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CHAPTER FOUR: ORDINARY DIFFERENTIAL EQUATIONS

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4. ORDINARY DIFFERENTIAL EQUATIONS

An ordinary differential equation, ODE is an equation that relates a function *y(x)*, to some of its derivatives, $y^r(x) = \frac{d^r y}{dx^r}$ where x is known as an *independent variable* and y is the *dependent variable*.

Generally, ordinary differential equations can be written as;

$$a_0(x)\frac{d^n y}{dx^n} + a_1(x)\frac{d^{n-1} y}{dx^{n-1}} + \cdots \dots\dots\dots\dots\dots a_{n-1}(x)\frac{dy}{dx} + a_n(x)y = f(x) \dots\dots\dots\dots\dots\dots\dots (1)$$

4.1 Order of an ODE

This is the highest derivative degree, n of y, the independent variable that occurs in the ODE.

The ODE in equation (1) is considered to be defined in some interval $a \leq x \leq b$

4.2 Common terms

- The functions $a_0(x)$, $a_1(x)$, ………$a_n(x)$ in equation (1) are called the *coefficients* of the ODE.
- The function $f(x)$ is called the *non – homogeneous* term or the *forcing function* of the ODE. It is called the forcing function because in real applications it represents the influence of an external input that drives a physical system represented by the ODE.
- Equation (1) is said to be homogenous if the forcing function, $f(x) = 0$, for the differential equation.

4.2.1 Variable and constant coefficient equations

The differential equation is said to be a *variable coefficient equation* if one or more of the coefficients, $a_0(x)$, $a_1(x)$, …….$a_n(x)$, depend(s) on x, the independent variable.

If all the coefficients do not depend on the independent variable, x, the differential equation is said to be a *constant coefficient equation* i.e. equation (2) with a_1, a_2,….a_n constants shows a constant coefficient equation:

$$a_0\frac{d^n y}{dx^n} + a_1\frac{d^{n-1} y}{dx^{n-1}} + \cdots \dots\dots\dots\dots\dots a_{n-1}\frac{dy}{dx} + a_n y = f(x) \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots (2)$$

4.2.2 Non linear ODE

Non linearity of an ODE arises because of the occurrence of a non linear function of the dependent variable, y, that sometimes occurs in form of a power or radical.

4.2.3 Degree of a differential equation

This is the greatest power to which the highest order derivative in the differential equation is raised after all radicals have been cleared from expressions involving the dependent, y.

Example 1

Classify the following differential equations as linear, non – linear, homogenous or non – homogenous indicating whether each is a constant coefficient or variable coefficient and its order:

(i) $\frac{dy}{dx} + 3xy = cosx$

This is a linear, variable coefficient non – homogenous first order equation.

(ii) $(2 - x^3)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + 5y = 0$

This is a linear, variable coefficient homogenous second order equation.

(iii) $3\frac{d^2y}{dx^2} + 7\frac{dy}{dx} + 6y = sin\omega x$; with $\omega - constant$

This is a linear constant coefficient non – homogenous second order equation.

(iv) $\frac{d^2\theta}{dt^2} + Ccos\theta = 0$, with, $C - constant$

This is a non – linear constant coefficient second order homogenous equation. It is non – linear because, θ, occurs non – linearly in the function $cos\theta$.

(v) $K\frac{d^2y}{dx^2} = f(x)\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}}, K > 0$

This is a non – linear variable coefficient second order differential equation of degree 2 involving both a power and a radical.

4.2.4 Solutions of an ODE

This is a function, $y = \varphi(x)$ that when substituted into the differential equation makes the equation identically zero over the interval on which the differential equation is defined.

General solution of an ODE

The solution of an nth order equation that contains *n* arbitrary constants is called the *general solution* of the equation.

Particular solution

This is the solution of an nth order differential equation in which the *n* arbitrary constants are assigned specific values.

Singular solution

This is a solution of a differential equation that cannot be obtained from the general solution for any choice of its arbitrary constants.

Note

Linear equations have no singular solutions, i.e. all its solutions can be obtained from the general solution.

Example 2

(a) The general solution of the constant, non – homogenous equation, $\frac{d^2y}{dx^2} - 4y = x$, is;

$y = Ae^{2x} + Be^{-2x} - \frac{x}{4}$, where A and B are constants i.e. substituting this in the equation gives, $x \equiv x$

(b) The differential equation, $\left(\frac{dy}{dx}\right)^2 + y^2 = 1$ has the general solution, $y = sin(x + A)$, because substitution of this into the differential equation gives, $1 \equiv 1$. However, $y = \pm 1$ are also solutions of the differential equation, though they cannot be obtained for any chosen value of A. Hence $y = \pm 1$ are singular solutions of the differential equation in question.

Example 3

Verify that the given, y, is a solution of the ODE and determine the particular solution that satisfies the given initial conditions:

(i) $y' = 0.5y, \quad y = Ce^{0.5x}, \quad y(2) = 2$

(ii) $y' = ytanx, \quad y = Csecx, \quad y(0) = \frac{1}{2}\pi$

(iii) $y' = 1 + 4y^2, \quad y = \frac{1}{2}\tan(2x + C), \quad y(0) = 0$

4.2.5 The linear operator

The linear equation (1) can be written as; $L(y) = f(x)$. Therefore, $L(y)$, is known as the linear operator of the equation. i.e.

$$L(y) = a_0(x)\frac{d^ny}{dx^n} + a_1(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots \dots\dots\dots\dots\dots . . a_{n-1}(x)\frac{dy}{dx} + a_n(x)y$$

4.3 Principle of linear superposition

If $y_1(x), y_2(x), \dots\dots\dots y_m(x)$ are solutions of the nth order homogenous equation, $L(y) = 0$, with $m \leq n$ and $C_1, C_2, \dots\dots C_m$ being arbitrary constants, then the linear combination;

$y(x) = C_1y_1(x) + C_2y_2(x) + \cdots \dots\dots\dots C_my_m(x)$ called the *linear superposition* of the m solutions is also a solution to the homogenous differential equation.

4.4 Initial conditions

These are auxiliary conditions specified at a single point $x = a$. An *initial value problem* (i.v.p) involves finding a solution of a differential equation that satisfies prescribed initial conditions.

4.5 Boundary conditions

Auxiliary conditions that are specified at two or more different points i.e. $x = a$ and $x = b$, are called *boundary conditions*. A boundary value problem (b.v.p) involves finding a solution of a differential equation that satisfies prescribed boundary conditions.

Example 5: b.v.p

The linear homogenous ordinary differential equation; $\frac{d^2y}{dx^2} + y = 0$ has a general solution, $y = Acosx + Bsinx$. Given y(0) = 0 and $y'\left(\frac{\pi}{3}\right) = 3$, find the particular solution of the equation.

Solution

Given, $y(0) = 0$, then;

$$0 = Acos0 + Bsin0 \Rightarrow A = 0$$

$$y'(x) = -Asinx + Bcosx, \qquad y'\left(\frac{\pi}{3}\right) = 3$$

$$\therefore 3 = Bcos\frac{\pi}{3} \Rightarrow B = 6$$

Therefore the particular solution is, $y = 6sinx$

Exercise 4.1

Determine the order and degree of the following differential equations and classify each as homogenous, linear, non – homogenous or non – linear:

(i) $y''' + 3y'' + 4y' - y = 0$

(ii) $\frac{y'}{y} + sinx = 3$

(iii) $(y'')^{3/2} + xy' = [(1 + x)y']$

(iv) $y' + 3xy = 1 + x^2$

4.6 Separable equations

Sometimes the function ,*f(x,y)*, in the first order equation, $\frac{dy}{dx} = f(x,y)$, can be written as the product of a function, *F(x),* depending on, *x*, only and, *G(y)*, depending on, *y*, only, i.e.

$$f(x,y) = F(x)G(y)$$

The differential equation can therefore be written as; $\frac{dy}{dx} = F(x)G(y)$

This can further be written as;

$\frac{1}{G(y)}dy = F(x)dx$, provided, $G(y) \neq 0$

This last equation can be solved by routine integration.

