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DEPARTMENT OF ELECTRICAL AND COMPUTER ENGINEERING

EMT 1201: ENGINEERING MATHEMATICS II LECTURE NOTES 2014/2015

CHAPTER THREE: VECTOR ALGEBRA

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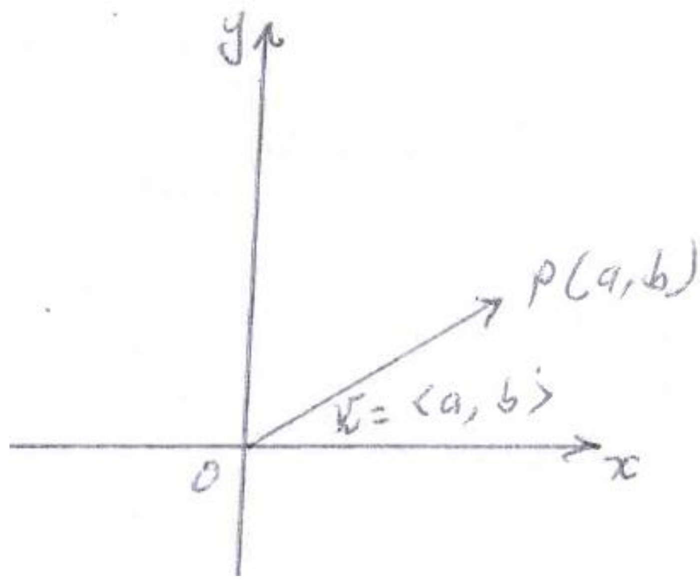
VECTOR ALGEBRA

Vectors are quantities that are defined by both magnitude and direction. E.g force, velocity, momentum, acceleration etc in contrast, quantities like mass, area, temperature, length, volume that can be completely specified by a real number, their magnitude are called scalar quantities.

A vector **v** in the Cartesian plane is an ordered pair of real numbers of the form $\langle a, b \rangle$. We write, $\boldsymbol{v} = \langle a, b\rangle$ and call, *a*, and, *b*, the components of the vector.

The direction line segment $\vec{OP}$ from the origin O to the point *P (a, b)* is one geometric representation of the vector **v**. for this reason, the vector $\boldsymbol{v} = \langle a, b\rangle$ is called the position vector.

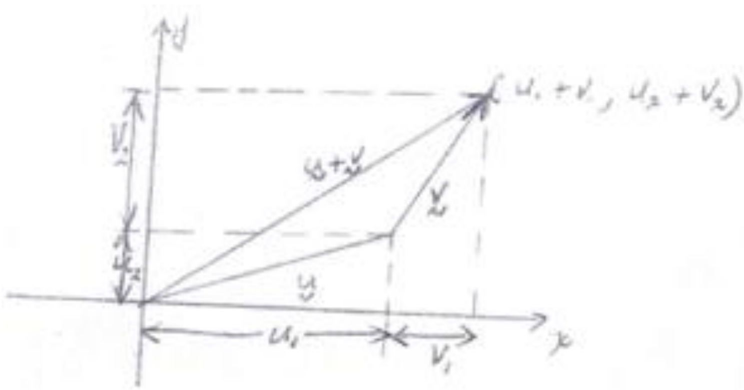


Vector operations

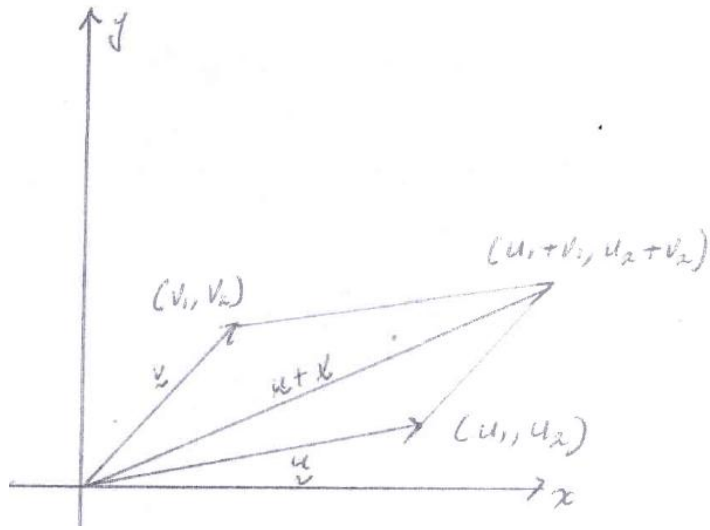
1. Addition of vectors

The sum $\boldsymbol{u} + \boldsymbol{v}$ of two vectors, $\boldsymbol{u} = \langle u_1, u_2\rangle$ and $\boldsymbol{v} = \langle v_1, v_2\rangle$ is the vector;

$\boldsymbol{u} = \langle u_1, u_2\rangle = \langle u_1 + v_1, u_2 + v_2\rangle$



Thus we add vectors by adding corresponding components i.e. by componentwise addition. The geometric interpretation of vector addition is the triangle law of addition. An equivalent interpretation is the parallelogram law of addition.



Parallelogram law for vector addition

2. Multiplication of a vector by a scalar

If $\boldsymbol{u} = \langle u_1, u_2 \rangle$ and c is a real number, then the scalar multiple, $c\boldsymbol{u}$ is the vector; $c\boldsymbol{u} = \langle cu_1, cu_2 \rangle$.

Note that:

$$|c\boldsymbol{u}| = \sqrt{(cu_1)^2 + (cu_2)^2} = |c|\sqrt{{u_1}^2 + {u_1}^2} = |c||\boldsymbol{u}|$$

Example

If $\boldsymbol{u} = \langle 4, -3 \rangle$ and $\boldsymbol{v} = \langle -2,3 \rangle$, then;

(i) $|\boldsymbol{u}| = \sqrt{4^2 + (-3)^2} = 5$

(ii) $\boldsymbol{u} + \boldsymbol{v} = \langle (4 \pm -2), (-3+3) \rangle = \langle 2,0 \rangle$

(iii) $\boldsymbol{u} - \boldsymbol{v} = \langle (4+2), (-3-3) \rangle = \langle 6, -6 \rangle$

(iv) $3\boldsymbol{u} = \langle 3(4), 3(-3) \rangle = \langle 12, -9 \rangle$

Many familiar algebraic properties of real numbers carry over to the following properties of vector addition and scalar multiplication. Let **a**, **b**, **c** be vectors and *r* and *s* be constants.

(i) $\boldsymbol{a} + \boldsymbol{b} = \boldsymbol{b} + \boldsymbol{a}$

(ii) $\boldsymbol{a} + (\boldsymbol{b} + \boldsymbol{c}) = (\boldsymbol{a} + \boldsymbol{b}) + \boldsymbol{c}$

(iii) $r(\boldsymbol{a} + \boldsymbol{b}) = r\boldsymbol{a} + r\boldsymbol{b}$

(iv) $(r + s)\boldsymbol{a} = r\boldsymbol{a} + s\boldsymbol{a}$

(v) $(rs)\boldsymbol{a} = r(s\boldsymbol{a}) = s(r\boldsymbol{a})$

The unit vectors *i* and *j*

A unit vector is one with length 1.0. I,e, a unit vector in a given direction is the one with modulus one and is denoted by, $\hat{\boldsymbol{u}}$

If $a = \langle a_1, a_2 \rangle \neq 0$, then

$$\hat{\boldsymbol{u}} = \frac{\boldsymbol{a}}{|\boldsymbol{a}|}$$

Example

Find the unit vector if; $\boldsymbol{a} = \langle 3, -4 \rangle$

$|\boldsymbol{a}| = \sqrt{3^2 + (-4)^2} = 5$

$\hat{\boldsymbol{a}} = \langle \frac{3}{5}, \frac{-4}{5} \rangle$.

Note: the unit vector $\hat{\boldsymbol{u}}$ has the same direction as ***u*.**

Two perpendicular unit vectors play a special role. They are the vectors; $\boldsymbol{i} = \langle 1,0 \rangle$ and $\boldsymbol{j} = \langle 0,1 \rangle$. The provide a useful alternative of writing vectors;

If $\boldsymbol{a} = \langle a_1, a_2 \rangle$, then;

$\boldsymbol{a} = \langle a_1, 0 \rangle + \langle 0, a_2 \rangle$, then;

$\boldsymbol{a} = \langle a_1, 0 \rangle + \langle 0, a_2 \rangle = a_1 \langle 1,0 \rangle + a_2 \langle 0,1 \rangle = a_1 \boldsymbol{i} + a_2 \boldsymbol{j}$

If $\boldsymbol{a} = a_1 \boldsymbol{i} + a_2 \boldsymbol{j}$ and $\boldsymbol{b} = b_1 \boldsymbol{i} + b_2 \boldsymbol{j}$, then;

$\boldsymbol{a} + \boldsymbol{b} = (a_1 + b_1)\boldsymbol{i} + (a_2 + b_2)\boldsymbol{j}$

Note

Any three dimensional vector in space can be represented by; ***i*, j, k.**

i.e. $\boldsymbol{a} = \langle a_1, a_2, a_3 \rangle$ can be written as;

$$\boldsymbol{a} = a_1 \boldsymbol{i} + a_2 \boldsymbol{j} + a_3 \boldsymbol{k}$$

The dot or scalar product

The dot product of two vectors, $\boldsymbol{a} = \langle a_1, a_2, a_3 \rangle$ and $\boldsymbol{b} = \langle b_1, b_2, b_3 \rangle$ is given by;

(i) $\boldsymbol{a}.\boldsymbol{b} = a_1 b_1 + a_2 b_2 + a_3 b_3$ (in terms of components.

