MAKERERE UNIVERSITY

COLLEGE OF ENGINEERING, DESIGN, ART AND TECHNOLOGY

SCHOOL OF ENGINEERING

DEPARTMENT OF ELECTRICAL AND COMPUTER ENGINEERING

EMT 1101: ENGINEERING MATHEMATICS I LECTURE NOTES 2015/2016

CHAPTER THREE: INTEGRAL CALCULUS

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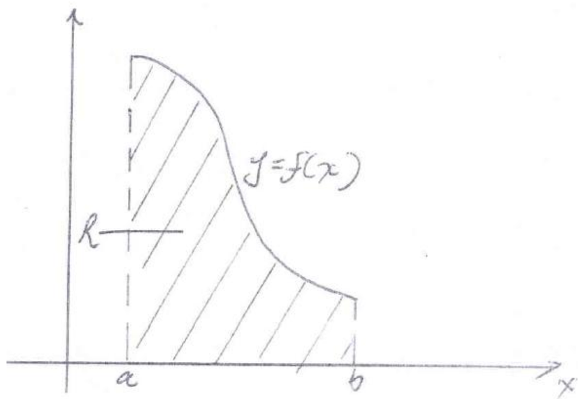
INTEGRAL CALCULUS

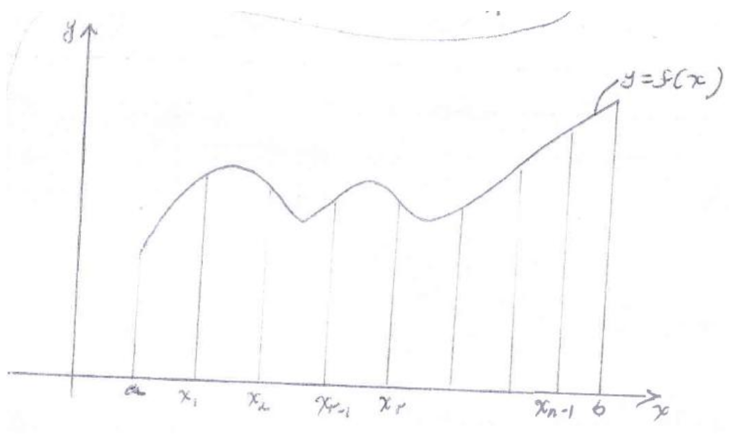
Integral calculus centers around the concept of the integral. The definition of the integral is motivated by the problem of defining and calculating the area of the region lying between the graph of a positive – valued function f and the x – axis over a closed interval$[a, b]$. The area of the region R of figure below is given by the integral of f from a to b, denoted by the symbol; $\int_a^b f(x)dx$. But the importance of the integral, like that of the derivative, is due to its application in many problems that may appear unrelated to its original motivation.



Areas under graphs

Let f(x) be continuous in the interval $[a, b]$. To calculate the area under the graph of y = f(x) over the interval$[a, b]$, we can divide the interval into n equal sub –intervals by choosing the points; $x_0, x_1, x_3, \ldots\ldots\ldots\ldots x_n$ where $a = x_0 < x_1 < x_3 \ldots\ldots\ldots\ldots < x_{n-1} < x_n = b$ with the length of the rth subinterval $[x_{r-1},\ x_r]$ given by:

$$\Delta x = x_r - x_{r-1} = \frac{b-a}{n}$$

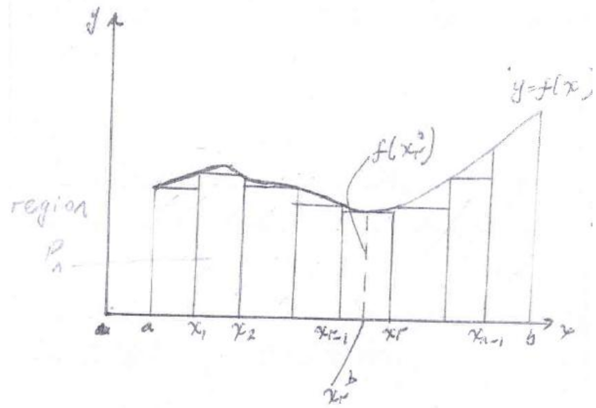


For each r ($1 \leq r \leq n$), the point x_r will be:

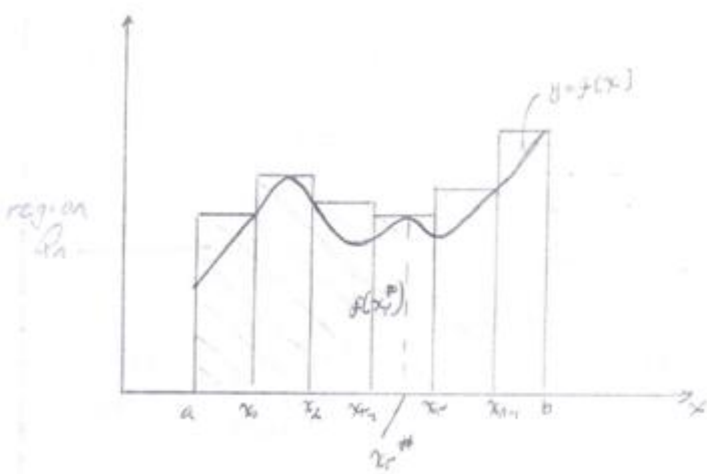
$$x_r = x_0 + r\Delta x = a + r\frac{(b-a)}{n}$$

$$x_r = a + \frac{r}{n}(b - a)$$

Now let $x_r{}^b$ be a point of $[x_{r-1}, x_r]$ at which f attains a minimum value such that $f(x_r{}^b)$ is the minimum value of f(x) on the r^th subinterval. The rectangle with base $[x_{r-1}, x_r]$ and height $f(x_r{}^b)$ then lies within the region R. The union of these rectangles for $r = 1, 2, \ldots\ldots\ldots, n,$ is the inscribed rectangular polygon P_n associated with our partition of $[a, b]$ into n equal subintervals.



In the figure above, the area of the inscribed polygon P_n is an underestimate of the area, $A = a(R)$



In the figure above the inscribed polygon gives an overestimate of the area $A = a(R)$

Its area is the sum of the areas of the n rectangles with base length Δx and heights $f(x_1{}^b), f(x_2{}^b), \ldots\ldots\ldots\ldots f(x_n{}^b)$, so:

$$a(P_n) = \sum_{r=1}^{n} f(x_r{}^b)\Delta x$$

If we let $x_r{}^*$ be the maximum point of $[x_{r-1}, x_r]$ such that $f(x_r{}^*)$ is the maximum value of f(x) on the r^th subinterval, its area is;

$$a(Q_n) = \sum_{r=1}^{n} f\left(x_r^*\right)\Delta x$$

Since region R contains P_n but lies within Q_n, we conclude that its area A satisfies the inequalities;

$$\sum_{r=1}^{n} f\left(x_r^b\right)\Delta x \le A \le \sum_{r=1}^{n} f\left(x_r^*\right)\Delta x$$

Now, if n is very large, and hence Δx is very small, then the areas $a(P_n)$ and $a(Q_n)$ will be very close to $A = a(R)$. The assumption that f is continuous is enough to ensure that this is so. This implies that:

$$A = \lim_{n\to\infty} \sum_{r=1}^{n} f\left(x_r^b\right)\Delta x = \lim_{n\to\infty} \sum_{r=1}^{n} f\left(x_r^*\right)\Delta x$$

Riemann sums and the integral

From the previous discussion, the area A under the graph from $x = a$ to $x = b$ of the continuous positive valued function f satisfies the inequalities;

$$\sum_{r=1}^{n} f\left(x_r^b\right)\Delta x \le A \le \sum_{r=1}^{n} f\left(x_r^*\right)\Delta x$$

Where $f(x_r^*)$ and $f\left(x_n^b\right)$ are the maximum and minimum values of the r^th subinterval $[x_{r-1}, x_r]$ of a partition of $[a, b]$ into n equal subintervals of length Δx.

The two approximating sums are both of the form;

$\sum_{r=1}^{n} f\left(x_r^*\right)\Delta x$ where x_r^* denotes an arbitrary point of the r^th subinterval $[x_{r-1}, x_r]$.

Sums of the form above appear as approximations in a wide range of applications and also constitute the basis for the definition of the integral.

Definition: Riemann Sum

Let f be a function defined on the interval $[a, b]$, if P is a partition of $[a, b]$ and S is selection for P, then the Riemann sum for f determined by P and S is; $R = \sum_{r=1}^{n} f\left(x_r^*\right)\Delta x_r$

Example

Use Riemann sums to estimate the area under the graph of $f(x) = 80 - 3x^2$ for $x = 1$ to $x = 5$.

$$R = \sum_{r=1}^{n} f\left(x_r^*\right)\Delta x_r$$

$$\Delta x_r = \frac{b-a}{n} = \frac{5-1}{n} = \frac{4}{n}$$

$$x_r = a + r(\Delta x) = 1 + \frac{4r}{n}$$

$$\therefore R=\sum_{r=1}^{n}\left[80-3\left(1+\frac{4r}{n}\right)^{2}\right].\frac{4}{n}=\sum_{r=1}^{n}\frac{320}{n}-\frac{12}{n}\left(1+\frac{8r}{n}+\frac{16r^{2}}{n^{2}}\right)$$

$$=\sum_{r=1}^{n}\left(\frac{320}{n}-\frac{12}{n}-\frac{96r}{n^{2}}-\frac{192r^{2}}{n^{3}}\right)$$

We now take the limit as $n\to\infty$, therefore;

$$A=\lim_{n\to\infty}\sum_{r=1}^{n}\left[\frac{308}{n}-\frac{96r}{n^{2}}-\frac{192r^{2}}{n^{3}}\right]$$

We need to recall that:

$$\sum_{r=1}^{n}1=n$$

$$\sum_{r=1}^{n}r=\frac{1}{2}n(n+1) \text{ and;}$$

$$\sum_{r=1}^{n}r^{2}=\frac{1}{6}n(n+1)(2n+1)$$

$$\therefore A=\lim_{n\to\infty}\left[\frac{308}{n}\sum_{r=1}^{n}1-\frac{96}{n^{2}}\sum_{r=1}^{n}r-\frac{192}{n^{3}}\sum_{r=1}^{n}r^{2}\right]=\lim_{n\to\infty}\left[308-48-\frac{48}{n}-64-\frac{96}{n}-\frac{32}{n^{2}}\right]$$

$$=308-48-64=196$$

Definition: The definite integral

The definite integral of the function *f* from *a* to *b* is number; $I=\lim_{\Delta x\to 0}\sum_{r=1}^{n}f(x_r^{*})\Delta x_r$, provided that this limit exists, in which case we say that *f* is integrable on $[a,b]$.

The customary notation for the integral is $I=\int_a^b f(x)dx=\lim_{\Delta x\to 0}\sum_{r=1}^{n}f(x_r^{*})\Delta x_r$

The above definition applies only if $a<b$, but it is convenient to include the cases $a=b$ and $a>b$. Then;

$$I=\int_a^b f(x)dx=0 \text{ and;}$$

$I=\int_a^b f(x)dx=-\int_b^a f(x)dx$, provided the right –hand integral exists. Thus interchanging the limits on integration reverses the sign of the integral.

