MAKERERE UNIVERSITY

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LINEAR TRANSFORMATION AND MATRICES

Definitions

A matrix is a rectangular array of numbers (real or complex) enclosed in brackets (or parenthesis) systematically arranged in rows and columns.

A matrix with m rows and n columns is called an $(m \times n)$matrix and each element in a matrix a_{jk} so denoted means *i* represents its row position and *j* its column position.

$$A = \begin{pmatrix} a_{11} & a_{12} & .... & a_{1n} \\ a_{21} & a_{22} & .... & a_{2n} \\ .. & .. & .... & .. \\ a_{m1} & a_{m2} & .... & a_{mn} \end{pmatrix}$$

If a matrix has only one row, it is called a *row matrix* or *row vector*;

$$a = \begin{pmatrix} a_1 & a_2 & .... & a_n \end{pmatrix}$$

Similarly, a *column matrix* or *column vector* is a matrix with only one column;

$$b = \begin{pmatrix} b_1 \\ b_2 \\ . \\ . \\ b_m \end{pmatrix}$$

A *square matrix* is a matrix that has as many rows as columns. If it has *n* rows and *n* rows, it is an $n \times n$ matrix of order *n*. The diagonal containing the entries $a_{11}, a_{22}, ..., a_{nn}$ in a square matrix $A = \lfloor a_{jk} \rfloor$ is called the *main diagonal* or *principal diagonal* or *leading diagonal* of this matrix.

e.g. $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is a square matrix of order 2, and *a* and *d* are on its main diagonal.

Any matrix obtained by omitting some rows and columns from a given $m \times n$ matrix A is called a *submatrix* of A.

Equality of matrices

Two matrices $A = [a_{il}]$ *and* $B = \lfloor b_{jk} \rfloor$are equal if and only if A and B have the same number of rows and the same number of columns and the corresponding elements are equal.

Example

$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ *and* $B = \begin{bmatrix} 1 & 3 \\ 8 & -1 \end{bmatrix}$, A = B, if and only if;

$$a_{11} = 1,\ a_{12} = 3, a_{21} = 8\, and\, a_{22} = -1$$

Addition of matrices

Addition is defined only for matrices with the same number of rows and the same number of columns and is defined as follows;

The sum of two $m \times n$ matrices, $A = [a_{il}] and B = \lfloor b_{jk} \rfloor$ is written as $A + B$ and is the $m \times n$ matrix obtained by adding the corresponding entries in A and B.

The entries in $A + B$ are; $a_{il} + b_{jk}$

Matrices with different numbers of rows or different number of columns cannot be added.

Example

$$A = \begin{pmatrix} -4 & 6 & 6 \\ 0 & 1 & 2 \end{pmatrix} \text{ and } B = \begin{pmatrix} 5 & -1 & 0 \\ 3 & 1 & 0 \end{pmatrix}$$

$$A + B = \begin{pmatrix} 1 & 5 & 3 \\ 3 & 2 & 2 \end{pmatrix}$$

Properties of matrix addition

(a) $A + B = B + A$

(b) $(U + V) + W = U + (V + W)$

(c) $A + 0 = A$

(d) $A + (-A) = 0$

Where $-A = \lfloor -a_{jk} \rfloor$ is the $m \times n$ matrix obtained by multiplying every entry in A by -1.

The $m \times n$ zero matrix is the $m \times n$ matrix with all entries zeros.

Instead of $A + (-B)$, we can write $A - B$ and call this matrix the difference of A and B.

Multiplication of matrices by scalars

The product of an $m \times n$ matrix $A = \lfloor a_{jk} \rfloor$ by a scalar c is cA or Ac and is the $m \times n$ matrix obtained by multiplying each entry in A by C:

$$cA = Ac = \begin{pmatrix} ca_{11} & ca_{12} & .... & ca_{1n} \\ ca_{21} & ca_{22} & .... & ca_{2n} \\ . & . & .... & . \\ ca_{m1} & ca_{m2} & .... & ca_{mn} \end{pmatrix}$$

Example

If $A = \begin{pmatrix} 2 & 5 \\ 8 & 3 \end{pmatrix}$, then $2A = \begin{pmatrix} 4 & 10 \\ 16 & 6 \end{pmatrix}$

Therefore, for any $m \times n$ matrices (with fixed m and n) and any scalars.

(a) $c(A+B) = cA + cB$

(b) $(c+k)A = cA + kB$

(c) $c(kA) = (ck)A$

Matrix multiplication

Let $A = \lfloor a_{jk} \rfloor$ be an $m \times n$ matrix and $B = \lfloor a_{jk} \rfloor$ an $r \times p$ matrix. Then the product AB (in this order) is defined only when $r = n$ i.e.

Number of row in B = Number of columns in A

AB is then the $m \times p$ matrix $C = \lfloor c_{jk} \rfloor$ whose entry c_{jk} is the dot product;

$c_{jk} = \left(j^{th} row vector of\ A\right).\left(k^{th}\ column vector of\ B\right)$ i.e. the product of two matrices is determined by multiplication of the rows of the first matrix and columns of the second matrix.

Example

$$\begin{pmatrix} 4 & 2 \\ 1 & 8 \end{pmatrix}\begin{pmatrix} 3 \\ 5 \end{pmatrix} = \begin{pmatrix} 12+10 \\ 3+40 \end{pmatrix} = \begin{pmatrix} 22 \\ 43 \end{pmatrix}$$

Note

Multiplication of matrices is not commutative, e.g.

$$\begin{pmatrix} 2 & 3 & 1 \\ 0 & 4 & 5 \end{pmatrix}\begin{pmatrix} 8 & 0 \\ 6 & 1 \\ 7 & 9 \end{pmatrix} = \begin{pmatrix} 41 & 12 \\ 59 & 49 \end{pmatrix} \text{ but;}$$

$$\begin{pmatrix} 8 & 0 \\ 6 & 1 \\ 7 & 9 \end{pmatrix}\begin{pmatrix} 2 & 3 & 1 \\ 0 & 4 & 5 \end{pmatrix} = \begin{pmatrix} 16 & 24 & 8 \\ 12 & 22 & 11 \\ 14 & 57 & 52 \end{pmatrix}$$

Hence, even if both AB and BA are defined; $AB \neq BA$, i.e. matrix multiplication is not commutative.