(ii) $\boldsymbol{a}.\boldsymbol{b} = |\boldsymbol{a}||\boldsymbol{b}| cos\theta$ where θ is the angle between the two vectors and $0 \leq \theta \leq \pi$

The result of this computation is a scalar quantity.

Note

(a) If θ = 0 (vectors in the same direction), then $cos\theta = 1$ and so the dot product of the vectors becomes the ordinary product of their magnitudes, $\boldsymbol{a}.\boldsymbol{b} = |\boldsymbol{a}||\boldsymbol{b}|$

(b) If $\theta = \frac{\pi}{2}$ (perpendicular vectors) then, $cos\theta = 0$ and so the dot product is zero.

(c) If θ = 180^0 (vectors in opposite directions), then $cos\theta = -1$ and so the dot product of the vectors is minus the product of their magnitudes; $\boldsymbol{a}.\boldsymbol{b} = -|\boldsymbol{a}||\boldsymbol{b}|$

Algebraic properties of the dot product

(i) $\boldsymbol{a}.\boldsymbol{a} = |\boldsymbol{a}|^2$

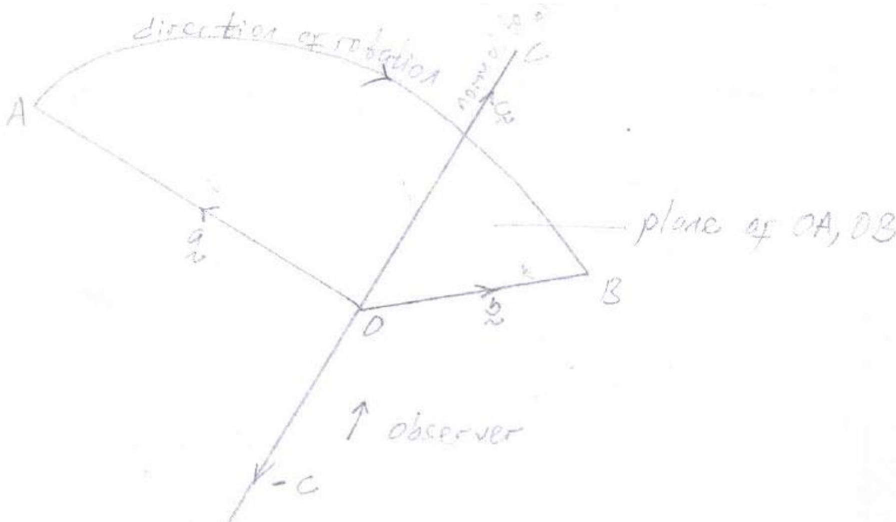
(ii) $\boldsymbol{a}.\boldsymbol{b} = \boldsymbol{b}.\boldsymbol{a}$

(iii) $\boldsymbol{a}.(\boldsymbol{b}+\boldsymbol{c}) = \boldsymbol{a}.\boldsymbol{b} + \boldsymbol{a}.\boldsymbol{c}$

(iv) $(r\boldsymbol{a}).\boldsymbol{b} = r(\boldsymbol{a}.\boldsymbol{b}) = \boldsymbol{a}(r\boldsymbol{b})$

The cross product

Let $\overrightarrow{OA}, \overrightarrow{OB}$ and $\overrightarrow{OC}$ be drawn from a point O such that, $\boldsymbol{a} = \overrightarrow{OA}, \boldsymbol{b} = \overrightarrow{OB}$ and $\boldsymbol{c} = \overrightarrow{OC}$;



Suppose that **c** is perpendicular to both **a** and **b**, then so also is **–c** as is for any real number k.

We say that the vectors **a**, **b**, and **c** are a right – handed set, if the direction of rotation from A to B is clockwise as seen by the observer looking in the direction, $\overrightarrow{OC}$.

It is conventional to choose the basis vectors ***i***, **j** and **k** so that they are a right handed set.

Definition of the vector product (cross product)

If **a** and **b** are two non – parallel vectors, there exists a unique vector **c** such that;

(a) **c** is perpendicular to both **a** and **b**.
(b) **a, b, c** is a right handed – set.
(c) $|\boldsymbol{c}| = |\boldsymbol{a}||\boldsymbol{b}|sin\theta$ where θ is the angle between **a** and **b**.

This vector **c** is called the vector product of, or cross product of **a** and **b**, written as;

$c = \boldsymbol{a} \times \boldsymbol{b}$ or $\boldsymbol{c} = \boldsymbol{a}\hat{}\boldsymbol{b}$

Note

(a) If a and b are parallel, then, $|\boldsymbol{c}| = 0$, i.e. $|\boldsymbol{c}| = |\boldsymbol{a}||\boldsymbol{b}|sin\theta = 0$.

In particular, $\boldsymbol{a} \times \boldsymbol{a} = 0$ for any vector **a** and so;

$i \times i = j \times j = k \times k = 0$

(b) If **a** and **b** are perpendicular, then, $\theta = \frac{\pi}{2}, sin\theta = 1$ and so;

$|\boldsymbol{c}| = |\boldsymbol{a} \times \boldsymbol{b}| = |\boldsymbol{a}||\boldsymbol{b}|$ it follows that;

$\boldsymbol{i} \times \boldsymbol{j} = \boldsymbol{k}$

$\boldsymbol{j} \times \boldsymbol{k} = \boldsymbol{i}$

$\boldsymbol{k} \times \boldsymbol{i} = \boldsymbol{j}$

(c) Unlike the dot product, which is a real number (a scalar), the vector product is a vector.

(d) The operation $\times$ (cross product) is non – commutative, i.e.;

$\boldsymbol{a} \times \boldsymbol{b} \neq \boldsymbol{b} \times \boldsymbol{a}$ but, $\boldsymbol{b} \times \boldsymbol{a} = -\boldsymbol{a} \times \boldsymbol{b}$

Since the definition requires that the **b**, **a** and **b × a** must be right – handed set.

(e) Distributive law

$\boldsymbol{a} \times (\boldsymbol{b} + \boldsymbol{c}) = (\boldsymbol{a} \times \boldsymbol{b}) + (\boldsymbol{a} \times \boldsymbol{c})$

And

$(\boldsymbol{a} + \boldsymbol{b}) \times \boldsymbol{c} = -\boldsymbol{c} \times (\boldsymbol{a} + \boldsymbol{b})$

$= (-\boldsymbol{c} \times \boldsymbol{a}) - (\boldsymbol{c} \times \boldsymbol{b}) = (\boldsymbol{a} \times \boldsymbol{c}) + (\boldsymbol{b} \times \boldsymbol{c})$

(f) Vector multiplication of brackets follows the usual pattern, e.g.

$(\boldsymbol{a} + \boldsymbol{b}) \times (\boldsymbol{c} + \boldsymbol{d}) = (\boldsymbol{a} \times \boldsymbol{c}) + (\boldsymbol{a} \times \boldsymbol{d}) + (\boldsymbol{b} \times \boldsymbol{c}) + (\boldsymbol{b} \times \boldsymbol{d})$

However, each term on the RHS must have the vectors in the correct order since; this multiplication is non – commutative.

Vector product in component form

If $a = x_1\boldsymbol{i} + y_1\boldsymbol{j} + z_1\boldsymbol{k}$ and $b = x_2\boldsymbol{i} + y_2\boldsymbol{j} + z_2\boldsymbol{k}$, then;

$a \times b = (y_1z_2 - y_2z_1)\boldsymbol{i} + (z_1x_2 - z_2x_1)\boldsymbol{j} + (x_1y_2 - x_2y_1)\boldsymbol{k}$

This can be best recalled by considering the cross product as a 3×3 determinant;

$$\begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{vmatrix} = (y_1z_2 - y_2z_1)\boldsymbol{i} + (z_1x_2 - z_2x_1)\boldsymbol{j} + (x_1y_2 - x_2y_1)\boldsymbol{k}$$

Examples

1. Find $a \times b$, given

(i) $\boldsymbol{a} = 2\boldsymbol{i} - \boldsymbol{j} + 3\boldsymbol{k}$ and $\boldsymbol{b} = \boldsymbol{i} + 2\boldsymbol{j} - 4\boldsymbol{k}$

$$\boldsymbol{a} \times \boldsymbol{b} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 2 & -1 & \mathbf{3} \\ 1 & 2 & -4 \end{vmatrix} = \boldsymbol{i}\begin{vmatrix} -1 & 3 \\ 2 & -4 \end{vmatrix} - \boldsymbol{j}\begin{vmatrix} 2 & 3 \\ 1 & -4 \end{vmatrix} + \boldsymbol{k}\begin{vmatrix} 2 & -1 \\ 1 & 2 \end{vmatrix}$$

$$= (4-6)\boldsymbol{i} - (-8-3)\boldsymbol{j} + (4+1)\boldsymbol{k} = -2\boldsymbol{i} + 11\boldsymbol{j} + 5\boldsymbol{k}$$

(ii) $a = 2\boldsymbol{i} + 2\boldsymbol{j} - \boldsymbol{k}$ and $b = 4\boldsymbol{i} - \boldsymbol{j} - \boldsymbol{k}$ (Ans: a × b = -3$\boldsymbol{i}$ – 2**j** – 10**k**)

