Chapter 10

Frequency Characteristics of AC Circuits

Chapter outline

- Introduction
- Two-port Networks
- The Decibel (dB)
- Frequency Response
- A High-Pass *RC* Network
- A Low-Pass *RC* Network
- A Low-Pass *RL* Network
- A High-Pass *RL* Network
- A Comparison of *RC* and *RL* Networks
- Bode Diagrams
- Combining the Effects of Several Stages
- *RLC* Circuits and Resonance
- Filters
- Stray Capacitance and Inductance

Introduction

- Having now looked at the AC behaviour of simple components, we can consider their effects on the frequency characteristics of simple circuits
- While the properties of a pure *resistance* are not affected by the frequency of the signal concerned, this is not true of *reactive* components
- We will start with a few basic concepts and then look at the characteristics of simple combinations of resistors, capacitors and inductors

Two-port Networks

- A two-port network has two ports:
  - an input port
  - an output port

(a) A two-port network

Two-port Networks

- We then define voltages and currents at the input and output

(b) A typical arrangement

- Then

$$\text{voltage gain } (A_v) = \frac{V_0}{V_i}$$

$$\text{current gain } (A_i) = \frac{I_0}{I_i}$$

$$\text{power gain } (A_p) = \frac{P_0}{P_i}$$

The Decibel (dB)

- The power gain of modern electronic amplifiers is often very high, gains of 10^6 or 10^7 being common
- With such large numbers it is often convenient to use a logarithmic expression for gain
- This is often done using **decibels**
- The decibel is a dimensionless figure for power gain

$$\text{Power gain (dB)} = 10\log_{10}\frac{P_2}{P_1}$$

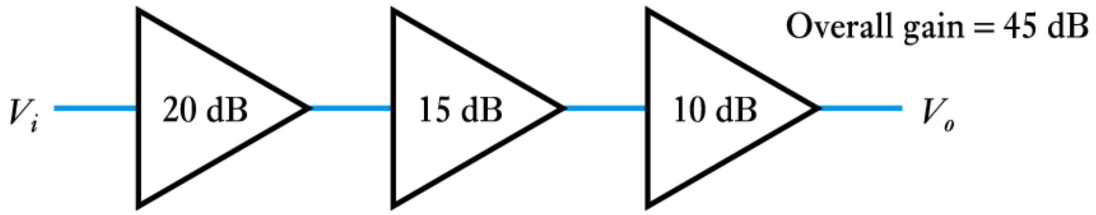
The Decibel (dB)

- Sample gains expressed in dBs

Power gain	Decibels (dBs)
100	20
10	10
1	0

Power gain	Decibels (dBs)
0.5	-3
0.1	-10
0.01	-20

- Using dBs simplifies calculation in cascaded circuits



The Decibel (dB)

- Power gain is related to voltage gain

$$\text{Power gain (dB)} = 10\log_{10}\frac{P_2}{P_1} = 10\log_{10}\frac{V_2^{\,2} / R_2}{V_1^{\,2} / R_1}$$

- If $R_1 = R_2$

$$\text{Power gain (dB)} = 10\log_{10}\frac{V_2^{\,2}}{V_1^{\,2}} = 20\log_{10}\frac{V_2}{V_1}$$

$$\text{Power gain (dB)} = 20\log_{10}(\text{Voltage gain})$$

- This expression is often used even when $R_1 \neq R_2$

The Decibel (dB)-Example

- Express gains of 20 dB, 30 dB and 40 dB as both power gains and voltage gains.

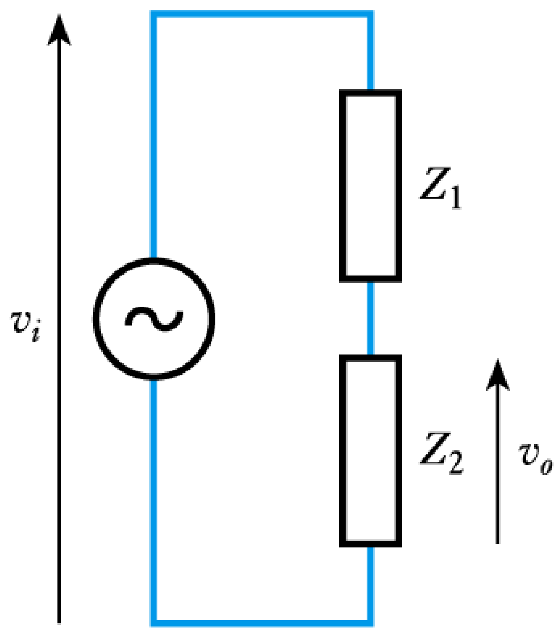
20 dB	$20 = 10 \log_{10}(\text{power gain})$ $\text{power gain} = 10^2$	$\text{power gain} = 100$
	$20 = 20 \log_{10}(\text{voltage gain})$ $\text{power gain} = 10$	$\text{voltage gain} = 10$
30 dB	$30 = 10 \log_{10}(\text{power gain})$ $\text{power gain} = 10^3$	$\text{power gain} = 1000$
	$30 = 20 \log_{10}(\text{voltage gain})$ $\text{power gain} = 10^{1.5}$	$\text{voltage gain} = 31.6$
40 dB	$40 = 10 \log_{10}(\text{power gain})$ $\text{power gain} = 10^4$	$\text{power gain} = 10{,}000$
	$40 = 20 \log_{10}(\text{voltage gain})$ $\text{power gain} = 10^2$	$\text{voltage gain} = 100$

Frequency response

- Since the characteristics of reactive components change with frequency, the behaviour of circuits using these components will also change
- The way in which the gain of a circuit changes with frequency is termed its **frequency response**
- These variations take the form of variations in the *magnitude* of the gain and in the *phase response*

Frequency response

- We will start by considering very simple circuits
- Consider the potential divider shown here



– from our earlier consideration of the circuit

$$v_o = v_i \times \frac{\mathbf{Z_2}}{\mathbf{Z_1} + \mathbf{Z_2}}$$

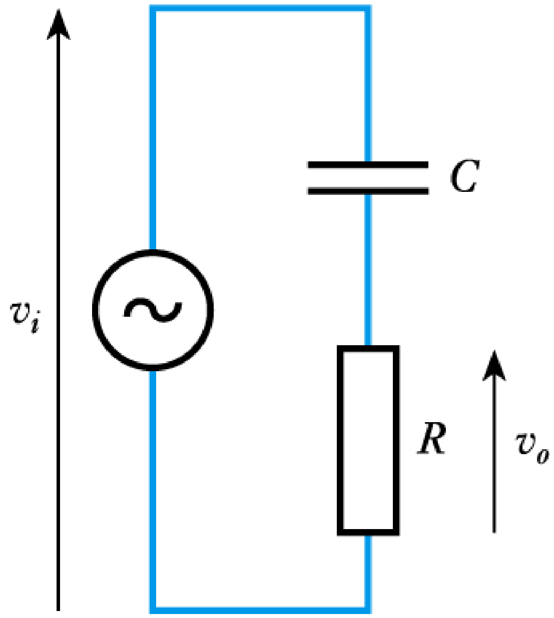
– rearranging, the gain of the circuit is

$$\frac{v_o}{v_i} = \frac{\mathbf{Z_2}}{\mathbf{Z_1} + \mathbf{Z_2}}$$

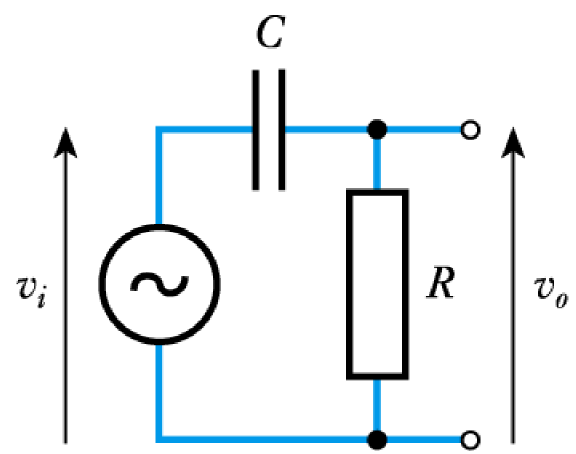
– this is also called the **transfer function** of the circuit

A High-Pass *RC* Network

- Consider the following circuit
  - which is shown re-drawn in a more usual form



(a)