Properties of matrix multiplication

(a) Matrix multiplication is associative and is distributive with respect to addition of matrices, i.e.

$$(kA)B = k(AB) = A(kB)$$

$$A(BC) = (AB)C$$

$$(A+B)C = AC + BC$$

$C(A+B) = CA + CB$, provided A, B and C are such that the expressions on the left are defined. k is any constant.

(b) Matrix multiplication is not commutative, i.e. if A and B are matrices such that both AB and BA are defined, then;

$AB \neq BA$

(c) $AB = 0$, does not necessarily imply that $A = 0$ or $B = 0$.

Special matrices

1. Triangular matrices

A square matrix whose entries above the main diagonal are all zero is called a *lower triangular matrix*.

e.g. $T_1 = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 3 & 0 \\ 5 & 0 & 2 \end{pmatrix}$

Similarly, an *upper triangular matrix* is a square matrix whose entries below the main diagonal are all zero.

e.g. $T_2 = \begin{pmatrix} 1 & 6 & -1 \\ 0 & 2 & 3 \\ 0 & 0 & 4 \end{pmatrix}$

2. Diagonal matrices

A square matrix $A = \lfloor a_{jk} \rfloor$ whose entries above and below the main diagonal, are all e.g. $\begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -4 \end{pmatrix} and \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$

A diagonal matrix whose entries on the main diagonal are all equal is called a *scalar matrix*. Thus a scalar matrix is of the form;

$S = \begin{pmatrix} c & 0 & .... & 0 \\ 0 & c & .... & 0 \\ . & . & .... & . \\ o & 0 & .... & c \end{pmatrix}$

Where c is any constant

3. Singular and non singular matrices

A matrix $A = \lfloor a_{jk} \rfloor$ is said to be *singular* if it does not have an inverse. However, if an inverse can be found for a given matrix A, then it is a *non singular matrix*. A matrix is singular if its determinant is zero.

4. Unit matrix (Identity matrix)

The $m \times n$ square matrix in which all the numbers on the leading diagonal are one and all other entries are zero is called the $n \times n$ unit matrix or identity matrix. E.g. $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} and \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ are 3×3 and 2×2 identity matrices respectively.

Unit matrices of all orders are usually denoted by I.

For any $m \times n$ matrix A, we have that; $IA = A$, where I is an $m \times m$ matrix.

Also $AI = A$, where I is the $n \times n$ unit matrix.

Transpose of a matrix

The transpose A^T of an $m \times n$ matrix A, is the $n \times m$ matrix that has the columns of A as rows. i.e.

If $A = \begin{pmatrix} a_{11} & a_{12} & .... & a_{1n} \\ a_{21} & a_{22} & .... & a_{2n} \\ . & . & .... & . \\ a_{m1} & a_{m2} & .... & a_{mn} \end{pmatrix}$, then; $A^T = \begin{pmatrix} a_{11} & a_{21} & .... & a_{m1} \\ a_{12} & a_{22} & .... & a_{m2} \\ . & . & .... & . \\ a_{1n} & a_{2n} & .... & a_{mn} \end{pmatrix}$

Example

If $A = \begin{pmatrix} 5 & -8 & 1 \\ 4 & 0 & 0 \end{pmatrix}, then A^T = \begin{pmatrix} 5 & 4 \\ -8 & 0 \\ 1 & 0 \end{pmatrix}$

Note:

(i) A real matrix A is said to be <u>*symmetric*</u> if it is equal to its transpose, A^T e.g. $A = \begin{pmatrix} -3 & 1 & 5 \\ 1 & 0 & -2 \\ 5 & -2 & 4 \end{pmatrix}$ is symmetric.

(ii) A real matrix, A, is said to be *skew – symmetric* if; $A^T = -A$ for examples, $B = \begin{pmatrix} 0 & -4 & 1 \\ 4 & 0 & 5 \\ -1 & 5 & 0 \end{pmatrix}$ is skew symmetric since;

$$B^T = \begin{pmatrix} 0 & 4 & -1 \\ -4 & 0 & 5 \\ 1 & -5 & 0 \end{pmatrix} = -\begin{pmatrix} 0 & 4 & -1 \\ 4 & 0 & -5 \\ -1 & 5 & 0 \end{pmatrix} = -B$$

Question

Prove that $(A+B)^T = A^T + B^T$

Note: The transpose of a product equals the product of the transposed factors, taken in the reverse order, i.e. $(AB)^T = B^T A^T$

Determinants

Consider systems of two linear equations;

$$\left.\begin{aligned} a_{11}x_1 + a_{12}x_2 &= b_1 \\ a_{21}x_1 + a_{22}x_2 &= b_2 \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

With x_1 and x_2 unknown

Solving these two linear equations, gives;

$$\left.\begin{aligned} x_1 &= \frac{b_1a_{22} - b_2a_{12}}{a_{11}a_{22} - a_{21}a_{12}} \\ x_2 &= \frac{a_{11}b_2 - a_{21}b_1}{a_{11}a_{22} - a_{21}a_{12}} \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2)$$

The expression in the denominators is written in the form; $\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$ and is called a determinant of second order, thus:

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{21}a_{12} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3)$$

The four entries a_{11}, a_{12}, a_{21}, and a_{22} are called the entries in the determinant or the elements of the determinant.