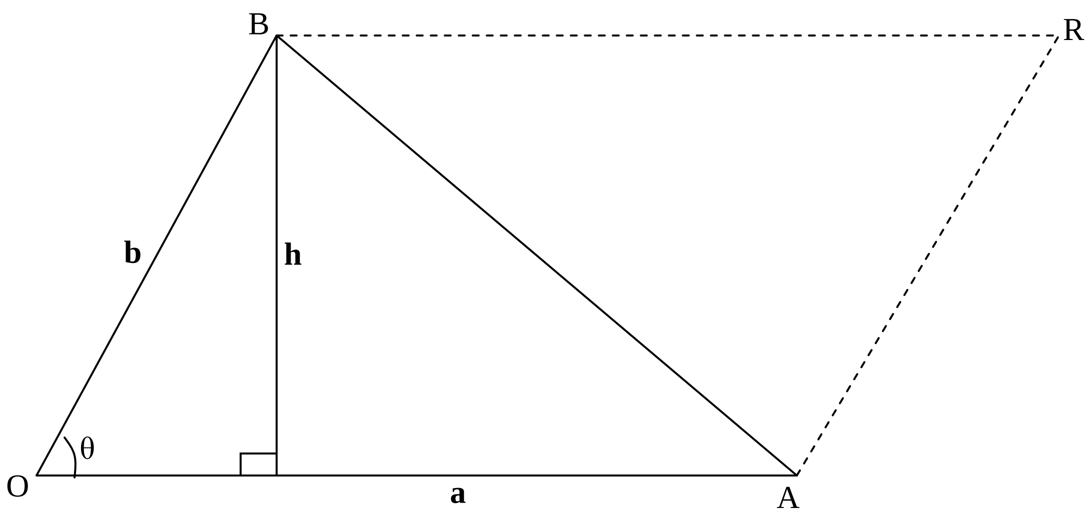
(iii)$a = 3\boldsymbol{i} - \boldsymbol{j} + 2\boldsymbol{k}$ and $\boldsymbol{u} = -3\boldsymbol{i} + 4\boldsymbol{k}$ (Ans: -4**i** – 18**j** – 3**k**)

2. Find the unit vector perpendicular to the plane of the vectors **a** and **b**, where **a** = 2**i** – 3**j** +**k** and **b = i** +2**j** – 4**k**

$$\boldsymbol{a}\times\boldsymbol{b}=\begin{vmatrix}\boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k}\\ 2 & -3 & 1\\ 1 & 2 & -4\end{vmatrix}=\boldsymbol{i}\begin{vmatrix}-3 & 1\\ 2 & -4\end{vmatrix}-\boldsymbol{j}\begin{vmatrix}2 & 1\\ 1 & -4\end{vmatrix}+\boldsymbol{k}\begin{vmatrix}2 & -3\\ 1 & 2\end{vmatrix}$$

$$=10\boldsymbol{i}+9\boldsymbol{j}+7\boldsymbol{k}$$

Application of vector product to area



Geometrically, the area of the parallelogram OARB is *OA.h*

$$Area = OA.h = |\boldsymbol{a}||\boldsymbol{b}|sin\theta$$

Thus the magnitude of **a** × **b** equals the area of the parallelogram with adjacent sided **a** and **b.**

i.e. $|\boldsymbol{a}\times\boldsymbol{b}| = |\boldsymbol{a}||\boldsymbol{b}|sin\theta$

Alternatively, for triangle OAB;

$$Area = \frac{1}{2}OA.h = \frac{1}{2}|\boldsymbol{a}||\boldsymbol{b}|sin\theta = \frac{1}{2}|\boldsymbol{a}\times\boldsymbol{b}|$$

Examples

1. Find the area of the triangle with adjacent sides representing the vectors, 2**i** – **j** + 3**k** and **i** +2**j** – 4**k.**

   Solution

$$Area = \frac{1}{2}|\boldsymbol{a}\times\boldsymbol{b}|$$

$$\boldsymbol{a}\times\boldsymbol{b}=\begin{vmatrix}\boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k}\\ 2 & -1 & 3\\ 1 & 2 & -\boldsymbol{4}\end{vmatrix}=\boldsymbol{i}\begin{vmatrix}-1 & 3\\ 2 & -4\end{vmatrix}-\boldsymbol{j}\begin{vmatrix}2 & -1\\ 1 & 2\end{vmatrix}+\boldsymbol{k}\begin{vmatrix}2 & -1\\ 1 & 2\end{vmatrix}$$

$$=(2-6)\boldsymbol{i}-(-8-3)\boldsymbol{j}+(4+1)\boldsymbol{k}=-2\boldsymbol{i}+11\boldsymbol{j}+5\boldsymbol{k}$$

$$Area=\frac{1}{2}|\boldsymbol{a}\times\boldsymbol{b}|=\frac{1}{2}\sqrt{4+121+25}=\frac{1}{2}\sqrt{150}$$

2. Find the area of a triangle having vertices P(1, 3, 2), Q(-2, 1, 3) and R(3, -2, -1)

$\overrightarrow{PQ} = OQ - OP = \langle -3, -2, -1 \rangle$

$\overrightarrow{PR} = OR - OP = \langle 2, -5, -3 \rangle$

$$\overrightarrow{PQ} \times \overrightarrow{PR} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ -3 & -2 & 1 \\ 2 & -5 & -3 \end{vmatrix} = \boldsymbol{i}\begin{vmatrix} -2 & 1 \\ -5 & -3 \end{vmatrix} - \boldsymbol{j}\begin{vmatrix} -3 & 1 \\ 2 & -3 \end{vmatrix} + \boldsymbol{k}\begin{vmatrix} -3 & -2 \\ 2 & -5 \end{vmatrix}$$

$$= 11\boldsymbol{i} - 7\boldsymbol{j} + 19\boldsymbol{k}$$

$$Area = \frac{1}{2}\left|\overrightarrow{PQ} \times \overrightarrow{PR}\right| = \frac{1}{2}\sqrt{11^2 + (-7)^2 + 19^2}$$

Application to analytic geometry

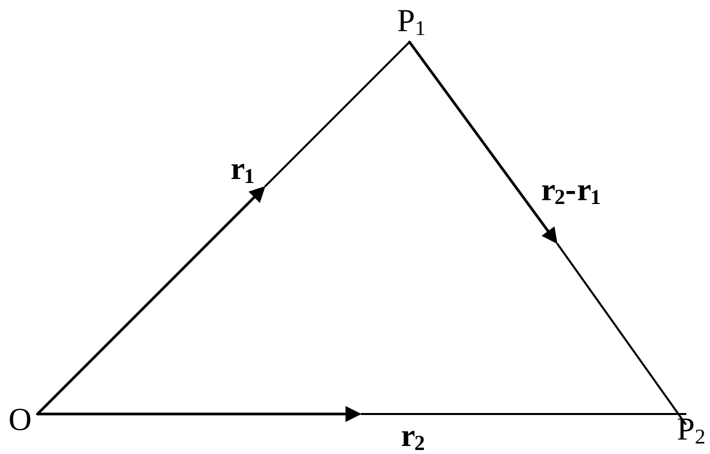
Vectors and coordinates

The vector **OP** drawn from the origin to the point P is called the position vector of P. if P has coordinates (x, y, z) then, $\overrightarrow{OP} = x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}$ where **i, j k** denote unit vectors in the directions Ox, Oy and Oz respectively.

$\boldsymbol{r} = \overrightarrow{OP}, r = |\boldsymbol{r}| = OP$

Let P_1 and P_2 have coordinates (x_1, y_1, z_1) and (x_2, y_2, z_2) respectively so that;

$\boldsymbol{r}_2 = x_1\boldsymbol{i} + y_1\boldsymbol{j} + z_1\boldsymbol{k}$ and $\boldsymbol{r}_2 = x_2\boldsymbol{i} + y_2\boldsymbol{j} + z_2\boldsymbol{k}$



Hence;

$\overrightarrow{P_1P_2} = (x_2 - x_1)\boldsymbol{i} + (y_2 - y_1)\boldsymbol{j} + (z_2 - z_1)\boldsymbol{k}$

This expression enables us to write down the vector joining any two points when we know the coordinates of these points e.g. the vector joining (1, 2, 3) to (4, -7, 11) is;

$(4 - 1)\boldsymbol{i} + (-7 - 2)\boldsymbol{j} + (11 - 3)\boldsymbol{k} = 3\boldsymbol{i} - 9\boldsymbol{j} + 8\boldsymbol{k}$

Distance between two points

The distance between $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$ equals the magnitude of $\overrightarrow{P_1P_2}$

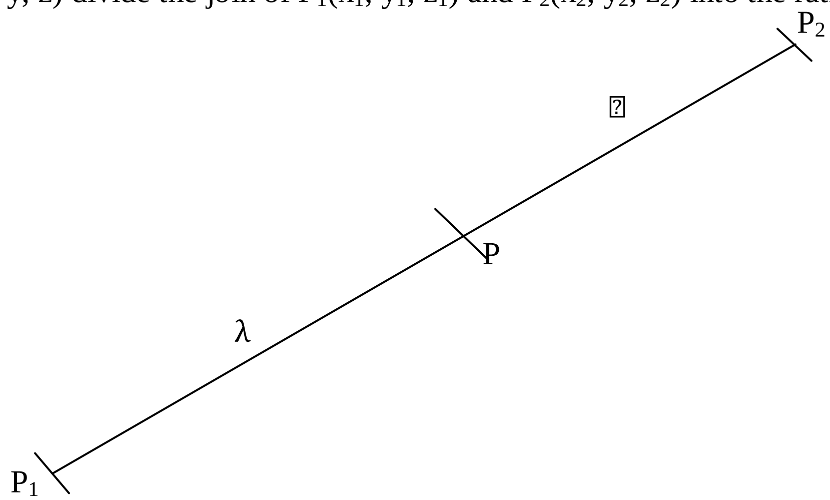
$$|\overrightarrow{P_1P_2}| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

e.g. the distance between the points (1, 2, 3) and (4, -7, 11), is;

$$\sqrt{3^2 + (-9)^2 + 8^2} = \sqrt{154}$$

Section formula

Let $P(x, y, z)$ divide the join of $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$ into the ratio $\lambda: \mu$.