Example 6

Solve the initial value problem; $x^2y^2dx - (1 + x^2)dy = 0$ given that *y(0) = 1*

Solution

The equation is separable since it can be written as; $\frac{dy}{y^2} = \frac{x^2}{1+x^2}dx$

Integrating on both sides leads to;

$$\int \frac{dy}{y^2} = \int \frac{x^2}{1 + x^2}dx \Rightarrow -\frac{1}{y} = \int \frac{x^2}{1 + x^2}dx$$

Letting $1 + x^2 = 1 + tan^2u \Rightarrow tanu = x \ \ and \ \ dx = sec^2udu$

$$\therefore -\frac{1}{y} = \int \frac{tan^2u}{sec^2u}.sec^2udu = \int (sec^2u - 1)du$$

$$\Rightarrow -\frac{1}{y} = tanu - u + C = x - tan^{-1}x + C$$

Given, $y(0) = 1 \Rightarrow -\frac{1}{1} = 0 - tan^{-1}(0) + C \ \ \therefore C = -1$

$$-\frac{1}{y} = x - tan^{-1}x - 1 \Rightarrow y = \frac{1}{tan^{-1}x + 1 - x}$$

Example 7

Find the general solution of; $\frac{dy}{dx} = (1 + 2x)(2 + 3y)$

Solution

This is a separable variable problem that can be written as;

$$\frac{dy}{2 + 3y} = (1 + 2x)dx \Rightarrow \int \frac{dy}{2 + 3y} = \int (1 + 2x)dx$$

$$\therefore \frac{1}{3}ln(2 + 3y) = x + x^2 + C \Rightarrow ln(2 + 3y) = 3x + 3x^2 + A$$

Example 8

Find the general solution of; $(1 + 3y^2)y' + 2yln|1 + x| = 0$

Solution

The equation can be written as; $(1+3y^2)\frac{dy}{dx}+2yln|1+x|=0$

This is a separable variable problem that can be written as;

$$\frac{1+3y^2}{2y}dy=-ln|1+x|dx \Rightarrow \int\left(\frac{1}{2y}+\frac{3}{2}y\right)dy=-\int ln|1+x|dx$$

$$\therefore \frac{1}{2}lny+\frac{3}{4}y^2=-\int ln|1+x|dx$$

If we let; $dv=1 \Rightarrow v=x$ and $u=ln|1+x| \Rightarrow du=\frac{1}{1+x}$

$$\therefore \int ln|1+x|dx=xln|1+x|-\int\frac{x}{1+x}dx$$

Letting, $p=1+x \Rightarrow dx=dp$

$$\therefore \int\frac{x}{1+x}dx=\int\frac{p-1}{p}dp=p-lnp+C$$

$$\Rightarrow \int ln|1+x|dx=xln|1+x|-(1+x)+ln|1+x|+C=(1+x)[ln|1+x-1|]+C$$

$$\therefore \frac{1}{2}lny+\frac{3}{4}y^2=-(1+x)[ln|1+x|-1]+B, \quad B=-C$$

Problems involving modeling

Example 9

The tank shown in Figure 1.1 below contains 1000 m^3 of water in which initially 100 kg of salt is dissolved. Brine runs in at a rate of 10m^3/min and each m^3 contain 5 kg of dissolved salt. The mixture in the tank is kept uniform by stirring. Brine runs out at 10 m^3/min. Find the amount of salt in the tank after 5 minutes.

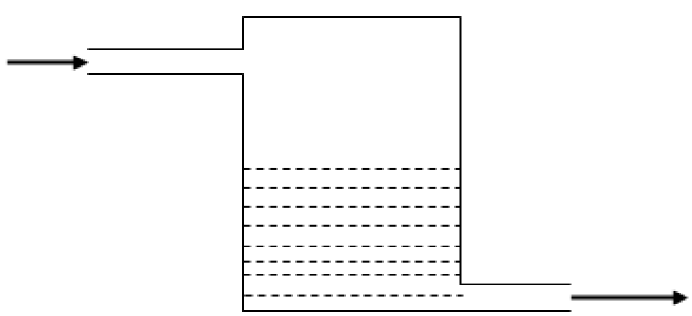

Figure1.1 Tank mixing model

Solution

Let, *y(t)*, denote the amount of salt in the tank at any time, *t*. It's rate of change is then given by;

$y' = salt\ inflow\ rate - salt\ out\ flow\ rate$

Inflow rate

$= 5(kg/m^3 * 10(m^3/min) = 50kg/min$

Outflow rate

10 m^3 of brine flow out. This is *10/1000 = 0.01* of the total brine in the tank. Hence *0.01* of the salt content, *y(t)*, i.e. *0.01y(t)* flows out per minute.

$$y' = 50 - 0.01y \Rightarrow \frac{dy}{dt} = 0.01(5000 - y)$$

$$\therefore \int \frac{1}{5000 - y} dy = \int 0.01dt \Rightarrow -ln(5000 - y) = 0.01t + C$$

$$\therefore 5000 - y = Ae^{-0.01t}, \qquad with\ \ A = e^C \Rightarrow y = 5000 - Ae^{-0.01t}$$

But initially at t = 0, $y(0) = 100$

$$\Rightarrow 100 = 5000 - A \therefore A = 4900$$

Therefore, $y = 5000 - 4900e^{-0.01t}$

At t = 5, $y = 5000 - 4900e^{-0.01*5} = 338.98\ kg$

Example 10: Heating an office

In winter the day time temperature in a certain office building is maintained at 70^{0}F. The heating is shut off at 10 PM and turned in again at 6 AM. On a certain day the temperature inside the building at 2 AM was found to be 65^{0}F. The outside temperature was 50^{0}F at 10 PM and had dropped to 40^{0}F by 6 AM. What was the temperature inside the building when the heat was turned on at 6 AM?

Solution

Let, *T(t)*, be the temperature inside the building and T_A the outside temperature.

By Newton's law;

$\frac{dT}{dt} = K(T - T_A)$, by separating variables;

$$\frac{dT}{T - T_A} = Kdt \Rightarrow \int \frac{dT}{T - T_A} = \int Kdt$$

$$\therefore ln|T - T_A| = Kt + C$$

T_A is determined as the average value of the given outside temperature, i.e. $T_A = \frac{40+50}{2} = 45^0F$

$$\therefore ln|T - 45| = Kt + C \Rightarrow T = 45 + Ae^{Kt}$$

At 10 PM, t = 0, T = 70^0F,

$\therefore 70 = 45 + Ae^0 \Rightarrow A = 25$

$\Rightarrow T = 45 + 25e^{Kt}$

To determine K we need to take the advantage of the other boundary condition, i.e.;

At 2 AM, t = 4 and T = 65^0F

$$\Rightarrow 65 = 45 + 25e^{4K} \Rightarrow K = \frac{ln0.8}{4} = -0.056$$

$\therefore T(t) = 45 + 25e^{-0.056t}$

At 6 AM, t = 8,

$\Rightarrow T(8) = 45 + 25e^{-0.056*8} = 61^0F$

Exercise 4.2

1. Solve the following differential equations:

   (i) $yy' = (x-1)e^{-y^2}, \qquad y(0) = 1$

   (ii) $2\sqrt{1+x^2}y' = \sqrt{1-y^2} \quad y(1) = 1$

2. Find the general solutions of the following differential equations:

   (i) $\sqrt{1+x^2}y' - 3x\sqrt{y^2-1} = 0$ Ans: $ln\left|y + \sqrt{y^2-1}\right| = 3(1+x^2)^{1/2} + C$

   (ii) $(1+x^2)yy' - x(y^2+y+1) = 0$ Ans: $\ln\left[\frac{(1+x^2)}{(y^2+y+1)}\right] + \frac{2}{\sqrt{3}}tan^{-1}\left(\frac{2y+1}{\sqrt{3}}\right) = C$

3. A simple model called Malthus' law for the change of silk worms population, *N(t)*, as a function of time, *t*, involves assuming that the rate of change is proportional to the population present at any time, *t*. Write down the differential equation governing, *N(t)*, if the constant of proportionality is $\lambda > 0$, and find an expression for, *N(t)*, given that initially $N(0) = N_0$. Find, λ, if $N(t_1) = N_1$ when $t = t_1$ and $N(t_2) = N_2$ when $t = t_2$ with $N_1 > N_2$ and $t_2 > t_1$.

   Ans: $\lambda = {ln\left(\frac{N_2}{N_1}\right)}\Big/{(t_2 - t_1)}$

4.7 Homogenous first order equations reducible to separable form

The first order ODE in differential form, $P(x,y)dx + Q(x,y)dy = 0$ is homogenous and reducible to a separable variable form if *P* and *Q* are homogenous functions of the same degree or if the differential equation is written in the form, $\frac{dy}{dx} = f(x,y)$, the function *f(x ,y)*, can be written as; $f(x,y) = g(y/x)$.

