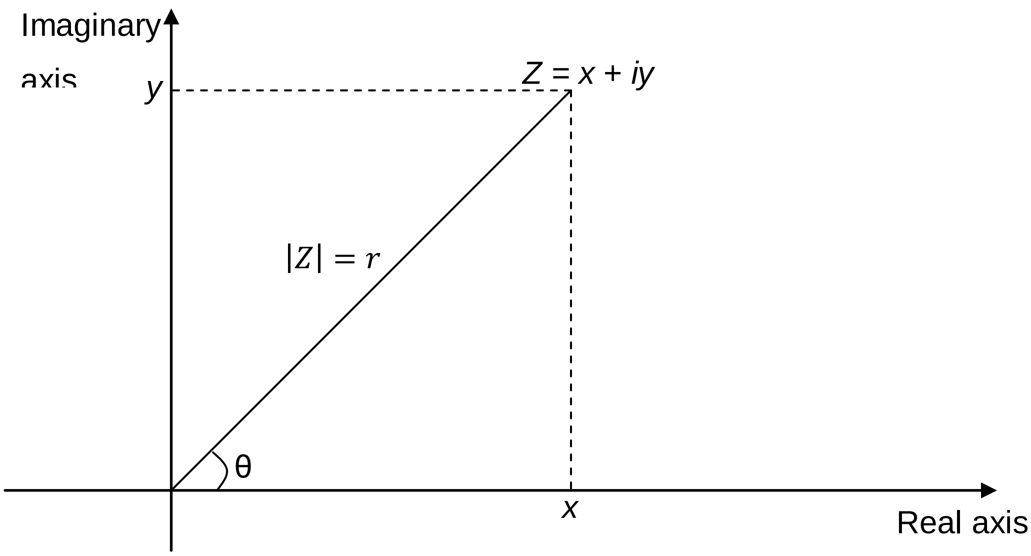
Equality of complex numbers

If $Z_1 = a + ib$ and $Z_2 = c + id$, then if $Z_1 = Z_2 \Rightarrow a = c\ and\ b = d$

Polar form of complex numbers

Given a complex number $Z = x + iy$, in polar coordinates it can written as;

$Z = r(cos\theta + isin\theta)$. See figure below:



$x = rcos\theta$ and $y = rsin\theta$

- r is called the ***absolute value*** or ***modulus*** of Z and is denoted $|Z|$. Hence;

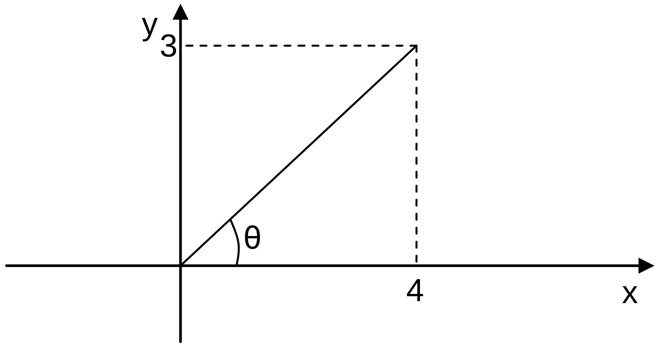
  $|Z| = r = \sqrt{x^2 + y^2} = \sqrt{Z\bar{Z}}$

- θ is called the argument of Z and is denoted *argZ.*

  $\theta = argZ = tan^{-1}\left(\frac{y}{x}\right)$. θ is the directed angle from the positive x – axis to OZ i.e. it is measured in the anticlockwise sense.

Example 1

Express $Z = 4 + 3i$ in polar form.



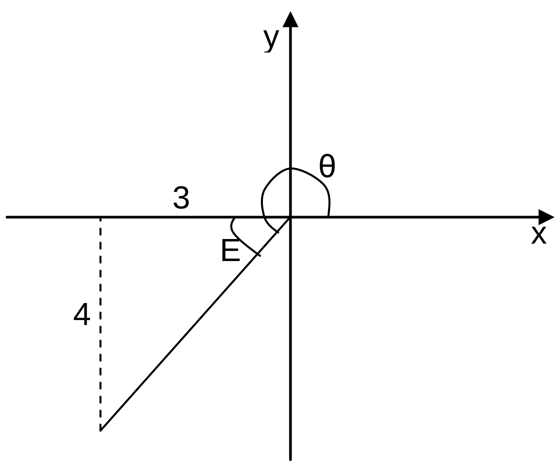
$r = \sqrt{3^2 + 4^2} = 5$ and $tan\theta = \frac{3}{4} \therefore \theta = 36^0 52^1$

Therefore the complex number can be written as;$Z = 5(cos36^0 52^1 + isin36^0 52^1)$

Note: There are two possible angles θ between 0^0 and 360^0, whose tangent, $tan\theta = \frac{y}{x}$. The Argand diagram must be drawn to ensure, we have the correct value.