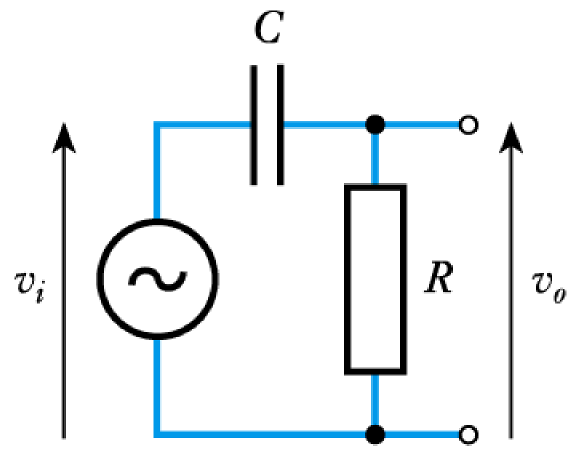


(b)

A High-Pass *RC* Network

- Clearly the transfer function is

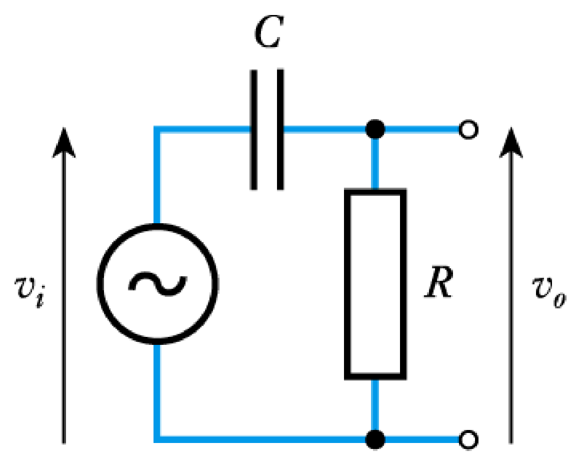
$$\frac{v_o}{v_i} = \frac{\mathbf{Z_R}}{\mathbf{Z_R} + \mathbf{Z_C}} = \frac{R}{R - j\dfrac{1}{\omega C}} = \frac{1}{1 - j\dfrac{1}{\omega CR}}$$



- At high frequencies
  - ω is large, voltage gain ≈ 1
- At low frequencies
  - ω is small, voltage gain $\rightarrow 0$

A High-Pass *RC* Network

- Since the denominator has real and imaginary parts, the *magnitude* of the voltage gain is



$$|\text{Voltage gain}| = \frac{1}{\sqrt{1^2 + \left(\frac{1}{\omega CR}\right)^2}}$$

- When $1/\omega CR = 1$

$$|\text{Voltage gain}| = \frac{1}{\sqrt{1+1}} = \frac{1}{\sqrt{2}} = 0.707$$

- This is a halving of power, or a fall in gain of 3 dB

A High-Pass *RC* Network

- The half power point is the **cut-off frequency** of the circuit
  - the angular frequency ω_C at which this occurs is given by

$$\frac{1}{\omega_c CR} = 1$$

$$\boxed{\omega_c = \frac{1}{CR} = \frac{1}{\mathrm{T}} \text{ rad/s}}$$

  - where T is the tim … CR network. Also

$$\boxed{f_c = \frac{\omega_c}{2\pi} = \frac{1}{2\pi CR} \text{ Hz}}$$

A High-Pass *RC* Network

- Substituting $\omega = 2\pi f$ and $CR = 1/\,2\pi f_C$ in the earlier equation gives

$$\frac{v_o}{v_i} = \frac{1}{1 - j\dfrac{1}{\omega CR}} = \frac{1}{1 - j\dfrac{1}{(2\pi f)\left(\dfrac{1}{2\pi f_c}\right)}} = \frac{1}{1 - j\dfrac{f_c}{f}}$$

- This is the general form of the gain of the circuit
- It is clear that both the *magnitude* of the gain and the *phase angle* vary with frequency

A High-Pass *RC* Network

- Consider the behaviour of the circuit at different frequencies:
- **When $f >> f_c$**
  - f_c/f << 1, the voltage gain ≈ 1
- **When $f = f_c$**

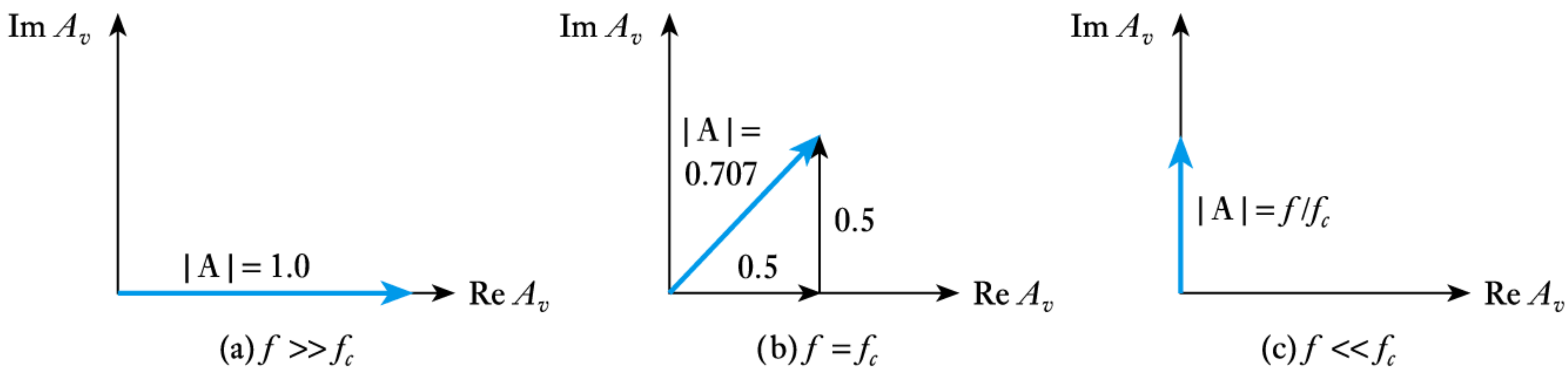
$$\frac{v_o}{v_i} = \frac{1}{1 - j\frac{f_c}{f}} = \frac{1}{1-j} = \frac{1\times(1+j)}{(1-j)\times(1+j)} = \frac{(1+j)}{2} = 0.5 + 0.5j$$

- **When $f << f_c$**

$$\frac{v_o}{v_i} = \frac{1}{1 - j\frac{f_c}{f}} \approx \frac{1}{-j\frac{f_c}{f}} = j\frac{f}{f_c}$$

A High-Pass *RC* Network

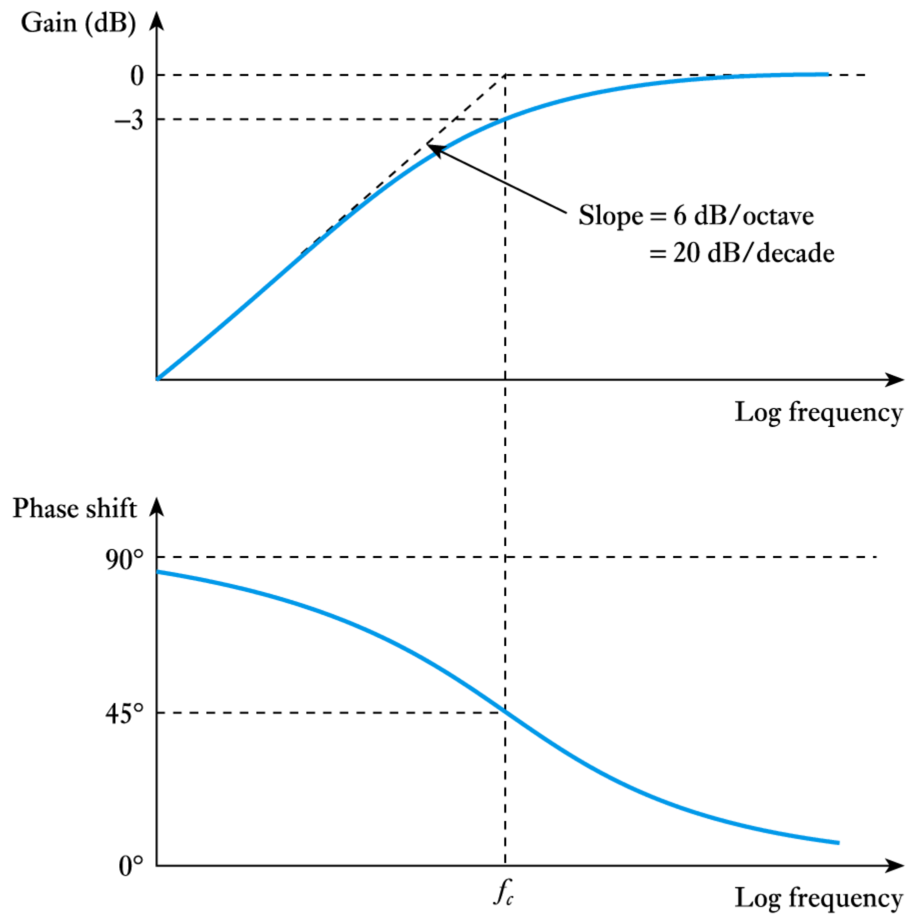
- The behaviour in these three regions can be illustrated using phasor diagrams