Note:

1. Existence of the integral: If the function f is continuous on $[a, b]$, then f is integrable on $[a, b]$.
2. The integral as a limit of a sequence: The function f is integrable on $[a, b]$ with integral I if and only if; $\lim_{n\to\infty} R_n = I$, for every sequence $\{R_n\}_1^\infty$ of Riemann sums associated with sequence partitions $\{P_n\}_1^\infty$ of $[a, b]$ such that $\Delta x \to 0 \ as \ n \to \infty$

Evaluation of integrals

If G is an antiderivative of the continuous function f on the interval $[a, b]$, then;

$$\int_a^b f(x)dx = G(b) - G(a) = [G(x)]_a^b$$

e.g. if $f(x) = x^n$ with $n \neq -1$, then an antiderivative of f is; $G(x) = \dfrac{x^{n+1}}{n+1}$, so;

$$\int_a^b x^n dx = \left[\frac{x^{n+1}}{n+1}\right]_a^b = \frac{b^{n+1} - a^{n+1}}{n+1}$$

Basic properties of integrals

We assume that f is integrable on $[a, b]$, then;

1. Integral of a constant; $\int_a^b C dx = C(b-a)$
2. Constant multiple property; $\int_a^b Cf(x)dx = C\int_a^b f(x)dx$
3. Interval union property
   If $a < c < b$, then;
   $$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$$
4. Comparison property;
   If $m \leq f(x) \leq M$ for all x in $[a, b]$, then;
   $$m(b-a) \leq \int_a^b f(x)dx \leq M(b-a)$$

Average value of a function

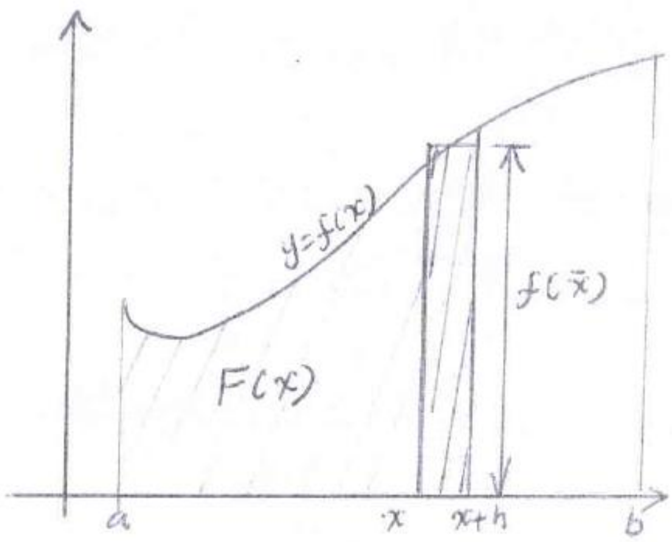
Suppose that the function f is integrable on $[a, b]$, then the average value $\bar{y}$ of $y = f(x)$ on $[a, b]$ is given by;

$$\bar{y} = \frac{1}{b-a}\int_a^b f(x)dx$$

Average value theorem

If f is continuous on$[a, b]$, then $f(\bar{x}) = \frac{1}{b-a}\int_a^b f(x)dx$, for some point $\bar{x}$ of $[a, b]$.

The fundamental theorem



If f is continuous on an interval I and has an antiderivative on I; in the case$f(x) > 0$, we let f(x) denote the area under the graph of f from a fixed point a in I to x, a point of I with $x > 0$. We define the function F by; $F(x) = \int_a^x f(t)dt$.

Theorem

Let f be a continuous function defined on $[a , b]$

Part1: If the function F is defined on $[a , b]$ by $F(x) = \int_a^x f(t)dt$, then F is an antiderivative of f i.e $F^1(x) = f(x)$ for x in$(a,\ b)$.

Part 2: If G is any antiderivative of f on$[a , b]$, then;

$$\int_a^b f(x)dx = [G(x)]_a^b = G(b) - G(a)$$

Proof of part 1

By the definition of the derivative;

$$F^1(x) = \lim_{h\to 0}\frac{F(x+h) - F(x)}{h}$$

$$= \lim_{h \to 0} \frac{1}{h} \left[\int_a^{x+h} f(t)dt - \int_a^{x} f(t)dt \right]$$

But $\int_a^{x+h} f(t)dt = \int_a^{x} f(t)dt + \int_x^{x+h} f(t)dt$

$$\therefore F^1(x) = \lim_{h \to 0} \frac{1}{h} \int_x^{x+h} f(t)dt$$

Using the average value theorem, $\frac{1}{h} \int_x^{x+h} f(t)dt = f\left(\bar{t}\right)$, for some number $\bar{t}$ in $[x, x+h]$, we note that $\bar{t} \to x \ as \ h \to o$.

Thus since *f* is continuous;

$$F^1(x) = \lim_{h \to 0} \frac{1}{h} \int_x^{x+h} f(t)dt = \lim_{h \to 0} f\left(\bar{t}\right) = \lim_{\bar{t} \to x} f\left(\bar{t}\right) = f(x)$$

The linearity property of the integral

$$\int_a^b [Af(x) + Bg(x)]dx = A\int_a^b f(x)dx + B\int_a^b g(x)dx$$

One consequence of linearity is the fact that integration preserves inequalities between functions. i.e if *f* and *g* are continuous functions with $f(x) \leq g(x)$ for all x in $[a, b]$, then;

$$\int_a^b f(x)dx \leq \int_a^b g(x)dx$$

Also;

$$\left| \int_a^b f(x)dx \right| \leq \int_a^b |f(x)|dx$$

The computation fundamental part of the fundamental theorem of calculus can be written in the form:

$$\int_a^b f(x)dx = \left[D^{-1} f(x)\right]_a^b$$

$$\int f(x)dx = D^{-1} f(x) = F(x) + C \text{, if } F^1(x) = f(x)$$

The expression, $\int f(x)dx$, with no limits on the integral symbol, is called the ***indefinite integral*** of the function *f*, in contrast with the definite integral, which has upper and lower limits. Thus the indefinite integral of *f* is simply the most general antiderivative of *f*, and indefinite integration is simply antedefifferetion.

Example

$$\int 3x^2 - 4\,dx = x^3 - 4x + C$$

A common sort of indefinite integral takes the form; $\int f(g(x))g^1(x)dx$

If we write $u = g(x)$, then $du = g^1(x)dx$, upon substitution we get;

$$\int f(g(x))g^1(x)dx = \int f(u)du$$

The above formula is the basis for the powerful technique of indefinite *integration by substitution*. It may be used whenever the integrand function is recognizable in the form $f(g(x))g^1(x)$.

Examples

1. $\int (x^3 + 2x + 6)^4 (3x^2 + 2)dx$

   Let $u = x^3 + 2x + 6$

   $du = (3x^2 + 2)dx$

   $\therefore \int u^4 du = \frac{1}{5}u^5 + C = \frac{1}{5}(x^3 + 2x + 6)^5 + C$

2. $\int \frac{\sin x \cos x}{\sqrt{1 + \sin^2 x}} dx$

   $u = \sin^2 x + 1 \Rightarrow du = 2\sin x \cos x dx$

   $\Rightarrow \int \frac{1}{2\sqrt{u}} du = \frac{1}{2}\int u^{-1/2} du$

   $= u^{1/2} + C = \sqrt{1 + \sin^2 x} + C$

3. $\int \sin^3 x \cos x dx$

   Let $u = \sin x \Rightarrow du = \cos x dx$

   $\int u^3 du = \frac{1}{4}u^4 + C = \frac{1}{4}\sin^4 x + C$

4. $\int x(x+1)^{18} dx$ $\qquad \left(Ans: \frac{(x+1)^{19}}{19} - \frac{(x+1)^{18}}{18} + C \right)$

Sometimes it is more convenient to determine the limits of integration with respect to the new variable.

5. $\int_0^3 x^2 \sqrt{x^3 + 9} dx$

   Let $u = x^3 + 9 \Rightarrow du = 3x^2 dx$

x	u
0	9
3	36

$$\int_0^3 x^2\sqrt{x^3+9}\,dx = \frac{1}{3}\int_9^{36} u^{1/2}du = \frac{1}{3}\left[\frac{2}{3}u^{3/2}\right]_9^{36} = 42$$

Computing areas by integration

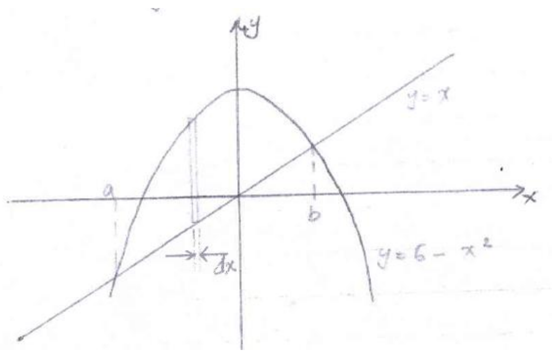
The area between two curves

Let *f* and *g* be continuous with $f(x) \geq g(x)$ for *x* in$[a, b]$. Then the area A of the region bounded by the two curves *y = f(x)* and *y = g(x)* and the vertical lines *x = a* and *x = b* is:

$$A = \int_a^b [f(x) - g(x)]dx$$

Examples

1. Find the area A of the region R that is bounded by the line *y = x* and the parabola $y = 6 - x^2$.



The limits *a* and *b* will be the x – coordinates of the two points of intersection of the line and the parabola.

$x = 6 - x^2$

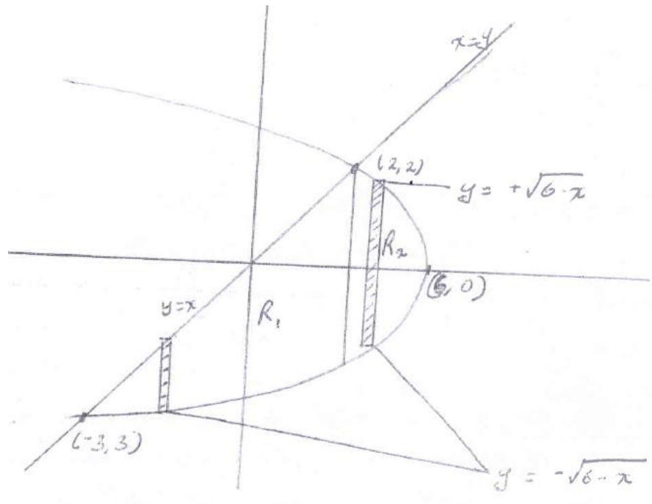
$x^2 + x - 6 = 0 \Rightarrow x = -3, 2$

Thus; $A = \int_{-3}^{2}(6 - x^2 - x)dx = \left[6x - \frac{x^3}{3} - \frac{x^2}{2}\right]_{-3}^{2} = \frac{125}{6}$

Note: Sometimes it is necessary to subdivide a region before finding the area.

Example

Find the area A of the region R bounded by the line y = x and the curve $y^2 = 6 - x$



Points of intersection

$x^2 = 6 - x \Rightarrow (x+3)(x-2) = 0$

$x = -3 \ and \ x = 2$

Points of intersection are: $(-3,-3) \, and \, (2,2)$

The lower boundary of R is given by $y = -\sqrt{6-x} \ on[-3,6]$

The upper boundary of R is given by $y = +\sqrt{6-x} \ on[2,6]$

We must therefore subdivide R into the two regions R_1 and R_2 as indicated.

$$A = \int_{-3}^{2} \left[x - \left(-\sqrt{6-x}\right)\right]dx + \int_{2}^{6} \left[\sqrt{6-x} - \left(-\sqrt{6-x}\right)\right]dx$$

$$= \int_{-3}^{2} \left(x + \sqrt{6-x}\right)dx + 2\int_{2}^{6} \sqrt{6-x}\,dx = \frac{125}{6}$$

The region of the previous example appears simpler if it is considered to be bounded by graphs of functions of *y* rather than functions of *x*.