Thus, now we can write;

$$x_1 = \frac{D_1}{D} \; and \; x_2 = \frac{D_2}{D} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(4)$$

Where;

$$D = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix},\; D_1 = \begin{vmatrix} b_1 & a_{12} \\ b_2 & a_{22} \end{vmatrix} \; and \; D_2 = \begin{vmatrix} a_{11} & b_1 \\ a_{21} & b_2 \end{vmatrix}$$

Formula (4) is called *crammer's rule.*

Note

- D_1 is obtained by replacing the first column of D, containing the coefficients on x_1 by the column with entries b_1 and b_2.
- Similarly is obtained by replacing the second column in D, containing the coefficients on x_2 by the column with entries b_1 and b_2.
- If both b_1 and b_2 are zero, the system is said to be homogeneous. In this case the system has at least the "trivial solution" $x_1 = 0$ or $x_2 = 0$.

Example

Solve the following set of equations using Crammer's rule;

$2x+3y-14=0$

$3x-2y+5=0$

$$D=\begin{vmatrix}2 & 3\\3 & -2\end{vmatrix}=-4-9=-13$$

$$D_1=\begin{vmatrix}14 & 3\\-5 & -2\end{vmatrix}=14*(-2)-(-5)*3=-13$$

$$D_2=\begin{vmatrix}2 & 14\\3 & -5\end{vmatrix}=-10-42=-52$$

Therefore, $x=\dfrac{D_1}{D}=\dfrac{-13}{-13}=1\ and\ y=\dfrac{-52}{-13}=4$

Determinants of third order

Minors of a determinant

Each of the nine positions for an element in a 3×3 determinant can best be described by stating the row and column to which the element belongs. Corresponding to each position, there is a 2×2 determinant, namely, the determinant obtained by when the entire row and entire column defining that position are both deleted.

The nine such 2×2 determinants are called the minors of D.

E.g. consider, $\begin{pmatrix}a_{11} & a_{12} & a_{13}\\a_{21} & a_{22} & a_{23}\\a_{31} & a_{32} & a_{33}\end{pmatrix}$

The minor of a_{11} is $\begin{vmatrix}a_{22} & a_{23}\\a_{32} & a_{33}\end{vmatrix}$

The minor of a_{22} is $\begin{vmatrix}a_{11} & a_{13}\\a_{31} & a_{33}\end{vmatrix}$ and,

The minor of a_{33} is $\begin{vmatrix}a_{21} & a_{22}\\a_{31} & a_{32}\end{vmatrix}$

Cofactors of a 3×3 matrix

The cofactor of the entry in the i^{th} row and the k^{th} column of D is defined as $(-1)^{i+k}$ times the minor of that entry. E.g. considering $\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$, then;

$$C_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix},\ C_{12} = -\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix},\ C_{13} = \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix},\ C_{21} = -\begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix},\ C_{22} = \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} \text{ etc.}$$

The signs $(-1)^{i+k}$ form a checkerboard pattern; $\begin{matrix} + & - & + \\ - & + & - \\ + & - & + \end{matrix}$

Evaluation of a 3×3 determinant

The determinant of a 3×3 matrix can be evaluated as the sum of the products of elements or entries in the top row and their respective cofactors.

Examples

1. $$\begin{vmatrix} 3 & 2 & 5 \\ 4 & 6 & 7 \\ 2 & 9 & 2 \end{vmatrix} = 3\begin{vmatrix} 6 & 7 \\ 9 & 2 \end{vmatrix} - 2\begin{vmatrix} 4 & 7 \\ 2 & 2 \end{vmatrix} + 5\begin{vmatrix} 4 & 6 \\ 2 & 9 \end{vmatrix}$$
$$= 3(12-63) - 2(8-14) + 5(36-12) = -21$$

2. $$\begin{vmatrix} 1 & 1 & 2 \\ 2 & 1 & 1 \\ 1 & 2 & 1 \end{vmatrix} = 1.\begin{vmatrix} 1 & 1 \\ 2 & 1 \end{vmatrix} - 1.\begin{vmatrix} 2 & 1 \\ 1 & 1 \end{vmatrix} + 2\begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix}$$
$$= 1(1-2) - 1(2-1) + 2(4-1) = 4$$

Note

We can, if we wish expand along any row or column in the same way, multiplying each element by its minor and assign to each product the appropriate + or – sign.

Example

Evaluate, $\begin{vmatrix} 2 & 7 & 5 \\ 4 & 6 & 3 \\ 8 & 9 & 1 \end{vmatrix}$

We can evaluate this by expanding down the middle column, remembering that the "place signs" for the 3[rd] determinant are; $\begin{matrix} + & - & + \\ - & + & - \\ + & - & + \end{matrix}$

Therefore, $\begin{vmatrix} 2 & 7 & 5 \\ 4 & 6 & 3 \\ 8 & 9 & 1 \end{vmatrix} = -7\begin{vmatrix} 4 & 3 \\ 8 & 1 \end{vmatrix} + 6\begin{vmatrix} 2 & 5 \\ 8 & 1 \end{vmatrix} - 9\begin{vmatrix} 2 & 5 \\ 4 & 3 \end{vmatrix} = -7(4-24) + 6(2-40) - 9(6-20) = 38$

But we can also decide to expand along the middle row as;

$$\begin{vmatrix} 2 & 7 & 5 \\ 4 & 6 & 3 \\ 8 & 9 & 1 \end{vmatrix} = -4\begin{vmatrix} 7 & 5 \\ 9 & 1 \end{vmatrix} + 6\begin{vmatrix} 2 & 5 \\ 8 & 9 \end{vmatrix} - 3\begin{vmatrix} 2 & 7 \\ 8 & 7 \end{vmatrix} = -4(7-45) + 6(2-40) - 3(18-56) = 38$$