Example 2

Express the complex number, $Z = -3 - 4i$ in polar form.
This can be represented as in the diagram below on an argand diagram:



$tanE = \frac{4}{3} \Rightarrow E = 53^0 8^1$ and $r = \sqrt{(-3)^2 + (-4)^2} = 5$

$\therefore \theta = E + 180^0 = 233^0 8^1$. Therefore, the complex number Z can be written as;

$Z = 5(cos233^0 8^1 + isin233^0 8^1)$

Example 3

Express $Z = -5 + 12i$ in polar form. (Ans: $Z = 13(cos112.6^0 + isin112.6^0)$)

Short hand $r\lfloor\theta$

The polar form of a complex number, $Z = r(cos\theta + isin\theta)$ can be written in the short hand form as, $Z = r\lfloor\theta$.

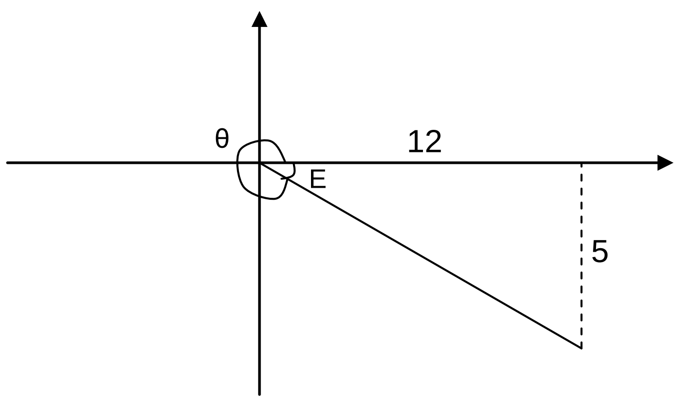
Cases when $\theta > 180^0$

θ is always measured in the anticlockwise sense, but for $\theta > 180^0$ it is always a good practice to measure it in the opposite sense, the clockwise direction such that it lies in the range $-180^0 \leq \theta \leq 180^0$. When this is done the angle θ is given a negative sign.

Example 4

Express $12 - 5i$ in polar form.

This can be represented on an argand diagram as below:



$tanE = \frac{5}{12} \Rightarrow E = 22.6^0$ and $r = \sqrt{5^2 + 12^2} = 13$

Therefore;

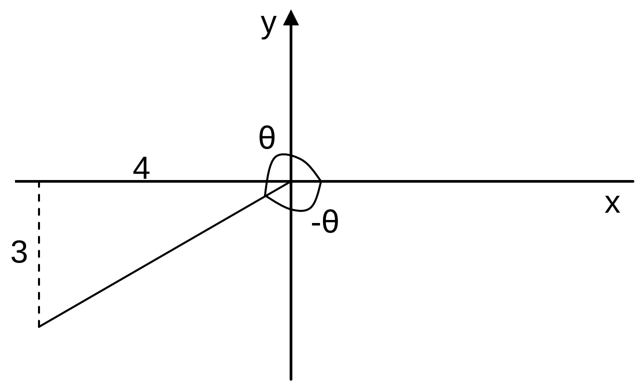
$Z = 13(\cos(-22.6^0) + isin(-22.6^0)) = 13(cos22.6^0 - isin22.6^0)$

Generally when θ is measured clockwise from the positive x – axis, then;

$Z = r(cos\theta - isin\theta)$, since $cos(-\theta) = cos\theta \; and \; sin(-\theta) = -sin\theta$. This is conveniently done if the positive θ is greater than 180^0 i.e. in the 3^{rd} or 4^{th} quadrants.