When we substitute, $y = ux$, the homogenous equation will be reduced to an equation involving the independent variable, *u*, in which variables are separable.

Example 11

Solve the differential equation, $(y^2 + 2xy)dx - x^2dy = 0$

Solution

The differential equation terms are all homogenous of degree 2 hence the deferential equation is homogenous and reducible.

Putting, $y = ux \Rightarrow \frac{dy}{dx} = x\frac{du}{dx} + u$

From $(y^2 + 2xy)dx - x^2dy = 0 \Rightarrow y^2 + 2xy - x^2\frac{dy}{dx} = 0$

Substituting for the variable, *u*,

$$u^2x^2 + 2x^2u - x^2\left(x\frac{du}{dx} + u\right) = 0 \Rightarrow u(u+2) - x\frac{du}{dx} - u = 0$$

This is separable variable problem, separating the variables leads to;

$$\int \frac{1}{u(u+1)}du = \int \frac{1}{x}dx$$

Decomposing the integrand on the left hand side into partial fractions leads to,

$$\int \frac{1}{u}du - \int \frac{1}{u+1}du = \int \frac{1}{x}dx \Rightarrow lnu - \ln(u+1) = \ln x + C$$

$$ln\left(\frac{u}{u+1}\right) = lnAx, \quad with\ C = lnA \Rightarrow \frac{u}{u+1} = Ax$$

$$\therefore u = \frac{Ax}{1-Ax}, \quad but\ u = \frac{y}{x} \quad \Rightarrow y = \frac{Ax^2}{1-Ax}$$

Example 12

Solve; $\frac{dy}{dx} = \frac{y^2}{xy-x^2}$

Solution

The equation is homogenous and reducible to separable form because it can be written in the form,

$$\frac{dy}{dx} = \frac{(y/x)^2}{y/x - 1}$$

Substituting, $y = ux \Rightarrow \frac{dy}{dx} = u + x\frac{du}{dx}$

$$\therefore u + x\frac{du}{dx} = \frac{u^2}{u-1} \Rightarrow x\frac{du}{dx} = \frac{u^2}{u-1} - u = \frac{u}{u-1}$$

Separating variables yields;

$$\int \frac{u-1}{u} du = \int \left(1 - \frac{1}{u}\right) du = \int \frac{1}{x} dx \quad \therefore u - lnu = lnx + C$$

Putting, $u = y/x$

$$\Rightarrow \frac{y}{x} = ln(y/x) + lnx + lnA \Rightarrow \frac{y}{x} = lnyA \qquad \therefore y = Be^{y/x}$$

Example 13

Find the general solution of, $\frac{dy}{dx} = \frac{2x + ycos^2(y/x)}{xcos^2(y/x)}$

Solution

This equation is homogenous and reducible to separable form since it can be written as;

$$\frac{dy}{dx} = \frac{2 + \frac{y}{x}cos^2(y/x)}{cos^2(y/x)}$$

Substituting for, $y = ux \quad and \quad \frac{dy}{dx} = u + x\frac{du}{dx}$, gives;

$$u + x\frac{du}{dx} = \frac{2 + ucos^2u}{cos^2u} \Rightarrow x\frac{du}{dx} = \frac{2}{cos^2u}$$

Separating the variables, we get;

$\int cos^2u \, du = \int \frac{2}{x} dx$, using a substitution; $cos^2u = \frac{1}{2}(cos2u + 1)$ gives;

$$\frac{1}{2}\left(\frac{1}{2}sin2u + u\right) = 2lnx + K \Rightarrow sin2u + 2u = 4lnx + C$$

$$\therefore 2cos\left(\frac{y}{x}\right)sin\left(\frac{y}{x}\right) + \frac{2y}{x} = 4lnx + C \Rightarrow y = xK + 2xlnx - xcos\left(\frac{y}{x}\right)sin\left(\frac{y}{x}\right)$$

Exercise 4.3

Find the general solutions of the following:

(i) $y'(2x + y) = y$ Ans: $\frac{x}{y^2} + \frac{1}{y} = C$

(ii) $y' = \frac{x-y}{x+2y}$ Ans: $-\frac{1}{2}x^2 + xy + y^2 = C$

4.8 Exact equations

The first order ODE, $M(x,y)dx + N(x,y)dy = 0$, is said to be exact if a function, *F(x,y)*, exists such that the total differential is;

$$d[F(x,y)] = M(x,y)dx + N(x,y)dy \qquad (3)$$

So since, $M(x,y)dx + N(x,y)dy = 0$, then, $d[F(x,y)] = 0$. Hence the general solution of equation 3 must be; $F(x,y) = constant$

Example 14

The total differential of $F(x,y) = 3x^3 + 2xy^2 + 4y^3 + 2x$ is given by;

$$d[F(x,y)] = \left(\frac{\partial F}{\partial x}\right)dx + \left(\frac{\partial F}{\partial y}\right)dy = (9x^2 + 2y^2 + 2)dx + (4xy + 12y^2)dy$$

So the equation, $(9x^2 + 2y^2 + 2)dx + (4xy + 12y^2)dy = 0$, is exact.

It can be shown that if an equation of the form 3 is not exact, it can be made exact by multiplying equation 3 by a suitable factor, $\mu(x,y)$, called an *integrating factor*.

4.8.1 Test for exactness

If, $F(x,y) = constant$, is a solution of the exact differential equation,

$M(x,y)dx + N(x,y)dx = 0$, then;

$M(x,y) = \frac{\partial F}{\partial x}$ and $N(x,y) = \frac{\partial F}{\partial y}$, so;

Provided that $\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial^2 F}{\partial x \partial y}$ and $\frac{\partial^2 F}{\partial y \partial x}$ are defined and continuous in the region with in which the differential equation is defined, the mixed derivatives must be equal, i.e. $\frac{\partial^2 F}{\partial x \partial y} = \frac{\partial^2 F}{\partial y \partial x}$

This result requires that $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, in order that the equation is exact.

Therefore, in short, the differential equation, $M(x, y)dx + N(x, y)dx = 0$ is exact if and only if; $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

Example 15

Test for exactness of the differential equations;

(a) $[\sin(xy+1) + xycos(xy+1)]dx + x^2\cos(xy+1)\,dy = 0$

(b) $(2x + siny)dx + (2xcosy + y)dy = 0$

Solutions

(a) $M = \sin(xy+1) + xycos(xy+1)$

$$\Rightarrow \frac{\partial M}{\partial y} = xcos(xy+1) + xcos(xy+1) - xy.xsin(xy+1)$$

$$= 2xcos(xy+1) - x^2ysin(xy+1)$$

$N = x^2\cos\,(xy+1)$

$$\Rightarrow \frac{\partial N}{\partial x} = 2xcos(xy+1) - x^2ysin(xy+1)$$

Since, $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, the differential equation is exact.

(b) $M = 2x + siny \Rightarrow \frac{\partial M}{\partial y} = 2 + cosy$

$$N = 2xcosy + y \Rightarrow \frac{\partial N}{\partial x} = 2cosy + 1$$

Since, $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, the equation is not exact.

Example 16

Show that the following equation is exact and find its general solution;

$[3x^2 + 2y + 2\cosh(2x+3y)]dx + [2x + 2y + 3\cosh(2x+3y)]dy = 0$

Solution

$$M = 3x^2 + 2y + 2\cosh(2x+3y) \Rightarrow \frac{\partial M}{\partial y} = 2 + 6\sinh(2x+3y)$$

$$N = 2x + 2y + 3\cosh(2x+3y) \Rightarrow \frac{\partial N}{\partial x} = 2 + 6\sinh(2x+3y)$$

Since, $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, the differential equation is exact.