Then $\overrightarrow{P_1P}$ and $\overrightarrow{PP_2}$ are vectors in the same direction and with magnitudes in the ratio $\lambda: \mu$, i.e.

$$(x - x_1)\boldsymbol{i} + (y - y_1)\boldsymbol{j} + (z - z_1)\boldsymbol{k} = \frac{\boldsymbol{\lambda}}{\mu}(x_2 - x)\mathbf{i} + \frac{\boldsymbol{\lambda}}{\mu}(y_2 - y)\boldsymbol{j} + \frac{\boldsymbol{\lambda}}{\mu}(z_2 - z)\boldsymbol{k}$$

This implies;

$$x - x_1 = \frac{\lambda}{\mu}(x_2 - x)$$

$$y - y_1 = \frac{\lambda}{\mu}(y_2 - y)$$

$$z - z_1 = \frac{\lambda}{\mu}(z_2 - z)$$

Solving we get;

$$x = \frac{\lambda x_2 + \mu x_1}{\lambda + \mu}$$

$$y = \frac{\lambda y_2 + \mu y_1}{\lambda + \mu}$$

$$z = \frac{\lambda z_2 + \mu z_1}{\lambda + \mu}$$

In particular, the coordinates of the mid – point of P_1P_2 are $\frac{1}{2}(x_1 + x_2), \frac{1}{2}(y_1 + y_2)$ and $\frac{1}{2}(z_1 + z_2)$

The point divides P_1P_2 internally or externally according as the ratio λ: μ is positive or negative; if the ratio is negative it is immaterial whether we assign a negative value to λ or μ.

Direction of a line

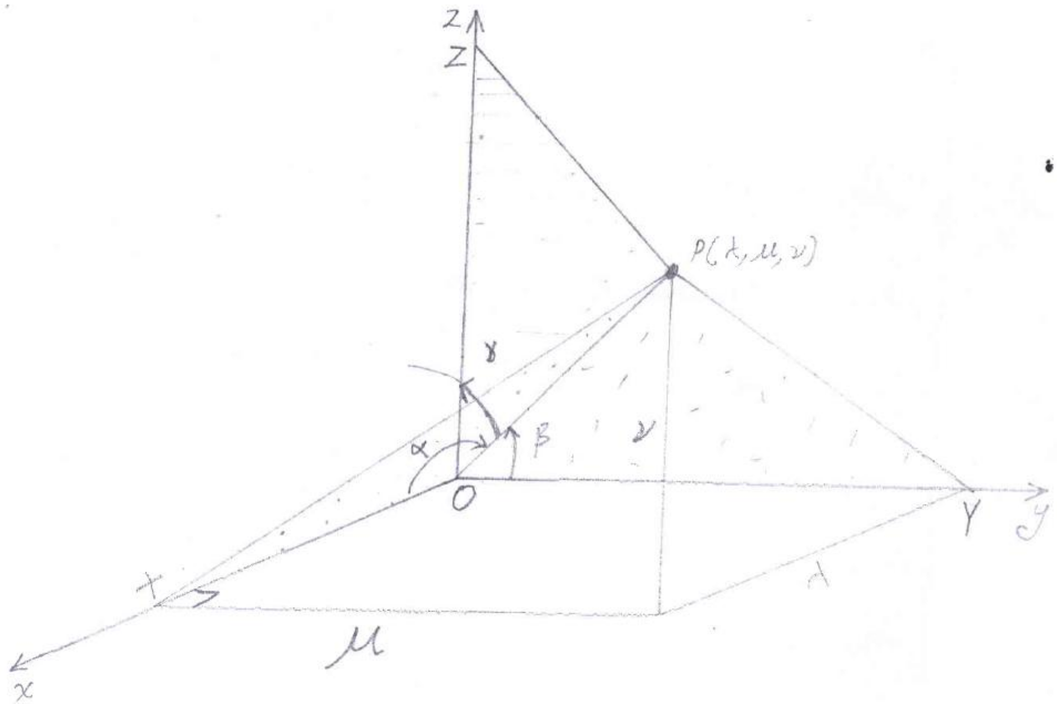
The direction of a line can be described by reference to a vector with the required direction. E.g. the line through the points, A(1, 2, 3) and B(4 , -7, 11) taken in the sense of AB has a direction $\overrightarrow{AB}$ i.e. 3**i** – 9**j** +8**k** or 6**i** -18**j** +16**k** or 3k**i** – 9k**j** +8k**k** for any positive k.

The line in the direction of the vector λ**i** +μ**j** +γ**k** is said to have direction ratios (DRs) [λ: μ: γ]. Thus the line AB has DRs as, [3: -9: 8].

It therefore follows that the line P_1P_2 joining $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$ has DRs;

$[(x_2\text{-}x_2), (y_2\text{-}y_1), (z_2\text{-}z_1)]$

Let $\overrightarrow{OP}$ be the vector $\lambda\boldsymbol{i} + \mu\boldsymbol{j} + \nu\boldsymbol{k}$ so that P is the point (λ, μ,ν). Let PX, PY, PZ be perpendiculars from P onto the x -, y -, z – axes respectively, and let OP make angles α, β, γ respectively with the positive directions of these axes.



From triangle OXP, right angled at X;

$\lambda = OX = OP cos\alpha$

$\mu = OZ = OP cos\beta$

$\nu = OZ = OP cos\gamma$

Thus $\lambda: \mu: \nu = cos\alpha: cos\beta: cos\gamma$

In particular, if $l\boldsymbol{i} + m\boldsymbol{j} + n\boldsymbol{k}$ is the unit vector in the direction of $\lambda\boldsymbol{i} + \mu\boldsymbol{j} + \nu\boldsymbol{k}$ then *l, m, n* can be found in terms of α, β, γ in the same way as for λ, μ, ν with the added simplification that *OP = 1*.

Thus,

$l = cos\alpha, m = cos\beta, n = cos\gamma$

$l^2 + m^2 + n^2 = 1$, which gives;

$cos^2\alpha + cos^2\beta + cos^2\gamma = 1$

The direction of a line may be specified by giving the special DRs [*l: m: n*] for which $l^2 + m^2 + n^2 = 1$. These are called the direction cosines (DCs) of the line and are written [*l, m, n*]. The DCs are the cosines of the angles the line makes with the coordinate axes, but are usually found much more easily from the associated unit vector in the direction of the line.

It has been shown that the unit vector in the direction of $\lambda\boldsymbol{i} + \mu\boldsymbol{j} + \nu\boldsymbol{k}$ is;

$$\frac{1}{\sqrt{\lambda^2+\mu^2+\nu^2}}(\lambda\boldsymbol{i}+\mu\boldsymbol{j}+\nu\boldsymbol{k})$$

It follows that if [λ: μ: ν] are the DRs of the line L, then the DCs of L are;

$$\left[\frac{\lambda}{\sqrt{\lambda^2+\mu^2+\nu^2}},\frac{\mu}{\sqrt{\lambda^2+\mu^2+\nu^2}},\frac{\nu}{\sqrt{\lambda^2+\mu^2+\nu^2}}\right]$$

Example

Given [3: -9: 8] are the DRs of a certain line, the DCs of the line are;

$$\left[\frac{3}{\sqrt{154}},\frac{-9}{\sqrt{154}},\frac{8}{\sqrt{154}}\right]$$

Angle between two lines (or vectors)

The angle θ between two lines can be determined by considering two vectors in the directions of the lines.

If [λ_1, μ_1, ν_1] and [λ_2, μ_2, ν_2] are the DRs of two lines, $\lambda_1\boldsymbol{i}+\mu_1\boldsymbol{j}+\nu_1\boldsymbol{k}$ and $\lambda_2\boldsymbol{i}+\mu_2\boldsymbol{j}+\nu_2\boldsymbol{k}$ are vectors in the direction of the lines and so using the dot product;

$$cos\theta=\frac{\lambda_1\lambda_2+\mu_1\mu_2+\nu_1\nu_2}{\sqrt{\left(\lambda_1{}^2+\mu_1{}^2+\nu_1{}^2\right)}\sqrt{\left(\lambda_2{}^2+\mu_2{}^2+\nu_2{}^2\right)}}$$

In particular, if [l_1, m_1, n_1], [l_2, m_2, n_2] are the DCs of the two lines, then;

$l_1^2 + m_1^2 + n_1^2 = 1$

$l_2^2 + m_2^2 + n_2^2 = 1$, then;

$$cos\theta=l_1l_1+m_1m_2+n_1n_2$$

The condition that the lines be perpendicular is; $\lambda_1\lambda_2+\mu_1\mu_2+\nu_1\nu_2=0$

Example

Show that the points; A(2, 4, 3), B(4, 1, 9), C(10, -1, 6) are vertices of an isosceles right – angled triangle.

Solutions

The DRs of AB are [(4-2): (1-4): (9-3)] i.e. [2: -3: 6]

Similarly, the DRs of BC and AC are [6: -2, -3] and [8: -5: 3] respectively,

Considering lines AB and BC, $\overrightarrow{AB}.\overrightarrow{BC}=12+6-18=0$

Therefore, the lines AB and BC are perpendicular.

Considering lines AB and AC, angle between them is given by;

$$cos\theta = \frac{16+15+18}{\sqrt{4+9+36}.\sqrt{64+25+9}} = \frac{49}{\sqrt{49}\sqrt{98}}$$

$$\therefore cos\theta = \frac{1}{\sqrt{2}} \Rightarrow \theta = 45^0$$

Hence triangle ABC is an isosceles right – angled triangle.