Example 5

Find the polar form of $Z = -4 - 3i$



$$tanE = \frac{3}{4} \Rightarrow E = 36.9^0 \Rightarrow -\theta = -(180 - 36.9) = -143.1^0$$

$r = \sqrt{3^2 + 4^2} = 5$, therefore;

$$Z = 5(cos143.1^0 - isin143.1^0)$$

Short form when $\boldsymbol{\theta} > 180$

When θ is measured in the clockwise sense, i.e. when the positive θ is greater than 180^0, polar form of Z can be written as;

$$Z = r(cos\theta - isin\theta) = r\lfloor -\theta = r\lceil \theta$$

e.g. $Z = -4 - 3i = 5(cos143.1^0 - isin143.1^0) = 5\lfloor -143.1^0 = 5\lceil 143.1$

Multiplication and Division in polar form

Multiplication

If $Z_1 = r_1(cos\theta_1 + isin\theta_1)$ and $Z_2 = r_2(cos\theta_2 + isin\theta_2)$

Then the product Z_1Z_2 is;

$$Z_1Z_2 = r_1r_2[(cos\theta_1cos\theta_2 - sin\theta_1sin\theta_2) + i(sin\theta_1cos\theta_2 + cos\theta_1sin\theta_2)]$$

Using addition rules of sines and cosines;

$Z_1Z_2 = r_1r_2[cos(\theta_1 + \theta_2) + isin(\theta_1 + \theta_2)]$

Therefore;

- $|Z_1Z_2| = |Z_1||Z_2|$, i.e. the modulus of the product of two complex numbers is equal to the product of their moduli.
- $\arg(Z_1Z_2) = argZ_1 + argZ_2$, i.e. the argument of the product of two complex numbers is equal to the sum of their arguments. However, it should be noted that the two arguments must be measured in the same direction i.e. anticlockwise, before they are added.

Division

Given, $Z_1 = r_1(cos\theta_1 + isin\theta_1)$ and $Z_2 = r_2(cos\theta_2 + isin\theta_2)$, then;

$\frac{Z_1}{Z_2} = \frac{r_1}{r_2}[cos(\theta_1 - \theta_2) + isin(\theta_1 - \theta_2)]$

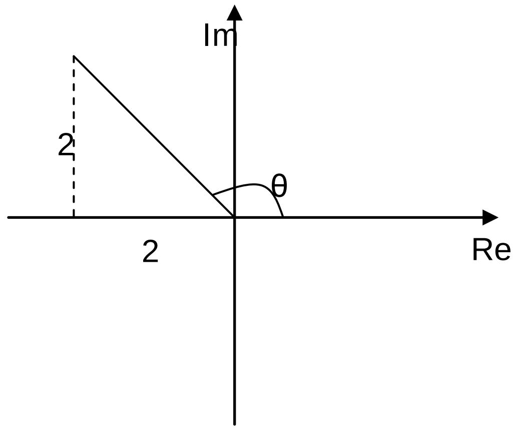
Therefore;

- $\left|\frac{Z_1}{Z_2}\right| = \frac{|Z_1|}{|Z_2|}$, provided $|Z_2| \neq 0$, i.e. the modulus of a division of two complex numbers is equal to the division of their respective moduli.
- $arg\left(\frac{Z_1}{Z_2}\right) = argZ_1 - argZ_2$, i.e. the argument of a division of two complex numbers is equal to the difference of their arguments provided the two arguments are measured in the same direction.

Example

If $Z_1 = -2 + 2i$ and $Z_2 = 3i$, express the two complex numbers in polar form and hence determine Z_1Z_2 and $\frac{Z_1}{Z_2}$

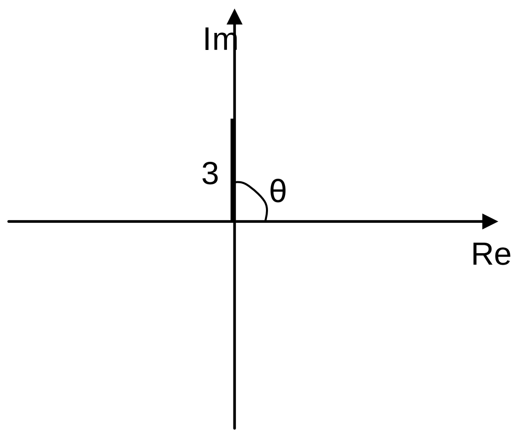
Z_1 can be drawn on an Argand diagram as shown in the figure below:



$\theta = 90^0 + tan^{-1}\left(\frac{2}{2}\right) = 135^0$ and $r = \sqrt{2^2 + 2^2} = 2\sqrt{2}$, therefore;

$Z_1 = 2\sqrt{2}(cos135^0 + isin135^0)$

Z_2 on an diagram is as shown in the figure below;



$\theta = 90^0$ and $r = 3$, therefore; $Z_2 = 3(cos90^0 + isin90^0)$

From this we can get Z_1Z_2 as;

$$Z_1Z_2 = 3 * 2\sqrt{2}(cos(135 + 90) + isin(135 + 90)) = 6\sqrt{2}(cos225^0 + isin225^0)$$

$$= -6 - 6i$$

And similarly;

$$\frac{Z_1}{Z_2} = \frac{2\sqrt{2}}{3}\left(cos(135 - 90) + isin(135 - 90)\right) = \frac{2\sqrt{2}}{3}(cos45 + isin45) = \frac{2}{3} + \frac{2}{3}i$$

Let's check our solution;

$(-2+2i)3i = -6i - 6$, as before.

$$\frac{-2+2i}{3i} = \frac{(-2+2i)3i}{(3i)(3i)} = \frac{2}{3} + \frac{2}{3}i$$

Exponential form of a complex number

Many functions can be expressed as series and among these are;

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots \ldots\ldots\ldots\ldots\ldots\ldots$$

$$sinx = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} + \cdots \ldots\ldots\ldots\ldots\ldots$$

$$cosx = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

Using the series of e^x for $x = i\theta$, we get;

$$e^{i\theta} = 1 + \theta i + \frac{(\theta i)^2}{2!} + \frac{(\theta i)^3}{3!} + \frac{(\theta i)^4}{4!} + \cdots \ldots\ldots\ldots\ldots\ldots\ldots$$

$$= 1 + \theta i + \frac{\theta^2 i^2}{2!} + \frac{\theta^3 i^3}{3!} + \frac{\theta^4 i^4}{4!} + \cdots \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$= \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} + \cdots \ldots\ldots\ldots\right) + i\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} + \cdots \ldots\ldots\ldots\ldots\right)$$

$\therefore\ e^{i\theta} = cos\theta + isin\theta$

Note: $e^{-i\theta} = cos\theta - isin\theta$

- The angle θ must be in radians before a complex number is expressed in exponential form.

Example

Express $Z = 5(cos60^0 + isin60^0)$ in exponential form.

$\theta = \frac{60\pi}{180} = \frac{\pi}{3}$, therefore, $Z = 5e^{\frac{\pi}{3}}$

Finding the natural logarithm of a complex number

From $Z = re^{i\theta}$

$\Rightarrow lnZ = lnre^{i\theta} = lnr + lne^{i\theta} = lnr + i\theta$; with θ is in radians.

Examples

(i) Find *lnZ* given, $Z = 5(cos60^0 + isin60^0)$

Solution

From the previous example, $Z = 5e^{\frac{\pi}{3}}$

Therefore, $lnZ = ln5 + i\frac{\pi}{3} = 1.609 + 1.047i$

(ii) Express $e^{1-i\frac{2\pi}{3}}$ in the form $a + ib$

Solution

$$e^{1-i\frac{2\pi}{3}} = e^1 e^{-i\frac{2\pi}{3}} = e\left(cos\frac{2\pi}{3} - isin\frac{2\pi}{3}\right)$$

$$= e\left(\frac{1}{2} - \frac{\sqrt{3}}{2}i\right) = \frac{e}{2}\left(1 - \sqrt{3}i\right)$$

Questions

1. Express $\frac{2+3i}{i(4-5i)} + \frac{2}{i}$ in the form $a + ib$
2. If $Z_o = \frac{2+i}{1-i}$, find Z in both Cartesian and polar forms given; $Z = Z_o + \frac{1}{Z_o}$. Hence determine lnZ
3. If *x* and *y* are real, solve the equation;

$$\frac{xi}{1 + yi} = \frac{3x + 4i}{x + 3y}$$

De Moivre's theorem

From;