The general solution of the differential equation is a function, *F(x,y)*, given by;

$$F(x,y) = \int M(x,y)dx = \int [3x^2 + 2y + 2\cosh(2x+3y)]\,dx$$

$$= x^3 + 2yx + \sinh(2x+3y) + C_1 + f(y)$$

And,

$$F(x,y) = \int N(x,y)dy = \int [2x + 2y + 3\cosh(2x+3y)]\,dy$$

$$= 2xy + y^2 + \sinh(2x+3y) + C_2 + g(x)$$

For these expressions to be identical then, $f(y) = y^2,\ g(x) = x^3\ \ and\ \ C_1 = C_2 = C$

$\therefore F(x,y) = x^3 + y^2 + 2xy + \sinh(2x+3y) + C$

Exercise 4.4

1. Test for exactness of the following and when the equation is exact, find its general solution:

(i) $[\sin(3y) + 4x^2y]dx + [3x\cos(3y) + y + 2x^3]dy = 0$

(ii) $[4x^3 + 3y^2 + \cos x]dx + [6xy + 2]dy = 0$ Ans: $F(x,y) = x^4 + \sin x + 3xy^2 + 2y + C$

(iii) $\left[x^2 + \frac{4}{y}\right]dy + \left[2xy + \frac{6}{x}\right]dx = 0;$ Ans: $x^2y + 6\ln x + 4\ln x = C$

(iv) $\left[\frac{2x}{2x+3y^2} - \frac{2x^2}{(2x+3y^2)^2} + 2\right]dx - \frac{6x^2y}{(2x+3y^2)^2}dy = 0$ Ans: $F(x,y) = \frac{x^2}{2x+3y^2} + 2x = C$

2. Under what conditions for the constants A, B, C, and D is the equation; $(Ax + By)dx + (Cx + Dy) = 0$ exact? Solve the exact solution.

4.9 Linear first order equations reducible to exactness

The standard form of the first order differential equation is; $\frac{dy}{dx} + P(x)y = Q(x)$, where, *P(x)*, and, *Q(x)*, are known functions.

To find the general solution of this equation, we multiply by a function, $\mu(x)$, still to be determined, hence;

$$\mu\frac{dy}{dx} + \mu P(x)y = \mu Q(x)$$

μ is chosen such that the left hand side of the equation becomes;

$\frac{d}{dx}(\mu y) = \mu Q(x)$……………………………………………………………………(a)

Hence integrating on both sides with respect to *x* gives;

$y(x) = \frac{C}{\mu(x)} + \frac{1}{\mu(x)}\int \mu(x)Q(x)\,dx$, where C is an arbitrary constant.

Note

μ(x), is known as the *integrating factor*, for the first order ODE.

From (a),

$$\mu\frac{dy}{dx} + y\frac{d\mu}{dx} = \mu\frac{dy}{dx} + \mu P(x)y \Rightarrow y\frac{d\mu}{dx} = \mu P(x)y$$

$\therefore \frac{d\mu}{dx} = P(x)dx \Rightarrow \frac{1}{\mu}d\mu = P(x)dx \Rightarrow \mu = Ae^{\int P(x)dx}$, where, *A,* is a constant.

Since μ multiplies the entire equation, the choice of *A* is immaterial, hence for simplicity we take, *A = 1*.

$\therefore \mu = e^{\int P(x)dx}$

4.9.1 Steps used to solve linear first order equation

1. If the equation is not in standard form and is written as, $a(x)\frac{dy}{dx} + b(x)y = c(x)$, divide through by, *a(x)*, to bring the equation to standard form. Hence,

   $P(x) = \frac{b(x)}{a(x)}$ and $Q(x) = \frac{c(x)}{a(x)}$

2. Find the integrating factor, $\mu = e^{\int P(x)dx}$
3. Re – write the original differential equation in the form; $\frac{d}{dx}(\mu y) = \mu Q(x)$
4. Integrate the equation in step 3 to obtain; $\mu(x)y(x) = \int \mu(x)Q(x)\,dx + C$
5. Divide the result of step 4 by, *μ(x)*, to obtain the required general solution.
6. If the initial condition, $y(x_0) = y_0$ is given, the required solution of the i.v.p is obtained by choosing the arbitrary constant, *C*, in the general solution.

Example 17

Solve the initial value problem, $\cos x\frac{dy}{dx} + y = \sin x, \quad y(0) = 2$

It is necessary to express the given equation in standard form;

$$\frac{dy}{dx} + \frac{1}{cosx} y = tanx$$

The integrating factor is given by;

$$\mu = e^{\int P(x)dx} = e^{\int secxdx} = e^{\ln(secx+tanx)} = secx + tanx$$

Therefore,

$$\frac{d}{dx}[(secx + tanx)y(x)] = (secx + tanx)tanx = secxtanx + tan^2 x$$

$$\Rightarrow \int \frac{d}{dx}[(secx + tanx)y(x)]dx = \int secxtanxdx + \int tan^2 xdx$$

Using the substitution, $tan^2 x = sec^2 x - 1$ and doing the subsequent integration gives;

$$(secx + tanx)y(x) = secx + tanx - x + C$$

$$\left(\frac{1+sinx}{cosx}\right)y(x) = \frac{1+sinx}{cosx} - x + c \Rightarrow y(x) = 1 + \frac{Ccosx}{1+sinx} - \frac{xcosx}{1+sinx}$$

Given $y(0) = 2$,

$$\Rightarrow 2 = 1 + C \Rightarrow C = 1$$

$$\therefore y(x) = 1 + \frac{(1-x)cosx}{1+sinx}$$

Example 18

Solve the i.v.p; $y' + ytanx = e^{-0.01x}cosx; \quad y(0) = 0$

Solution

The equation can be written as; $\frac{dy}{dx} + ytanx = e^{-0.01x}cosx$, this is already in standard form with, $P(x) = tanx$.

The integrating factor, $\mu(x) = e^{\int tanxdx} = e^{-lncox} = e^{\ln(cosx)^{-1}} = \frac{1}{cosx}$

$$\therefore \frac{d}{dx}(ysecx) = (e^{-0.01x}cosx)secx \Rightarrow \frac{d}{dx}(ysecx) = e^{-0.01x}$$

Integrating on both sides gives; $ysecx = -\frac{1}{0.01}e^{-0.01x} + C = -100e^{-0.01x} + C$

Putting, $y(0) = 0$,

$$0 = -100 + C \Rightarrow C = 100$$

$$\therefore ysecx = -100e^{-0.01x} + 100 = 100cosx(1 - e^{-0.01x})$$

Exercise 4.5

1. Solve the following equations:

   (i) $y' = 6(y - 2.5)tanh1.5x$

   (ii) $x^3y' + 3xy = 5sinh10x$

   (iii)$y' + 4ycot2x = 6cos2x; \quad y\left(\frac{\pi}{4}\right) = 2$

   (iv)$x\frac{dy}{dx} - y = x^2cosx; \quad y\left(\frac{\pi}{2}\right) = \pi$ Ans: $y = xsinx + x$

   (v) $sinx\frac{dy}{dx} + ycosx = 2sin^2x; \quad y\left(\frac{\pi}{2}\right) = 0$ Ans: $y = \frac{x}{sinx} - \frac{\pi}{2sinx} - cosx$

2. A particle of unit mass moves horizontally in a resisting medium with velocity, *v(t)*, at time, *t*, with a resistance opposing the motion given by, *Kv(t)*, with $K > 0$. If the particle is also subject to an additional resisting force, *Kt*, write down the differential equation for, *v(t)*, and hence find the value of K if the motion starts with velocity, $v(0) = v_0$, and at time, $t = \frac{1}{K}$, its velocity is, $v\left(\frac{1}{K}\right) = \frac{1}{4}v_0$. Ans: $v(t) = \frac{(v_0K-1)}{K}e^{-Kt} + \frac{1}{K} - t \quad and \quad K = \frac{4}{(4-e)v_0}$

4.10 The Bernoulli equation

The Bernoulli equation is a non – linear first order differential equation with the standard form; $\frac{dy}{dx} + P(x)y = Q(x)y^n; \quad n \neq 1.$

The substitution of, $u = y^{1-n}$, reduces the equation to a linear first order ODE of the form;

$\frac{1}{1-n}\frac{du}{dx} + P(x)u = Q(x)$, which can be solved by the *method of integrating factor*.

Once the solution *u(x)* is found, the general solution, *y(x)*, can also be obtained by returning to the expression, $u = y^{1-n}$

Note

It is important to first write the Bernouli equation in standard form before identifying *P(x)*, *Q(x)*, and *n*.

Example 18

Solve the Bernoulli equation; $\frac{da}{dt} + a = ta^2$

Solution

The equation is already in standard form with, $P(t) = 1$, $Q(t) = t$ and $n = 2$.