Let f and g be continuous with $f(y) \le g(y)$ for y in $[c,d]$. Then the area A of the region bounded by the curves *x = f(y)* and *x = g(y)* and the horizontal lines *y = c* and *y = d* is:

$$A = \int_{c}^{d} [f(y) - g(y)]dy$$

Example

By integrating with respect to *y*, find the area of the region R bounded by the line $y = x$ and the curve $y^2 = 6 - x$.

The intersection points are;

$$6 - y^2 = y \Rightarrow y = -3, 2$$

$$A = \int_{-3}^{2} \left[6 - y^2 - y\right] dy = \left[6y - \frac{y^3}{3} - \frac{y^2}{2}\right]_{-3}^{2} = \frac{125}{6}$$

Volumes by the method of cross sections

The method of cross sections is a way of computing the volume of a solid that is conveniently described in terms of its cross sections in planes perpendicular to a fixed reference line L. This reference line will ordinarily be either the x – axis or the y – axis.

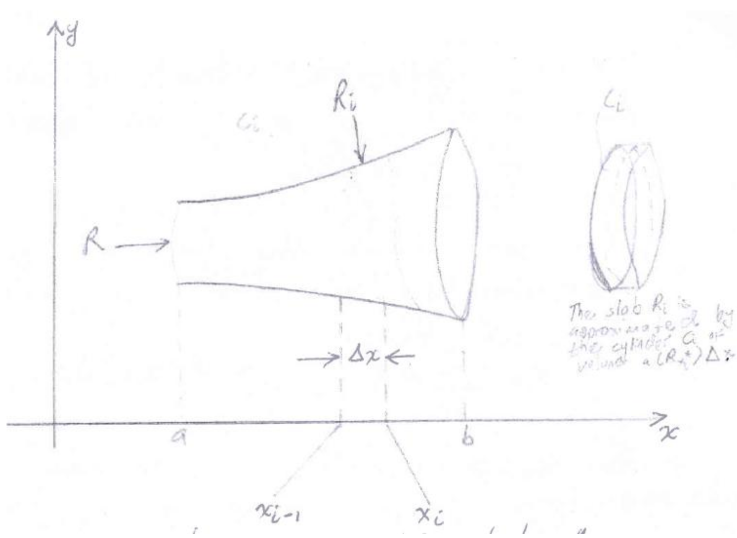
We assume that every reasonably nice solid region R has a volume v(R), a nonnegative real number with the following properties:

(a) If the solid region R is contained in the solid region S, then $v(R) \le v(S)$

(b) If R and S are non overlapping solid regions, then the volume of either union is $v(RUS) = v(R) + v(S)$.

(c) If the solid regions R and S are congruent (have the same size and shape), then $v(R) = v(S)$

(d) The volume of any cylinder, circular or not, is the product of its height and its cross section area.



Now for each *x* in $[a, b]$, let *A(x)* denote the cross section R_x ;

$$A(x) = a(R_x)$$

Assume that this cross section area function *A* is continuous and hence integrable.

To set up an integral formula for V = v(R), we begin with a regular partition of $[a, b]$ into n equal subintervals each with length; $\Delta x = \dfrac{b-a}{n}$

Let R_i denote the slab (slice) of the solid R that lies opposite the i[th] subinterval $\left[x_{i-1}, x_i\right]$.

The volume of this i[th] slice of R is denoted by $\Delta V_i = v(R_i)$ so that;

$$V = \sum_{i=1}^{n} \Delta V_i$$

To approximate ΔV_i, we select an arbitrary point $x_r{}^*$ in $[x_{i-1}, x_i]$ and consider the cylinder C_i with height Δx and whose base is the cross section area $R_{x_i{}^*}$ of R at $x_r{}^*$.

This suggests that if Δx is small then $V(C_i)$ is a good approximation to $\Delta V_i = V(R_i)$

Hence, $\Delta V_i \approx V(i) = a\left(R_{x_i{}^*}\right) = A(x_i{}^*)$

Then we add the volumes of these approximating cylinders for $i = 1, 2, 3, \ldots\ldots\ldots, n$. We find that;

$$V = \sum_{i=1}^{n} \Delta V_i \approx \sum_{i=1}^{n} A(x_i^*)\Delta x$$

The approximating sum on the right is a Riemann sum that approaches $\int_a^b A(x)dx$ as $n \to \infty$. But this sum should also approach the actual volume as $n \to \infty$.

Definition: Volume by cross sections

If the solid R lies opposite the interval $\left[a, b\right]$ on the x – axis and has continuous cross – sectional area function $A(x)$ then its volume, $V = v(R)$ is;

$$V = \int_a^b A(x)dx$$

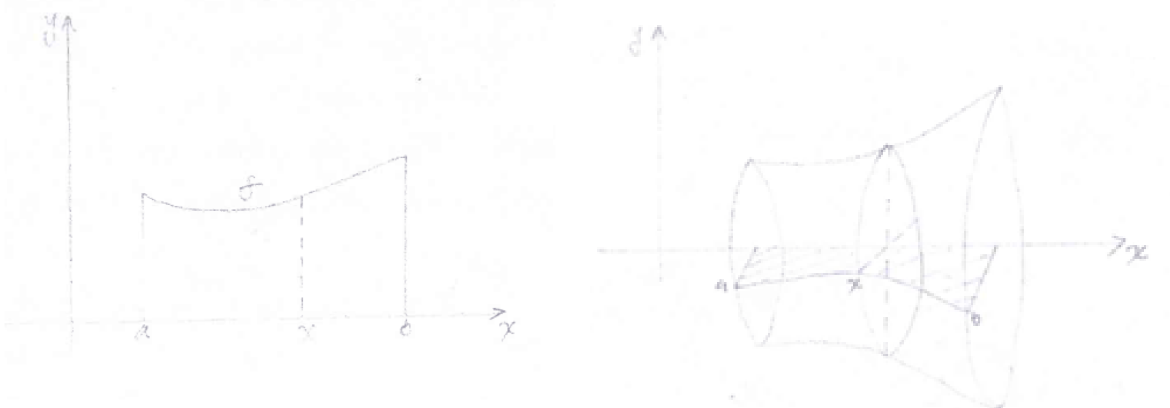
This formula is sometimes called *Cavalieri's principle*.

Note

In case the solid R lies opposite the interval $\left[c, d\right]$ on the y – axis, we denote $A(y)$ the area if its cross section R_y in the same plane that is perpendicular to the y – axis at the point y of $\left[c, d\right]$ leading to the volume formula:

$$V = \int_c^d A(y)dy$$

Solids of revolution



An important special case of Cavalieri's principle gives the volume of a solid of revolution e.g. let the solid R be obtained by revolving about the x axis, as shown in the figure above, the region under the graph $y = f(x)$ over the interval $[a, b]$ where $f(x) \geq 0$.

Since the solid R is obtained by revolution, each cross section of R at x is a circular disc of radius $f(x)$. The cross section area function is then; $A(x) = \pi\left[f(x)\right]^2$, so;

$V = \int_a^b \pi y^2 dx = \int_a^b \pi\left[f(x)\right]^2 dx$, for a solid of revolution around the x – axis.

By a similar argument, if the region bounded by the curve $x = g(y)$, the y – axis and the horizontal lines $y = c$ and $y = d$ is rotated about the y – axis, then the volume of the resulting solid is given by;

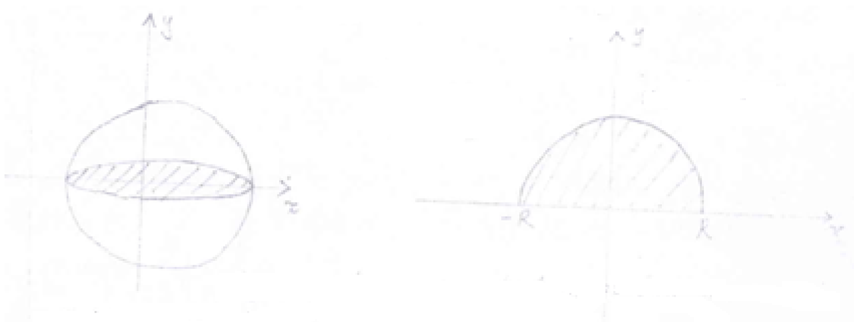
$$V = \int_c^d \pi x^2 dy = \int_c^d \pi\left[f(y)\right]^2 dy$$

Examples

Use the method of cross sections to verify the familiar formula $V = \frac{4}{3}\pi R^3$ for the volume of a sphere of radius R.

Solution

We can think of the sphere as the solid of revolution obtained by revolving the semi – circular plane region around the x – axis. This is the region bounded by the semi circle $y^2 + x^2 = R^2$ by the interval $\left[-R, R\right]$; see the proceeding figure:



$$y = f(x) = \sqrt{R^2 - x^2}$$

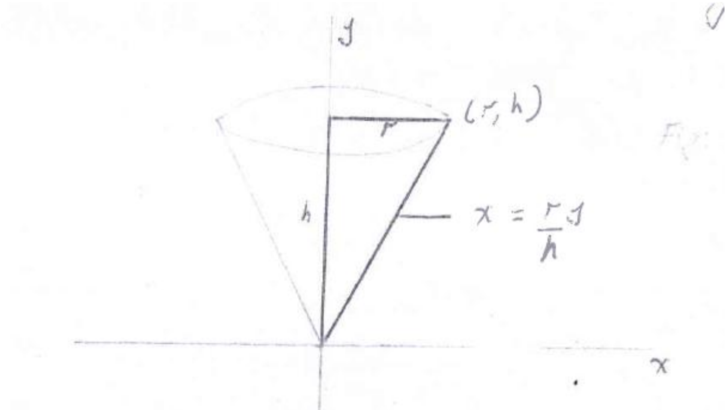
$$V = \int_a^b \pi y^2 dx = \pi \int_{-R}^{R} \left(R^2 - x^2\right) dx = \pi \left[R^2 x - \frac{x^3}{3} \right]_{-R}^{R}$$

$$= \pi \left[\left(R^3 - \frac{R^3}{3} \right) - \left(-R^3 + \frac{R^3}{3} \right) \right] = \frac{4}{3} \pi R^3$$

2. Use the method of cross sections to verify the familiar formula $V = \frac{1}{3}\pi r^2 h$ for the volume of a right circular cone with base radius r and height h.