$Z_1Z_2 = r_1r_2[cos(\theta_1 + \theta_2) + isin(\theta_1 + \theta_2)]$ and

$\frac{Z_1}{Z_2} = \frac{r_1}{r_2}[cos(\theta_1 - \theta_2) + isin(\theta_1 - \theta_2)]$, we have;

$Z^2 = r^2(cos2\theta + isin2\theta)$ and $Z^{-2} = r^{-2}[cos(-2\theta) + isin(-2\theta)]$

More generally, for any integer, *n*;

$Z^n = r^n(\cos(n\theta) + i\sin(n\theta))$

For $|Z| = r = 1$ the above formula yields what is known as the *De Moivre's theorem*. i.e.

$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$

Applications of De Moivre's theorem

(a) $\boldsymbol{\cos n\theta}$ and $\boldsymbol{\sin n\theta}$

De Moivre's theorem can be used to express $\cos n\theta$ and $\sin n\theta$ in terms of $\cos\theta$ and $\sin\theta$.

From;

$\cos n\theta + i\sin n\theta = (\cos\theta + i\sin\theta)^n$. The right hand side can be expanded using the Binomial series and then the imaginary and real parts equated.

Example 1

Use De Moivre's theorem to determine the expansions of $\cos 3\theta$ and $\sin 3\theta$.

Solution

$\cos 3\theta + i\sin 3\theta = (\cos\theta + i\sin\theta)^3$

Let $\cos\theta = c$ and $\sin\theta = s$

$$\begin{aligned}\cos 3\theta + i\sin 3\theta &= (c + is)^3 \\ &= c^3 + 3c^2si - 3cs^2 - s^3i \\ &= c^3 - 3cs^2 + i(3c^2s - s^3)\end{aligned}$$

$$\begin{aligned}\Rightarrow \cos 3\theta = c^3 - 3cs^2 &= \cos^3\theta - 3\cos\theta\sin^2\theta \\ &= \cos^3\theta - 3\cos\theta(1 - \cos^2\theta)\end{aligned}$$

$\therefore \cos 3\theta = 4\cos^3\theta - 3\cos\theta$

Similarly equating imaginary parts, gives;

$\sin 3\theta = 3\sin\theta - 4\sin^3\theta$

Example 2

Use De Moivre's theorem to express $tan5\theta$ in terms of $tan\theta$.

(b) Expansions of $cos^n\theta$ and $sin^n\theta$

By De Moivre's theorem, with $|Z| = 1$;

$Z^n = cosn\theta + isinn\theta$

$\frac{1}{Z^n} = Z^{-n} = cosn\theta - isinn\theta$, therefore;

$Z^n + \frac{1}{Z^n} = 2cosn\theta$ and $Z^n - \frac{1}{Z^n} = i2sinn\theta$

With $n = 1$;

$z = cos\theta + isin\theta$ and $\frac{1}{Z} = cos\theta - isin\theta$, therefore;

$Z + \frac{1}{Z} = 2cos\theta$ and $Z - \frac{1}{Z} = i2sin\theta$

Examples

1. Show that $cos^3\theta = \frac{1}{4}(cos3\theta + 3cos\theta)$

Solution

From $Z + \frac{1}{Z} = 2cos\theta \Rightarrow (2cos\theta)^3 = \left(Z + \frac{1}{Z}\right)^3 = Z^3 + 3Z^2 * \frac{1}{Z} + 3Z * \frac{1}{Z^2} + \frac{1}{Z^3}$

$= Z^3 + 3Z + \frac{3}{Z} + \frac{1}{Z^3}$

The RHS can be re – written by collecting terms in pairs;

$$\therefore 8cos^3\theta = \left(Z^3 + \frac{1}{Z^3}\right) + 3\left(Z + \frac{1}{Z}\right)$$

But;

$Z^3 + \frac{1}{Z^3} = 2cos3\theta$ and $Z + \frac{1}{Z} = 2cos\theta$

$\Rightarrow 8cos^3\theta = 2cos3\theta + 3(2cos\theta)$

$$\therefore cos^3\theta = \frac{1}{4}(cos3\theta + 3cos\theta)$$

2. Show that $sin^4\theta = \frac{1}{8}(cos4\theta - 4cos2\theta + 3)$

Solution

For this case we use;

