SciComp Project 1
Weeks 1+2

September 6, 2021
To be handed in Monday September 20 before 12:00 noon

# 1 Background

Many of you will know the phenomenon that a prism refracts light, i.e. splits it up in different colors, because the refractive index of the prism varies with the frequency $\omega$ of light. The underlying property of the molecules forming the material is the frequency dependent polarizability, $\alpha(\omega)$. The polarizability, like many other properties of molecules and materials, can be calculated from the basic laws of physics, here the the time-dependent Schrödinger equation.

In the end, the polarizability for a given frequency $\omega$ of the incoming light is obtained as the following scalar product of two column vectors $\mathbf{z}$ and $\mathbf{x}$:

$$\alpha(\omega) = \mathbf{z}^T\mathbf{x} \tag{1}$$

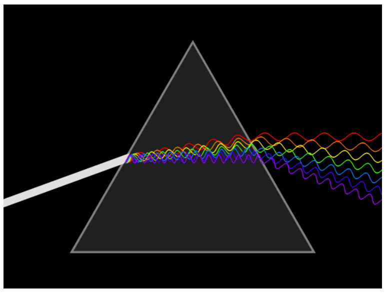

Figure 1: A dispersive prism slows light at different rates depending on the wave-length, causing refraction.

where $\mathbf{z}$ is a vector that can be calculated from the Schrödinger equation, and $\mathbf{x}$ is the solution to the following system of linear equations:

$$(\mathbf{E} - \omega\mathbf{S})\,\mathbf{x} = \mathbf{z} \tag{2}$$

Here, $\mathbf{E}$ and $\mathbf{S}$ are two square matrices, and $\omega$ is the frequency of the incoming light. Like the column vector $\mathbf{z}$, the matrices $\mathbf{E}$ and $\mathbf{S}$ are calculated from the Schrödinger equation, which we will not discuss further in this course.

# 2 Data

In this project we consider the water molecule, $H_2O$, and its frequency dependent polarizability, $\alpha(\omega)$. It turns out that the matrices $\mathbf{E}$ and $\mathbf{S}$, and the column vector $\mathbf{z}$, have a structure that lets us decompose the matrices into submatrices as follows:

$$\mathbf{E} = \begin{bmatrix}\mathbf{A} & \mathbf{B}\\ \mathbf{B} & \mathbf{A}\end{bmatrix}, \qquad \mathbf{S} = \begin{bmatrix}\mathbf{I} & \mathbf{0}\\ \mathbf{0} & -\mathbf{I}\end{bmatrix}, \qquad \mathbf{z} = \begin{bmatrix}\mathbf{y}\\ -\mathbf{y}\end{bmatrix} \tag{3}$$

The Python-file `watermatrices.py` and the Matlab-file `watermatrices.mat` contain the submatrices $\mathbf{A}$ and $\mathbf{B}$, and the subvector $\mathbf{y}$ for a water molecule, obtained by an approximate solution to the Schrödinger equation.

# 3 Questions for Week 1

a. (1) Write a small function that computes the condition number of a matrix under the max-norm:

$$\mathrm{cond}_\infty(\mathbf{M}) = \|\mathbf{M}\|_\infty \left\|\mathbf{M}^{-1}\right\|_\infty$$

Use a library matrix inversion routine[1] for $\mathbf{M}^{-1}$, but do program the max-norm yourself using `sum`, `abs`, and `max`. (2) For three frequencies, $\omega = \{1.300, 1.607, 2.700\}$, calculate the condition number for the matrix $\mathbf{E} - \omega\mathbf{S}$. The right-hand-side $\mathbf{z}$ is given with 8 significant digits. How many significant digits could we guarantee in the solution $\mathbf{x}$ if everything else were assumed exact? Why?

b. (1) For each of the three $\omega$, determine a bound on the relative forward error in the max-norm:

$$\frac{\|\Delta\mathbf{x}\|_\infty}{\|\hat{\mathbf{x}}\|_\infty} \leq \mathrm{cond}_\infty\left(\mathbf{E} - \omega\mathbf{S}\right) \frac{\|\delta\omega\mathbf{S}\|_\infty}{\|\mathbf{E} - \omega\mathbf{S}\|_\infty}$$

for the perturbation that the frequency $\omega$ is changed by $\delta\omega = \frac{1}{2} \cdot 10^{-3}$. (2) As $\omega$ is given with 3 digits after the comma, how many significant digits could we be guarantee in $\mathbf{x}$ if everything else were exact? Why?

c. Implement three separate functions

1. `L,U = lu_factorize(A)`, which takes a square matrix $\mathbf{M}$ as input and returns two square matrices: A triangular matrix $\mathbf{L}$ and upper triangular matrix $\mathbf{U}$ such that $\mathbf{M} = \mathbf{LU}$.
2. `y = forward_substitute(L,z)`, which takes a square lower triangular matrix $\mathbf{L}$ and a vector $\mathbf{b}$ as input, and returns the solution vector $\mathbf{y}$ to $\mathbf{Ly} = \mathbf{b}$.
3. `x = back_substitute(U,y)`, which takes a square upper triangular matrix $\mathbf{U}$ and a vector $\mathbf{y}$ as input, and returns the solution vector $\mathbf{x}$ to $\mathbf{Ux} = \mathbf{y}$.