Solution

We may think of the cone as the solid of revolution obtained by revolving the plane region bounded by the y – axis and the lines $y = h$ and $y = 0$; see figure below:

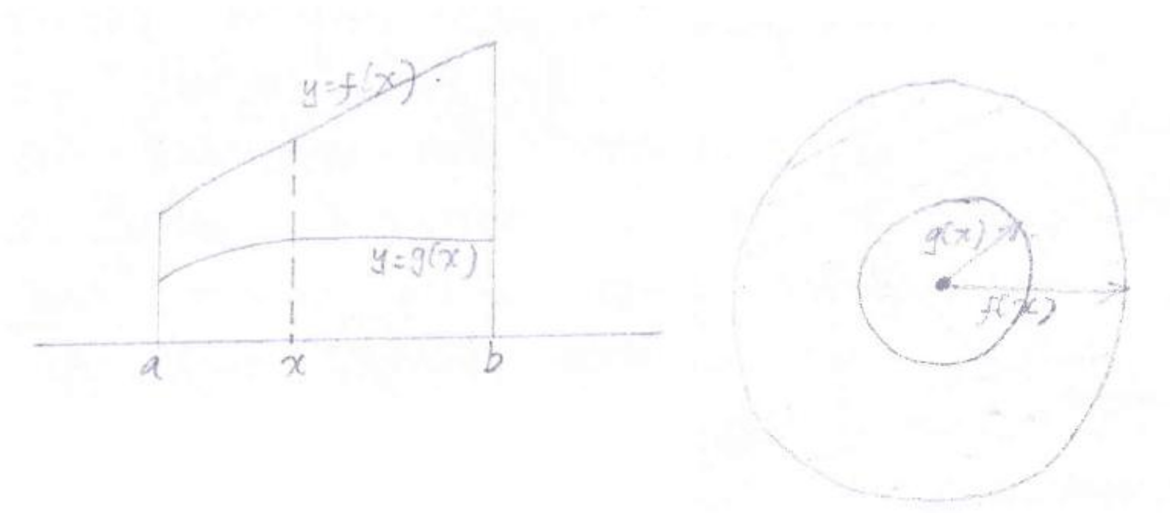


Therefore, $y = \frac{h}{r} x \Rightarrow x = \frac{r}{h} y = g(y)$

$$V = \int_c^d \pi x^2 dy = \int_0^h \pi \left(\frac{ry}{h} \right)^2 dy = \frac{\pi r^2}{h} \int_0^h y^2 dy$$

$$= \frac{\pi r^2}{h^2} \left[\frac{y^3}{3} \right]_0^h = \frac{1}{3} \pi r^2 h$$

Sometimes we need to calculate the volume of a solid generated by revolution of a plane region lying between two given curves. Suppose that $f(x) > g(x) > 0$ for x in the interval $[a,\ b]$ and that the solid R is generated by revolving the region between $y = f(x)$ and $y = g(x)$ about the x – axis.



Then the cross section at x is an annular ring bounded by two circles; see figure above.

The ring has inner radius $r_i = g(x)$ and the outer ring has radius $r_0 = f(x)$, so our formula for cross section area R at x is;

$$A(x) = \pi(r_0)^2 - \pi(r_i)^2 = \pi\left[(f(x))^2 - (g(x))^2\right]$$

Thus the volume V of R is:

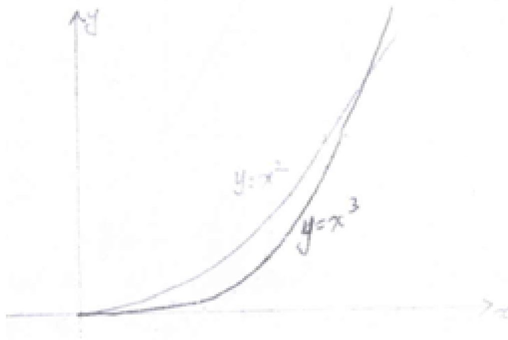
$$\therefore V = \int_a^b \pi\left[[f(x)]^2 - [g(x)]^2\right]dx$$

Similarly, if $f(y) > g(y) > 0$ for y in $[c, d]$, then the volume of the solid obtained by revolving the region between $x = f(y)$ and $x = g(y)$ about the y – axis is:

$$\therefore V = \int_c^d \pi\left[[f(y)]^2 - [g(y)]^2\right]dy$$

Example

Consider the region shown in the figure below, bounded by the curves y = x^2 and y = x^3 over the interval $0 \leq x \leq 1$. If this region is rotated about the x – axis, then the volume with $r_0 = x^2$ and $r_i = x^3$ gives the volume swept out as:



$$V = \int_0^1 \pi\left[\left(x^2\right)^2 - \left(x^3\right)^2\right]dx = \int_0^1 \left(x^4 - x^6\right)dx$$

$$= \pi\left[\frac{x^5}{5} - \frac{x^7}{7}\right]_0^1 = \frac{2\pi}{35}$$

If the same region is rotated about the y – axis, then each cross section perpendicular to the y – axis is an annular ring with an outer radius of $x = y^{1/3}$ and $x = y^{1/2}$, hence;

$$V = \int_0^1 \pi\left[\left(y^{1/3}\right)^2 - \left(y^{1/2}\right)^2\right]dy = \pi\int_0^1 \left[y^{2/3} - y\right]dy$$

$$= \left[\frac{3}{5}y^{5/3} - \frac{y^2}{2}\right]_0^1 = \frac{\pi}{10}$$

Volume by the method of cylindrical shells

The method of cylindrical shells is a second way of computing volumes of solids of revolution. It is a technique of approximating a solid of revolution with a collection of thin cylindrical shells, and it sometimes leads to simpler computations.

Suppose that we want to find the volume V of revolution generated by revolving around the y – axis the region under $y = f(x)$ from $x = a$ to $x = b$. We assume that $0 \leq a < b$ and that $f(x)$ is non negative on $[a, b]$.

To find V, we begin with a regular partition of $[a, b]$ into n equal subintervals each of length $\Delta x = \frac{b-a}{n}$.

Let x_i^* denote the midpoint of the subinterval $[x_{i-1}, x_i]$, and consider the rectangle with base $[x_{i-1}, x_i]$ and height $f(x_i^*)$. When this rectangle is revolved about the y – axis, it sweeps a cylindrical shell, with average radius x_i^*, thickness Δx and height $f(x_i^*)$. This cylindrical shell approximates the solid with volume ΔV_i that is obtained by revolving the region under $y = f(x)$ over $[x_{i-1}, x_i]$, and thus gives:

$$\Delta V_i \approx 2\pi x_i^* f\left(x_i^*\right)\Delta x$$

This sum should approximate V, since the union of the shells should approximate the solid of revolution.

$$V = \sum_{i=1}^{n} \Delta V_i \approx \sum_{i=1}^{n} 2\pi x_i^* f\left(x_i^*\right)\Delta x$$

This approximation to the volume V is a Riemann sum that approaches $\int_a^b 2\pi x f(x)dx$ as $\Delta x \to 0$.

$$V = \int_a^b 2\pi x f(x)dx$$

Example

Find the volume of the solid generated by revolving around the y – axis the region under $y = 3x^2 - x^3$ from $x = 0$ to $x = 3$.

Area of surfaces of Revolution

Suppose that the surface S has area A and is generated by revolving around the x – axis the smooth arc $y = f(x)$, $a \leq x \leq b$; suppose also that f(x) is never negative on $[a, b]$.

The area A of the surface generated by revolving the smooth arc y = f(x), $a \leq x \leq b$, around the x – axis is defined by;

$$A = \int_a^b 2\pi f(x)\sqrt{1+\left[f^1(x)\right]}dx \text{...(1)}$$

If we write y = f(x) and $ds = \sqrt{1+\left(\frac{dy}{dx}\right)^2}dx$, then we get;

$$A = \int 2\pi y ds$$

If our smooth arc being revolved around the x – axis is given by $x = g(y)$, $c \leq y \leq d$, then on approximation based on a regular partition of $[c, d]$ leads to the area formula:

$$A = \int_c^d 2\pi g(y)\sqrt{1+\left(g^1(y)\right)^2}dy$$

Now let us consider the surface generated by revolving our smooth arc generated by revolving our smooth arc around the y – axis. This suggests the abbreviated formula:

$$A = \int 2\pi x ds$$

If the smooth arc is given by $y = f(x)$, $a \leq x \leq b$, then the symbolic substituting of $ds = \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx$ gives;

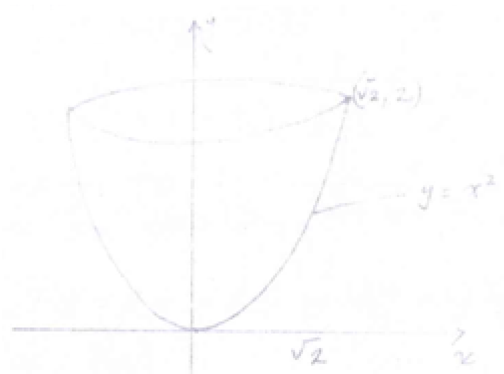
$$A = \int_a^b 2\pi x\sqrt{1+\left[f^1(x)\right]^2}\, dx$$

But if our smooth curve is given by x = g(y), $c \leq y \leq d$, then the symbolic substitution of $ds = \sqrt{1+\left(\frac{dx}{dy}\right)^2}\, dy$ gives;

$$A = \int_c^d 2\pi g(y)\sqrt{1+\left(g^1(y)\right)^2}\, dy$$

Example

Find the area of the paraboloic shown in the figure below, obtained by revolving the parabolic arc $y = x^2$, $0 \leq x \leq \sqrt{2}$, around the y – axis



$$A = \int_a^b 2\pi x\sqrt{1+\left[f^1(x)\right]^2}\, dx = \int_0^{\sqrt{2}} 2\pi x\sqrt{1+(2x)^2}\, dx = \int_0^{\sqrt{2}} 2\pi x\sqrt{1+4x^2}\, dx$$

$$\int_0^{\sqrt{2}} \frac{\pi}{4}\left(1+4x^2\right)^{1/2} 8x dx = \left[\frac{\pi}{6}\left(1+4x^2\right)^{3/2}\right]_0^{\sqrt{2}} = \frac{13\pi}{3}$$

Alternatively, we could describe our parabola as $x = y^{1/2}$, $0 < y < 2$.

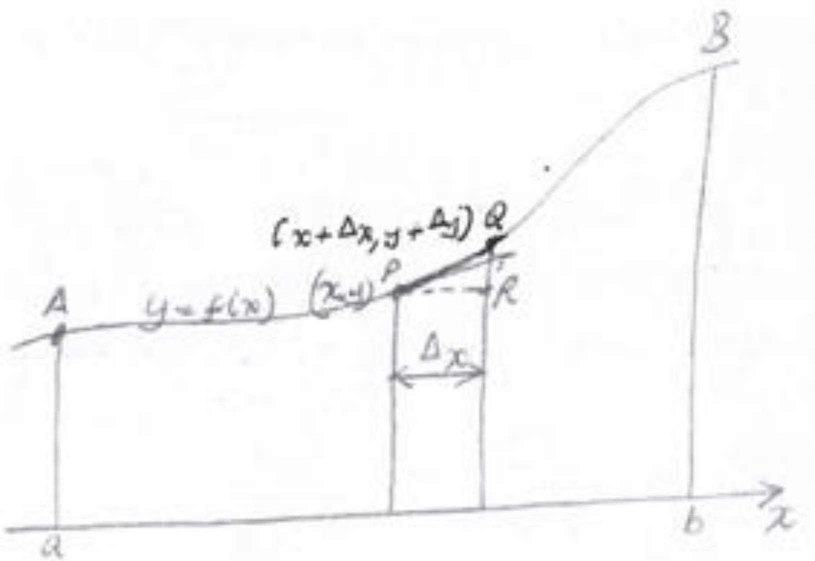
$$\therefore A = \int_0^2 2\pi x\sqrt{1+\left(\frac{dx}{dy}\right)^2}\, dy = \int_0^2 2\pi y^{1/2}\sqrt{1+\left(\frac{1}{2}y^{-1/2}\right)^2}\, dy$$

$$= \int_0^2 2\pi y^{1/2}\sqrt{1+\frac{1}{4y}}dy = \int_0^2 2\pi y^{1/2}.\frac{1}{2y^{1/2}}\sqrt{4y+1}dy$$

$$= \int_0^2 \pi\sqrt{1+4y}\,dy = \left[\frac{\pi}{6}(1+4y)^{3/2}\right] = \frac{13\pi}{3}$$

Arc length

Let $S_{a,b}$ denote the length of the curve $y = f(x)$ from point A where x = a to point B where x = b. The arc AB of the curve may be divided into small pieces by lines parallel to the y – axis, and we denote by $S_{x,\ x+\Delta x}$ the length of the typical piece PQ shown in the figure below.