- At *low* frequencies the gain is linearly related to frequency. It falls at -6dB/octave (-20dB/decade)

A High-Pass *RC* Network

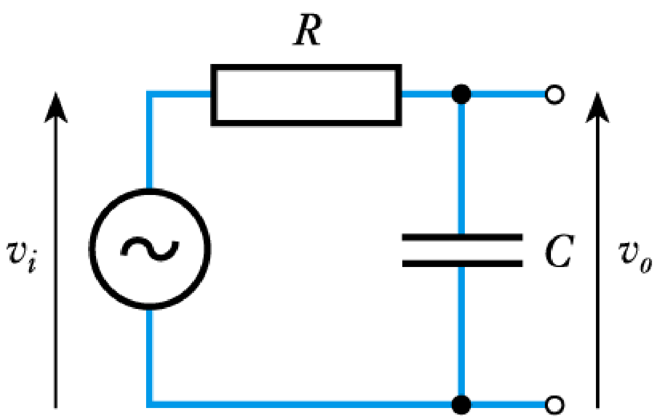
- Frequency response of the high-pass network
  - the gain response has two *asymptotes* that meet at the cut-off frequency
  - figures of this form are called **Bode diagrams**



A Low-Pass *RC* Network

- Transposing the *C* and *R* gives

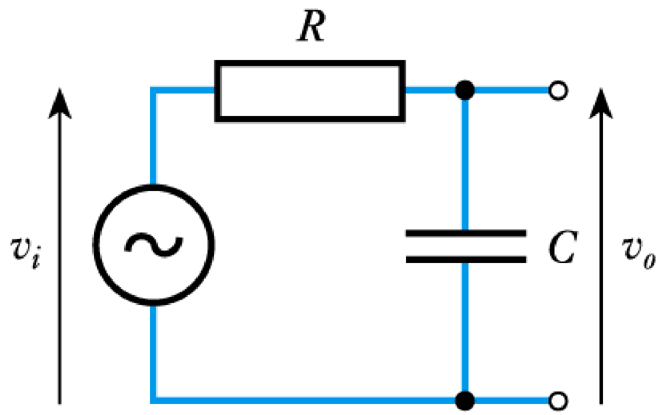
$$\frac{v_o}{v_i} = \frac{\mathbf{Z_C}}{\mathbf{Z_R} + \mathbf{Z_C}} = \frac{-\mathrm{j}\dfrac{1}{\omega C}}{R - \mathrm{j}\dfrac{1}{\omega C}} = \frac{1}{1 + \mathrm{j}\omega CR}$$



- At high frequencies
  - ω is large, voltage gain $\rightarrow 0$
- At low frequencies
  - ω is small, voltage gain ≈ 1

A Low-Pass *RC* Network

- A similar analysis to before gives



$$|\text{Voltage gain}| = \frac{1}{\sqrt{1+(\omega CR)^2}}$$

- Therefore when, when $\omega CR = 1$

$$|\text{Voltage gain}| = \frac{1}{\sqrt{1+1}} = \frac{1}{\sqrt{2}} = 0.707$$

- Which is the cut-off frequency

A Low-Pass *RC* Network

- Therefore
  - the angular frequency ω_C at which this occurs is given by

$$\omega_c CR = 1$$

$$\omega_c = \frac{1}{CR} = \frac{1}{\mathrm{T}} \text{ rad/s}$$

  - where T is the time constant of the *CR* network, and as before

$$f_c = \frac{\omega_c}{2\pi} = \frac{1}{2\pi CR} \text{ Hz}$$

A Low-Pass *RC* Network

- Substituting $\omega = 2\pi f$ and $CR = 1/\, 2\pi f_C$ in the earlier equation gives

$$\frac{v_o}{v_i} = \frac{1}{1 + j\omega CR} = \frac{1}{1 + j\dfrac{\omega}{\omega_C}} = \frac{1}{1 + j\dfrac{f}{f_C}}$$

- This is similar, but not the same, as the transfer function for the high-pass network

A Low-Pass *RC* Network

- Consider the behaviour of this circuit at different frequencies:
- **When $f \ll f_c$**
  - $f/f_c \ll 1$, the voltage gain ≈ 1
- **When $f = f_c$**

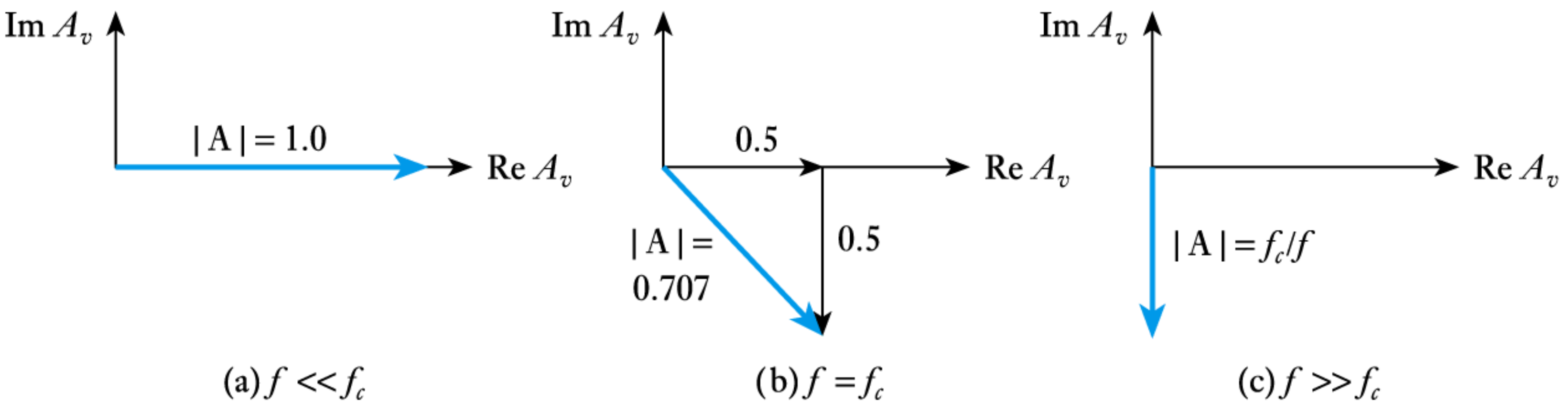
$$\frac{v_o}{v_i} = \frac{1}{1+\mathrm{j}\dfrac{f}{f_c}} = \frac{(1-\mathrm{j})(1+\mathrm{j})}{(1+\mathrm{j})} = \frac{(1-\mathrm{j})}{2} = 0.5+0.5\mathrm{j}$$

- **When $f \gg f_c$**

$$\frac{v_o}{v_i} = \frac{1}{1+\mathrm{j}\dfrac{f}{f_c}} \approx \frac{1}{\mathrm{j}\dfrac{f}{f_c}} = -\mathrm{j}\frac{f_c}{f}$$