Putting, $u = a^{1-n} = a^{1-2} = \frac{1}{a}$

$$\Rightarrow \frac{da}{dt} = \frac{da}{du}.\frac{du}{dt} = -\frac{1}{u^2}.\frac{du}{dt}$$

$$\therefore -\frac{1}{u^2}.\frac{du}{dt} + \frac{1}{u} = t.\frac{1}{u^2} \Rightarrow \frac{du}{dt} - u = -t$$

This is a linear first order differential equation with, $P(t) = -1,\ and\ Q(t) = -t$

The integrating factor, $\mu(t) = e^{\int -dt} = e^{-t}$

$$\therefore \int \frac{d}{dt}(e^{-t}u)dt = -\int te^{-t}dt$$

Let, $u = t \Rightarrow du^* = 1 \quad and\ \ dv = e^{-t} \Rightarrow v = -e^{-t}$

By integration by parts;

$$(e^{-t}u) = -\left[-te^{-t} - \int -e^{-t}dt\right] = te^{-t} + e^{-t} + C$$

$$\therefore u = t + 1 + Ce^{t} \Rightarrow a(t) = \frac{1}{Ce^{t} + t + 1}$$

Example 19

Find the general solution of; $x\frac{dy}{dx} + y = 2xy^{1/2}$

Solution

The equation can be written in standard form as; $\frac{dy}{dx} + \frac{y}{x} = 2y^{1/2}$

Hence, $P(x) = \frac{1}{x},\ Q(x) = 2\ and\ \ n = \frac{1}{2}$

Letting, $u = y^{1-1/2} = y^{-1/2} \Rightarrow y = u^{-2}$

$$\frac{dy}{dx} = \frac{dy}{du}.\frac{du}{dx} = -2u^{-3}.\frac{du}{dx} = -\frac{2}{u^3}.\frac{du}{dx}$$

$\Rightarrow -\frac{2}{u^3}.\frac{du}{dx} + \frac{u^{-2}}{x} = 2(u^{-2})^{1/2} \therefore \frac{du}{dx} - \frac{1}{2x}u = -u^2$, this is also a Bernoulli equation.

Putting $v = u^{1-2} = u^{-1} \Rightarrow u = v^{-1}\ and\ \frac{du}{dv} = -\frac{1}{v^2}$

$$\frac{du}{dx} = \frac{du}{dv}.\frac{dv}{dx} = -\frac{1}{v^2}.\frac{dv}{dx}$$

$\Rightarrow -\frac{1}{v^2}\frac{dv}{dx} - \frac{1}{2x}.\frac{1}{v} = v^{-2} \Rightarrow \frac{dv}{dx} + \frac{1}{2x}v = -1$, which is a linear first order ODE with $P(x) = \frac{1}{2x}, and\ Q(x) = -1$

The integrating factor, $\mu = e^{\int \frac{1}{2x}dx} = x^{1/2}$

$$\therefore \frac{d}{dx}\left(x^{\frac{1}{2}}v\right) = -1 \Rightarrow v = -x^{\frac{1}{2}} + \frac{C}{x^{\frac{1}{2}}} = \frac{C-x}{\sqrt{x}}$$

$$\Rightarrow u(x) = \frac{1}{v} = \frac{\sqrt{x}}{C-x} \quad and\ y(x) = \frac{1}{[u(x)]^2} = \frac{(C-x)^2}{x}$$

Exercise 4.6

Find the general solutions of the following equations:

(i) $\frac{dy}{dx} - y = 2xy^{3/2}$ Ans: $y^{1/2} = \frac{1}{4-2x+Ce^{-x/2}}$

(ii) $x\frac{dy}{dx} - 2y = xy^{3/2}$ Ans: $y^{1/2} = \frac{4x}{A-x^2}$

4.11 Some problems leading to ordinary differential equations

4.11.1 Chemical reactions and radioactive decay

In many circumstances, the rate of reaction of a chemical process can be considered to be proportional only to the amount, *Q*, of the chemical that is present at a given time, *t*. This leads to a differential equation of the form;

$$\frac{dQ}{dt} = KQ, \quad where\ \ K \geq 0$$

A similar scenario applies to a radioactive decay of an isotope for which the decay is proportional to the amount, *Q*, of the radioactive isotope that is present. But instead of the amount growing as in the previous case, it is decreasing; hence the governing differential equation is;

$$\frac{dQ}{dt} = -\mu Q$$

4.11.2 The logistic equation: population growth

The rate of increase in population of animals that compute for limited food resources is influenced by both the population present at any given instant and by the limitation of the resource that is necessary (e.g. food). Similar situations occur in manufacturing where there is competition for scarce resources and in a variety of similar situations.

Let *P* represent the amount of quantity e.g, population present at any time, *t*, and, *M*, represents the amount of resources available at the start, then;

$\frac{dP}{dt} = KP(M-P)$, i.e. the rate of increase, $\frac{dP}{dt}$, is proportional to the amount, *P*, that is present at time, t, and to the amount $M-P$ that remains.

4.11.3 Differential equations that model damped oscillations

Mechanical and electrical systems can exhibit oscillatory behavior that an initial disturbance slowly decays to zero. The process producing the decay is a ***dissipative one*** that removes energy from the system, and it is called *damping*.

Consider a mass, *m*, on a rough horizontal surface attached by a spring of negligible mass to a fixed point. The system is caused to oscillate when, *m*, is displaced slightly from its equilibrium position.

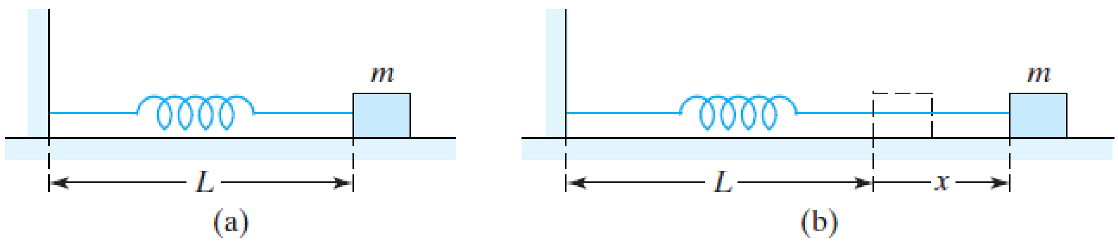

Figure1.2 Mass - spring system

The acceleration of the mass is, $\frac{d^2x}{dt^2}$.

So the force driving the motion is, $m\frac{d^2x}{dt^2}$, from Newton's 2nd law.

The opposing forces are the spring force (which is proportional to the displacement, *x*, and frictional force which is oftenly assumed to be proportional to the velocity, $\frac{dx}{dt}$.

If the spring constant is, K_s and the frictional constant of proportionality is, K_f, then the two opposing forces are, $K_s x$, and, $K_f\frac{dx}{dt}$.

Therefore, if the driving force applied in the direction of mass motion is, *f(t)*, by Newton's 2nd law, the governing differential equation is;

$$m\frac{d^2x}{dt^2} = f(t) - K_s x - K_f\frac{dx}{dt}$$

Dividing through by, *m*, and rearranging gives;

$$\frac{d^2x}{dt^2} + a\frac{dx}{dt} + bx = \frac{1}{m}f(t)$$

An equation similar to that above governs the oscillation of charge, *q*, in the R – L – C electric circuit.

