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exam-1-b0e22
#270 2 of 12

## INSTRUCTIONS

1. The duration of the exam is 110 minutes, closed book and notes.
2. No space other than the pages on the exam booklet will be scanned for grading! Do not write your solutions on the back of the pages.
3. If you require an additional page for a question, you can use the extra page provided within this booklet. However please indicate clearly that you are continuing the solution on the additional page.
4. You may not exit the exam room until the expiration of the 110 minutes alloted for the examination.

Q1 15

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exam-1-b0e22
#270 3 of 12

**1. (15 pts)** Show that for any **odd** positive integer $n$, there is some strictly positive integer $x > 0$, such that $2^x - 1$ is divisible by $n$.

let $A = \{2^1, 2^2, 2^3, \dots, 2^{n+1}\}$

pigeons: numbers in set $A$, with $n+1$ total count

pigeons holes: $1 \sim n$, $n$ possible remainders. 0 cannot be the remainder since $n$ is odd, $2^x$ cannot be divisible by an odd number. let's call this lemma 1.

Proof of lemma 1: $n$ is odd $\Rightarrow 2^x$ is not divisible by $n$. contrapositive. $2^x$ is divisible by $n \Rightarrow n$ is even This is true.

by PHP, there must be two elements in set $A$ such that they have the same remainder.

let's call them $2^a$ and $2^b$, where $0 < a < b \leq n$

so $2^b - 2^a$ is divisible by $n$

$= 2^a(2^{b-a} - 1)$

we know that $2^a$ is not divisible by $n$, so $2^{b-a} - 1$ is divisible by $n$, and this is exactly the number that $2^x - 1$ is divisible by $n$. to show

Q2 10

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exam-1-b0e22
#270 4 of 12

**2.** **(10 pts)** Determine if the following statement

$$\neg(p \Rightarrow q) \Rightarrow (p \wedge (\neg p \vee \neg q))$$

is a tautology. Show your work and explain your reasoning of each step by using logical equivalences. Do not use truth tables.

$\neg(p \Rightarrow q) \Rightarrow (p \wedge (\neg p \vee \neg q))$

$\equiv (p \Rightarrow q) \vee (p \wedge (\neg p \vee \neg q))$

$\equiv (\neg p \vee q) \vee (p \wedge (\neg p \vee \neg q))$

$\equiv (\neg p \vee q) \vee ((p \wedge \neg p) \vee (p \wedge \neg q))$

$\equiv (\neg p \vee q) \vee (F \vee (p \wedge \neg q))$

$\equiv \neg p \vee q \vee (p \wedge \neg q)$

$\equiv \neg p \vee ((q \vee p) \wedge (q \vee \neg q))$

$\equiv \neg p \vee q \vee p$

$\equiv T \vee q$

$\equiv T$

so this is a tautology.

Q3a 5



**3. (30 pts)**

a) **(5 pts)** Write the contrapositive of the following predicate:

$$(\exists x \in \mathbb{R}\, \forall y \in \mathbb{R},\ y \leq x) \Rightarrow (\forall z \in \mathbb{R}\, \exists w > 0, z < w).$$

$\neg(\forall z \in R\ \exists w > 0, z < w) \Rightarrow \neg(\exists x \in R\ \forall y \in R, y \leq x)$

$\equiv\ (\exists z \in R\ \forall w > 0,\ z \geq w) \Rightarrow (\forall x \in R\ \exists y \in R,\ y > x)$

Q3b 5

b) **(5 pts)** Write the negation of the following statement:

*For all $x, y \in \mathbb{R}$ such that $x < y$, there exists $z \in \mathbb{R}$ such that $x < z < y$.*

original statement: $\forall x \in R\ \forall y \in R (x < y)\ \exists z \in R,\ x < z < y$

negation: $\exists x \in R\ \exists y \in R\ (x < y)\ \forall z \in R,\ \neg(x < z < y)$

$x < z < y \equiv (x < z) \wedge (z < y)$

$\therefore$ the negation becomes:

$\exists x \in R\ \exists y \in R\ (x < y)\ \forall z \in R,\ (x \geq z) \vee (z \geq y)$

well done

Q3c 10

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exam-1-b0e22
#270 6 of 12

c) **(10 pts)** Write the negation of the following predicate:

$$\forall x \forall \varepsilon(\varepsilon > 0 \Rightarrow \exists \delta(\delta > 0 \wedge (\forall y((|y - x| < \delta) \Rightarrow (|f(y) - f(x)| < \varepsilon))))).$$

$\exists x \exists \varepsilon, \neg(\varepsilon > 0 \Rightarrow \exists \delta(\delta > 0 \wedge (\forall y((|y-x| < \delta) \Rightarrow (|f(y) - f(x)| < \varepsilon)))))$

$\equiv \exists x \exists \varepsilon, (\varepsilon > 0 \wedge \forall \delta \neg(\delta > 0 \wedge (\forall y((|y-x| < \delta) \Rightarrow (|f(y) - f(x)| < \varepsilon)))))$

$\equiv \exists x \exists \varepsilon, (\varepsilon > 0 \wedge \forall \delta (\delta \leq 0 \vee \neg(\forall y((|y-x| < \delta) \Rightarrow (|f(y) - f(x)| < \varepsilon)))))$

$\equiv \exists x \exists \varepsilon, (\varepsilon > 0 \wedge \forall \delta (\delta \leq 0 \vee \exists y((|y-x| < \delta) \wedge (|f(y) - f(x)| \geq \varepsilon))))$