The common perpendicular to two given directions

Let $[\lambda_1: \mu_1: \nu_1]$ and $[\lambda_2: \mu_2: \nu_2]$ be the DRs of the given directions. Finding a direction perpendicular to both of these is equivalent to finding a vector perpendicular to the two vectors.

i.e. such a vector is given by;

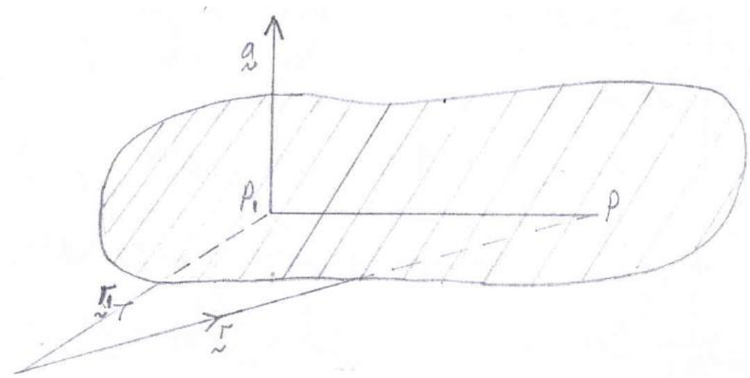
$(\lambda_1\boldsymbol{i} + \mu_1\boldsymbol{j} + \nu_1\boldsymbol{k}) \times (\lambda_2\boldsymbol{i} + \mu_2\boldsymbol{j} + \nu_2\boldsymbol{k})$, giving;

$[(\mu_1\nu_2 - \mu_2\nu_1):(\nu_1\lambda_2 - \nu_2\lambda_1):(\lambda_1\mu_2 - \lambda_2\mu_1)]$ as DRs for the common perpendicular; these DRS treat the perpendicular as pointing in the direction that makes $[\lambda_1, \mu_1, \nu_1]$, $[\lambda_2, \mu_2, \nu_2]$ and the perpendicular a right – handed set.

Equation of a plane

(a) Given a point on the plane and the direction of the normal to the plane

Let $\mathbf{r_1}$ be the position vector of the point P_1 on the plane and let $\mathbf{n}$ be a vector normal to the plane.



If $\mathbf{r}$ is the position vector of any other point **P** on the plane, then $\mathbf{r} - \mathbf{r_1}$ is the vector $\overrightarrow{P_1P}$. The basic property of the normal to a plane is that it is perpendicular to any line in the plane, hence;

$(\boldsymbol{r} - \boldsymbol{r}_1)\boldsymbol{n} = 0$

This equation is satisfied by the position vector $\mathbf{r}$ of any point in the given plane and is the vector equation of the plane.

Let P_1 and P be the points (x_1, y_1, z_1) and (x, y, z) respectively, so that;

$\mathbf{r_1} = x_1\mathbf{i} + y_1\mathbf{j} + z_1\mathbf{k}$ and $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$. Putting $\mathbf{n} = \lambda\mathbf{i} + \mu\mathbf{j} + \nu\mathbf{k}$ so that $[\lambda: \mu: \nu]$ are the DRs of the normal to the plane, we then get;

$$\{(x-x_1)\boldsymbol{i}+(y-y_1)\boldsymbol{j}+(z-z_1)\boldsymbol{k}\}.\{\lambda\boldsymbol{i}+\mu\boldsymbol{j}+\nu\boldsymbol{k}\}=0$$

i.e.

$$\lambda(x-x_1)+\mu(y-y_1)+\nu(z-z_1)=0$$

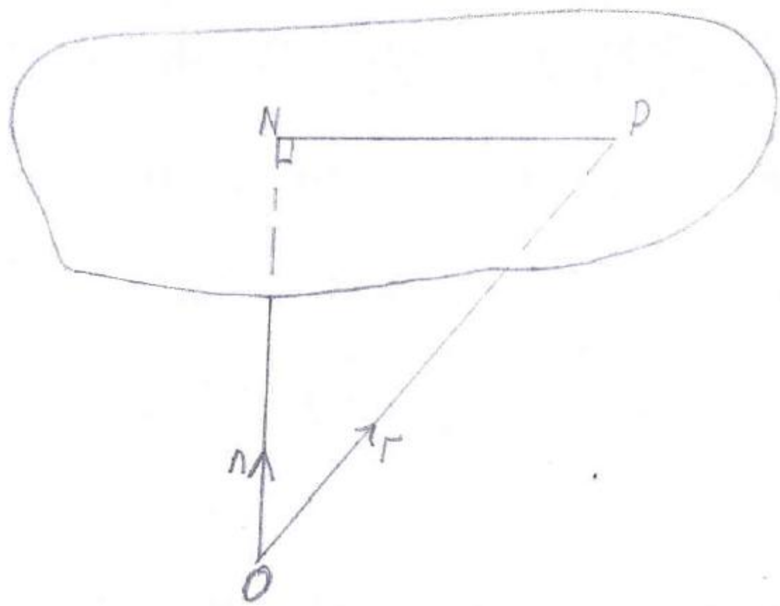
This is the equation of the plane through the point (x_1, y_1, z_1) with normal in the direction

(λ: μ: ν)

(b) Given the direction of the normal of the normal and the perpendicular distance from the origin onto the plane

Let $\hat{\boldsymbol{n}}$ be a unit vector normal to the plane and let $ON = p$ where N is the foot of the perpendicular from the origin O onto the plane.

If *p* is any point on the plane, *PN* lie in the plane and so it is perpendicular to *ON*.



Hence; $\boldsymbol{r}.\hat{\boldsymbol{n}}=p$

Where **r** is the position vector of p. This equation is the vector equation of the plane, e.g.

If $\boldsymbol{r}=x\boldsymbol{i}+y\boldsymbol{j}+z\boldsymbol{k}, \boldsymbol{n}=l\boldsymbol{i}+m\boldsymbol{j}+n\boldsymbol{k},$ we get;

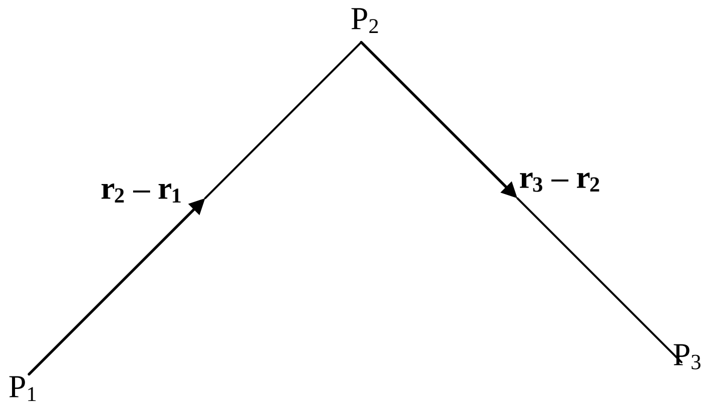
$$lx+my+nz=p$$

This is the equation of the plane distant *p* from the origin when the direction [l, m, n] is normal to the plane and pointing away from the origin. If the direction [l, m, n] is normal to the plane and pointing towards the origin, the equation is;

$$lx+my+nz=-p$$

(c)Given three points on the plane

Let the given points P_1, P_2, P_3 have position vectors, $\mathbf{r_1}$, $\mathbf{r_2}$, $\mathbf{r_3}$ respectively. Since P_1P_2 and P_2P_3 lie in the plane it follows that $\overrightarrow{P_1P_2} \times \overrightarrow{P_2P_3}$ is normal to plane.



Thus a vector normal to the plane is found by obtained the vector product;

$(\boldsymbol{r}_2 - \boldsymbol{r}_1) \times (\boldsymbol{r}_3 - \boldsymbol{r}_2)$

The vector equation of the plane becomes;

$(\boldsymbol{r} - \boldsymbol{r}_1)(\boldsymbol{r}_2 - \boldsymbol{r}_1) \times (\boldsymbol{r}_3 - \boldsymbol{r}_2) = 0$

$\mathbf{r}_2$ or $\mathbf{r}_3$ could replace $\mathbf{r}_1$ in the first bracket.

Consider the case where the given points are; (a, 0, 0), (0, b, 0) and (0, 0, c), then;

$\boldsymbol{r}_1 = a\boldsymbol{i}$, $\boldsymbol{r}_2 = b\boldsymbol{j}$ and $\boldsymbol{r}_3 = c\boldsymbol{k}$ hence;

$$(\boldsymbol{r}_2 - \boldsymbol{r}_1) \times (\boldsymbol{r}_3 - \boldsymbol{r}_2) = (b\boldsymbol{j} - a\boldsymbol{i}) \times (c\boldsymbol{k} - b\boldsymbol{j})$$

$$= bc\boldsymbol{i} + ac\boldsymbol{j} + ab\boldsymbol{k}$$

Also $\boldsymbol{r} - \boldsymbol{r}_1 = (x - a)\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}$

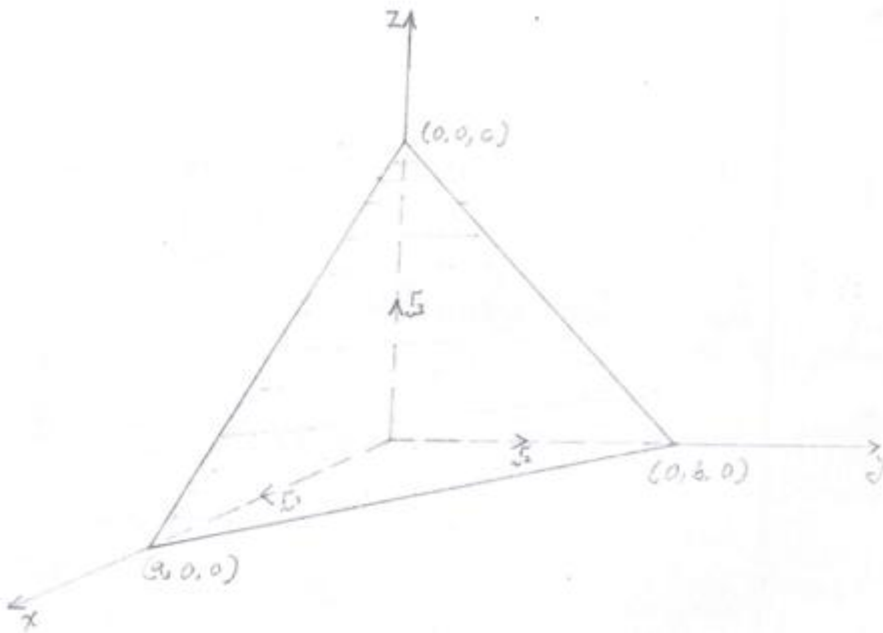
The equation thus becomes;

$bc(x - a) + acy + abz = 0;$

$bcx + acy + abz = abc$

$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$

This is the equation of the plane that makes intercepts of lengths a, b, c on Ox, Oy and Oz ewspectively.