Crammer's rule for a system of three linear equations

Consider systems of three linear equations;

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1$$
$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2$$
$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3$$

With x_1, x_2, and x_3 unknown

Upon simplification, the determinant of third order is obtained by;

$$D = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

By an analogy with the system of two linear equations;

$Dx_1 = D_1$, where; $D_1 = \begin{vmatrix} b_1 & a_{12} & a_{13} \\ b_2 & a_{22} & a_{23} \\ b_3 & a_{32} & a_{33} \end{vmatrix}$

$Dx_2 = D_2$, where; $D_2 = \begin{vmatrix} a_{11} & b_1 & a_{13} \\ a_{21} & b_2 & a_{23} \\ a_{31} & b_3 & a_{33} \end{vmatrix}$ and;

$Dx_3 = D_3$, where; $D_3 = \begin{vmatrix} a_{11} & a_{12} & b_1 \\ a_{21} & a_{23} & b_2 \\ a_{31} & a_{32} & b_3 \end{vmatrix}$

It follows that if $D \neq 0$, the system has the unique solution;

$x_1 = \frac{D_1}{D}$, $x_2 = \frac{D_2}{D}$ and $x_3 = \frac{D_3}{D}$

If the system of three linear equations is homogeneous, i.e. $b_1 = b_2 = b_3 = 0$, it has at least the trivial solution $x_1 = x_2 = x_3$, and non – trivial solutions exist if and only if $D = 0$.

If it is non homogeneous and $D \neq 0$, it has precisely one solution.

Example

Solve by Crammer's rule:

$2x - y + 2z = 2$

$x + 10y - 3z = 5$

$-x + y + z = -3$

The determinant of the system is;

$$D = \begin{vmatrix} 2 & -1 & 2 \\ 1 & 10 & -3 \\ -1 & 1 & 1 \end{vmatrix} = 2\begin{vmatrix} 10 & -3 \\ 1 & 1 \end{vmatrix} - (-1)\begin{vmatrix} 1 & -3 \\ -1 & 1 \end{vmatrix} + 2\begin{vmatrix} 1 & 10 \\ -1 & 1 \end{vmatrix}$$

$$= 2(10 - (-3)) - (-1)(1-3) + 2(1 - (-10)) = 46$$

The determinants in the numerators are:

$$D_1 = \begin{vmatrix} 2 & -1 & 2 \\ 5 & 10 & -3 \\ -3 & 1 & 1 \end{vmatrix} = 2(10+3) - (-1)(5-9) + 2(5+30) = 92$$

$$D_2 = \begin{vmatrix} 2 & 2 & 2 \\ 1 & 5 & -3 \\ -1 & -3 & 1 \end{vmatrix} = 2(5-9) - 2(1-3) + 2(-3+) = 0$$

$$D_3 = \begin{vmatrix} 2 & -1 & 2 \\ 1 & 10 & 5 \\ -1 & 1 & -3 \end{vmatrix} = 2(-30-5) - (-1)(-3+5) + 2(1+10) = -46$$

Therefore;

$x = \frac{92}{46} = 2$, $y = \frac{0}{46} = 0$ and $z = \frac{-46}{46} = -1$

Questions

Solve the following set of equations using Crammer's rule:

(i) $$\left.\begin{array}{l} 2x+3y-z-4=0 \\ 3x+y+2z-13=0 \\ x+2y-5z+11=0 \end{array}\right\} Ans: x=2, y=1 \, and \, z=3$$

(ii) $$\left.\begin{array}{l} 4x-5y+7z=-14 \\ 9x+2y-3z=47 \\ x-y-5z=11 \end{array}\right\} Ans: x=3, y=-2 \, and \, z=-1$$

(iii) $$\left.\begin{array}{l} 4x-3y+2z=-7 \\ 6x+2y-3z=33 \\ 2x-4y-z=-3 \end{array}\right\} Ans: x=2.5, y=3, z=-4$$

General properties of determinants

1. The value of a determinant is not altered if its rows are written as columns in the same order;

   e.g. $$\begin{vmatrix} 2 & 7 & 5 \\ 4 & 6 & 3 \\ 8 & 9 & 1 \end{vmatrix} = \begin{vmatrix} 2 & 4 & 8 \\ 7 & 6 & 9 \\ 5 & 3 & 1 \end{vmatrix} = 38$$

2. If any two rows (or columns) of a determined are interchanged, the value of the determined is multiplied by -1. E.g.

   $$D = \begin{vmatrix} 2 & 6 & 4 \\ 1 & 3 & 0 \\ -1 & 0 & 2 \end{vmatrix} = 2(6)-6(2)+4(3)=12$$

   $$D = \begin{vmatrix} 1 & 3 & 0 \\ 2 & 6 & 4 \\ -1 & 0 & 2 \end{vmatrix} = 1(12)-3(4+4)+0=-12$$

3. If one row of a determinant is proportional to another, the value of the determinant is zero.
4. A factor of the entries in any row or column can be placed before the determinant;

   $$D = \begin{vmatrix} 1 & 3 & 0 \\ 2 & 6 & 4 \\ -1 & 0 & 2 \end{vmatrix} = 2\begin{vmatrix} 1 & 3 & 0 \\ 1 & 3 & 2 \\ -1 & 0 & 2 \end{vmatrix} = 2.3\begin{vmatrix} 1 & 1 & 0 \\ 1 & 1 & 2 \\ -1 & 0 & 2 \end{vmatrix} = 2.3.2\begin{vmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ -1 & 0 & 1 \end{vmatrix}$$

5. The value of a determine remains unaltered if the elements of one row or column are altered by adding to them any constant multiple of the corresponding elements in any other row or column.