A Low-Pass *RC* Network

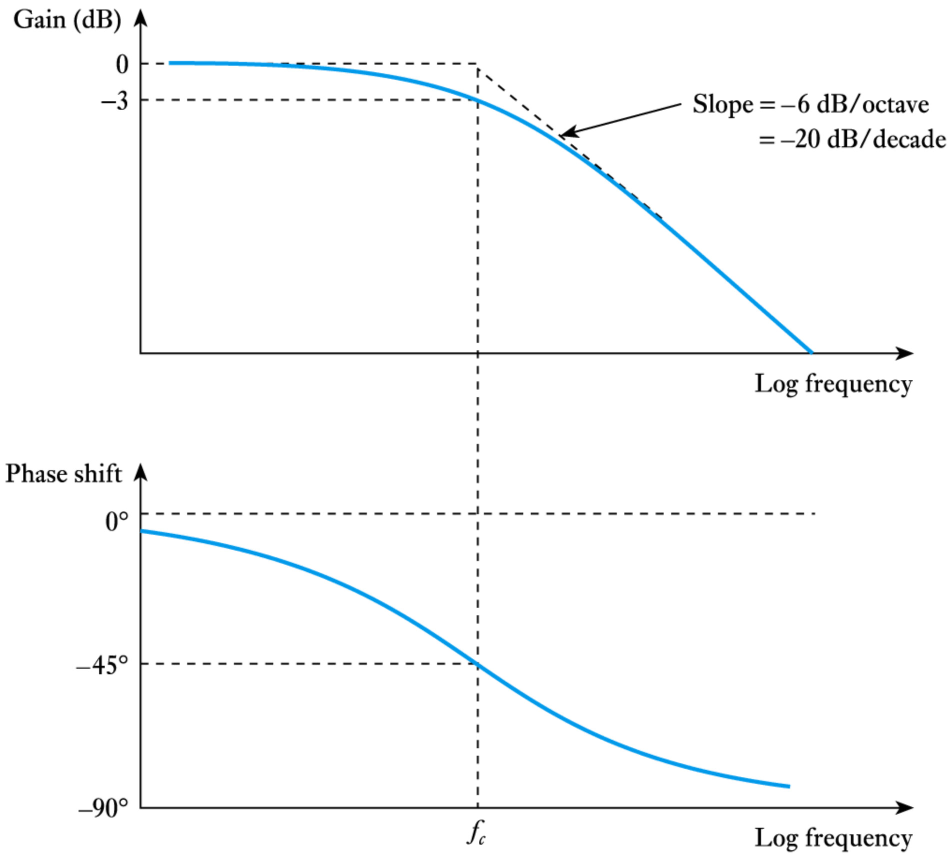
- The behaviour in these three regions can again be illustrated using phasor diagrams



- At *high* frequencies the gain is linearly related to frequency. It falls at 6dB/octave (20dB/decade)

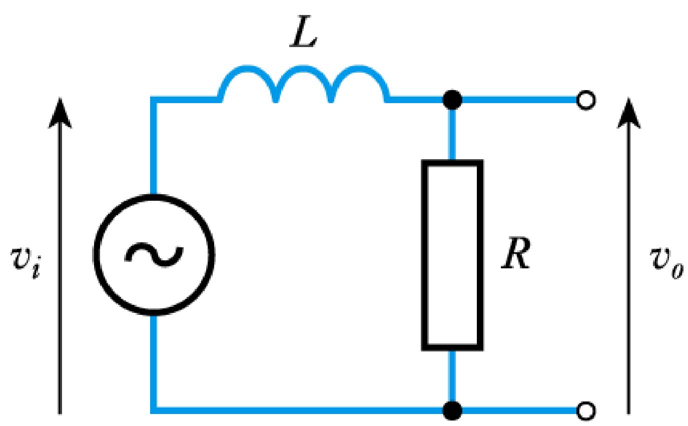
A Low-Pass *RC* Network

- Frequency response of the low-pass network
  - the gain response has two *asymptotes* that meet at the cut-off frequency
  - you might like to compare this with the Bode Diagram for a high-pass network



A Low-Pass *RL* Network

- Low-pass networks can also be produced using *RL* circuits
  - these behave similarly to the corresponding *CR* circuit
  - the voltage gain is



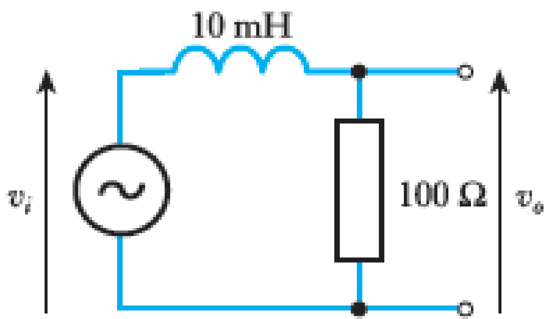
$$\frac{v_o}{v_i} = \frac{\mathbf{Z_R}}{\mathbf{Z_R} + \mathbf{Z_L}} = \frac{R}{R + j\omega L} = \frac{1}{1 + j\omega \frac{L}{R}}$$

  - the cut-off frequency is

$$\omega_c = \frac{R}{L} = \frac{1}{\mathrm{T}} \text{ rad/s} \qquad f_c = \frac{\omega_c}{2\pi} = \frac{R}{2\pi L} \text{ Hz}$$

A Low-Pass *RL* Network

- Calculate the time constant T, the angular cut-off frequency ω_c and the cyclic cut-off frequency f_c of the following arrangement.



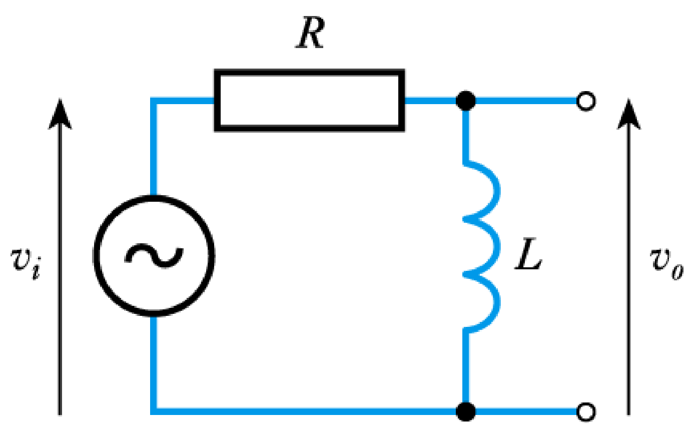
$$T = \frac{L}{R} = 10^{-4}S$$

$$\omega_c = \frac{1}{T} = 10^4 rad/s$$

$$f_c = \frac{\omega_c}{2\pi} = 1.59kHz$$

A High-Pass *RL* Network

- High-pass networks can also be produced using *RL* circuits
  - these behave similarly to the corresponding *CR* circuit
  - the voltage gain is



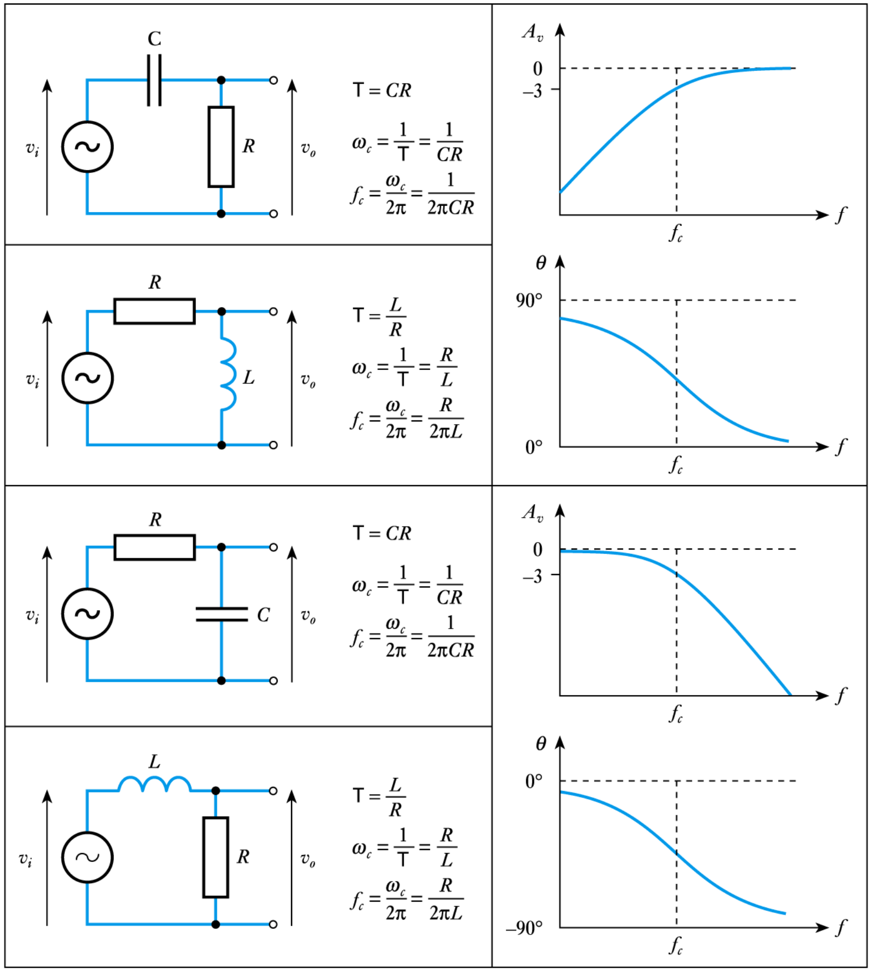
$$\frac{v_o}{v_i} = \frac{\mathbf{Z_L}}{\mathbf{Z_R} + \mathbf{Z_L}} = \frac{\mathrm{j}\omega L}{R + \mathrm{j}\omega L} = \frac{1}{1 + \frac{R}{\mathrm{j}\omega L}} = \frac{1}{1 - \mathrm{j}\frac{R}{\omega L}}$$