The general equation of the first degree

The general equation of the first degree can be written as;

$$ax+by+cz+d=0$$

This may be written as;

$$\frac{ax+by+cz}{\sqrt{a^2+b^2+c^2}}=\frac{-d}{\sqrt{a^2+b^2+c^2}}$$

i.e. $lx+my+nz=p$, where;

$$l=\frac{\pm a}{\sqrt{a^2+b^2+c^2}},\ m=\frac{\pm b}{\sqrt{a^2+b^2+c^2}},\ n=\frac{\pm c}{\sqrt{a^2+b^2+c^2}} \text{ and } p=\frac{\pm d}{\sqrt{a^2+b^2+c^2}}$$

And the upper or lower sign is chosen according to which is required to make $p\geq 0$

Since $l^2+m^2+n^2=1$, it follows that [l, m, n] are the DCs of some line.

Therefore, the equation $lx+my+nz=p$ is the equation of the plane, distant p from the origin, whose normal in the direction away from the origin has DCs [l, m, n].

Example

The equation $2x-y+2z+6=0$ can be written as;

$$\frac{2}{3}x-\frac{1}{3}y+\frac{2}{3}z=-2 \text{ i.e. } -\frac{2}{3}x+\frac{1}{3}y-\frac{2}{3}z=2$$

Which is the equation of the plane, distant 2 from the origin whose normal in the direction away from the origin has DCs $\left[-\frac{2}{3}, \frac{1}{3}, -\frac{2}{3}\right]$. Thus the equation of a plane is a first – degree equation and conversely any first – degree equation is an equation of a plane.

Example 1

Find the equation of the plane through the point (1, 2, 3) and parallel to plane $2x-3y+4z=12$

Solution

The direction [2: -3: 4] is normal to the plane required. It follows that the equation of the required plane can be written as;

$2x-3y+4z=k$, for some constant k

Since this plane passes through (1, 2, 3), then;

$2(1)-3(2)+4(3)=k \Rightarrow k=8$

Therefore, the equation of the plane is; $2x-3y+4z=8$

Example 2

Find the equation of the plane that passes through the points (3, 4, 1), (1, 1, -7) and (2, 2, -4).

Solutions

Two methods can be employed.

Method 1

Given, $\boldsymbol{r}_1 = 3\boldsymbol{i} + 4\boldsymbol{j} + \boldsymbol{k}$, $\boldsymbol{r}_2 = \boldsymbol{i} + \boldsymbol{j} - 7\boldsymbol{k}$ and $\boldsymbol{r}_3 = 2\boldsymbol{i} + 2\boldsymbol{j} - 4\boldsymbol{k}$, then normal vector to required plane is given by;

$$\boldsymbol{n} = (\boldsymbol{r}_2 - \boldsymbol{r}_1) \times (\boldsymbol{r}_3 - \boldsymbol{r}_2) = (-2\boldsymbol{i} - 3\boldsymbol{j} - 8\boldsymbol{k}) \times (\boldsymbol{i} + \boldsymbol{j} + 3\boldsymbol{k}) = -\boldsymbol{i} - 2\boldsymbol{j} + \boldsymbol{k}$$

Therefore, the equation of the plane is given by;

$$(r - \boldsymbol{r}_1)\boldsymbol{n} = 0 \Rightarrow [(x-3)\boldsymbol{i} + (y-4)\boldsymbol{j} + (z-1)\boldsymbol{k}].(-\boldsymbol{i} - 2\boldsymbol{j} + \boldsymbol{k}) = 0$$

$$\therefore -(x-3) - 2(y-4) + (z-1) = 0$$

$$-x-2y+z=-10$$

Hence the equation of the required plane is; $x+2y-z=10$

Method 2

The plane $ax+by+cz+d=0$ passes through the given points, if;

$3a+4b+c+d=0$...(1)

$a+b-7c+d=0$...(2)

$2a+2b-4c+d=0$...(3)

From (2) and (3);

$c = \frac{1}{10}d$

From (1) and (2)

$b = \frac{-2}{10}d \ and \ a = \frac{-1}{10}d$

Hence the equation of the plane is given by;

$$\left(\frac{-1}{10}d\right)x + \left(\frac{-2}{10}d\right)y + \left(\frac{1}{10}d\right)z + d = 0$$

Hence, $x + 2y - z = 10$

The angle between two planes

The angle θ between two planes is the angle between their respective normals. If the planes have equations;

$$a_1x + b_1y + c_1z + d_1 = 0$$
$$a_2x + b_2y + c_2z + d_2 = 0$$

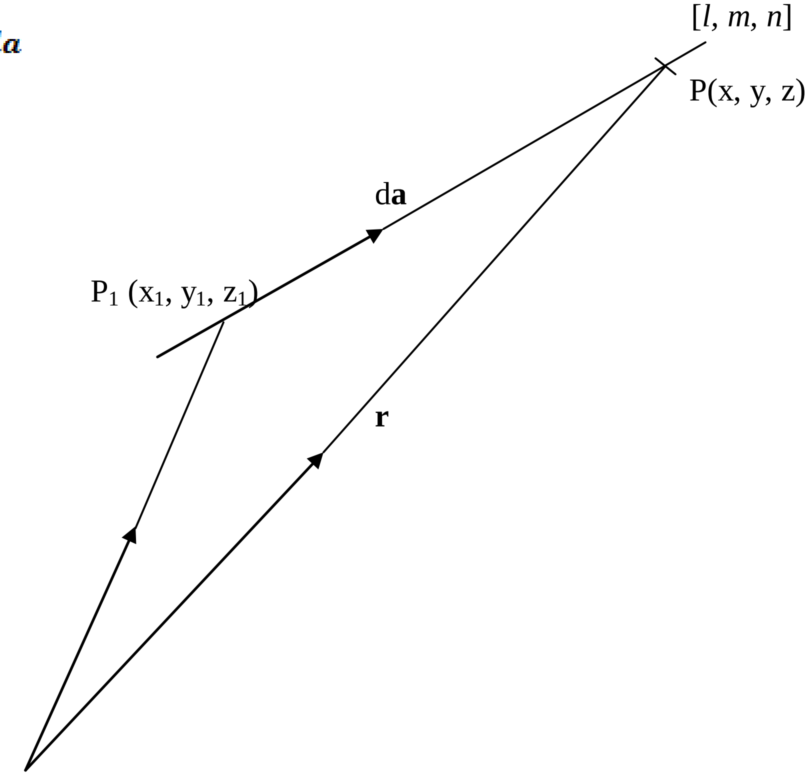
Then;

$$\cos\theta = \frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_1^{\,2} + b_1^2 + c_1^2}.\sqrt{a_2^{\,2} + b_2^2 + c_2^2}}$$

Equations of a straight line

Let $P_1(x_1, y_1, z_1)$ be a point on a given straight line and let **a** be a unit vector in the direction of the line. Let P (x, y, z) be any other point on the line distant *d* from point P_1 so that $\overrightarrow{P_1P} = d\boldsymbol{a}$. Then the position vector $\mathbf{r}_1$ and **r** of the points P_1 and P are related by the equation;

$\boldsymbol{r} = \boldsymbol{r}_1 + d\boldsymbol{a}$



If $d = 0$ the points P_1 and P coincide while positive values of d give points P on one side of P_1 and negative values of d give points of P on the other side.

If we regard d as a variable parameter, then the equation $\boldsymbol{r} = \boldsymbol{r}_1 + d\boldsymbol{a}$, in which **r** is now a variable position vector representing all points on the given line. This is the vector equation of the line.

If [*l*, *m*, *n*] are the DCs of the line, we have that; $\boldsymbol{a} = l\boldsymbol{i} + m\boldsymbol{j} + n\boldsymbol{k}$ and by definition, of;

$\boldsymbol{r}_1 = x_1\boldsymbol{i} + y_2\boldsymbol{j} + z_1\boldsymbol{k}$ and $\boldsymbol{r} = x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}$, the equation $\boldsymbol{r} = \boldsymbol{r}_1 + d\boldsymbol{a}$ becomes;

$x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k} = (x_1 + dl)\boldsymbol{i} + (y_1 + dm)\boldsymbol{j} + (z_1 + dn)\boldsymbol{k}$

Hence;

$x = x_1 dl$

$y = y_1 + dm$

$z = z_1 + dn$

Which are parametric equations for the line, with *d* as the parameter, we have;

$\frac{x-x_1}{l}=\frac{y-y_1}{m}=\frac{z-z_1}{n}$, which gives the equation of the line in symmetric (or standard) form.

The above equations are equivalent to the equations;

$\frac{x-x_1}{\lambda}=\frac{y-y_1}{\mu}=\frac{z-z_1}{\gamma}$, where [λ: μ: ν] are the DRs of the line.

In particular, the equations of the line joining $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$ are;

$$\frac{x-x_1}{x_2-x_1}=\frac{y-y_1}{y_2-y_1}=\frac{z-z_1}{z_2-z_1}$$

Example

Find the equations of the line joining A(1, 2, 4) and B(2, 4, 2). Show that AB meets the xy – plane in a point of trisection of the joining C(3, -4, 2) and D(12, 14, 2).