and test them with the linear equation

$$\begin{bmatrix} 2 & 1 & 1 \\ 4 & 1 & 4 \\ -6 & -5 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 11 \\ 4 \end{bmatrix}$$

You can test your solution against a library routine.[2]

If you know how, try to use vector operations instead of for-loops where possible (orders of magnitude faster in Python and Matlab).

d. Implement a function $\alpha$ = `solve_alpha(omega)` for calculating the frequency-dependent polarizability $\alpha(\omega) = \mathbf{z}^T\mathbf{x}$ for water in the given approximation. This routine should solve Equation (2) by LU-factorization using your own three routines from (c). (1) Using your routine, make a table of the polarizabilities for the frequencies given in (a) and their perturbations, i.e. for $\omega = \{1.300 \pm \delta\omega, 1.607 \pm \delta\omega, 2.700 \pm \delta\omega\}$, with $\delta\omega = \frac{1}{2} \cdot 10^{-3}$ as before. (2) Which error-bound is the correct one to understand the variation of the calculated polarizabilities due to the perturbation: (a) or (b) or both? Explain why. Do your calculated values fall within the bounds you calculated above?

e. (1) Compute a table of $\alpha(\omega)$ for 1000 evenly spaced values in the interval $[1.2, 4]$[3] using your routine from (d), and plot the values. (2) Can you explain what happens to the linear system of Equation (2) around the frequency $\omega = 1.60686978$, and how is this reflected in $\alpha(\omega)$?

[1] E.g. `inv` from `numpy.linalg` for Python, `inv` for Matlab. But only here.
[2] E.g. `numpy.linalg.solve` in Python, or `linsolve` in Matlab.
[3] Use `linspace` in both Python and Matlab.

# 4 Questions for Week 2

f. Implement two routines and test them with the matrix $\mathbf{A}_{\text{test}}$ and right-hand-side $\mathbf{b}$:

$$\mathbf{A}_{\text{test}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 1 \end{bmatrix} \text{ and } \mathbf{b}_{\text{test}} = \begin{bmatrix} 1237 \\ 1941 \\ 2417 \\ 711 \\ 1177 \\ 475 \end{bmatrix}$$

1. `Q,R = householder_QR(A)`, which takes as input a rectangular matrix $\mathbf{A}\colon m \times n$ and uses the Householder method to compute its QR decomposition. Check that $\mathbf{Q}\colon m \times m$ is orthogonal, i.e., $\mathbf{Q}^T\mathbf{Q} = \mathbf{Q}\mathbf{Q}^T = \mathbf{I}$, and that the upper triangular matrix $\mathbf{R}\colon m \times n$ satisfies $\mathbf{A} = \mathbf{QR}$.
2. `x̃ = least_squares(A,b)`, which combines this routine with your back-substitution from (c.3) to compute a linear least squares fitting. It should take as input a rectangular $m \times n$ matrix $\mathbf{A}$ and an $m \times 1$ right-hand-side vector $\mathbf{b}$, returning an $n \times 1$ approximate solution vector $\tilde{\mathbf{x}}$ to $\mathbf{A}\tilde{x} \simeq \mathbf{b}$ as output.

g. We now want to approximate $\alpha(\omega)$ in the interval $[1.2, \omega_p]$ using a polynomial

$$P(\omega) = \sum_{j=0}^{n} a_j \omega^{2j} \tag{4}$$

(1) Suggest a suitable value of $\omega_p < 4$. (2) Find the coefficients of the polynomial using your linear least squares routine from (f), the table computed above, and $n = 4$. (3) Repeat the computation for $n = 6$ and compare the accuracies of the two polynomials: Plot the magnitude of the relative error (in a $\log_{10}$-scale) of the polynomial approximation as the difference between the $P(\omega)$ and $\alpha(\omega)$ values. (4) How many significant digits does each approximation yield?

h. We would now like to approximate $\alpha(\omega)$ everywhere in $[1.2, 4]$, and choose the *rational* approximating function, which is able to represent singularities:

$$Q(\omega) = \frac{\sum_{j=0}^{n} a_j \omega^j}{1 + \sum_{j=1}^{n} b_j \omega^j} \tag{5}$$

(1) Find the coefficients $a_j$ and $b_j$ using your linear least squares routine, the table of $\alpha$-values computed above, and $n = 2$. You need to reformulate the expression as an linear approximation so that you can use a linear least squares fitting. Plot the error of the the rational-function approximation $Q(\omega)$ compared to $\alpha(\omega)$ calculated by Equations (1) and (2).[4] (2) Repeat the computation for $n = 4$ and compare the accuracies of the two approximations quantitatively. (3) If you look at $\alpha(\omega)$ in the extended interval $\omega \in [-4; 4]$, you will notice that $\alpha(\omega)$ has multiple singularities. Are you able to modify your approximation to accurately approximate the full interval $[-4; 4]$, in particular so that it reproduces the singularities correctly? Explain your solution.

[4] Once you have computed the coefficients $\mathbf{a}$ and $\mathbf{b}$, be sure to finish the construction of $Q(\omega)$ using Eq. (5), and to use this for the error. It is tempting (but wrong) to just use the linear approximation you used to find $\mathbf{a}$ and $\mathbf{b}$, but that is linear and hence cannot represent singularities.