  - the cut-off frequency is

$$\omega_C = \frac{R}{L} = \frac{1}{\mathrm{T}} \text{ rad/s} \qquad f_C = \frac{\omega_C}{2\pi} = \frac{R}{2\pi L} \text{ Hz}$$

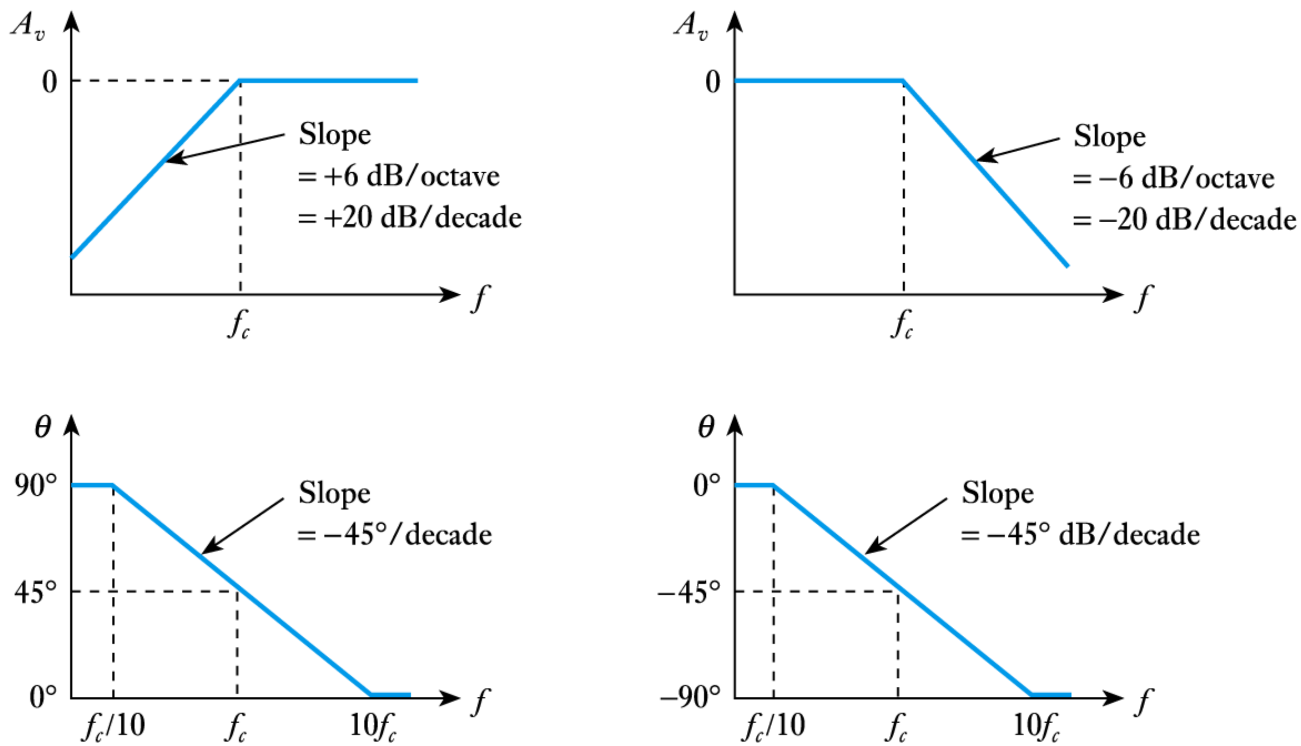
A Comparison of *RC* and *RL* Networks

- Circuits using RC and RL techniques have similar characteristics



Bode Diagrams

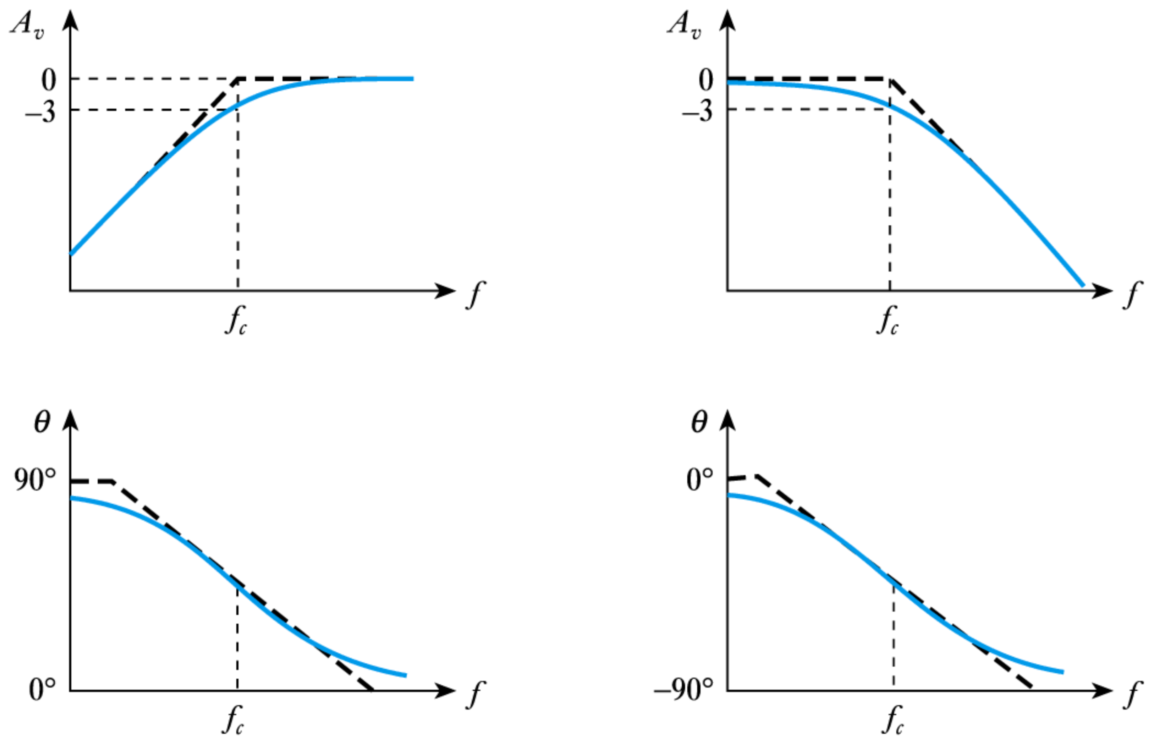
- Straight-line approximations



(a) High-pass circuit (b) Low-pass circuit

Bode Diagrams

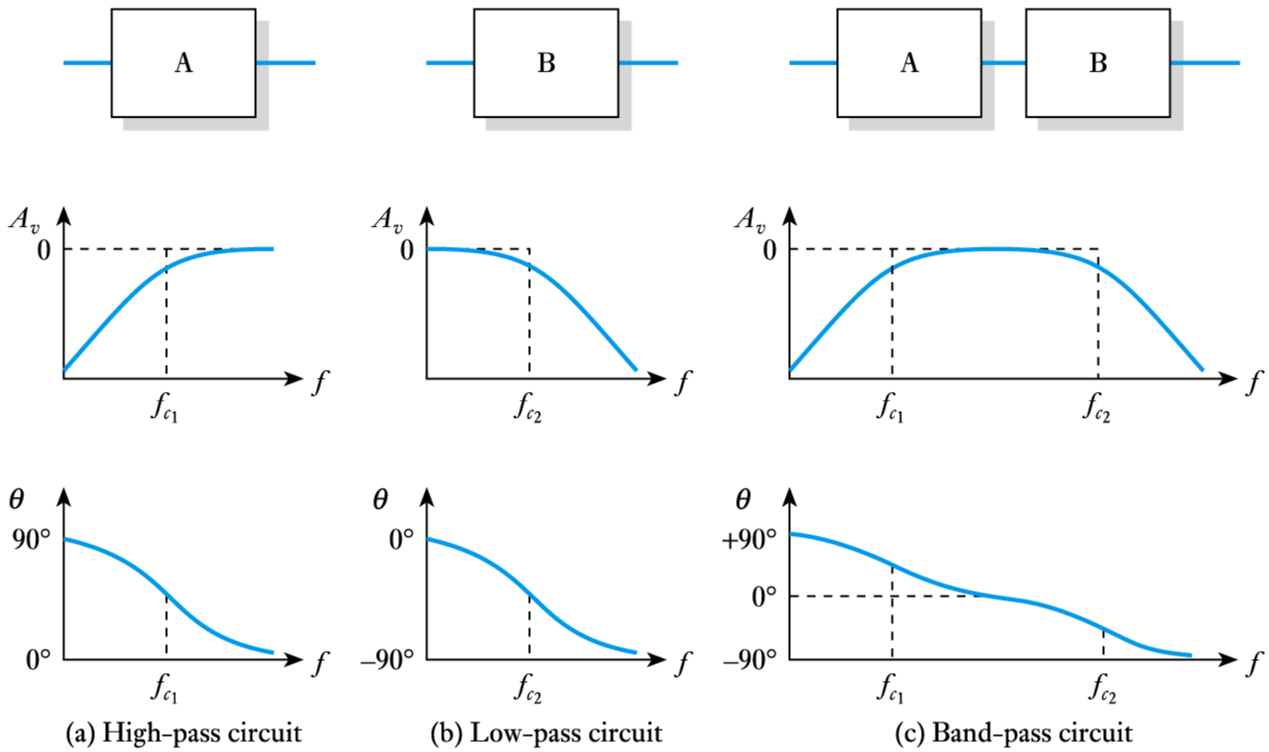
- Creating more detailed Bode diagrams



(a) High-pass circuit (b) Low-pass circuit

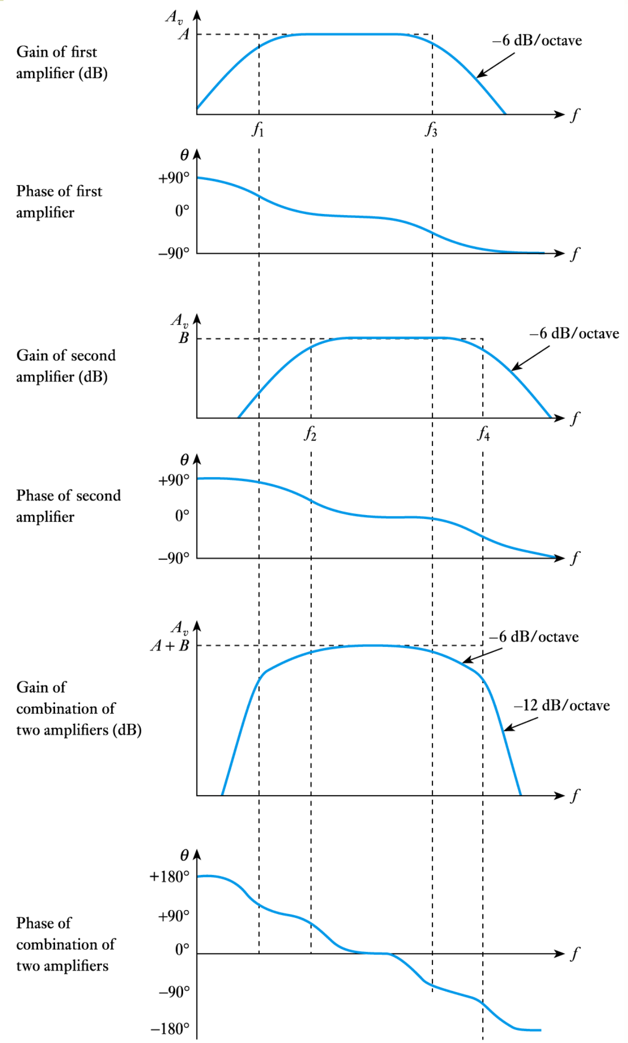
Combining the Effects of Several Stages

- The effects of several stages ‘add’ in bode diagrams



Combining the Effects of Several Stages

- Multiple high- and low-pass elements may also be combined



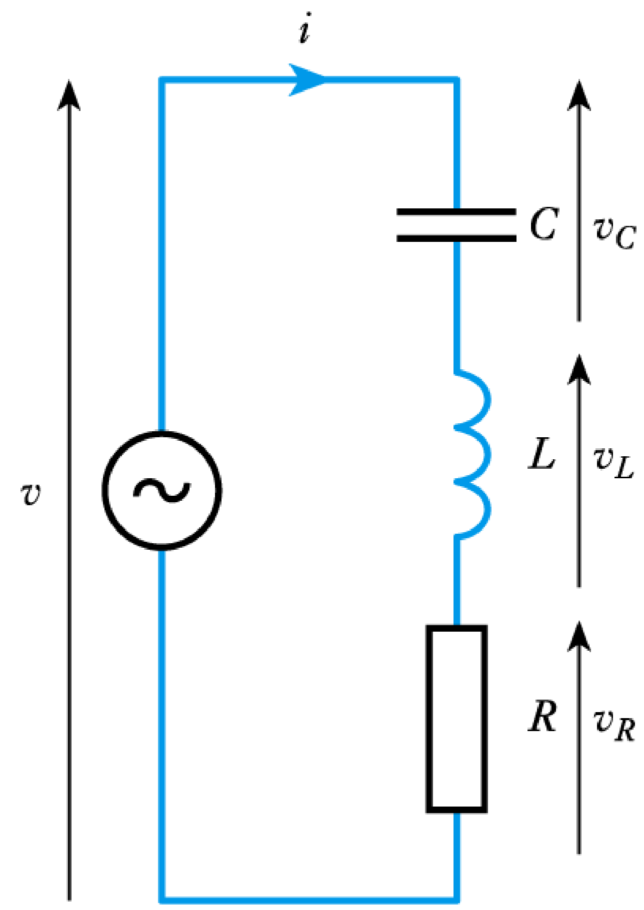
RLC Circuits and Resonance

- **Series *RLC* circuits**
  - the impedance is given by

$$\mathbf{Z} = R + j\omega L + \frac{1}{j\omega C} = R + j(\omega L - \frac{1}{\omega C})$$

  - if the magnitude of the reactance of the inductor and capacitor are equal, the imaginary part is zero, and **the impedance is simply *R***