Solution

The equation of the line is simply given by;

$$\frac{x-1}{(2-1)}=\frac{y-2}{(4-2)}=\frac{z-4}{(2-4)};hence$$

$$\frac{x-1}{1}=\frac{y-2}{2}=\frac{z-4}{-2}$$

AB meets the xy – plane when $z = 0$, and therefore, when;

$$\frac{x-1}{1}=\frac{y-2}{2}=2\Rightarrow x=3 \text{ and } z=6$$

Hence it meets the xy – plane at; (3, 6, 0)

The point of trisection of CD is the point that divides it in the ratio, 1:2, which is given by;

$$\left[\frac{1}{3}(12-3),\frac{1}{3}(14+4),\frac{1}{3}(2-2)\right]=(3,6,0)$$

Hence AB meets the xy –plane at the point of trisection of CD.

The line of intersection of two planes

Consider the planes whose equations are;

$$a_1x+b_1y+c_1z+d_1=0$$
$$a_2x+b_2y+c_2z+d_2=0$$

These planes are parallel if, $a_1:b_1:c_1=a_2:b_2:c_2$

If they are not parallel, they intersect in a line ξ. All points on ξ must satisfy both equations of the two planes.

Example 2

Find in symmetric form the equations of the line of intersection of the planes $x+y+z=5$ *and* $4x+y+2z=15$

Method 1

The DRs of the normals are [*1: 1: 1*] and [*4: 1: 2*]. Hence the line has direction [*1: 2: -3*], this is determined by finding the vector product of the two normals since the required line is perpendicular to both normals.

To find one point on the line we can put z = 0, in both equations and solve for *x* and *y*. i.e.

$x+y=5$

$4x+y=15$

Hence, $x=\frac{10}{3}$ *and* $y=\frac{5}{3}$, thus the point $\left(\frac{10}{3},\frac{5}{3},0\right)$ lies on the line.

We can now write down the equations of the line as;

$$\frac{x-10/3}{1}=\frac{y-5/3}{2}=\frac{z}{-3}$$

Method 2

Eliminating y and z between the equations of the planes, we obtain;

$x+y+z=5$

$4x+y+2z=15$

Hence;

$$3x+z=10 \Rightarrow x=\frac{1}{3}(10-z)$$

$$2x-y=5 \Rightarrow x=\frac{1}{2}(5+y)$$

Hence the equations may be written in the form; $x=\frac{y+5}{2}=\frac{z-10}{-3}$

Check: the denominators 1, 2, -3 agree with the DRs found in method 1. Verify that $\left(0,\frac{-5}{3},\frac{10}{3}\right)$ is a point on the line.

Question 2

Through the point (-1, 1, 2) a line is drawn parallel to line of intersection of the planes, $x-2y+z=3$ and $x+6y-5z=0$. Find the equations of this line and the coordinates of the points where it meets the plane $x-3y+2z=2$.

Solution

The direction of the line is perpendicular to [1: -2: 1] and [1: 6: -5]. Hence the line has DRs [4: 6: 8] i.e. [2: 3: 4]

Hence the equations of the line can be written in the form;

$$\frac{x+1}{2}=\frac{y-1}{3}=\frac{z-2}{4}=d\ (say)$$

So that;

$$x=2d-1$$
$$y=3d+1$$
$$z=4d+2$$

Substituting in the equation $x-3y+2z=2$, gives;

$$2d-1-3(3d+1)+2(4d+2)=2 \Rightarrow d=2$$

This gives the point of intersection as; (3, 7, 10)

The equation of a plane through a line of intersection of two given planes

Let the planes,

$$a_1x+b_1y+c_1z+d_1=0 \quad \text{...(1)}$$
$$a_2x+b_2y+c_2z+d_2=0 \quad \text{...(2)}$$

Intersect in a line ξ. For any constant k the equation;

$$a_1x+b_1y+c_1z+d_1+k(a_2x+b_2y+c_2z+d_2)=0 \quad \text{...(3)}$$

Represents a plane since it is of first degree. Points that satisfy (1) and (2) simultaneously must satisfy (3), and so (3) is the equation of a plane passing through ξ

Examples

1. Find the equation of the plane that contains the line $\frac{x-4}{2}=\frac{y-3}{5}=\frac{z+1}{-2}$ and passes through the point (2, -4, 2).

   Solution

   The line, $\frac{x-4}{2}=\frac{y-3}{5}=\frac{z+1}{-2}$ may be regarded as the line of intersection of the planes;

$$\frac{x-4}{2}=\frac{y-3}{5}\ i.e.\ 5x-2y=14$$

And,

$$\frac{y-3}{5}=\frac{z+1}{-2}\ i.e. 2y+5z=1$$

But $5x-2y-14+k(2y+5z-1)$, passes through the point (2, -4, 2) if;

$$5(2)-2(-4)-14+k(2(-4)+5(2)-1)=0 \Rightarrow k=-4$$

Hence the equation of the required plane is;

$$5x-2y-14-4(2y+5z-1)=0$$

$$\therefore x-2y-4z=2$$

2. The two planes $2x-y-z=3\ and\ 2x+y-2z=1$ meet in a line *l*. Find;

(a) The equation of the plane which contains the line, $\frac{1}{2}x=y+1=1-z$ parallel to *l*; and;

(b) The equation of the plane containing *l* and passes through the origin.

Solution

(a) The DRs of *l* are [3: 2:4].

The line $\frac{1}{2}x=y+1=1-z$ is the line of intersection of the planes; $x-2y-2=0\ and\ y+z=0$.

The equation of any plane through this line is of the form;

$$x-2y-2+k(y+z)=0$$

i.e. $x+(k-2)y+kz=2$

This line is parallel to *l* if its normal is perpendicular to *l* i.e. if the line with DRs [*l: k-2: k*] is perpendicular to *l*

i.e. $3+2(k-2)+4k=0 \Rightarrow k=\frac{1}{6}$, and the equation of the required plane is;

$$x+\left(\frac{1}{6}-2\right)y+\frac{1}{6}z=2$$

$$\therefore 6x-11y+z=12$$

(b) The plane, $2x-y-z-3+k(2x+y-2z-1)=0$ passes through *l*. It will pass through the origin if *k = -3* i.e. for *x = 0, y = 0* and *z = 0*

Hence the equation of the required plane is;

$$2x-y-z-3-3(2x+y-2z-1)=0$$

$$\Rightarrow 4x+4y-5z=0$$

The perpendicular distance from a point onto a plane

Let the equation of the plane be $lx+my+nz=p; p\geq 0$, and let P_0 be the point (x_0, y_0, z_0). To find the distance of P_0 from the plane, we change the origin to (x_0, y_0, z_0). The equation of the plane becomes;

$l(x+x_0)+m(y+y_0)+n(z+z_0)=p$, that is;

$lx+my+nz=p^1$, where, $p^1=p-(lx_0+my_0+nz_0)$, is the distance of the plane from P_0, the new origin.

It follows that the distance of (x_0, y_0, z_0) from the plane $ax+by+cz+d=0$ is;

$$\pm\frac{ax_0+by_0+cz_0+d}{\sqrt{a^2+b^2+c^2}}$$

Examples

1. Find the perpendicular distance from the point (1, 3, 5) onto the plane x – y +2z +4 = 0.

   The distance is;

$$\frac{1-3+10+4}{\sqrt{1+1+4}}=\frac{12}{\sqrt{6}}=2\sqrt{6}$$

2. Find the foot of the perpendicular from the point (1, 3, 5) onto the plane x – y + 2z +4 = 0.

   Solution

   This perpendicular passes through the point (1, 3, 5) and has DRs [*1: -1: 2*]. Hence its equations are;

$$\frac{x-1}{1}=\frac{y-3}{-1}=\frac{z-5}{2}=t$$

   Thus any point on the line has coordinates (t+1, -t+3, 2t+5) for some t, and the point where the line meets the plane is given by;

   $t+1+t-3+4t+10+4=0\Rightarrow t=2$

   Thus, the required point is; (-1, 5, 1)

3. Find the perpendicular distance of the point (-1, 0, 7) from the line; $\frac{x-1}{1}=\frac{y-2}{-1}=\frac{z-3}{-2}$, the equations of the perpendicular and the coordinates of the foot.

   Solution

   Putting;

$\frac{x-1}{1}=\frac{y-2}{-1}=\frac{z-3}{-2}=t$, we see that the points on the line have coordinates given by, $(t+1,-t+2,-2t+3)$. The line joining the point (-1, 0, 7) to a point on the given line has DRs, [*t+2: -t+2: -2t-4*], and is perpendicular to the given line if;

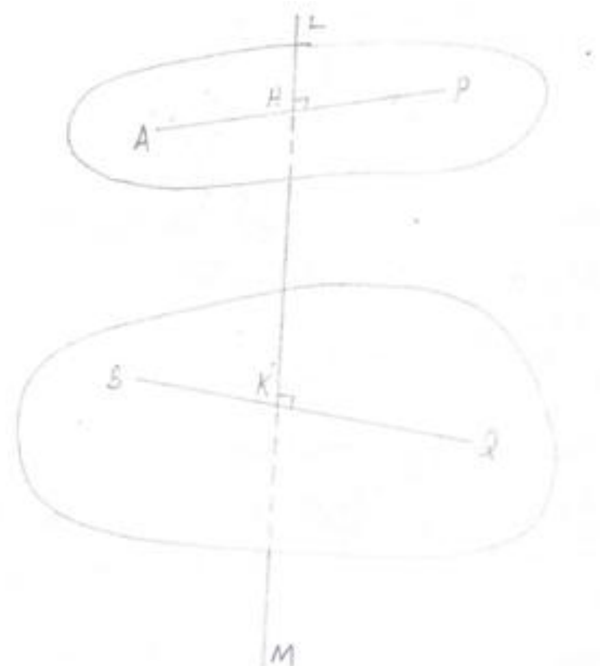
$$1(t+2)-1(-t+2)-2(-2t-4)\Rightarrow t=-\frac{4}{3}$$

Therefore, the coordinates of the foot of the perpendicular are $\left(\frac{-1}{3},\frac{10}{3},\frac{17}{3}\right)$ and the length of the perpendicular is;

$$\sqrt{\frac{120}{9}}=\frac{2}{3}\sqrt{30}$$

The shortest distance between two skew lines

Two straight lines are said to be skew if they are not coplanar, i.e. if they neither intersect nor parallel.