Taking the length of a piece of tangent PT as an approximation to $S_{x,\ x+\Delta x}$

$TQ = RQ - RT = \Delta y - f^1(x)\Delta x$, so that;

$$\frac{TQ}{\Delta x} = \frac{\Delta y}{\Delta x} - f^1(x) \to 0 \ as\, \Delta x \to 0$$

The chord PQ is also a sufficiently close approximation to $S_{x,\ x+\Delta x}$

From the right – angled triangle PRT;

$$PT^2 = (\Delta x)^2 + \left(f^1(x)\Delta x\right)^2$$

$$PT = \sqrt{\left\{1 + \left(f^1(x)\right)^2\right\}}\Delta x$$

Since $S_{x,x+\Delta x} = \sqrt{\left\{1 + \left(f^1(x)\right)^2\right\}}\Delta x$ we get:

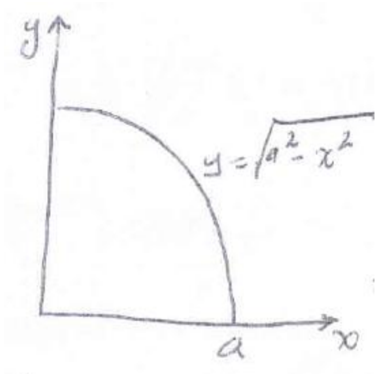
$$S_{a,b} = \int_a^b \sqrt{\left\{1 + \left(f^1(x)\right)^2\right\}}dx$$

Examples

1. Find the circumference of a circle of radius a, using integral calculus.

   Solution

   The curve $y^2 + x^2 = a^2$ from x = 0 to x = a represents a quarter of the total circumference.



From the equation of the curve, we get; $1+\left(\frac{dy}{dx}\right)^2 = \frac{a^2}{a^2 - x^2}$

Hence total circumference is given by;

$$4\int_0^a \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx$$

$$= 4\int_0^a \frac{a}{\sqrt{a^2 - x^2}}\, dx$$

$$= 4a\left[\sin^{-1}\left(\frac{x}{a}\right)\right]_0^a = 2a\pi$$

Likewise the length of the curve $x = g(y)$ from the point where $y = c$ to the point where $y = d$ is;

$$S_{c,d} = \int_c^d \sqrt{\left[1+\left(\frac{dx}{dy}\right)\right]}dy$$

In example 1, the equation of the curve may be written as $x = \sqrt{a^2 - y^2}$, giving;

$$1+\left(\frac{dx}{dy}\right)^2 = \frac{a^2}{a^2 - y^2}$$

So the total circumference is; $= 4\int_0^a \frac{a}{\sqrt{a^2 - y^2}}\, dy = 4a\left[\sin^{-1}\left(\frac{y}{a}\right)\right]_0^a = 2a\pi$

2. Find the length of the curve; $y=(x+1)(x+2)-\frac{1}{8}\ln(2x+3)$ between the points where *x =1* and *x = 2*.

From;

$$y=(x+1)(x+2)-\frac{1}{8}\ln(2x+3) \Rightarrow \frac{dy}{dx}=2x+3-\frac{1}{4(2x+3)}$$

$$\therefore 1+\left(\frac{dy}{dx}\right)^2=\left[(2x+3)+\frac{1}{4(2x+3)}\right]^2$$

The required length is

$$\int_1^2 (2x+3)+\frac{1}{4(2x+3)}\,dx=\left[x^2+3x+\frac{1}{8}\ln(2x+3)\right]_1^2$$

$$=6+\frac{1}{8}\ln\left(\frac{7}{5}\right)$$

Techniques of integration

Integration tables and simple substitution

All integral tables can be expected to include only a small fraction of the integrals that may need to be evaluated. Thus it is necessary to learn techniques for deriving new formulas and for transforming a given integral either into one that is already familiar or into one appearing in an accessible table.

A short table of integrals

Integrand	Result
u^n	$\frac{u^{n+1}}{n+1}+C$
$\frac{1}{u}$	$ln\lvert u\rvert+C$
e^u	e^u+C
$cosu$	$sinu+C$
$sinu$	$-cosu+C$
sec^2u	$tanu+C$
$secutanu$	$secu+C$
$cosec^2u$	$-cotu+C$
$\frac{1}{\sqrt{1-u^2}}$	$sin^{-1}u+C$
$\frac{1}{1+u^2}$	$tan^{-1}u+C$
$\frac{1}{u\sqrt{u^2-1}}$	$sec^{-1}\lvert u\rvert+C$

Method of substitution

Recall that if $\int f(u)du = F(u)+C$, then;

$$\int f\big(g(x)\big)g^1(x)dx = F\big(g(x)\big)+C$$

Thus the substitution of $u=g(x)\,, du=g^1(x)dx$ transforms the integrand into the simpler integral $\int f(u)du$

Examples

1. Find $\int \frac{1}{x}(1+\ln x)^5 dx$

Let $u = 1 + lnx \Rightarrow du = \frac{1}{x}dx$

$$\therefore \int \frac{1}{x}(1+lnx)^5 dx = \int u^5 du$$

$$= \frac{u^6}{6} + C = \frac{1}{6}(1+lnx)^6 + C$$

2. Find $\int \frac{x}{1+x^4}dx$

   Let $u=x^2 \Rightarrow du = 2xdx$

   $$\therefore \int \frac{x}{1+x^4}dx = \frac{1}{2}\int \frac{du}{1+u^2} = \frac{1}{2}\tan^{-1}u + C$$

   $$= \frac{1}{2}\tan^{-1}x^2 + C$$

3. Find $\int \frac{\sin^2 \cos x}{\sqrt{25-16\sin^2 x}}dx$

   Let, $u = 4\sin x \Rightarrow du = 4\cos xdx$

   $$\therefore \int \frac{\sin^2 x\cos x}{\sqrt{25-16\sin^2 x}}dx = \int \frac{(u/4)^2(1/4)du}{\sqrt{25-u^2}} = \frac{1}{64}\int \frac{u^2du}{\sqrt{25-u^2}}$$

   $$= \frac{1}{64}\left[\frac{25}{2}\sin^{-1}\left(\frac{u}{5}\right) - \frac{u}{2}\sqrt{25-u^2}\right] + C$$

   $$= \frac{1}{64}\left[\frac{25}{2}\sin^{-1}\left(\frac{4\sin x}{5}\right) - \frac{\sin x}{2}\sqrt{25-\sin^2 x}\right] + C$$

Trigonometric integrals

Case 1: $\int \sin^m x \cos^n x dx$ **with at least one of m or n is an odd positive integer**

We substitute for u such that the odd integer is reduced to an even one and then the identity $\sin^2 x + \cos^2 x = 1$ can be employed.

Examples

Find the following:

1. $\int \sin^3 x \cos x dx$

   Using the substitution; $u = \sin x$; $du = cosxdx$

   $\therefore \int \sin^3 x \cos x dx = \int u^3 du$

   $$= \frac{u^4}{4} + C = \frac{1}{4}\sin^4 x + C$$

2. $\int \sin^3 x \cos^2 x dx$

   Let $u = \cos x \Rightarrow du = -\sin x dx$

   $$\int \sin^3 x \cos^2 x dx = \int (1-\cos^2 x)\cos^2 x \sin x dx$$
   $$= \int (1-u^2)u^2(-du) = \int (u^4 - u^2)du$$
   $$= \frac{u^5}{5} - \frac{u^3}{3} + C = \frac{1}{5}\cos^5 x - \frac{1}{3}\cos^3 x + C$$

3. $\int \cos^5 x dx$

   $$\int \cos^5 x dx = \int \cos^4 x \cos x dx = \int (\cos^2 x)^2 \cos x dx$$
   $$= \int (1-\sin^2 x)^2 \cos x dx$$

   Let $u = \sin x \Rightarrow du = \cos x dx$

   $$\therefore \int \cos^5 x dx = \int (1-u^2)^2 du = \int (1-2u^2+u^4)du$$
   $$= u - \frac{2}{3}u^3 + \frac{1}{5}u^5 + C = \sin x - \frac{2}{3}\sin^3 x + \frac{1}{5}\sin^5 x + C$$

Case 2: $\int \sin^m x \cos^n x dx$ **with both m and n even**

When both m and n are nonnegative even integers, we use the half – angle formulas;

$\sin^2\theta = \frac{1}{2}(1-\cos 2\theta)$ and,

$\cos^2\theta = \frac{1}{2}(1+\cos 2\theta)$

Examples

1. $\int \sin^2 x \cos^2 x dx = \int \frac{1}{2}(1-\cos 2x).\frac{1}{2}(1+\cos 2x)dx$

$$= \frac{1}{4}\int (1-\cos^2 2x)dx = \frac{1}{4}\int\left[1-\frac{1}{2}(1+\cos 4x)\right]dx$$

$$= \frac{1}{8}\int (1-\cos 4x)dx = \frac{1}{8}x - \frac{1}{32}\sin 4x + C$$

2. $\int \cos^4 3x dx = \int \frac{1}{4}(1+\cos 6x)^2 dx$

$$= \frac{1}{4}\int (1+2\cos 6x+\cos^2 6x)dx = \frac{1}{4}\int 1+2\cos 6x+\frac{1}{2}(1+\cos 12x)dx$$

$$= \frac{1}{4}\int\left(\frac{3}{2}+2\cos 6x+\frac{1}{2}\cos 12x\right)dx = \frac{3}{8}x+\frac{1}{12}\sin 6x+\frac{1}{96}\sin 12x + C$$

Integral of the form $\int \tan^m x \sec^n x dx$

This can be evaluated in the following two cases:

Case 1: When m is an odd positive integer

In this case we split off the factor $secxtanx$to form, along with dx, the differential $secxtanxdx$ of $secx$. We then use the identity $\tan^2 x = \sec^2 x - 1$ to convert the remaining even power of $tanx$ into powers of $secx$. This prepares us for the substitution $u = \sec x$.

Example

$$\int \tan^3 x \sec^3 x dx = \int (\sec^2 x - 1)\sec^2 x \sec x \tan x dx$$

Let $u = \sec x \Rightarrow du = \sec x \tan x dx$

$$\int \tan^3 x \sec^3 x dx = \int (u^2-1)u^2 du = \int (u^4 - u^2)du$$

$$= \frac{1}{5}u^5 - \frac{1}{3}u^3 + C = \frac{1}{5}\sec^5 x - \frac{1}{3}\sec^3 x + C$$

Case 2: n is an even positive integer

Here we split off sec^2x to form, along with dx, the differential of $tanx$. Use of the identity $\sec^2 x = 1 + \tan^2 x$ to convert the remaining even power to powers of $tanx$ then this prepares us for the substitution $u = \tan x$.