  - this occurs when

$$\omega L = \frac{1}{\omega C} \qquad \omega^2 = \frac{1}{LC} \qquad \omega = \frac{1}{\sqrt{LC}}$$

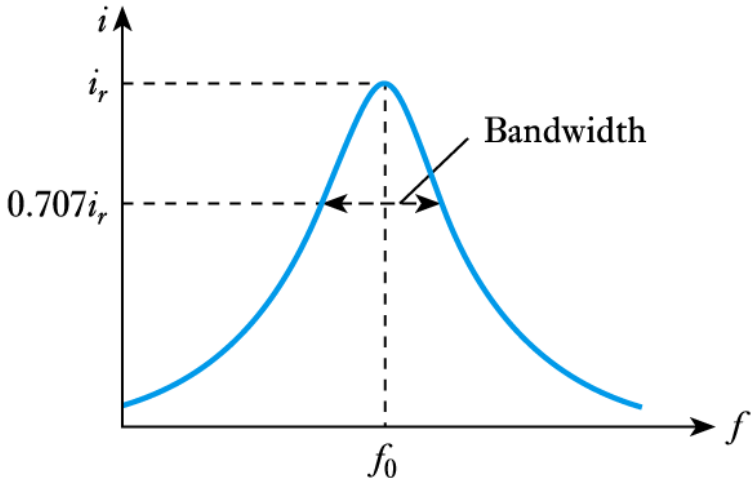


RLC Circuits and Resonance

- This situation is referred to as **resonance**
  - the frequency at which is occurs is the **resonant frequency**

$$\omega_o = \frac{1}{\sqrt{LC}} \qquad f_o = \frac{1}{2\pi\sqrt{LC}}$$

  - in the **series resonant circuit**, the *impedance* is at a minimum at resonance
  - the *current* is at a maximum at resonance



RLC Circuits and Resonance

- The resonant effect can be quantified by the **quality factor, Q**
  - this is the ratio of the energy dissipated to the energy stored in each cycle
  - it can be shown that

$$\text{Quality factor } Q = \frac{X_L}{R} = \frac{X_C}{R}$$

  - and

$$Q = \frac{1}{R}\sqrt{\left(\frac{L}{C}\right)}$$

RLC Circuits and Resonance

- The series *RLC* circuit is an **acceptor circuit** since
- it passes signals at frequencies close to its resonant frequency but rejects signals at other frequencies
- the narrowness of bandwidth is determined by the quality factor, Q