Consistency of a set of equations

Equations are said to be consistent if they have a common solution, otherwise they are non – constant.

e.g. consider the set of equations;

$$3x - y - 4 = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(a)$$
$$2x + 3y - 8 = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(b)$$
$$x - 2y + 3 = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(c)$$

Solving (b) and (c) gives; x = 1 and y = 2. But on substituting x = 1 and y = 2 into (a); gives;

$3(1) - 2 - 4 = -3$, and not 0 as the equation states. Thus the three equations are non – consistent.

But if;

$$3x - y - 5 = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(a)$$
$$2x + 3y - 8 = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(b)$$
$$x - 3y + 3 = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(c)$$

The solution of (b) and (c) gives; x = 1 and y = 2 as before. Substituting these in (a) gives;

$3x + y - 5 = 3 + 2 - 5 = 0$. Hence the three equations are consistent.

Taking the general case;

$$a_1x + b_1y + d_1 = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(a)$$
$$a_2x + b_2y + d_2 = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(b)$$
$$a_3x + b_3y + d_3 = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(c)$$

Solving for (b) and (c), gives;

$$\frac{x}{D_1} = \frac{-y}{D_2} = \frac{1}{D_0}$$

Where, $D_1 = \begin{vmatrix} b_2 & d_2 \\ b_3 & d_3 \end{vmatrix}$, $D_2 = \begin{vmatrix} a_2 & d_2 \\ a_3 & d_3 \end{vmatrix}$ $and\ D_0 = \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}$

If these results satisfy equation (a), then;

$$a\frac{D_1}{D_0} + b_1\left(-\frac{D_2}{D_0}\right) + d_1 = 0$$

Therefore, $a_1D_1 - b_1D_2 + d_1D_0 = 0$

$$\Rightarrow a_1\begin{vmatrix} b_2 & d_2 \\ b_3 & d_3 \end{vmatrix} - b_1\begin{vmatrix} a_2 & d_2 \\ a_3 & d_3 \end{vmatrix} + d_1\begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix} = 0$$

$$\therefore \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix} = 0$$

Which is therefore, the condition that the three equations are consistent. So three simultaneous equations in two unknowns are consistent if the determinant of the coefficients is zero.

Examples

1. Test for consistency:

$$2x + y - 5 = 0$$
$$x + 4y + 1 = 0$$
$$3x - y - 10 = 0$$

Solution

For consistency; $\begin{vmatrix} 2 & 1 & -5 \\ 1 & 4 & 1 \\ 3 & -1 & -10 \end{vmatrix} = 0$

$\begin{vmatrix} 2 & 1 & -5 \\ 1 & 4 & 1 \\ 3 & -1 & -10 \end{vmatrix} = 2\begin{vmatrix} 4 & 1 \\ -1 & -10 \end{vmatrix} - \begin{vmatrix} 1 & 1 \\ 3 & -10 \end{vmatrix} - 5\begin{vmatrix} 1 & 4 \\ 3 & -1 \end{vmatrix} = 0$; hence, the equations are consistent.

2. Given;

$$x + (k+1)y + 1 = 0$$
$$2kx + 5y - 3 = 0$$
$$3x + 7y + 1 = 0$$

Find the values of k for which the equations are consistent.

Solutions

$$\begin{vmatrix} 1 & k+1 & 1 \\ 2k & 5 & -3 \\ 3 & 7 & 1 \end{vmatrix} = 0$$

$$\begin{vmatrix} 5 & -3 \\ 7 & 1 \end{vmatrix} - (k+1)\begin{vmatrix} 2k & -3 \\ 3 & 1 \end{vmatrix} + \begin{vmatrix} 2k & 5 \\ 3 & 7 \end{vmatrix} = 0$$

$$(5+21) - (k+1)(2k+9) + (14k-15) = 0$$

$$26 - (2k^2 + 11k + 9) + 14k - 15 = 0$$

$$\therefore 2k^2 - 3k - 2 = 0 \Rightarrow (2k+1)(k-2) = 0$$

Hence, $k = \frac{-1}{2} \; or \; k = 2$

Question

Find the values of β for which the following equations are consistent;

$$\left.\begin{array}{l} 5x+(\beta+1)y=5 \\ (\beta-1)x+7y=-5 \\ 3x+5y=-1 \end{array}\right\} Ans: \beta=4\, or\, \beta=-14$$

Adjoint of a matrix

The adjoint of a matrix A is the transpose of its matrix of cofactors. It is denoted by **adjA** i.e. $adjA = C^T$

Inverse of a matrix

The inverse of a square matrix A, is given by; $A^{-1}=\frac{1}{|A|}.adjA=\frac{1}{|A|}C^T$, provided $|A|\neq 0$

Note

$A.A^{-1}=I=A^{-1}A=\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, i.e the product of a matrix and its inverse is equal to an identity matrix.

Example

Given $A=\begin{pmatrix} 13 & -3 & -2 \\ -3 & 5 & 6 \\ -1 & 3 & 14 \end{pmatrix}$, find A^{-1}, hence show that $A^{-1}A=I$.

$$D=\begin{vmatrix} 13 & -3 & -2 \\ -3 & 5 & 6 \\ -1 & 3 & 14 \end{vmatrix}=13(70-18)+3(-42+6)-2(-9+5)=576$$