$Z - \frac{1}{Z} = i2sin\theta$ and $\left(Z - \frac{1}{Z}\right)^n = (i2sin\theta)^n$

Therefore;

$$(i2sin\theta)^4 = \left(Z - \frac{1}{Z}\right)^4 = Z^4 - 4Z^3 * \frac{1}{Z} + 6Z^2 * \frac{1}{Z^2} - 4Z * \frac{1}{Z^3} + \frac{1}{Z^4}$$

$$= \left(Z^4 + \frac{1}{Z^4}\right) - 4\left(Z^2 + \frac{1}{Z^2}\right) + 6$$

Using; $Z^n + \frac{1}{Z^n} = 2cosn\theta$;

$$16sin^4\theta = 2cos4\theta - 4(2cos2\theta) + 6 \Rightarrow sin^4\theta = \frac{1}{8}(cos4\theta - 4cos2\theta + 3)$$

(c) Determination of Z^n

De Moivres's theorem can be applied to any complex number provided the number is first expressed in polar form.

Example

Evaluate the following using Demoivre's theorem.

(i) $\left(1 + \sqrt{3}i\right)^5$

Solution

Since $1 + \sqrt{3}i = 2(cos60^0 + isin60^0)$, then

$$\left(1 + \sqrt{3}\right)^5 = 2^5\left(cos(5 * 60) + isin(5 * 60)\right) = 32(cos300^0 + isin300^0)$$

$$= 16\left(1 - \sqrt{3}i\right)$$

(ii) $\left(1 + \sqrt{3}i\right)^{\frac{3}{4}}$

Solution

$$\left(1 + \sqrt{3}i\right)^{\frac{3}{4}} = 2^{\frac{3}{4}}\left[cos\left(\frac{3}{4} * 60\right) + isin\left(\frac{3}{4} * 60\right)\right]$$

$$= 2^{\frac{3}{4}}[cos45^0 + isin45^0] = 2^{\frac{3}{4}}\left(\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}\right) = 2^{\frac{3}{4}} * \frac{\sqrt{2}}{2}(1 + i)$$

$$= 2^{1/4}(1 + i)$$

(d) Roots of complex numbers

If $Z = w^n$(n = 1, 2, …………..), then to each value of *w*, there corresponds one value *Z*. to a given $Z \neq 0$, there corresponds precisely *n* distinct values of *w*. Each of these values is called an *n^th^* root of Z, and we write; $w = \sqrt[n]{Z}$.

Hence, *w* is multi - valued, namely n – valued.

In terms of polar form for Z and

$w = R(cos\emptyset + isin\emptyset)$

The equation, $w^n = Z$ becomes;

$w^n = R^n(cosn\emptyset + isinn\emptyset) = Z = r(cos\theta + isin\theta)$

By equating the absolute values on both sides, we have;

$R^n = r$, thus $R = \sqrt[n]{r}$

Where the root is real positive and thus uniquely determined

By equating the arguments we obtain;

$n\emptyset = \theta + 2k\pi$, thus $\emptyset = \frac{\theta}{n} + \frac{2k\pi}{n}$, where k is an integer and θ is in radians.

For k = 0, 1,………….,n-1 we get n distinct values of w. Further integers of k would give values already obtained, e.g. for $k = n$ gives $\frac{2k\pi}{n} = 2\pi$ hence w value corresponding to k = 0, etc. Consequently, $\sqrt[n]{Z}$, for $Z \neq 0$, has the n distinct values of:

$\sqrt[n]{Z} = \sqrt[n]{r}\left[cos\frac{\theta+2k\pi}{n} + isin\frac{\theta+2k\pi}{n}\right]$, for *k = 0, 1, ………….n-1*

These n values lie on a circle of radius $\sqrt[n]{r}$ with centre at the origin and constitute the vertices of a regular polygon of n – sides.

The values of $\sqrt[n]{Z}$ obtained by taking the principal value of $argZ$ (i.e. a value of arg lying between $-\pi \leq arg \leq \pi$) and *k = 0* is called the principal value of $w = \sqrt[n]{Z}$.