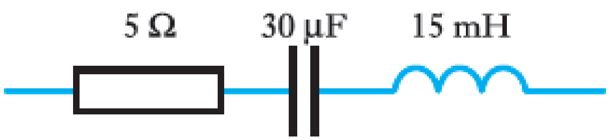
$$\text{Quality factor } Q = \frac{\text{Resonant frequency}}{\text{Bandwidth}} = \frac{f_o}{B}$$

- combining this equation with the earlier one gives

$$B = \frac{R}{2\pi L}\ \text{Hz}$$

RLC Circuits and Resonance

- For the following arrangement, calculate the resonant frequency $f0$, the impedance of the circuit at this frequency, the quality factor Q of the circuit and its bandwidth B.



$$f_0 = \frac{1}{2\pi\sqrt{LC}} = 237\ Hz$$

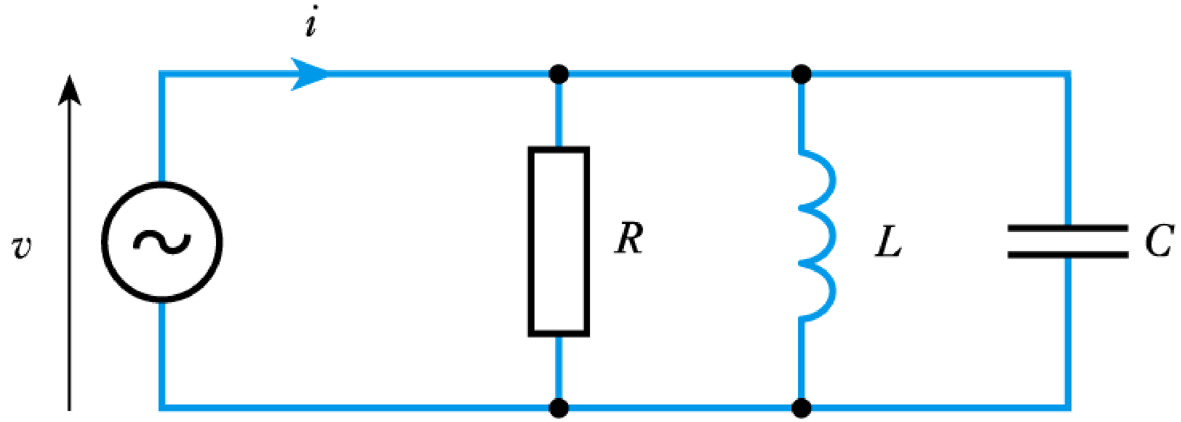
At the resonant frequency the impedance is equal to R, so Z = 5Ω

$$Q = \frac{1}{R}\sqrt{\left(\frac{L}{C}\right)} = 4.47$$

$$B = \frac{R}{2\pi L} = 53\ Hz$$

RLC Circuits and Resonance

- **Parallel *RLC* circuits**

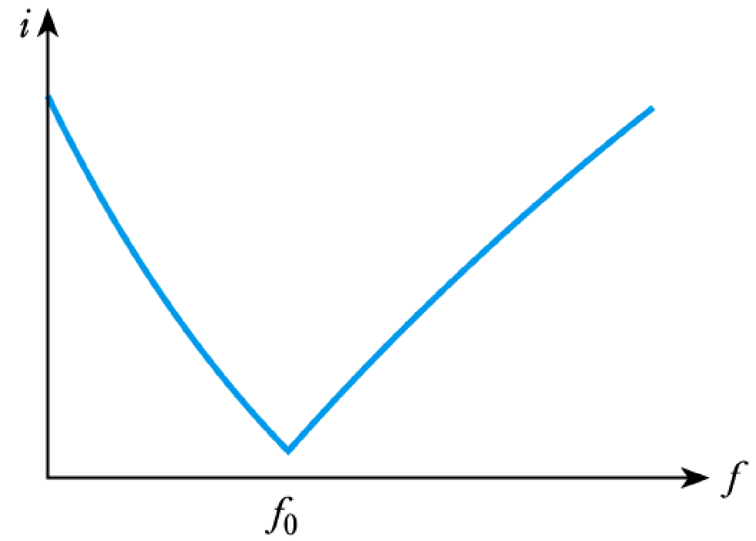


- as before

$$\omega_o = \frac{1}{\sqrt{LC}} \qquad f_o = \frac{1}{2\pi\sqrt{LC}}$$

RLC Circuits and Resonance

- The parallel arrangement is a **rejector circuit**
  - in the **parallel resonant circuit**, the *impedance* is at a maximum at resonance
  - the *current* is at a minimum at resonance
  - in this circuit



$$Q = R\sqrt{\left(\frac{C}{L}\right)}$$

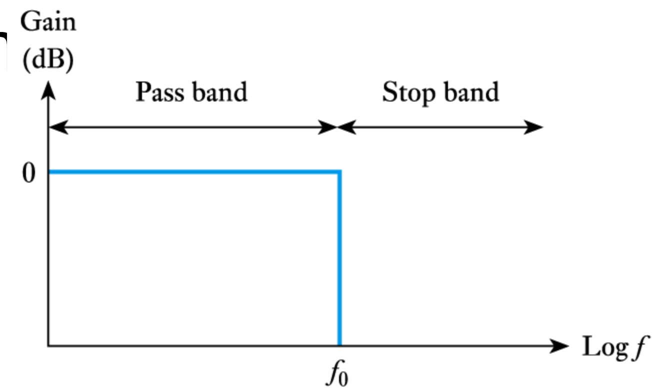
$$B = \frac{1}{2\pi RC}\ \text{Hz}$$

Filters

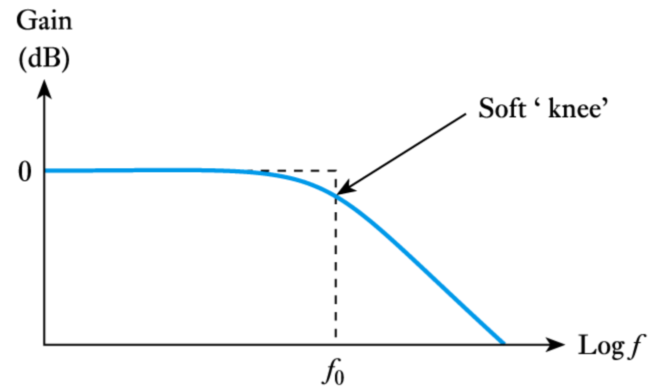
- ***RC* Filters**
- The *RC* networks considered earlier are **first-order** or **single-pole** filters
  - these have a maximum roll-off of 6 dB/octave
  - they also produce a maximum of 90° phase shift
- Combining multiple stages can produce filters with a greater ultimate roll-off rates (12 dB, 18 dB, etc.) but such filters have a very soft ‘knee’

Filters

- An ideal filter would have constar phase shift for frequencies within zero gain for frequencies outside (its **stop band**)
- Real filters do not have these idealised characteristics