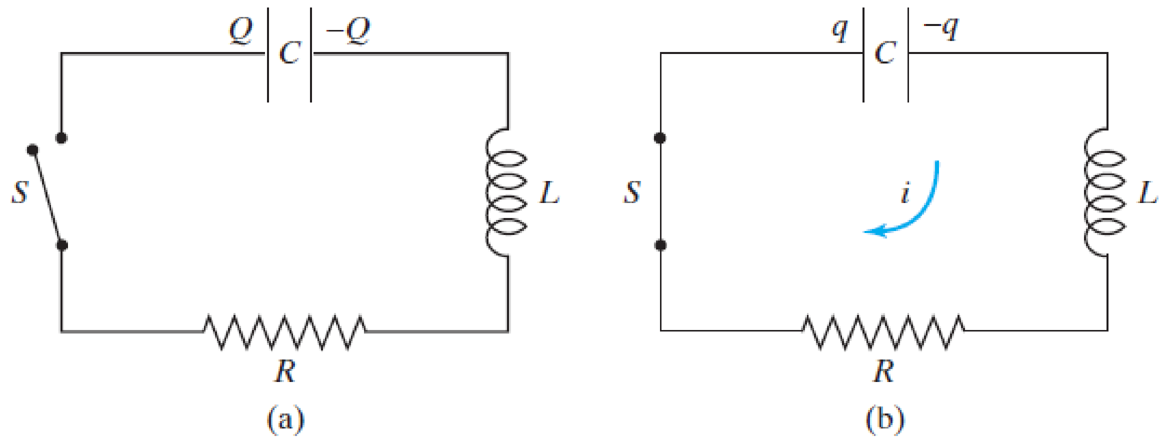

Figure1.3 An R - L - C circuit

The respective potential drops across the resistor, *R*, the inductance, *L*, and the capacitance, *C*, are; $V = iR,\ L\frac{di}{dt},\ and\ \frac{q}{C}$ respectively, where, $i = \frac{dq}{dt}$

By Kirchoff's law, i.e. the sum of potential drops around a closed loop is zero;

$\Rightarrow L\frac{di}{dt} + Ri + \frac{q}{C} = 0$, putting $i = \frac{dq}{dt}$ gives;

$$LC\frac{d^2q}{dt^2} + RC\frac{dq}{dt} + q = 0$$

If there is a total driving potential of, *E(t)*, then the governing equation becomes;

$$LC\frac{d^2q}{dt^2} + RC\frac{dq}{dt} + q = E(t)$$

4.12 Second order linear ODEs

Linear ODEs of second order are more important than non – linear ones because of their application in most Engineering systems.

A second order ODE is called linear if it can be written in the form;

$$y'' + P(x)y' + q(x)y = r(x)$$

If the equation begins with say, $f(x)y''$, then it is necessary to divide by, *f(x)*, to the equation have in the *standard form*.

If $r(x) \equiv 0$, then the equation reduces to; $y'' + P(x)y' + q(x)y = 0$ and is called *homogenous* but if $r(x) \neq 0$, the equation is then *non – homogenous*.

4.12.1 Homogenous linear 2nd order ODEs

The solution of the homogenous linear ODEs greatly depends on the *superposition principle* or the *linearity principle*.

Theorem

For a homogenous linear ODE of 2nd order, any linear combination of two solutions on an open interval I is again a solution of the equation.

4.12.2 Homogeneous linear ODEs with constant terms

Considering a differential equation of the form; $y" + ay' + by = 0$, with *a*, *b* being constants. To solve this equation we remember that the first order linear ODE; $y' + ky = 0$, has a solution, $y = Ce^{-kx}$.

This gives us an idea to try, $y = e^{\lambda x}$ as a solution of the equation in question. Substituting this and its derivatives; $y' = \lambda e^{\lambda x}$ and $y'' = \lambda^2 e^{\lambda x}$ into the differential equation, gives;

$(\lambda^2 + a\lambda + b)e^{\lambda x} = 0$

Hence, λ, is a solution of the equation, $\lambda^2 + a\lambda + b = 0$, known as the *characteristic equation*.

The two possible solutions of λ are; $\lambda_1 = \frac{1}{2}\left(-a + \sqrt{a^2 - 4b}\right)$ *and* $\lambda_2 = \frac{1}{2}\left(-a - \sqrt{a^2 - 4b}\right)$

Hence the two possible solutions of the ODE are; $y_1 = e^{\lambda_1 x}$ and $y_2 = e^{\lambda_2 x}$

There are three cases that can be encountered;

- Case I: Two real roots i.e. $a^2 - 4b > 0$
- Case II: A real double root i.e. $a^2 - 4b = 0$
- Case III: Complex conjugate roots i.e. $a^2 - 4b < 0$

Case I: Two distinct real roots

The two possible solutions are $y_1 = e^{\lambda_1 x}$ and $y_2 = e^{\lambda_2 x}$. By the principle of linearity, the general solution becomes; $y = C_1 e^{\lambda_1 x} + C_2 e^{\lambda_2 x}$

Example 1.12.1

Solve the i.v.p; $y'' + 6y' + 8y = 0; \quad y(0) = 1 \text{ and } y'(0) = 0$

Solution

The characteristic equation is; $\lambda^2 + 6\lambda + 8 = 0 \Rightarrow (\lambda + 2)(\lambda + 4) = 0$

$\therefore \lambda_1 = -2 \text{ and } \lambda_2 = -4$

The general solution is therefore, $y = C_1 e^{-2x} + C_2 e^{-4x}$

Given *y'(0) = 0*; $y' = -2C_1e^{-2x} - 4C_2e^{-4x}$

$\therefore 0 = -2C_1 - 4C_2 \Rightarrow C_1 = -2C_2$

Given, *y(0) = 1*,$\Rightarrow 1 = C_1 + C_2 \therefore C_1 = -1 \;\; and \;\; C_2 = 2$

Therefore, the particular solution is; $y = 2e^{-2x} - e^{-4x}$

Case II: Real double roots

If the discriminant, $a^2 - 4b = 0$, we see directly that $\lambda = \lambda_1 = \lambda_2 = -\frac{a}{2}$, hence the only solution we have is $y_1 = e^{-(a/2)x}$

But we very well know that a 2nd order ODE must yield atleast two independent solutions and substituting the solution, $y_2 = xe^{-(a/2)x}$, shows that it also satisfies the ODE. Therefore, the required general solution is; $y = C_1e^{-(a/2)x} + C_2xe^{-(a/2)x} = e^{-(a/2)x}(C_1 + xC_2)$

Example 20

Solve the initial value problem; $y'' + y' + 0.25y = 0; \quad y(0) = 3 \;\; and \;\; y'(0) = -3.5$

Solution

The characteristic equation is; $\lambda^2 + \lambda + 0.25 = (\lambda + 0.5)^2 = 0$

It has a double root, $\lambda = -0.5$

The general solution is thus; $y = (C_1 + xC_2)e^{-0.5x}$

Given, *y(0) = 3*, $\Rightarrow 3 = C_1$

$y' = -0.5C_1e^{-0.5x} + C_2e^{-0.5x} - 0.5C_2xe^{-0.5x}$

Since, $y'(0) = -3.5$

$\Rightarrow -3.5 = -0.5 * 3 + 1.5 + C_2 \Rightarrow C_2 = -2.0$

Hence the particular solution is; $y = (3 - 2x)e^{-0.5x}$

Case III: Complex roots

This occurs when the descriminant, $a^2 - 4b < 0$.

Let the solutions of the characteristic equation, $\lambda^2 + a\lambda + b = 0$ be, $\lambda_1 = \alpha + i\beta \;\; and \;\; \lambda_2 = \alpha - i\beta$, then it can be shown that the two possible independent solutions of the ODE are;

$y_1 = e^{\alpha x}cos\beta x \;\; and \;\; y_2 = e^{\alpha x}sin\beta x$, by use of the Euler formula (check “complex numbers, engineering mathematics 1”).

There follows that the general solution is; $y = e^{\alpha x}(Acos\beta x + Bsin\beta x)$

Example 21

Find the general solution of the ODE; $y'' + 2.4y' + 4y = 0$

Solution

The characteristic equation is; $\lambda^2 + 2.4\lambda + 4 = 0$, solving this gives;

$\lambda_1 = -1.2 + 1.6i \;\; and \;\; \lambda_2 = -1.2 - 1.6i$

The general solution is; $y = e^{-1.2x}(Acos1.6x + Bsin1.6x)$

Special case of $y'' \pm n^2 y = 0$

(a) $y'' + n^2 y = 0$

In this case, the characteristic equation is; $\lambda^2 + n^2 = 0 \therefore \lambda = \pm ni$

This is similar to the complex roots case, however, with $\alpha = 0$

The general solution is thus, $y = Acosnx + Bsinnx$

(b) **For** $y'' - n^2 y = 0$

The characteristic equation is $\lambda^2 - n^2 = 0 \Rightarrow \lambda = \pm n$

This is case is similar to the distinct real roots case; $\lambda_1 = n \;\; and \;\; \lambda_2 = -n$

Therefore the general solution is; $y = C_1 e^{nx} + C_2 e^{-nx}$

But from $sinhnx = \frac{e^{nx} - e^{-nx}}{2}$ and $coshnx = \frac{e^{nx} + e^{-nx}}{2}$

$e^{nx} - e^{-nx} = 2sinhnx$………………………………………….(*)

$e^{nx} + e^{-nx} = 2coshnx$………………………………………….(**)