Matrix of cofactors;

$$C=\begin{pmatrix} \begin{vmatrix} 5 & 6 \\ 3 & 14 \end{vmatrix} & -\begin{vmatrix} -3 & 6 \\ -1 & 14 \end{vmatrix} & \begin{vmatrix} -3 & 5 \\ -1 & 3 \end{vmatrix} \\ -\begin{vmatrix} -3 & -2 \\ 3 & 14 \end{vmatrix} & \begin{vmatrix} 13 & -2 \\ -1 & 14 \end{vmatrix} & -\begin{vmatrix} 13 & -3 \\ -1 & 3 \end{vmatrix} \\ \begin{vmatrix} -3 & -2 \\ 5 & 6 \end{vmatrix} & -\begin{vmatrix} 13 & -2 \\ -3 & 6 \end{vmatrix} & \begin{vmatrix} 13 & -3 \\ -3 & 5 \end{vmatrix} \end{pmatrix}=\begin{pmatrix} 52 & 36 & -4 \\ 36 & 180 & -36 \\ -8 & -72 & 56 \end{pmatrix}=4\begin{pmatrix} 13 & 9 & -1 \\ 9 & 45 & -9 \\ -2 & -18 & 14 \end{pmatrix}$$

$$\therefore C^T = 4\begin{pmatrix} 13 & 9 & -2 \\ 9 & 45 & -18 \\ -1 & -9 & 14 \end{pmatrix}$$

Thus the inverse matrix is;

$$\frac{4}{576}\begin{pmatrix} 13 & 9 & -2 \\ 9 & 45 & -18 \\ -1 & -9 & 14 \end{pmatrix} = \frac{1}{144}\begin{pmatrix} 13 & 9 & -2 \\ 9 & 45 & -18 \\ -1 & -9 & 14 \end{pmatrix}$$

Determinant of a product of matrices

For any $n \times n$ matrices, A and B;

$\det(AB) = \det(BA) = \det A.\det B$

Solution of systems of linear equations

There are two basic equations that can be applied;

1. Crammer's rule; already looked at.
2. Guass elimination method

Guass elimination

A system of m linear equations in n unknowns, x_1, x_2, ………………x_m is a set of equations of the form;

$$\left.\begin{array}{l} a_{11}x_1 + \dots\dots\dots\dots\dots + a_{1n}x_n = b_1 \\ a_{21}x_1 + \dots\dots\dots\dots\dots + a_{2n}x_n = b_2 \\ \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \\ a_{m1}x_1 + \dots\dots\dots\dots\dots + a_{mn}x_n = b_m \end{array}\right\} \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots(1)$$

The a_{jk} are given constants called the coefficients of the system. The b_i are also given constants. If the b_i are all zero, then (1) is called a <u>*homogeneous system*</u>. If at least one b_i is not zero, then (1) is called a <u>*non – homogeneous*</u> system.

A <u>*solution*</u> of (1) is a set of values; $x_1, x_2, \dots\dots\dots, x_n$ which satisfy all the m equations. A vector solution of (1) is a column vector whose entries are the values of x_1, x_2,…..,x_n corresponding to the solution of the system. If the system (1) is homogenous, it has at least the trivial solution, $x_1 = x_2 = \dots\dots\dots = x_n = 0$

From the definition of matrix multiplication, the m equations of (1) may be written as a single vector equation; $Ax = b$……………………………………………………………(2)

Where the coefficient matrix, $A = \lfloor a_{jk} \rfloor$ is the $m \times n$ matrix

$$A = \begin{pmatrix} a_{11} & a_{12} & .... & a_{1n} \\ a_{21} & a_{22} & .... & a_{2n} \\ . & . & .... & . \\ a_{m1} & a_{m2} & .... & a_{mn} \end{pmatrix}, \ x = \begin{pmatrix} x_1 \\ x_2 \\ . \\ . \\ x_n \end{pmatrix} \text{and } b = \begin{pmatrix} b_1 \\ b_2 \\ . \\ . \\ b_m \end{pmatrix}$$

We can augment A by column b, as;

$$\overline{A} = \begin{pmatrix} a_{11} & ... & a_{1n} & b_1 \\ . & ... & . & . \\ . & ... & . & . \\ a_{m1} & ... & a_{mn} & b_m \end{pmatrix}$$. This is the augmented matrix of the system (1)

Each row in the augmented matrix contains the coefficients to represent the coefficients corresponding to one equation in the system. The linear system can be solved by performing row operations on the augmented matrix as follows:

1. Use a_{11} as the *pivot entry* and use it to eliminate all other entries from the first column.
2. Continue this process of reduction until your main matrix is reduced to a triangular form also referred to as the *echelon form*.
3. Use back substitution to solve for $x_1, x_2, \dots\dots\dots, x_n$.

Note

The pivot entry must never be zero. If such a situation occurs, rows and columns should be interchanged to get a non – zero pivot entry.

Example

Use Guass elimination to solve this system;

$$3y + 2z = 9$$
$$2x + 2y - 4z = -1$$
$$-4x + y + 3z = 4.5$$

Solution

This can be written in matrix form as;

$$\begin{pmatrix} 0 & 3 & 2 \\ 2 & 2 & -4 \\ -4 & 1 & 4 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 9 \\ -1 \\ 4.5 \end{pmatrix}$$

Since the pivot entry is 0, interchange the rows;

$$\begin{pmatrix} 2 & 2 & -4 \\ 0 & 3 & 2 \\ -4 & 1 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -1 \\ 9 \\ 4.5 \end{pmatrix}$$

The augmented matrix is;

$$\begin{pmatrix} 2 & 2 & -4 & -1 \\ 0 & 3 & 2 & 9 \\ -4 & 1 & 3 & 4.5 \end{pmatrix}$$

2R₁ + R₃

$$\begin{pmatrix} 2 & 2 & -4 & -1 \\ 0 & 3 & 2 & 9 \\ 0 & 5 & -5 & 2.5 \end{pmatrix}$$

$5R_2 - 3R_3$; $\begin{pmatrix} 2 & 2 & -4 & -1 \\ 0 & 3 & 2 & 9 \\ 0 & 0 & 25 & 37.5 \end{pmatrix}$

Therefore; $\begin{pmatrix} 2 & 2 & -4 \\ 0 & 3 & 2 \\ 0 & 0 & 2.5 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -1 \\ 9 \\ 37.5 \end{pmatrix}$

By back substitution;

$25z = 37.5 \Rightarrow z = 1.5$

$3y + 2z = 9 \Rightarrow y = 2$ and;

$2x + 2y - 4z = -1 \Rightarrow x = \dfrac{1}{2}$

The system (1) is called over determined if it has more equations than unknowns, determined if m = n and underdetermined if (1) has fewer equations than unknowns.