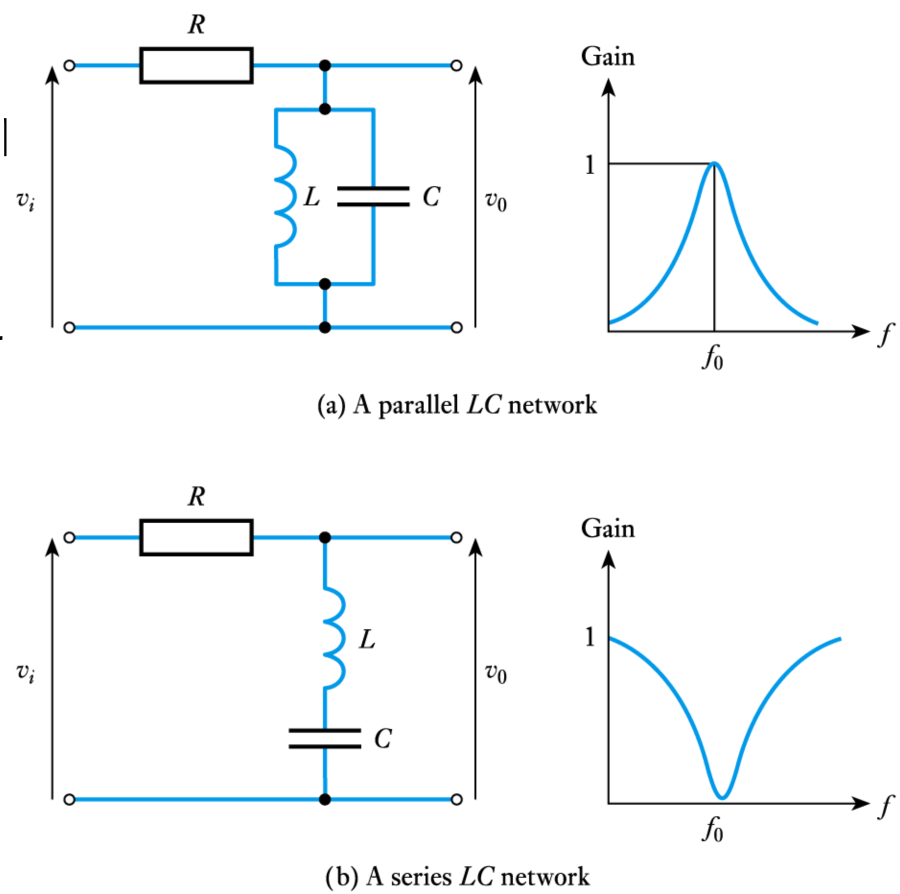
(a) An ideal low-pass filter



(b) A multi-stage *RC* filter

Filters

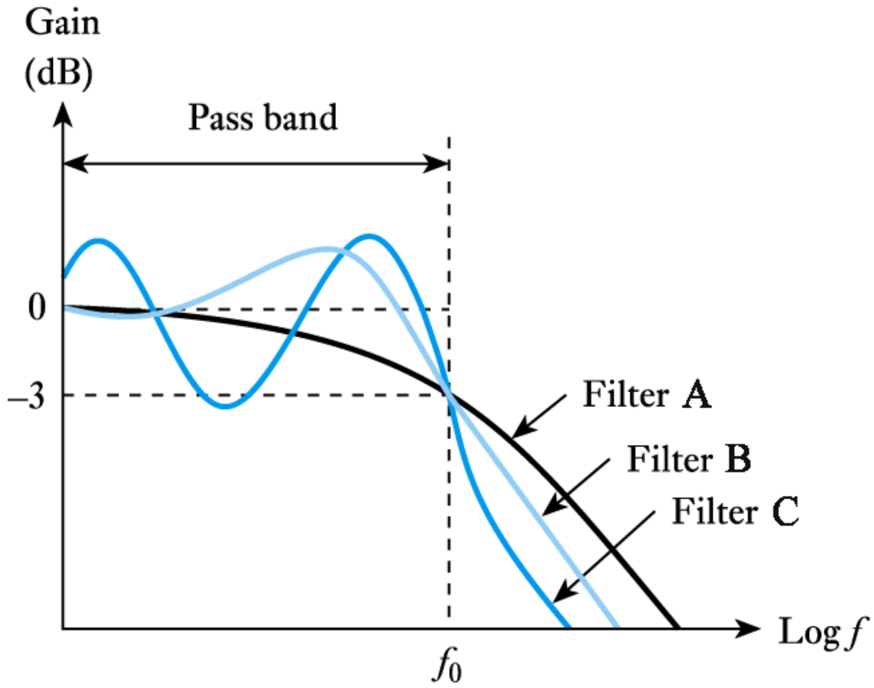
- ***LC* Filters**
- Simple LC filters can be pı
parallel tuned circuits
  - these produce narrow-ba
frequency f_o



$$f_o = \frac{1}{2\pi\sqrt{LC}}$$

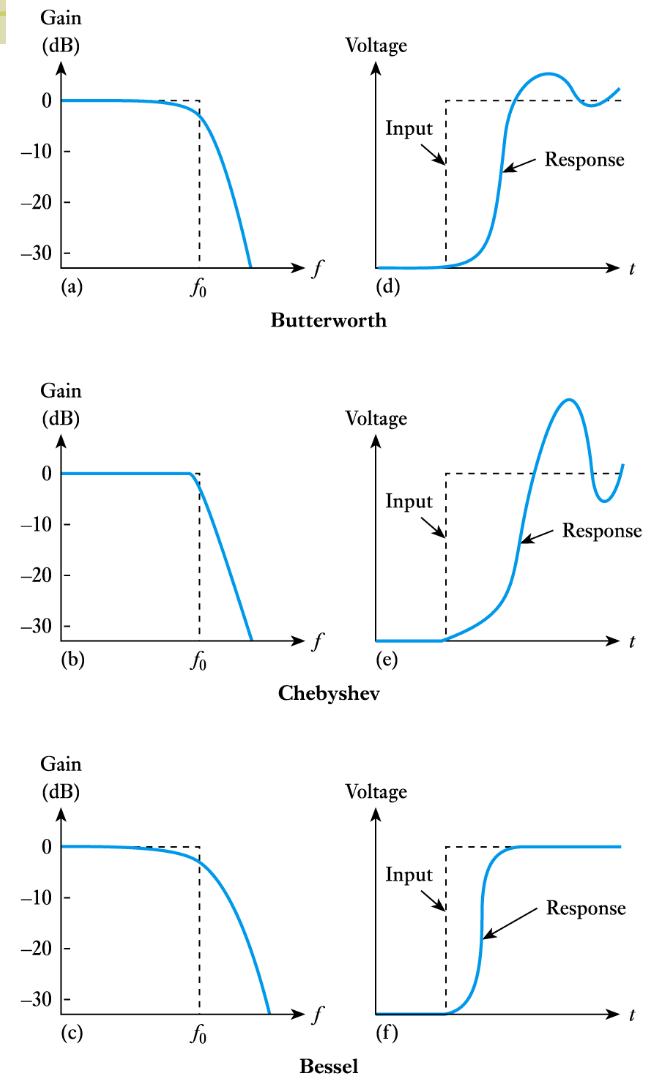
Filters

- **Active filters**
  - combining an op-amp with suitable resistors and capacitors can produce a range of filter characteristics
  - these are termed **active filters**



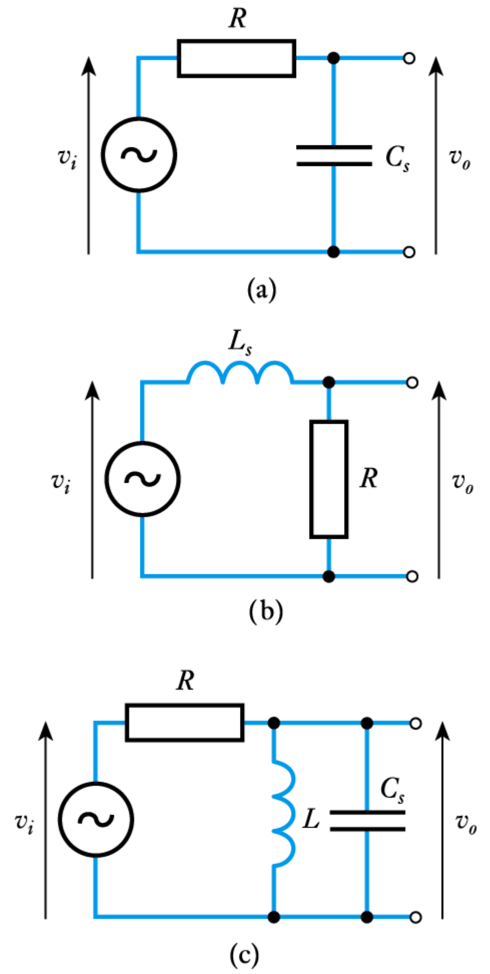
Filters

- Common forms include:
- **Butterworth**
  - optimised for a flat response
- **Chebyshev**
  - optimised for a sharp ‘knee’
- **Bessel**
  - optimised for its phase response



Stray Capacitance and Inductance

- All circuits have stray capacitance and stray inductance
  - these unintended elements can dramatically affect circuit operation
  - for example:
    - (a) C_s adds an unintended low-pass filter
    - (b) L_s adds an unintended low-pass filter
    - (c) C_s produces an unintended resonant circuit and can produce instability



Questions

Key Points

- The reactance of capacitors and inductors is dependent on frequency
- Single *RC* or *RL* networks can produce an arrangement with a single upper or lower cut-off frequency
- In each case the angular cut-off frequency ω_o is given by the reciprocal of the time constant T
- For an *RC* circuit $T = CR$, for an *RL* circuit $T = L/R$
- Resonance occurs when the reactance of the capacitive element cancels that of the inductive element
- Simple *RC* or *RL* networks represent single-pole filters
- Active filters produce high performance without inductors
- Stray capacitance and inductance are found in all circuits