From equations (*) and (**),

$e^{nx} = coshnx + sinhnx$ and $e^{-nx} = coshnx - sinhnx$

The general solution, $c_1 e^{nx} + C_2 e^{-nx} = (C_1 + C_2)coshnx + (C_1 - C_2)sinhnx$

$$= Acoshnx + Bsinhnx$$

Example 22

Solve the differential equations;

(i) $\frac{d^2y}{dx^2} + 9y = 0$

(ii) $\frac{d^2y}{dx^2} - 25y = 0$

Solutions

(i) The characteristic equation is; $\lambda^2 + 9 = 0 \Rightarrow \lambda = \pm in$

The general solution is; $y = Acosnx + Bsinnx$

(ii) The characteristic equation; $\lambda^2 - 25 = 0 \Rightarrow \lambda = 5$

The general solution is, $y = Acoshnx + Bsinhnx$

Summary

Table 1.1 shows the summary of the of the three cases of solution of second order homogenous ODEs

Table1.1 Summary of solutions of homogenous second order ODEs

Case	Roots	Basis	General solution
I	Distinct roots λ_1 and λ_2	$e^{\lambda_1 x}$ and $e^{\lambda_2 x}$	$y = C_1e^{\lambda_1 x} + C_2e^{\lambda_2 x}$
II	Real double $\lambda = -\frac{1}{2}a$	$e^{\lambda x}$ and $xe^{\lambda x}$	$y = (C_1 + C_2x)e^{\lambda x}$
III	Complex conjugates $\lambda_1 = \alpha + i\beta$ and $\lambda_2 = \alpha - i\beta$	$e^{\alpha x}cos\beta x$ $e^{\alpha x}sin\beta x$	$y = e^{\alpha x}(Acos\beta x + Bsin\beta x)$

Exercise 4.7

1.Find the general solutions of the following:

$$y'' - 6y' - 7y = 0$$

$$y'' + 4\pi y' + 4\pi^2 y = 0$$

$$y'' + 9\pi^2 y = 0$$

$$y'' - 144y = 0$$

$$10\frac{d^2\theta}{dt^2} - 7\frac{d\theta}{dt} + 1.2\theta = 0$$

2.Solve the i.v.p; $\frac{d^2z}{dt^2} + 2k\frac{dz}{dt} + (k^2 + \omega^2)z = 0; \quad z(0) = 1 \; and \; z'(0) = -k$

4.13 Non – homogenous ODEs

The non – homogenous linear second order ODE is of the form, $y'' + p(x)y' + q(x)y = r(x)$, with $r(x) \neq 0$. Such an equation has a solution of the form; $y(x) = y_h(x) + y_p(x)$, where;

- $y_h = C_1y_1 + C_2y_2$ is a general solution of the corresponding homogenous equation.
- $y_p(x)$ is any solution on the interval of the equation containing no arbitrary constants.
- $y_h(x)$, is normally referred to as the complementary solution of the ODE and, $y_p(x)$, the particular solution of the equation.

4.13.1 Determining of, y_p, using the method of undetermined coefficients

The function $y_p(x)$ is chosen according to Table 1.2 below, and then substituted with its derivatives, y_p' and y_p'' to determine any involved constants.

Table1.2 Choice of the solution y_p

Term in *r(x)*	**Choice of $y_p(x)$**
K	C
Ke^{ax}	Ce^{ax}

Kx^n (n = 0, 1, 2,………)	$K_nx^n + K_{n-1}x^{n-1} + \cdots \ldots\ldots\ldots.. + K_1x + K_0$
$Kcos\omega x$	$Acos\omega x + Bsin\omega x$
$Ksin\omega x$	
$Ke^{ax}sin\omega x$	$e^{ax}(Acos\omega x + Bsin\omega x)$
$Ke^{ax}cos\omega x$	

4.13.2 Choice rules

1. If a term in your choice of y_p, happens to be a solution of the homogenous ODE, multiply your choice, $y_p(x)$, by x (or by x^2 if this solution corresponds to a double root of the characteristic equation of the homogenous ODE).
2. If, $r(x)$, is a sum of functions in the first column of Table 1.2, choose y_p as the sum of the corresponding lines in the second column of Table 1.2.

Example 23

Solve the equation; $y'' + y = 0.001x^2;\ \ y(0) = 0\ \ and\ \ y'(0) = 1.5$

Solution

The characteristic equation of the associated homogenous equation is; $\lambda^2 + 1 = 0 \Rightarrow \lambda = \pm i$

Hence the complementary solution; $y_h = Acosx + Bsinx$

Given, $r(x) = 0.001x^2$

From Table 1.2, $y_p = K_2x^2 + K_1x + K_0$

$$\Rightarrow y_p' = 2K_2x + K_1 \text{ and } y_p'' = 2K_2$$

Substituting into the Ode in question, gives;

$2K_2 + (K_2x^2 + K_1x + K_0) = 0.001x^2 \Rightarrow K_2x^2 + K_1x + (2K_2 + K_0) = 0.001x^2$

Comparing coefficients, we have; $K_2 = 0.001, K_1 = 0\ \ and\ K_0 = -0.002$

The general solution is given by; $y = y_h + y_p$

$\therefore y = Acosx + Bsinx + 0.001x^2 - 0.002$

Given , $y(0) = 0$, $\Rightarrow 0 = A - 0.002 \Rightarrow A = 0.002$

$y'(x) = -Asinx + Bcosx + 0.002x$, given, $y'(0) = 1.5 \Rightarrow B = 1.5$

Therefore, the required solution is; $y = 1.5sinx + 0.001(cosx + x^2 - 2)$

Example 24

Solve, $y'' + 3y' + 2.25y = -10e^{-1.5x}; \;\; y(0) = 1, \; y'(0) = 0$

Solution

Solving the corresponding homogenous equation, gives; $y_h = (C_1 + C_2x)e^{-1.5x}$

From Table 1.2, with $r(x) = -10e^{-1.5x}$, we would choose, $y_p = Ce^{-1.5x}$, but this corresponds to a homogenous equation solution of double roots. Hence the right choice is; $y_p = Cx^2e^{-1.5x}$

Therefore, $y = (C_1 + C_2x)e^{-1.5x} + Cx^2e^{-1.5x}$

The solution, y_p, must also satisfy the ODE,

$y_p' = 2xCe^{-1.5x} - 1.5Cx^2e^{-1.5x} = C(2x - 1.5x^2)e^{-1.5x}$

$y_p'' = 2Ce^{-1.5x} - 3xCe^{-1.5x} - 3Cxe^{-1.5x} + 2.25Cx^2e^{-1.5x} = C(2 - 6x + 2.25x^2)e^{-1.5x}$

Substituting into the ODE, leads us to;

$C(2 - 6x + 2.25x^2)e^{-1.5x} + 3C(2x - 1.5x^2)e^{-1.5x} + 2.25Cx^2e^{-1.5x} = -10e^{-1.5x}$

Comparing coefficients on x^0; $2C = -10 \Rightarrow C = -5$

The general solution is thus; $y = (C_1 + C_2x)e^{-1.5x} - 5x^2e^{-1.5x}$

Using the initial conditions, $y(0) = 1 \;\; and \; y'(0) = 0$, gives; $C_1 = 1 \;\; and \;\; C_2 = 1.5$

Therefore, the particular solution is; $y = (1 + 1.5x)e^{-1.5x} - 5x^2e^{-1.5x}$

Example 25

Find the general solution of, $y'' + 4y' + 5y = 25x^2 + 13sin2x$

The characteristic equation of the homogenous equation is; $\lambda^2 + 4\lambda + 5 = 0$

$\Rightarrow \lambda_1 = -2 + i \;\;\; and \;\; \lambda_2 = -2 - i$

The complementary solution is thus; $y_h = e^{-2x}(Acosx + Bsinx)$

From the equation in question, $r(x) = 25x^2 + 13sin2x$, using Table 1.2 and choice rule 2;

$y_p = K_2x^2 + K_1x + K_0 + Csin2x + Dcos2x$

$\Rightarrow y_p' = 2K_2x + K_1 + 2Ccos2x - 2Dsin2x$ and,

$$y_p'' = 2K_2 - 4C\sin 2x - 4D\cos 2x$$

Substituting into the given ODE;

$$2K_2 - 4C\sin 2x - 4D\cos 2x + 4(2K_2x + K_1 + 2C\cos 2x - 2D\sin 2x) + 5(K_2x^2 + K_1x + K_0 + C\sin 2x + D\cos 2x) \quad = 25x^2 + 13\sin 2x$$