Example

$$\int \sec^6 2x dx = \int (1+\tan^2 2x)^2 \sec^2 2x dx$$

Let $u = \tan 2x \Rightarrow du = 2\sec^2 2x dx$

$$\therefore \int \sec^6 x dx = \frac{1}{2}\int (1+u^2)^2 du = \frac{1}{2}\int (1+2u^2+u^4)du$$

$$= \frac{1}{2}u + \frac{u^3}{3} + \frac{u^5}{10} + C = \frac{1}{2}\tan 2x + \frac{1}{3}\tan^3 2x + \frac{1}{10}\tan^5 2x + C$$

Integration by parts

The formula for integration by parts is a simple consequence of the product rule for derivatives:

$$\frac{d}{dx}(uv) = v\frac{du}{dx} + u\frac{dv}{dx}$$

If this formula is written in the form:

$u(x)v^1(x) = \frac{d}{dx}[u(x)v(x)] - v(x)u^1(x)$, then antidifferentiation gives:

$$\int u(x)v^1(x)dx = u(x)v(x) - \int v(x)u^1(x)dx$$

This is the formula for integration by parts. With $du = u^1(x)dx\, and\ dv = v^1(x)dx$, then;

$$\int udv = uv - \int vdu$$

Examples

1. Find $\int \ln xdx$

   Let $u = \ln x \Rightarrow du = \frac{1}{x}dx$ and;

   $dv = dx \Rightarrow v = x$

   $\therefore \int \ln xdx = x\ln x - \int x.\frac{1}{x}dx = x\ln x - \int dx$

   $= x\ln x - x + C$

2. Find $\int \sin^{-1} xdx$

   $u = \sin^{-1} x \Rightarrow du = \frac{dx}{\sqrt{1-x^2}}$ and $dv = dx \Rightarrow v = x$

   $\therefore \int \sin^{-1} xdx = x\sin^{-1} x - \int \frac{x}{\sqrt{1-x^2}}dx$

   $= x\sin^{-1} x + \sqrt{1-x^2} + C$

3. Find $\int xe^{-x}dx$ $\quad \left(Ans: -xe^{-x} - e^{-x} + C\right)$

Trigonometric substitution

The method of trigonometric substitution is often effective in dealing with integrals when the integrands involve certain algebraic expressions such as $\left(a^2-u^2\right)^{1/2}, \left(u^2-a^2\right)^{1/2}, and\ \frac{1}{a^2+u^2}$

There are three basic trigonometric substitutions:

If the integrand involves	The substitute	Identity used
a^2-u^2	$u = asin\theta$	$1-sin^2\theta = cos^2\theta$
a^2+u^2	$u = atan\theta$	$1+tan^2\theta = sec^2\theta$
u^2-a^2	$u = asec\theta$	$sec^2\theta-1 = tan^2\theta$

Example

$$\int \frac{x^3}{\sqrt{1-x^2}}dx, where |x| < 1$$

In case *a = 1* and *u = x*, so we substitute; $x = \sin\theta \Rightarrow dx = \cos\theta d\theta$

$$\therefore \int \frac{x^3}{\sqrt{1-x^2}}dx = \int \frac{\sin^3\theta\cos\theta}{\sqrt{1-\sin^2\theta}}d\theta$$

$$= \int \sin^3\theta d\theta = \int \sin\theta\left(1-\cos^2\theta\right)d\theta = \frac{1}{3}\cos^3\theta - \cos\theta + C$$

But $\cos\theta = \sqrt{1-\sin^2\theta} = \sqrt{1-x^2}$

$$\Rightarrow \int \frac{x^3}{\sqrt{1-x^2}}dx = \frac{1}{3}\left(1-x^2\right)^{3/2} - \sqrt{1-x^2} + C$$

Hyperbolic substitutions

Hyperbolic substitutions can also be used in a similar way;

If the integrand involves	The substitute	Identity used
a^2-u^2	$u = atanh\theta$	$1-tanh^2\theta = sech^2\theta$
a^2+u^2	$u = asinh\theta$	$1+sinh^2\theta = cosh^2\theta$
u^2-a^2	$u = acosh\theta$	$cosh^2\theta-1 = sinh^2\theta$

Example

Find $\int \frac{dx}{\sqrt{x^2-1}}, x > 1$

We can use the two forms of substitutions:

(a) Let $x = \sec\theta \Rightarrow dx = \sec\theta\tan\theta d\theta$

$\tan\theta = \sqrt{x^2-1}$

$$\therefore \int \frac{dx}{\sqrt{x^2-1}} = \int \frac{\sec\theta\tan\theta}{\tan\theta}d\theta$$

$$= \int \sec\theta d\theta$$

$$= \ln\left|\sec\theta + \tan\theta\right| + C = \ln\left|x + \sqrt{x^2 - 1}\right| + C$$

(b) Using hyperbolic substitution;

$x = \cosh\theta \Rightarrow dx = \sinh\theta$

$\sqrt{x^2 - 1} = \sqrt{\cosh^2\theta} = \sinh\theta$

We take the positive square root because x > 1 implies that $\theta = \cosh^{-1} x > 0$ and thus $\sinh\theta > 0$.

$$\int \frac{dx}{\sqrt{x^2 - 1}} = \int \frac{\sinh\theta}{\sinh\theta} d\theta = \theta + C = \cosh^{-1} x + C$$

Integrals involving quadratic polynomials

An integral involving a square root or negative power of a quadratic polynomial $ax^2 + bx + c$ can often be simplified by the process of completing squares.

Example

1. Find $\int \frac{dx}{9x^2 + 6x + 5}$

$$9x^2 + 6x + 5 = 9\left(x^2 + \frac{2}{3}x\right) + 5 = 9\left(x^2 + \frac{2}{3}x + \frac{1}{9} - \frac{1}{9}\right) + 5$$

$$= 9\left(x + \frac{1}{3}\right)^2 + 5 - 1 = 9\left(x + \frac{1}{3}\right)^2 + 4 = (3x+1)^2 + 4$$

$$\Rightarrow \int \frac{dx}{9x^2 + 6x + 5} = \int \frac{dx}{(3x+1)^2 + 4}$$

Let $u = 3x + 1 \Rightarrow du = 3dx$, then;

$$\int \frac{dx}{9x^2 + 6x + 5} = \frac{1}{3}\int \frac{du}{u^2 + 4} = \frac{1}{6}\int \frac{du/2}{(u/2)^2 + 1}$$

Let $v = u/2 \Rightarrow dv = du/2$

$$= \frac{1}{6}\int \frac{dv}{v^2 + 1} = \frac{1}{6}\tan^{-1} v + C = \frac{1}{6}\tan^{-1}\left(\frac{3x+1}{2}\right) + C$$

An integral involving a quadratic expression can sometimes be split into two simpler integrals.

Example

Find $\int \frac{2x+3}{9x^2 + 6x + 5} dx$

Because $\frac{d}{dx}(9x^2 + 6x + 5) = 18x + 6$, this would be simpler if the numerator, *2x+3* were a constant multiple of *18x + 6*, so;

$$2x + 3 = A(18x + 6) + B$$

So we can split the given integral into a sum of two integrals, one of which has a numerator *18x+6* in its integrand.

By matching coefficients we have that;

A = 1/9 and B = 7/3

$$\therefore \int \frac{2x+3}{9x^2+6x+5}dx = \frac{1}{9}\int \frac{18x+6}{9x^2+6x+5}dx + \frac{7}{3}\int \frac{dx}{9x^2+6x+5}$$

$$= \frac{1}{9}\ln\left(9x^2+6x+5\right) + \frac{7}{18}\tan^{-1}\left(\frac{3x+1}{2}\right) + c$$

Rational fractions and partial fractions

The method of partial fractions involves decomposing *R(x)* into a sum of terms;

$$R(x) = \frac{P(x)}{Q(x)} = P(x) + F_1(x) + F_2(x) + \cdots \ldots \ldots \ldots \ldots + F_k(x)$$

Where P(x) is a polynomial and each expression $F_i(x)$ is a fraction that can be integrated by methods discusses earlier.

Rule 1: Linear factor partial fractions

The part of the partial – fraction decomposition of *R(x)* corresponding to the linear factor *ax+b* of multiplicity n is a sum of n partial fractions having the form;

$$\frac{A_1}{ax+b} + \frac{A_2}{(ax+b)^2} + \ldots\ldots\ldots\ldots\ldots + \frac{A_n}{(ax+b)^n}$$

Where A_1, A_2, …………., A_n are constants.

Note

R(x) must be a proper fraction before it is decomposed into its various partial fractions, otherwise if it is improper it should first be changed into a mixed fraction by long division.

Example

Find $\int \frac{4x^2-3x-4}{x^3+x^2-2x}dx$

The rational fraction to be integrated is proper, so we proceed to factor its denominator;

$$x^3+x^2-2x = x\left(x^2+x-2\right) = x(x-1)(x+2)$$

$$\frac{4x^2-3x-4}{x^3+x^2-2x} \equiv \frac{A}{x} + \frac{B}{x-1} + \frac{C}{x+2}$$

$$4x^3-3x-4 = A(x-1)(x+2) + Bx(x+2) + Cx(x-1)$$

Hence; *A = 2, B = -1, C = 3*

$$\int \frac{4x^2-3x-4}{x^3+x^2-2x}dx = \int\left(\frac{2}{x} - \frac{1}{x-1} + \frac{3}{x+2}\right)dx$$

$$= 2\ln x - \ln(x-1) + 3\ln(x+2) + C$$

Rule 2: Quadratic factor partial fractions

The part of the partial – fraction decomposition of R(x) corresponding to the irreducible quadratic factor ax^2+bx+c of multiplicity n is a sum of n partial fractions, having the form;

$$\frac{B_1x+C_1}{ax^2+bx+c}+\frac{B_2x+C_2}{(ax^2+bx+c)^2}+\ldots\ldots\ldots\ldots\ldots\ldots+\frac{B_nx+C_n}{(ax^2+bx+c)^n}$$

Where $B_1, B_2, \ldots\ldots\ldots, B_n$ and $C_1, C_2, \ldots\ldots\ldots, C_n$ are constants.

Example

Find $\int \frac{5x^3-3x^2+2x-1}{x^4+x^2}dx$

The denominator $x^4+x^2=x^2(x^2+1)$ has both a quadratic factor and a repeated linear factor.

$$\frac{5x^3-3x^2+2x-1}{x^4+x^2}=\frac{A}{x}+\frac{B}{x^2}+\frac{Cx+D}{x^2+1}$$

$$\therefore 5x^3-3x^2+2x-1=Ax(x^2+1)+B(x^2+1)+(Cx+D)x^2$$

From which A = 2, B = -1, C = 3 and D = -2

Thus, $\int \frac{5x^3-3x^2+2x-1}{x^4+x^2}dx=\int\left(\frac{2}{x}+\frac{-1}{x^2}+\frac{3x-2}{x^2+1}\right)dx$

$$=2\ln x+\frac{1}{x}+\frac{3}{2}\int\frac{2x}{x^2+1}dx-2\int\frac{dx}{x^2+1}$$

$$=2\ln x+\frac{1}{x}+\frac{3}{2}\ln(x^2+1)-2\tan^{-1}x+C$$

Rationalizing substitutions

Here, a substitution that eliminates the radical is made. This method succeeds with any integral of the form;

$\int P(x)\sqrt{ax+b}\,dx$ Where *P(x)* is a polynomial. Either the substitution $u = ax+b$ or the substitution $u=(ax+b)^{1/2}$ may be used.