Square root

From;

$\sqrt[n]{Z} = \sqrt[n]{r}\left[cos\frac{\theta+2k\pi}{n} + isin\frac{\theta+2k\pi}{n}\right]$, for *k = 0, 1, ………….n-1*

It follows that $w = \sqrt{Z}$ has two values of;

$w_1 = \sqrt{r}\left(cos\frac{\theta}{2} + isin\frac{\theta}{2}\right)$ and;

$w_2 = \sqrt{r}\left[cos\left(\frac{\theta}{2} + \pi\right) + isin\left(\frac{\theta}{2} + \pi\right)\right] = -w_1$, which lie symmetric with respect to the origin.

- Therefore; $\sqrt{Z} = \pm\sqrt{r}\left[cos\frac{\theta}{2} + isin\frac{\theta}{2}\right]$…………………………………………..*

Example

Find the square root of $4i$.

Solution

$$4i = 4\left(cos\frac{\pi}{2} + isin\frac{\pi}{2}\right)$$

$$\therefore \sqrt{4i} = \pm 2\left(cos\frac{\pi}{4} + isin\frac{\pi}{4}\right) = \pm\sqrt{2}(1+i)$$

From equation (*) we can obtain a much more important and practical formula;

$$\sqrt{Z} = \pm\left[\sqrt{\tfrac{1}{2}(|Z|+x)} + (signy)i\sqrt{\tfrac{1}{2}(|Z|-x)}\right]$$

Where $(sign\, y) = 1$ if $y \geq 0$ and $(signy) = -1$ when $y < 0$, and all square roots of positive numbers are taken with the positive sign. This formula is very important when solving complex quadratic equations.

Examples

1. Solve the equation; $Z^2 - (5+i)Z + 8 + i = 0$

   Solution

   From;

   $Z = \frac{-b\pm\sqrt{b^2-4ac}}{2a}$, putting; $a = 1, b = -(5+i) and\ c = (8+i)$, gives;

   $$Z = \frac{1}{2}(5+i) \pm \sqrt{\frac{1}{4}(5+i)^2 - 8 - i}$$

   $$= \tfrac{1}{2}(5+i) \pm \sqrt{-2+\tfrac{3}{2}i} = \tfrac{1}{2}(5+i) \pm \left[\sqrt{\tfrac{1}{2}\left(\tfrac{5}{2}+(-2)\right)} + i\sqrt{\tfrac{1}{2}\left(\tfrac{5}{2}-(-2)\right)}\right]$$

   $$= \tfrac{1}{2}(5+i) \pm \left(\tfrac{1}{2}+\tfrac{3}{2}i\right)$$

   $\therefore Z = 3 + 2i\ or\ 2 - i$

Finding higher roots e.g cube roots, forth roots etc

Generally the n – roots of the n[th] root of a number Z can be determined according to the following simplified procedure;

(a) We use De Moivre's theorem to obtain the first root.

(b) The other roots have the same modulus but with their arguments distributed around the Argand diagram and separated by an angle $\frac{360^0}{n}$

Note: The root nearest to the positive x – axis is known as the ***principal root*** of the complex number.

Examples

(i) Find the forth roots of $Z = 7(cos80^0 + isin80^0)$

Solution

$Z^{1/4} = 7^{1/4}\left(cos\left(\frac{80}{4}\right) + isin\left(\frac{80}{4}\right)\right) = 1.627\lfloor 20^0$

The other roots are at intervals of $\frac{360^0}{4} = 90^0$

Therefore, the four roots are;

$Z_1 = 1.627\lfloor 20^0,\ 1.627\lfloor 110^0,\ 1.627\lfloor 200^0\ and\ 1.627\lfloor 290^0$

Sketching these on an the same Argand diagram, the principal root is $1.627\lfloor 20^0 = 1.627(cos20 + isin20) = 1.529 + 0.556i$

(ii) Find the cube roots of $Z = 6(cos240^0 + isin240^0)$. Express the principal root, Z_0 in exponential form and hence determine its lnZ_0.