Q3d 10

d) **(10 pts)** Convert $\neg(z \Rightarrow (\neg x \wedge y))$ to DNF by use of a truth table. Simply write down the truth table and deduce your DNF.

| $x$ | $y$ | $z$ | $\neg x$ | $\neg x \wedge y$ | $z \Rightarrow (\neg x \wedge y)$ | $\neg(z \Rightarrow (\neg x \wedge y))$ | DNF |
|---|---|---|---|---|---|---|---|
| 0 | 0 | 0 | 1 | 0 | 1 | 0 | |
| 1 | 0 | 0 | 0 | 0 | 1 | 0 | |
| 0 | 1 | 0 | 1 | 1 | 1 | 0 | |
| 0 | 0 | 1 | 1 | 0 | 0 | 1 | $\neg x \wedge \neg y \wedge z$ |
| 1 | 1 | 0 | 0 | 0 | 1 | 0 | |
| 1 | 0 | 1 | 0 | 0 | 0 | 1 | $x \wedge \neg y \wedge z$ |
| 0 | 1 | 1 | 1 | 1 | 1 | 0 | |
| 1 | 1 | 1 | 0 | 0 | 0 | 1 | $x \wedge y \wedge z$ |

DNF is $(\neg x \wedge \neg y \wedge z) \vee (x \wedge \neg y \wedge z) \vee (x \wedge y \wedge z)$

Q4 15

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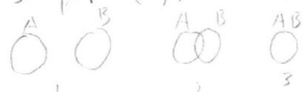
exam-1-b0e22
#270 7 of 12

**4. (15 pts)** Let $A$ and $B$ be finite sets. Prove by contradiction that $|A \setminus B| \leq |A|$.

let's assume $|A \setminus B| > |A|$

so $|A - (A \cap B)| > |A|$




case 1: $A \cap B = \phi$

$|A - \phi| = |A| \not> |A|$ there is a contradiction.

case 2: $A \cap B \neq \phi$ $A \cap B < A$

$|A - (A \cap B)| < |A|$ there is a contradiction

cas 3: $A \cap B = A$

$|A - (A \cap B)| = |\phi| = 0$ ...

In any cases, there are a... so

$|A \setminus B| \leq |A|$ is proven.

Cases 2 and 3 could technically be combined if you used <= in the first line of case 2. Also should use a subset symbol instead of an inequality

Q5 15

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exam-1-b0e22
#270 8 of 12

**5. (15 pts)** Prove by mathematical induction that

$$7^{2n+1} + 6 \times 4^n \times 9^n$$

is divisible by 13 for all integers $n \geq 0$. Let P(n) be the above statement

base case: $n=0$

$7^1 + 6 \times 4^0 \times 9^0 = 13$, which is divisible by 13, so the base case is true.

IH: P(i) is true for $0 \leq i \leq k$, $i \in N$ $k \in N$

IS: Prove P(k+1):

$7^{2(n+1)+1} + 6 \times 4^{n+1} \times 9^{n+1}$

$= 7^{2n+3} + 6 \times 4 \cdot 4^n \times 9 \cdot 9^n$

$= 49 \cdot 7^{2n+1} + 36 \cdot 6 \times 4^n \times 9^n$

$= 13 \cdot 7^{2n+1} + 36 \cdot 7^{2n+1} + 36 \cdot 6 \times 4^n \times 9^n$

$= 13 \cdot 7^{2n+1} + 36 \cdot (7^{2n+1} + 6 \times 4^n \times 9^n)$

by IH we know $7^{2n+1} + 6 \times 4^n \times 9^n$ is divisible by 13, and $13 \cdot 7^{2n+1}$ is also divisible by 13, so the above is divisible by 13.

The P(k+1) is proven. By mathematical induction, the original statement is proven.

Q6 15

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exam-1-b0e22
#270 9 of 12

**6. (15 pts)** Let $\mathbb{Z}$ denote the set of all integers i.e. $\mathbb{Z} := \{\dots, -2, -1, 0, 1, 2, \dots\}$. Let $x \neq 0$ be any real number such that $x + 1/x \in \mathbb{Z}$. Prove by strong mathematical induction that

$$x^n + \frac{1}{x^n} \in \mathbb{Z}, \quad \text{for any } n \geq 0.$$

let $P(n)$ be the above statement.

base case: $n=0$

$x^0 + \frac{1}{x^0} = 1+1 = 2 \in \mathbb{Z}$. when $n=1$, $x + \frac{1}{x} \in \mathbb{Z}$ by definition.

IH: $P(i)$ is true for $0 \leq i \leq k$, $i \in N$, $k \in N$

IS: prove $P(k+1)$

let $k+1 = a+b$ where $n \leq a \leq b \leq k$, $a \in N$, $b \in N$

$(x^a + x^{-a})(x^b + x^{-b})$

$= x^{a+b} + x^{a-b} + x^{b-a} + x^{-(a+b)}$

$= x^{k+1} + x^{-(k+1)} + x^{b-a} + x^{-(b-a)}$

by IH we know that $x^a + x^{-a}$, $x^b + x^{-b}$ are all integer, so $(x^a + x^{-a})(x^b + x^{-b})$ is also integer.

because $0 \leq b-a \leq k$, IH again $x^{b-a} + x^{-(b-a)}$ is also a integer.

so $x^{k+1} + x^{-(k+1)}$ plus a integer equals an integer, that means $x^{k+1} + x^{-(k+1)} \in \mathbb{Z}$, so $P(k+1)$ is proven.

By mathematical induction, the original statement is proven.

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exam-1-b0e22
#270 12 of 12

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