Note

The method of Guass elimination can also be applied to a system of more than three unknowns.

Questions

Solve the following set of equations using Guass elimination:

(i) $\left.\begin{array}{l} 3x_1 + 2x_2 + x_3 = 1 \\ x_1 - x_2 + 3x_3 = 5 \\ 2x_1 + 5x_2 - 2x_3 = 0 \end{array}\right| \; Ans: x_1 = -2, x_2 = 2, x_3 = 3$

(ii) $\left.\begin{array}{l} 2b + c = 1 \\ a - b + c = 5 \\ 2a + b + 3c = 10 \end{array}\right| \; Ans: a = 1, b = -1, c = 3$

Rank of a matrix

The maximum number of linear independent row vectors of a matrix A is the rank of A and is denoted, *rankA*

Example

If, $A = \begin{pmatrix} 3 & 0 & 2 & 2 \\ -6 & 42 & 24 & 54 \\ 21 & -21 & 0 & -15 \end{pmatrix}$, has a *rank = 2* since the last row is a linear combination of the two others, which are linearly independent i.e. $R_3 = 6R_1 - \frac{R_2}{2}$

The rank of a general $m \times n$ matrix is equal to the size of the largest square sub – matrix of A whose determinant is non zero.

The following are important features associated with the rank of a matrix:

- The only matrices which have zero rank are those with all their elements equal to zero.
- If the matrix is an $n \times n$ matrix and non – singular then *rank = n*
- If A is an $m \times n$ matrix, then the rank cannot exceed the smaller of m and n e.g. given a 3×2 matrix, the rank cannot exceed 2.

Calculation of rank

The simple way of calculating the rank of a matrix is by using tow operations;

E.g. $\begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix}$

Det = 2 – 2 = 0; therefore, *rank = 1*

2. If $A = \begin{pmatrix} 1 & 1 & 0 & -2 \\ 2 & 0 & 2 & 2 \\ 4 & 1 & 3 & 1 \end{pmatrix}$, all 3×3 give det = 0 and all 2×2 give $det \neq 0$. Therefore, *rank = 2*

Eigen values and Eigen vectors

From the statement of Engineering applications, Eigen value problems are the most important problems in connection with matrices.

Suppose a linear operator, A, transform vectors **x** into other vectors A**x**, the possibility then arises that there will exist vectors **x** each of which is transformed by A simply into a multiple itself. Such a vector, **x**, must therefore satisfy; $Ax = \lambda x$.

For non – zero solutions of this system, the values of λ are called *Eigen values*, *characteristic values* or *latent roots* of a matrix A. The corresponding solutions of, **x**, are called the *Eigen vectors* or *characteristic vectors* of A.

In general, the operator A has n independent Eigen vectors denoted $\mathbf{x_i}$ with Eigen values λ_i.

The set of equations;

$$\begin{pmatrix} a_{11} & a_{12} & .... & a_{1n} \\ a_{21} & a_{22} & .... & a_{2n} \\ . & . & .... & . \\ . & . & .... & . \\ a_{n1} & a_{n1} & .... & a_{nn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ . \\ . \\ x_n \end{pmatrix} = \lambda \begin{pmatrix} x_1 \\ x_2 \\ . \\ . \\ x_n \end{pmatrix}$$

This implies;

$$\begin{pmatrix} (a_{11}-\lambda) & a_{12} & ... & a_{1n} \\ a_{21} & (a_{22}-\lambda) & ... & a_{2n} \\ . & . & ... & . \\ . & . & ... & . \\ a_{n1} & a_{n2} & ... & (a_{nn}-\lambda) \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ . \\ . \\ x_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ . \\ . \\ 0 \end{pmatrix}$$

i.e. A**x** = λx, becomes; *(A – λI)**x** = **0***

A unit matrix is introduced to enable subtraction of the two matrices.

For this set of homogeneous equations to have non – trivial solutions, then;

$$|A - \lambda I| = 0$$

This is the *<u>characteristic determinant</u>* of A and $|A - \lambda I| = 0$ is the characteristic equation whose solution gives the Eigen values of A.

Examples

1. Find the Eigen values and corresponding vectors of $A = \begin{pmatrix} 2 & 3 \\ 4 & 1 \end{pmatrix}$

   $\begin{pmatrix} 2 & 3 \\ 4 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \lambda \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$, then;

   $$\begin{pmatrix} 2-\lambda & 3 \\ 4 & 1-\lambda \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

   From the characteristic equation $|A - \lambda I| = 0$, i.e.

   $$\begin{vmatrix} 2-\lambda & 3 \\ 4 & 1-\lambda \end{vmatrix} = 0$$

   $$\Rightarrow (2-\lambda)(1-\lambda) - 12 = 0$$

   $$\lambda^2 + 2\lambda - 5\lambda - 10 = 0 \Rightarrow \lambda = -2 \, and \, \lambda = 5$$

Substituting each value of λ into the equation *(A – λI)**x** = **0***, gives;

For *λ* = *-2*

$$\left\{\begin{pmatrix}2 & 3\\ 4 & 1\end{pmatrix} - (-2)\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\right\}\begin{pmatrix}x_1\\ x_2\end{pmatrix} = 0 \Rightarrow \left\{\begin{pmatrix}2 & 2\\ 4 & 1\end{pmatrix} + \begin{pmatrix}2 & 0\\ 0 & 2\end{pmatrix}\right\}\begin{pmatrix}x_1\\ x_2\end{pmatrix} = \begin{pmatrix}0\\ 0\end{pmatrix}$$

$$\Leftrightarrow \begin{pmatrix}4 & 3\\ 4 & 3\end{pmatrix}\begin{pmatrix}x_1\\ x_2\end{pmatrix} = \begin{pmatrix}0\\ 0\end{pmatrix}$$