Solution

From De Moivre's theorem, the first root, $Z_1 = 6^{1/3}\left(cos\left(\frac{240}{3}\right) + isin\left(\frac{240}{3}\right)\right) = 1.817\lfloor 80^0$

The other roots have arguments at intervals of $\frac{360^0}{3} = 120^0$

Therefore, the three roots are; $1.817\lfloor 80^0,\ 1.817\lfloor 200^0\ and\ 1.817\lfloor 320^0$

By plotting the three roots on the same Argand diagram, it is clear that the principal root is, $1.817\lfloor 320^0$

$$Z_0 = 1.817\lfloor 320^0 = 1.817\lfloor \tfrac{16\pi}{9} = 1.817e^{\frac{16\pi}{9}i}$$

$$\therefore lnZ_0 = ln1.817 + \frac{16\pi}{9}i = 0.597 + 5.585i$$

Questions

1. Express $2+3i$ and $1-2i$ in polar form and apply De Moivre's theorem to evaluate $\frac{(2+3i)^4}{1-2i}$. Express the result in the form $a+ib$ and in exponential form, hence find the value of their natural logarithm.
2. Find the fifth roots of $-3+3i$ in polar and in exponential form.
3. Using De Moivre's, solve the equation, $x^4+256=0$, and determine the natural logarithm of the principal root of the equation.

Exponential and hyperbolic of the complex variable

The complex exponential function is written as e^Z and is one of the most important analytic functions.

The definition of e^Z in terms of the real functions $e^x, cosy\ and\ siny$ is;

$e^Z = e^x(cosy + isiny)$

Let $Z_1 = x_1 + iy_1$ and $Z_1 = x_2 + iy_2$ then;

$e^{Z_1 Z_2} = e^{x_1}(cosy_1 + isiny_1)e^{x_2}(cosy_2 + isiny_2)$

Since $e^{x_1}e^{x_2} = e^{x_1+x_2}$ for these real functions, by an application of the addition formulas for the cosine and sine functions, we find that;

$e^{Z_1}e^{Z_2} = e^{(x_1+x_2)}[cos(y_1+y_2) + isin(y_1+y_2)]$

An interesting special case is when $Z_1 = x$ and $Z_2 = iy$; $e^Z = e^x e^{iy}$

Furthermore, for $Z = iy$, we have from $e^Z = e^x(cosy + isiny)$ that;

$e^{iy} = cosy + isin$y, this is called the Euler's formula and from which we have;

$|e^{iy}| = |cosy + isiny| = \sqrt{cos^2y + sin^2y} = 1$, i.e. for pure imaginary exponents, the exponential function has an absolute value of one.

Trigonometric functions

Using the Euler's formula;

$e^{ix} = cosx + isinx$ and $e^{-ix} = cosx - isinx$

By addition and subtraction, we get;

$cosx = \frac{1}{2}\left(e^{ix} + e^{-ix}\right)$ and $sinx = \frac{1}{2i}\left(e^{ix} - e^{-ix}\right)$

This suggests the following definitions for complex values of $Z = x + iy$

- $cosZ = \frac{1}{2}\left(e^{iZ} + e^{-iZ}\right)$ and
- $sinZ = \frac{1}{2i}\left(e^{iZ} - e^{-iZ}\right)$

Just like the definitions from real calculus, we define:

- $tanZ = \frac{sinZ}{cosZ}$
- $cotZ = \frac{cosZ}{sinZ}$
- $secZ = \frac{1}{cosZ}$
- $cosecZ = \frac{1}{sinZ}$

Hyperbolic functions

The complex hyperbolic cosine and sine are defined by the formulas,

$coshZ = \frac{1}{2}(e^{Z} + e^{-Z})$ and $sinhZ = \frac{1}{2}(e^{Z} - e^{-Z})$

The other hyperbolic functions are defined by;

- $tanhZ = \frac{sinhZ}{coshZ}$
- $cothZ = \frac{coshZ}{sinhZ}$
- $sechZ = \frac{1}{coshZ}$
- $cosechZ = \frac{1}{sechZ}$

Complex trigonometric and hyperbolic functions are related:

$coshiZ = cosZ$ and $sinhiZ = isinZ$ and;

$cosiZ = coshZ$ and $siniZ = isinhZ$

Question 1

Show that if $Z = x + iy$,then:

(a) $cosZ = cosxcoshy - isinxsinhy$

(b) $sinZ = sinxcoshy + icosxsinhy$

(c) $|cosZ|^2 = cos^2x + sinh^2y$

(d) $|sinZ|^2 = sin^2x + sinh^2y$

Question 2

Using the relationships above, determine;

(a) $sinh(4 - 3i)$

(b) $cosh\left(1 + \frac{1}{2}\pi i\right)$

END