More generally, the substitution $u^n = f(x)$ may succeed when the integrand involves $\sqrt[n]{f(x)}$

It always succeeds in the case of an integral of the form $\int P(x)\sqrt[n]{\frac{ax+b}{cx+d}}dx$, where *P(x)* is a polynomial. The substitution, $u^n=\frac{ax+b}{cx+d}$ converts the integrand into a rational function of u which can then be integrated by the method of partial fractions.

Reduction formulae

Consider the trigonometric identity $\tan^2 x = \sec^2 x - 1$, it follows that;

$$\int \tan^2 x dx = \tan x - x + C$$

And in general;

$$\int \tan^n x dx = \int \tan^{n-2} x\left(\sec^2 x - 1\right)dx$$

$$= \frac{1}{n-1}\tan^{n-1} x - \int \tan^{n-1} x dx \quad (n \geq 2)$$

On writing $I_n = \int \tan^n x dx$ the above expression becomes;

$$I_n = \frac{1}{n-1}\tan^{n-1} x - I_{n-2} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

The above expression (1) expresses one such integral in terms of another with a lower value of *n*; such a formula is called a ***reduction formula***. Successive applications of this formula reduce the problem of finding I_n to that of finding I_1, if n is odd, or I_0 if n is even.

Example

$$\int \tan^5 x dx = I_5$$

$$= \frac{1}{4}\tan^4 x - I_3 = \frac{1}{4}\tan^4 x - \frac{1}{2}\tan^2 x + I_1$$

$$= \frac{1}{4}\tan^4 x - \frac{1}{2}\tan^2 x - \ln\cos x + C$$

The reduction formula (1) was obtained simply by using an identity. Integration by parts is particularly useful for finding reduction formula.

Example

Obtain a reduction formula for; $\int x^n e^x dx$

$I_n = \int x^n e^x dx$, using integration by parts, gives;

$$I_n = x^n e^x - \int n x^{n-1} e^x dx$$

$$= x^n e^x - nI_{n-1}$$

Note

Most reduction formulae involve the use of integration by parts followed by use of an identity.

Questions

Find the reduction formulae for:

(i) $\int \sin^n x dx$

(ii) $\int \cos^n x dx$

(iii) $\int \sin^m x \cos^n x dx$

Integration of functions of two variables

Given $f(x, y)$, and suppose we wish to find $F(x, y)$ such that;

$$\frac{\partial F}{\partial x} = f(x, y)$$

Clearly $F(x, y) + g(y)$, where g is an arbitrary function, is also a solution, since;

$$\frac{\partial}{\partial x}[F(x, y) + g(y)] = f(x, y)$$

Thus $\int f(x, y)dx = F(x, y) + g(y)$

Examples

$$\int x^2 y + xy^2 dx = \frac{1}{3}x^3 y + \frac{1}{2}x^2 y^2 + g(y)$$

We are integrating with respect to x in the usual way, treating y as a constant.

Similarly;

$\int f(x, y)dy = G(x, y) + g(x)$, where g is an arbitrary function

Limits of integration may also be used, i.e.:

$\int_a^b f(x, y)dx = F(b, y) - F(a, y)$ and;

$$\int_c^d f(x, y)dy = G(x, d) - G(x, c)$$

Since $\int f(x, y)dx$ is a function of two variables x and y, it may be integrated with respect to y to give the repeated integral;

$\int \left[\int f(x, y)dx\right]dy$, which is usually written as;

$$\int dy \int f(x, y)dx \; or \; \int\int f(x, y)dxdy$$

Examples

1. Find $\int_0^1 dy \int_{y^2}^{y} \left(x^2 y + xy^2\right) dx$

$$= \int_0^1 \left[\frac{1}{3}x^3 y + \frac{1}{2}x^2 y^2\right]_{x=y^2}^{x=y} dy$$

$$= \int_0^1 \left(\frac{1}{3}y^4 + \frac{1}{2}y^4 - \frac{1}{3}y^7 - \frac{1}{2}y^6\right) dy = \int \left(\frac{5}{6}y^4 - \frac{1}{3}y^7 - \frac{1}{2}y^6\right) dy$$

$$= \left[\frac{1}{6}y^5 - \frac{1}{24}y^8 - \frac{1}{14}y^7\right]_0^1 = \frac{3}{56}$$

2. $\int_0^1 \int_{y-1}^{y+1} x^2 y^3 dxdy$

$$= \int_0^1 \left[\frac{1}{3}x^3 y^3\right]_{y-1}^{y+1} dy = \frac{1}{3}\int_0^1 y^3\left((y+1)^3 - (y-1)^3\right) dy$$

Question

Show that $\int_0^1 \int_0^y \frac{xy^2}{\sqrt{x^2+y^2}} dxdy = \frac{1}{4}\left(\sqrt{2} - 1\right)$

Note

If *a* and *b* do not depend on y, then $\int_a^b f(x)dx$ is a constant and does not enter into the process of integration with respect to *y*. Thus;

$\int_c^d dy \int_a^b f_1(x) f_2(y) dx = \left[\int_c^d f_2(y)dy\right] \times \left[\int_a^b f_1(x)dx\right]$, if *a*, *b*, *c*, and *d* are constants, i.e. the integral may be evaluated as a product of two independent integrals.

Example

$$\int_0^1 \int_0^{\frac{\pi}{2}} e^{3y} \sin 2x dxdy = \int_0^1 e^{3y} dy \int_0^{\frac{\pi}{2}} \sin 2x dx$$

$$= \left[\frac{1}{3}e^{3y}\right]_0^1 \left[-\frac{1}{2}\cos 2x\right]_0^{\pi/2} = \frac{1}{3}\left(e^3 - 1\right)$$

Double integrals

Definition: The double integral of the bounded function f over the plane region R is the number;

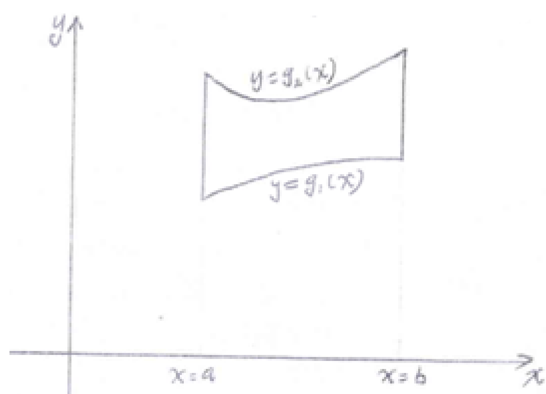
$I = \iint_R f(x, y)dA$, provided that, for every $\varepsilon > 0$, there exists a number $\delta > 0$ such that;

$\left|\sum_{i=1}^{k} f\left(x_i^*, y_i^*\right)\Delta A\right| < \varepsilon$ for every inner partition $\{R_1, R_2, R_3, ..., R_n\}$, of R having mesh $|\rho| < \delta$ and every selection of points $\left(x_i^*, y_i^*\right)$ in $R_i(i = 1,2,3,.........k)$

For certain common types of regions, we can evaluate double integrals by using integrated integrals, in much the same way as when the region is a rectangle.

The region R is called vertically simple if it is described by means of the inequalities;

$a \le x \le b$ and $g_1(x) \le y \le g_2(x)$.

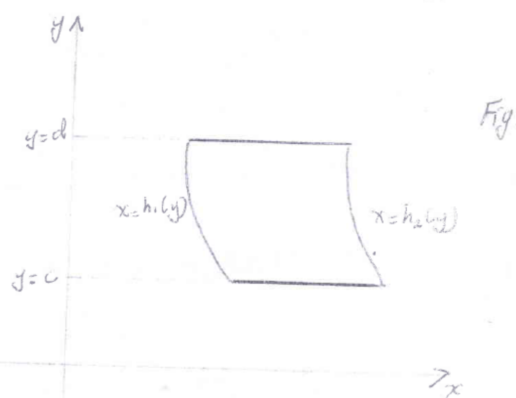


A vertically simple region R

Where g_1 and g_2 are continuous on $[a, b]$.

The region R is called horizontally simple if it is described by the inequalities;

$c \le y \le d$ and $h_1(y) \le x \le h_2(y)$, where h_1 and h_2 are continuous on $[c, d]$.



A horizontally simple region R

Suppose that f(x, y) is continuous on the region R. If R is the vertically simple region then;

$$\iint_R f(x,y)dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)dydx$$

If R is the horizontally simple region then;

$$\iint_R f(x,y)dA = \int_c^d \int_{h_1(x)}^{h_2(x)} f(x,y)dydx$$

Some useful properties of double integrals

Let *C* be a constant and *f* and *g* continuous functions on the region *R* on which *f(x,y)* attains a minimum value *m* and a maximum value *M*. let *a(R)* denote the area of the region *R*. If the integrated integrals all exist, then;

$$\iint_R Cf(x,y)dA = C\iint_R f(x,y)dA \qquad (1)$$

$$\iint_R [f(x,y)+g(x,y)]dA = \iint_R f(x,y)dA + \iint_R g(x,y)dA \qquad (2)$$

$$ma(R) \le \iint_R f(x,y)dA \le Ma(R) \qquad (3)$$

$$\iint_R f(x,y)dA = \iint_{R_1} f(x,y)dA + \iint_{R2} f(x,y)dA \qquad (4)$$

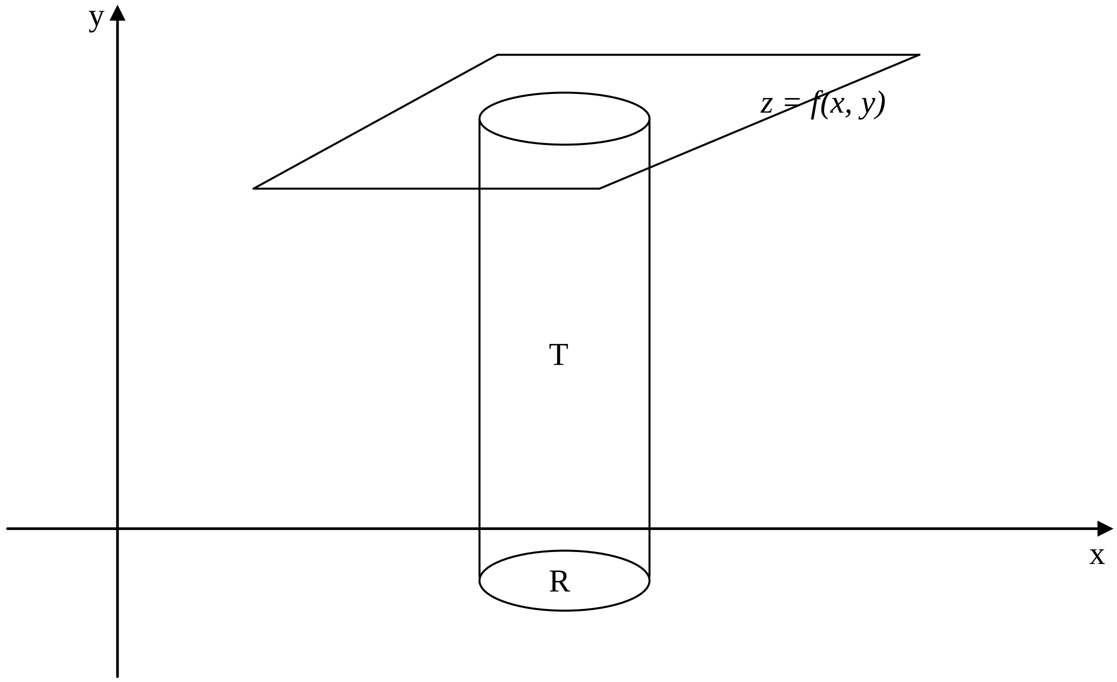
In equation (4), R_1 and R_2 are two non overlapping regions (regions with disjoint interiors) with union R.