Comparing coefficients on x^2; $5K_2 = 25 \Rightarrow K_2 = 5$

Coefficients on *x*; $8K_2 + 5K_1 = 0 \Rightarrow K_1 = -8$

Coefficients on x^0, $2K_2 + 4K_1 + 5K_0 = 0 \Rightarrow K_0 = 4.4$

Coefficients on *cos2x*, $-4D + 8C + 5D = 0 \Rightarrow D = -8C$

Coefficients on *sin2x*, $-4C - 8D + 5C = 13 \ \Rightarrow C = \frac{1}{5}\ and\ D = -\frac{8}{5}$

Therefore, the general solution is,

$$y = e^{-2x}(A\cos x + B\sin x) + 5x^2 - 8x + 4.4 + \frac{1}{5}\sin 2x - \frac{8}{5}\cos 2x$$

Exercise 4.8

Solve the following i.v.ps:

$$y'' + 4y = 16\cos 2x;\ \ y(0) = 0\ \ and\ y'(0) = 0$$

$$y'' - 2y' = 12e^{2x} - 8e^{-2x};\ \ y(0) = -2\ \ and\ y'(0) = 12$$

$$y'' + 2y' + 10y = 17\sin x - 37\sin 3x;\ \ y(0) = 6.6\ \ and\ \ y'(0) = -2.2$$

Find the general solutions of the following ODEs:

$y'' + 2y' - 3y = 4 + x + 4e^{2x}$ Ans: $y = -\frac{14}{7} - \frac{1}{3}x + \frac{4}{5}e^{2x} + C_1e^x + C_2e^{-3x}$

$y'' + 9y = 2\cos 3x + \sin 3x;$

Ans: $y = \frac{1}{18}\cos 2x + \frac{1}{36}\sin 3x - \frac{1}{6}x\cos 3x + \frac{1}{3}x\sin 3x + C_1\cos 3x + C_2\sin 3x$

$y'' + 2y' + y = 5 + x^2e^x$

$$Ans: y = 5 + \frac{3}{8}e^x - \frac{1}{2}xe^x + \frac{1}{4}x^2e^x + C_1e^{-x} + C_2xe^{-x}$$

4.14 Higher order homogenous equations

Higher order homogenous differential equations are solved in a similar way to second order ODEs using the 3 basic cases discussed in section 1.13.1 and the principle of *linear superposition*.

Example 26

Find the general solution of; $y''' - 2y'' - 5y' + 6y = 0$

Solution

The characteristic equation is; $\lambda^3 - 2\lambda^2 - 5\lambda + 6 = 0$

By inspection, $\lambda = 1$, is a root of the equation. Using long division, it can be shown that the other roots are solutions of the quadratic, $\lambda^2 - \lambda - 6 = 0$

Therefore, $(\lambda - 1)(\lambda^2 - \lambda - 6) = (\lambda - 1)(\lambda - 3)(\lambda + 2) = 0$

$$\lambda = 1,\ \lambda = 3\ \ and\ \ \lambda = -2$$

Example 27

Solve the initial value problem; $\frac{d^4y}{dt^4} - \frac{d^2y}{dt^2} - 6y = 0;\ \ y(0) = 0, y'(0) = 1, y''(0) = 0\ \ and\ y'''(0) = 0$

Solution

The characteristic equation is; $\lambda^4 - \lambda^2 - 6 = 0$, putting $\lambda^2 = m$

$\Rightarrow m^2 - m - 6 = 0 \Rightarrow m = 3.0\ or\ m = -2.0$

Therefore, $\lambda^2 = 3 \Rightarrow \lambda = \pm\sqrt{3}\ \ or\ \ \lambda^2 = -2 \Rightarrow \lambda = \pm\sqrt{2}i$

Hence, the general solution is;

$$y = C_1e^{\sqrt{3}x} + C_2e^{-\sqrt{3}x} + Acos\sqrt{2}x + Bsin\sqrt{2}x$$

$$= Ccosh\sqrt{3}x + Dsinh\sqrt{3}x + Acosh\sqrt{2}x + Bsinh\sqrt{2}x$$

The values of the constants *A*, *B*, *C* and *D* can be determined using the given initial conditions.

Exercise 4.9

Solve the following i.v.ps;

$$y''' + y'' - 4y = 0;\quad y(0) = 1, y'(0) = 1\ and\ y''(0) = 0$$

Ans: $y = \frac{3}{4} + \frac{1}{68}\left[9\sqrt{17}sinh\left(\frac{\sqrt{17}}{2}x\right) + 17cosh\left(\frac{\sqrt{17}}{2}x\right)\right]e^{-x/2}$

$$y^{iv} - y'' - 2y = 0,\ y(0) = 1, y'(0) = 0, y''(0) = 0\ and\ y'''(0) = 0$$

4.15 Higher order non – homogenous equations

Higher order non – homogenous equations are solved in a similar was to second order equations. The following notes should be considered, however;

1. The solution y_p is chosen according to Table 1.2
2. If a term in your choice of y_p, happens to be a solution of the homogenous ODE, multiply your choice, $y_p(x)$, by x (or by x^2, x^3, x^4,.....x^n, if this solution corresponds to a double root, triple repeated root, …..n repeated roots of the characteristic equation of the corresponding homogenous ODE).
3. If, $r(x)$, is a sum of functions in the first column of Table 1.2, choose y_p as the sum of the corresponding lines in the second column of Table 1.2.

Example 28

Solve the i.v.p; $y''' + y'' - y' - y = 2 + e^{-x};\ \ y(0) = 1, y'(0) = 1\ and\ y''(0) = 0$

Solution

The characteristic equation of the corresponding homogenous equation is;

$$\lambda^3 + \lambda^2 - \lambda - 1 = 0$$

$\lambda = -1$ makes the equation identical, hence $(\lambda + 1)$, is a factor of the equation. Other factors can be determined after long division, i.e. $(\lambda + 1)(\lambda^2 - 1) = (\lambda - 1)(\lambda + 1)^2 = 0$

Hence the characteristic values are; $\lambda = 1\ and\ \lambda = -1$ (a repeated double root)

The complementary general solution is; $y_h = Ae^x + (B + Cx)e^{-x}$

The forcing function is; $2 + e^{-x}$, we would choose $y_p = K + K_1 e^{-x}$, as the corresponding particular solution, but since e^{-x} is a solution of the homogenous equation corresponding to a repeated double root, the right choice of y_p is; $y_p = K + K_1 x^2 e^{-x}$.

$$y_p' = 2K_1 xe^{-x} - K_1 x^2 e^{-x}$$

$$y_p'' = 2K_1 e^{-x} - 2K_1 xe^{-x} - 2K_1 xe^{-x} + K_1 x^2 e^{-x} = 2K_1 e^{-x} - 4K_1 xe^{-x} + K_1 x^2 e^{-x}$$

$$y_p''' = -2K_1 e^{-x} - 4K_1 e^{-x} + 4K_1 xe^{-x} + 2K_1 xe^{-x} - K_1 x^2 e^{-x}$$

$$= -6K_1 e^{-x} + 6K_1 xe^{-x} - K_1 x^2 e^{-x}$$

Substituting into the ODE and rearranging gives;

$$(-6K_1 + 2K_1)e^{-x} - K = 2 + e^{-x}$$

$$\Rightarrow K_1 = -\frac{1}{4} \;\; and \;\; K = -2$$

The general solution becomes; $y = y_h + y_p = Ae^{\bar{x}} + (B + Cx)e^{-x} - 2 - \frac{1}{4}x^2e^{-x}$

$$y' = Ae^{x} - Be^{-x} + Ce^{-x} - Cxe^{-x} - \frac{1}{2}xe^{-x} + \frac{1}{4}x^2e^{-x}$$

$$y'' = Ae^{x} + Be^{-x} - 2Ce^{-x} + Cxe^{-x} - \frac{1}{2}e^{-x} + xe^{-x} - \frac{1}{4}x^2e^{-x}$$

Using the initial conditions given, i.e. $y(0) = 1$, $y'(0) = 0$ and $y''(0) = 0$;

$$A = \frac{11}{8}, \qquad B = \frac{13}{8} \; and \; C = \frac{5}{4}$$

The particular solution is thus; $y(x) = \frac{11}{8}e^{x} + \frac{13}{8}e^{-x} + \frac{5}{4}xe^{-x} - 2 - \frac{1}{4}x^2e^{-x}$