Therefore, $4x_1 + 3x_2 = 0$

This relationship does not give specific values of $x_1\, and\, x_2$

$$x_2 = -\frac{4}{3}x_1$$

When, $x_1 = 1$, $x_2 = -\frac{4}{3}$, so $\mathbf{x_1}$ is of the form; $\begin{pmatrix}1\\ -\frac{4}{3}\end{pmatrix}$

Hence, $x_1 = \alpha\begin{pmatrix}3\\ -4\end{pmatrix}$, where α is a constant multiplier. The simplest result, with α = 1, is normally quoted.

$$\therefore\ for\ \lambda = -1, x_1 = \begin{pmatrix}3\\ -4\end{pmatrix}$$

Similarly for $\lambda_2 = 5$,

$$\left\{\begin{pmatrix}2 & 3\\ 4 & 1\end{pmatrix} - (5)\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\right\}\begin{pmatrix}x_1\\ x_2\end{pmatrix} = 0 \Rightarrow \begin{pmatrix}-3 & 3\\ 4 & -4\end{pmatrix}\begin{pmatrix}x_1\\ x_2\end{pmatrix} = \begin{pmatrix}0\\ 0\end{pmatrix}$$

$$-3x_1 + 3x_2 = 0 \Leftrightarrow x_2 = x_1$$

The corresponding Eigen vector is, $\beta\begin{pmatrix}1\\ 1\end{pmatrix}$

Taking β =1, then for $\lambda_2 = 5$, $x_2 = \begin{pmatrix}1\\ 1\end{pmatrix}$

The required Eigen vectors are:

$$x_1 = \begin{pmatrix}3\\ -4\end{pmatrix}\ for\ \lambda_1 = -2$$

$$x_2 = \begin{pmatrix}1\\ 1\end{pmatrix}\ for\ \lambda_2 = 5$$

2. Find the Eigen values and Eigen vectors of the matrix; $A=\begin{pmatrix}-2 & 2 & -3\\ 2 & 1 & -6\\ -1 & -2 & 0\end{pmatrix}$

From, $|A-\lambda I|=0$

$$\begin{vmatrix}-2-\lambda & 2 & -3\\ 2 & 1-\lambda & -6\\ -1 & -2 & -\lambda\end{vmatrix}=0$$

$$(-2-\lambda)\begin{vmatrix}1-\lambda & -6\\ -2 & -\lambda\end{vmatrix}-2\begin{vmatrix}\lambda & -6\\ -1 & -\lambda\end{vmatrix}-3\begin{vmatrix}2 & 1-\lambda\\ -1 & -2\end{vmatrix}=0$$

$$(-2-\lambda)(-\lambda+\lambda^2-12)-2(-2\lambda-6)-3(-4-(-1+\lambda))=0$$

$$\Rightarrow -\lambda^3-\lambda^2+21\lambda+45=0$$

The roots of A are:

$\lambda_1=5, \lambda_2=-3 \, and \, \lambda_3=-3$

For $\lambda_1 = 5$;

$$\left\{\begin{pmatrix}-2 & 2 & -3\\ 2 & 1 & -6\\ -1 & -2 & 0\end{pmatrix}-5\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}\right\}\begin{pmatrix}x_1\\ x_2\\ x_3\end{pmatrix}=\begin{pmatrix}0\\ 0\\ 0\end{pmatrix}\Rightarrow\begin{pmatrix}-7 & 2 & -3\\ 2 & -4 & -6\\ -1 & -2 & -5\end{pmatrix}\begin{pmatrix}x_1\\ x_2\\ x_3\end{pmatrix}=\begin{pmatrix}0\\ 0\\ 0\end{pmatrix}$$

Using Guassian elimination;

The augmented matrix form is;

$$\begin{pmatrix}-7 & 2 & -3 & . & 0\\ 2 & -4 & -6 & . & 0\\ -1 & -2 & -5 & . & 0\end{pmatrix}$$

$R_2 + 2R_3$ and $2R_1 + 7R_2$, gives;

$$\begin{pmatrix}-7 & 2 & -3 & 0\\ 0 & -24 & -48 & 0\\ 0 & -8 & -16 & 0\end{pmatrix}$$

$-8x_2-16x_3=0\Rightarrow x_2=-2x_3$ and, $-7x_1+2(-2x_3)-3x_3\Rightarrow x_1=-x_3$

Therefore, x_1, x_2, and x_3 are in the ratio; $1:2:-1$

Therefore, $x_1=\begin{pmatrix}1\\ 2\\ -1\end{pmatrix}$

Similarly for λ = -3; $x_2 = \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}$

Questions

Find the Eigen values and t he corresponding Eigen vectors of the following matrices.

(i) $A = \begin{pmatrix} 2 & 7 & 0 \\ 1 & 3 & 1 \\ 5 & 0 & 8 \end{pmatrix} \Bigg| \ Ans: \lambda = 1,3,9 \ and \ x = \begin{pmatrix} 7 \\ -1 \\ -5 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ -7 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 5 \end{pmatrix}$

(ii) $A = \begin{pmatrix} 5 & -6 & 1 \\ 1 & 1 & 0 \\ 3 & 0 & 1 \end{pmatrix} \Bigg| \ Ans: \lambda = 1,2,4 \ x = \begin{pmatrix} 0 \\ 1 \\ 6 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix}, \begin{pmatrix} 3 \\ 1 \\ 3 \end{pmatrix}$

END