Areas and volumes by Double integration

Definition: Volume under $z = f(x,y)$

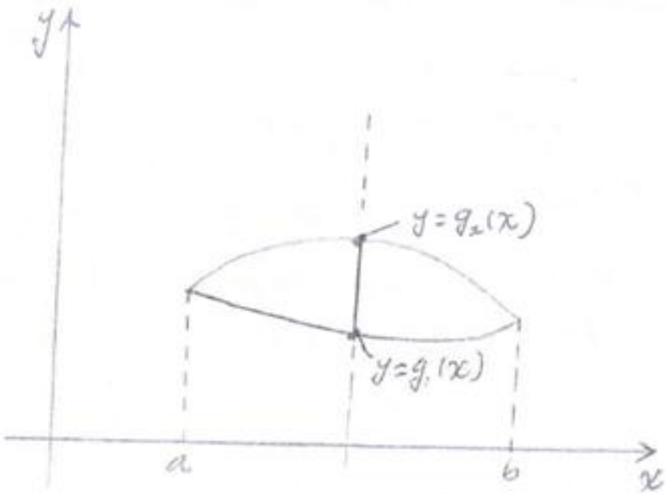
Suppose that the function f is continuous and non negative on the bounded plane R, then the volume V of the solid that lies under the surface z = f(x, y) and above the region R is defined to be:

$$V = \iint_R f(x, y)\,dA$$



A three – dimensional region T is typically described in terms of the surfaces that bound it. The first step in computing its volume V is to determine the region R in the xy – plane over which T lies. The second step is to determine the appropriate order of iterated integration. This may be done in the following way:

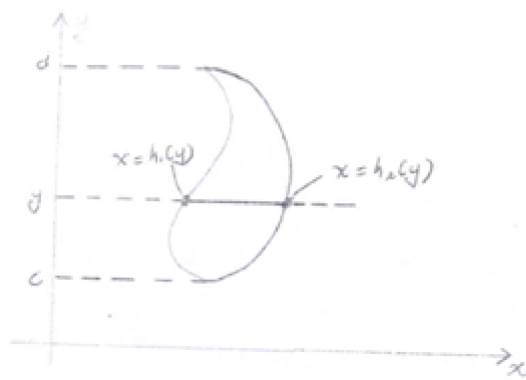
1. If each vertical line in the *xy – plane* meets *R* in a single line segment, then *R* is vertically simple and one may integrate first with respect to *y*. The limits on *y* will be the y – coordinates $g_1(x)$ and $g_2(x)$ of the end points of this line segment. i.e. a vertically simple region.



2. If each horizontal line in the *xy – plane* meets *R* in a single line segment, then *R* is horizontally simple.

   $$V = \iint_R f(x,y)\,dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\,dx\,dy$$

   $h_1(y)$ and $h_2(y)$ are the *x – coordinates* of the end points of this horizontal line segment, and c and d are the end points of the corresponding interval *y – axis*.



3. If the region *R* is both vertically simple and horizontally simple, then you have the pleasant option of choosing the order of integration that will lead to the simpler subsequent computations. If *R* is neither vertically simple nor horizontally simple, then you must first subdivide *R* into simple regions before proceeding with iterated integration.

Examples

1. Find the volume of the solid that lies under the surface $z = 1 + xy$ and the surface R in the xy – plane consisting of those points (x, y) for which $0 \le x \le 2$ and $0 \le y \le 1$

   Solution

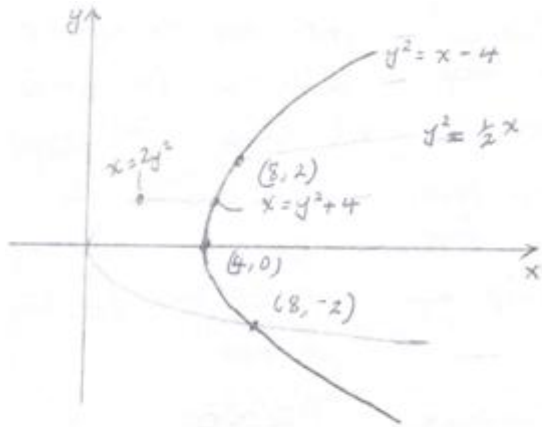
   $$f(x,y) = 1 + xy$$

   $$V = \int_0^2 \int_0^1 (1 + xy)\,dy\,dx$$

   $$= \int_0^2 \left[y + \frac{xy^2}{2} \right]_0^1 dx = \int_0^2 \left(1 + \frac{1}{2}x \right) dx = \left[x + \frac{1}{4}x^2 \right]_0^2 = 3$$

2. Compute the area of the region R in the xy – plane that is bounded by the two parabolas $y^2 = \frac{x}{2}$ and $y^2 = x - 4$

Solution



$$y^2 = x - 4 \; and \; y^2 = \frac{x}{4} \Rightarrow \frac{1}{2}x = x - 4 \therefore x = 8 \; and \; y = \pm 2$$

$$a(R) = \int_{-2}^{2}\int_{2y^2}^{y^2+4} 1dxdy = \int_{-2}^{2}[x]_{2y^2}^{y^2+4} dy = \int_{-2}^{2}\left(4 - y^2\right)dy$$

$$= \left[4y - \frac{1}{3}y^3\right]_{-2}^{2} = \frac{32}{3}$$

Improper integrals and infinite limits of integration

Suppose that the function *f* is continuous and non negative on the unbounded interval $[a, +\infty)$. Then for any $t > 0$, the area $A(t)$ of the region under $y = f(x)$ from $x = a$ to $x = t$ is given by the definite integral,

$$A(t) = \int_a^t f(x)dx$$

If we let $t = +\infty$, and find that the limit of $A(t)$ exists. Then we may regard this limit as the area of the unbounded region lying under $y = f(x)$ and over$[a, +\infty)$. For *f* continuous on$[a, +\infty)$, we therefore define;

$\int_a^\infty f(x)dx = \lim_{x\to\infty}\int_a^t f(x)dx$ provided that this limit exists, In this case we say that the improper integral on the left **converges,** otherwise, we say that it **diverges**. If f(x) is nonnegative on$[a, +\infty)$, then the limit either exists or is infinite, and in the latter case we write,

$\int_a^\infty f(x)dx = +\infty$, and we say that the improper integral **diverges to infinity**.

If the function f has both positive and negative values on $[a, +\infty)$, then the improper integral can diverge by oscillation; that is, without diverging to infinity. This occurs with $\int_a^{\infty} \sin x dx$, because it easy to notice that this integral is zero if t is an even multiple of π but is 2 if t is an odd multiple of π. Thus $\int_a^{\infty} \sin x dx$ oscillates between 0 and 2 as $t \to \infty$, and so the limit $\int_a^{\infty} f(x)dx$ does not exist.

The infinite lower limit of integration can be treated similarly which is defined as;

$\int_{-\infty}^{b} f(x)dx = \lim_{x \to -\infty} \int_{t}^{b} f(x)dx$, provided that the limit exists. If f is continuous on the whole real line, then;

$\int_{-\infty}^{\infty} f(x)dx = \int_{-\infty}^{c} f(x)dx + \int_{c}^{\infty} f(x)dx$, provided that both improper integrals on the right converge

Note

- It makes no difference what value of *c* is used since if *c* < *d*, then,

$$\int_{-\infty}^{c} f(x)dx + \int_{c}^{\infty} f(x)dx = \int_{-\infty}^{c} f(x)dx + \int_{c}^{d} f(x)dx + \int_{d}^{\infty} f(x)dx = \int_{-\infty}^{d} f(x)dx + \int_{d}^{\infty} f(x)dx$$

- $\int_{-\infty}^{\infty} f(x)dx$ is not necessary equal to $\lim_{t \to \infty} \int_{-t}^{t} f(x)dx$.

Example 1

Investigate the improper integrals;

(a) $\int_{1}^{\infty} \frac{1}{x^2} dx$

(b) $\int_{-\infty}^{0} \frac{1}{\sqrt{1-x}} dx$

(c) $\int_{-\infty}^{\infty} \frac{1}{1+x^2} dx$

Solution

(a) $\int_1^\infty \frac{1}{x^2}\,dx = \lim_{t\to\infty}\int_1^t \frac{1}{x^2}\,dx$

$= \lim_{t\to\infty}\left[-\frac{1}{x}\right]_1^t = \lim_{t\to\infty}\left(-\frac{1}{t}+1\right) = 1$, hence the integral converges to 1.

(b) $\int_{-\infty}^0 \frac{1}{\sqrt{1-x}}\,dx = \lim_{t\to-\infty}\int_t^0 \frac{1}{\sqrt{1-x}}\,dx = \lim_{t\to-\infty}\left[-2\sqrt{1-x}\right]_t^0$

$= \lim_{t\to-\infty}\left(2\sqrt{1-t}-2\right) = +\infty$, thus this integral diverges to $+\infty$.

(c) We shall choose *c = 0*.

$$\int_{-\infty}^{\infty}\frac{1}{1+x^2}\,dx = \int_{-\infty}^{0}\frac{1}{1+x^2}\,dx + \int_{0}^{\infty}\frac{1}{1+x^2}\,dx$$

$$= \lim_{s\to-\infty}\int_s^0 \frac{1}{1+x^2}\,dx + \lim_{t\to\infty}\int_0^t \frac{1}{1+x^2}\,dx$$

$$= \lim_{s\to-\infty}\left[\tan^{-1}x\right]_s^0 + \lim_{t\to\infty}\left[\tan^{-1}x\right]_0^t = \lim_{s\to-\infty}\left(-\tan^{-1}s\right) + \lim_{t\to\infty}\left(\tan^{-1}t\right) = \frac{\pi}{2}+\frac{\pi}{2} = \pi$$

Exercises

1. Determine whether or not the following improper integrals converge, and evaluate those that do converge.

   (i) $\int_4^\infty \frac{1}{x^{3/2}}\,dx$ (Ans: 1.0)

   (ii) $\int_1^\infty \frac{1}{1+x}\,dx$ (Ans, $+\infty$)

   (iii) $\int_2^\infty \frac{1}{(x-1)^{1/3}}\,dx$ (Ans, $+\infty$)

   (iv) $\int_{-\infty}^{-2} \frac{1}{(x+1)^3}\,dx$ (Ans: -1/2)

   (v) $\int_{-\infty}^{\infty} \frac{x}{x^2+4}\,dx$

2. Show that $\int_{-\infty}^{\infty}\frac{1+x}{1+x^2}\,dx$ diverges, but that $\lim_{t\to\infty}\int_{-t}^{t}\frac{1+x}{1+x^2}\,dx = \pi$