The figure above shows two skew lines AP and BQ which pass through the points P(x_1, y_1, z_1) and Q(x_2, y_2, z_2). The shortest distance between them is the intersect HK that they make on their common perpendicular LM.

Since LM is perpendicular to AP and BQ, HK is the projection of PQ on LM hence if [*l, m, n*] are the DCs of LM,

$$HK=\left|l(x_2-x_1)+m(y_2-y_1)+n(z_2-z_1)\right|$$

Example

1. Show that the length of the common perpendicular to the lines whose equations referred to the rectangular axes are, $\frac{x-5}{1}=\frac{y}{2}=\frac{z+1}{-1}$ *and* $\frac{x-2}{1}=\frac{y-4}{-1}=\frac{z}{1}$ is $\sqrt{14}$.

Solution

The line;

$$\frac{x-5}{1}=\frac{y}{2}=\frac{z+1}{-1}$$ passes through P(5, 0, -1) and the line;

$$\frac{x-2}{1}=\frac{y-4}{-1}=\frac{z}{1}$$, passes through Q(2, 4, 0).

The common perpendicular to the lines has DRs [*1: -2: -3*], which is obtained by crossing the vectors parallel to the two lines.

Hence, the DCs of the common perpendicular are;

$$[l,m,n]=\left[\frac{1}{\sqrt{1+4+9}},\frac{-2}{\sqrt{1+4+9}},\frac{-3}{\sqrt{1+4+9}}\right]=\left[\frac{1}{\sqrt{14}},\frac{-2}{\sqrt{14}},\frac{-3}{\sqrt{14}}\right]$$

The projection of PQ on this line is;

$$3\left(\frac{1}{\sqrt{14}}\right)+(-4)\left(\frac{-2}{\sqrt{14}}\right)+(-1)\left(\frac{-3}{\sqrt{14}}\right)=\frac{14}{\sqrt{14}}=\sqrt{14}$$

2. Find the length and the equations of the shortest distance between the lines, $x=y-1=4-z$ *and* $x-2y+9=0, x+z-10=0$

Solution

Writing the lines in standard form, gives;

$$\frac{x}{1}=\frac{y-1}{1}=\frac{z-4}{-1}=d\text{...}(1)$$

$$\frac{x}{2}=\frac{y-\frac{9}{2}}{1}=\frac{z-10}{-2}=t\text{...}(2)$$

The coordinates of any point P on (1) may be written as $(d, 1+d, 4-d)$ and similarly Q, any point on (2) is written as, (*2t*, *t + 9/2*, *10 -2t*).

The DRs of PQ are $\left[d-2t: d-t-\frac{7}{2}: 2t-d-6\right]$

The DRs of the common perpendicular to (1) and (2) are [-1: 0: -1]

Hence PQ is the common perpendicular if;

$$\frac{d-2t}{-1}=\frac{d-t-7/2}{0}=\frac{2t-d-6}{-1}\Rightarrow d=10\, and\, t=\frac{13}{2}$$

With these values P is the point, $(10, 11, -6)$ and Q is (*13*, *11*, -*3*), and;

$PQ = \sqrt{3^2 + 0^2 + 3^2} = \sqrt{18} = 3\sqrt{2}$

The equations of PQ are;

$$\frac{x-10}{-1} = \frac{y-11}{0} = \frac{z+6}{-1}$$

The sphere

If the sphere has centre $P_0(x_0, y_0, z_0)$ and radius a, then any point on the sphere is distant a from P_0. If $\mathbf{r}_0$ is the position vector of P_0, it follows that the equation of the sphere is;

$$|\boldsymbol{r} - \boldsymbol{r}_0| = a$$

Which becomes, in Cartesian form;

$$(x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2 = a^2$$

In particular, the equation of a sphere with centre, the origin and radius a is;

$$x^2 + y^2 + z^2 = a^2$$

The general equation,

$x^2 + y^2 + z^2 2ux + 2vy + 2wz + k = 0$, can be written as;

$$(x+u)^2 + (y+v)^2 + (z+w)^2 = u^2 + v^2 + w^2 - k$$

And also represents a sphere with centre ($-u$, $-v$, $-w$) and radius, $\sqrt{u^2 + v^2 + w^2 - k}$

If the position vectors of P_0, P_1 are $\mathbf{r}_0$, and $\mathbf{r}_1$ respectively, then $\mathbf{r}$ is the position vector of any point on the tangent plane it follows that $\mathbf{r} - \mathbf{r}_1$ is perpendicular to $\mathbf{r}_1 - \mathbf{r}_0$.

Hence the equation of the tangent plane at (x_1, y_1, z_1) on the sphere with centre (x_0, y_0, z_0) is;

$$(x-x_1)(x_1-x_0) + (y-y_0)(y_1-y_0) + (z-z_0)(z_1-z_0)$$

Examples

Find the centre and radius of the sphere whose equation is, $x^2 + y^2 + z^2 - 2x - 4y - 6z - 2 = 0$. Show that the intersection of this sphere and the plane $x + 2y + 2x - 20 = 0$ is a circle whose centre is the point (2, 4, 5), and find the radius of this circle.

The centre of the sphere is (1, 2, 3) and its radius is $\sqrt{1+4+9+2} = 4$

The normal through the centre, C to the given plane has equations,

$\dfrac{x-1}{1} = \dfrac{y-2}{2} = \dfrac{z-3}{2} = t$, and meets the plane at A where,

$$(t+1) + 2(2t+2) + 2(2t+3) - 20 = 0 \Rightarrow t = 1$$

Hence A, the centre of the circle of the section is the point (*2, 4, 5*),

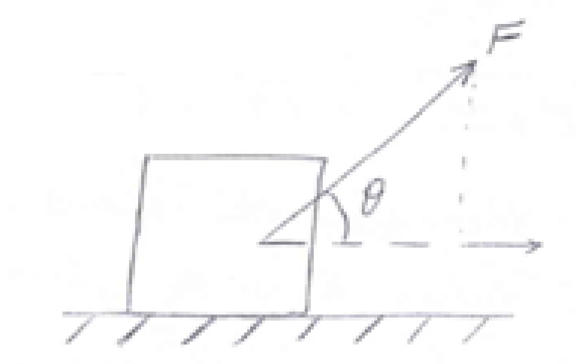
$$AC = \sqrt{1^2 + 2^2 + 2^2} = 3$$

By Pythagoras theorem, the radius r of the circle and the radius R of the sphere are connected by the equation, $R^2 = AC^2 + r^2$

Therefore, $16 = 9 + r^2 \Rightarrow r = \sqrt{7}$

Physical applications of vectors

1. Work done by a force



$W = \boldsymbol{F}.\boldsymbol{d}$

$= \boldsymbol{F}cos\theta.\boldsymbol{d} = |\boldsymbol{F}||\boldsymbol{d}|cos\theta$

Example

If a body is displaced point A(40, 10, 30) to B(50, 20, 50) by a force $\boldsymbol{F} = 10\boldsymbol{i} + 10\boldsymbol{j} - 10\boldsymbol{k}$, calculate the work done.

The displacement AB is given as;

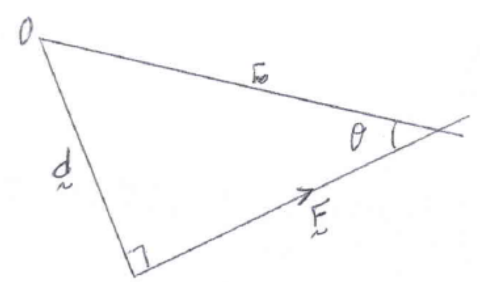
$\overrightarrow{AB} = (50\boldsymbol{i} + 20\boldsymbol{j} + 50\boldsymbol{k}) - (40\boldsymbol{i} + 10\boldsymbol{j} + 30\boldsymbol{k}) = 10\boldsymbol{i} + 10\boldsymbol{j} + 20\boldsymbol{k}$

Work done is then given by;

$W = \boldsymbol{F}.\boldsymbol{d} = (10\boldsymbol{i} + 10\boldsymbol{j} - 10\boldsymbol{k}).(10\boldsymbol{i} + 10\boldsymbol{j} + 20\boldsymbol{k})$

$= 100 + 100 - 200 = 0$

2. Moment



The moment about O is given by;

$M_O = \mathbf{F} \times \mathbf{d}$

But $d = rsin\theta$.

Therefore, $M_O = |\mathbf{F}||\mathbf{r}|\sin\theta = |\mathbf{F} \times \mathbf{r}|$

The value of $|\mathbf{F}\times\mathbf{r}|$ is called the torque or moment vector.

Example

If a force $3\mathbf{i} - \mathbf{j} + 2\mathbf{k}$, acts on a line through a point A(0, -1, 4), find the moment about Q(3, 0, 2).

$\overrightarrow{QA} = -3\boldsymbol{i} - \boldsymbol{j} + 2\boldsymbol{k}$

$$M_O = |\mathbf{F} \times \mathbf{r}| = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & -1 & 2 \\ -3 & -1 & 2 \end{vmatrix}$$

END