Lec18: Greedy Algorithms II

Algorithm I
COMP319-003
Spring 2023

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Last time

- **Greedy algorithms**
  - Make a series of choices.
    - Choose this activity, then never backtrack.
  - Show that, at each step, your choice does not rule out success.
    - At every step, there exists an optimal solution consistent with the choices we've made so far.
  - At the end:
    - You've built only one solution, never having ruled out success,
    - so your solution must be correct.

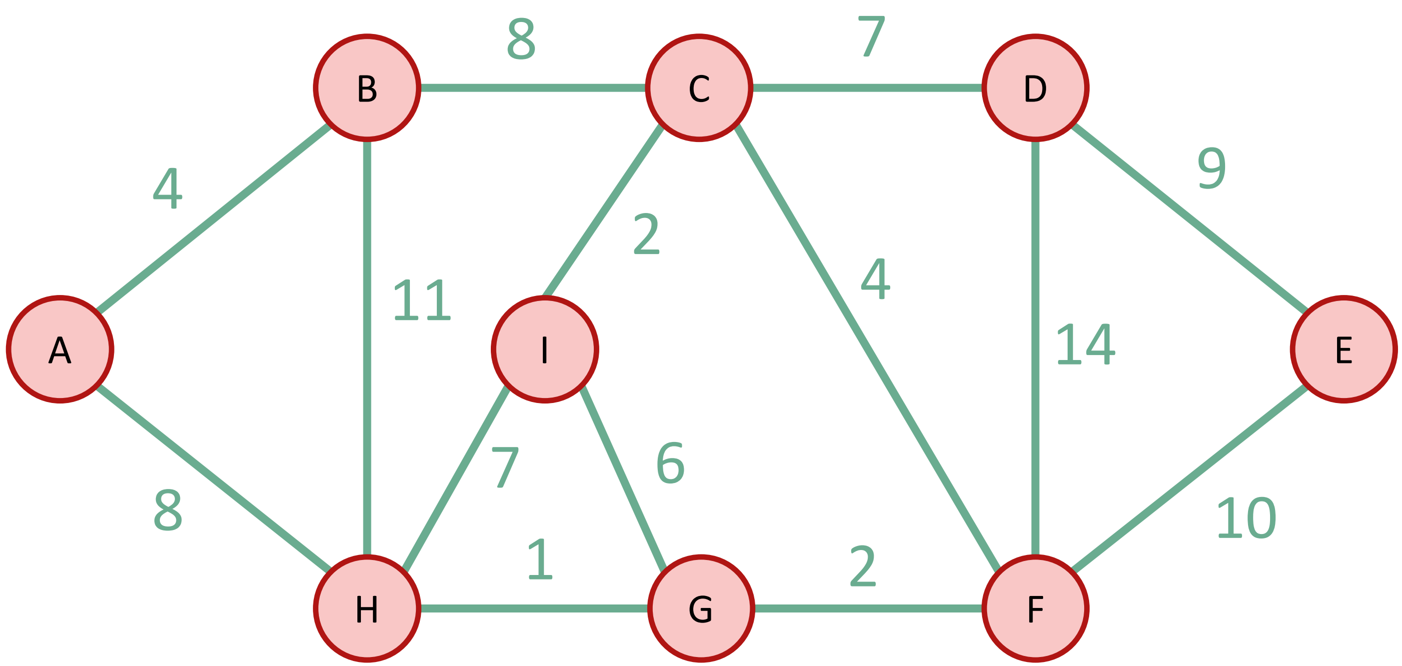
Today

- Greedy algorithms for **Minimum Spanning Tree.**

- Agenda:
  1. What is a Minimum Spanning Tree?
  2. Short break to introduce some graph theory tools
  3. Prim's algorithm
  4. Kruskal's algorithm

Minimum Spanning Tree

- Say we have an undirected weighted graph.



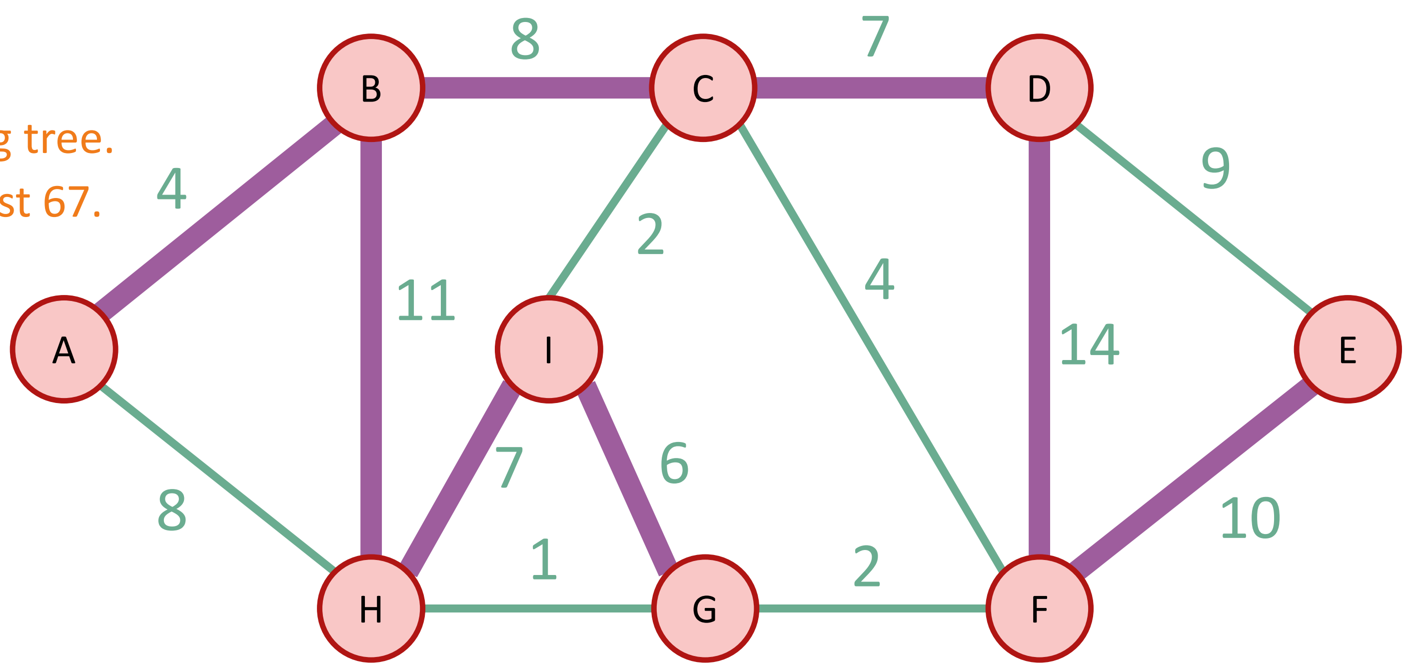
*A **tree** is a connected graph with no cycles!

- A **spanning tree** is a **tree** that connects all of the vertices.

Minimum Spanning Tree

- Say we have an undirected weighted graph.
- The **cost** of a spanning tree is the sum of the weights on the edges.

This is a spanning tree.
It has cost 67.



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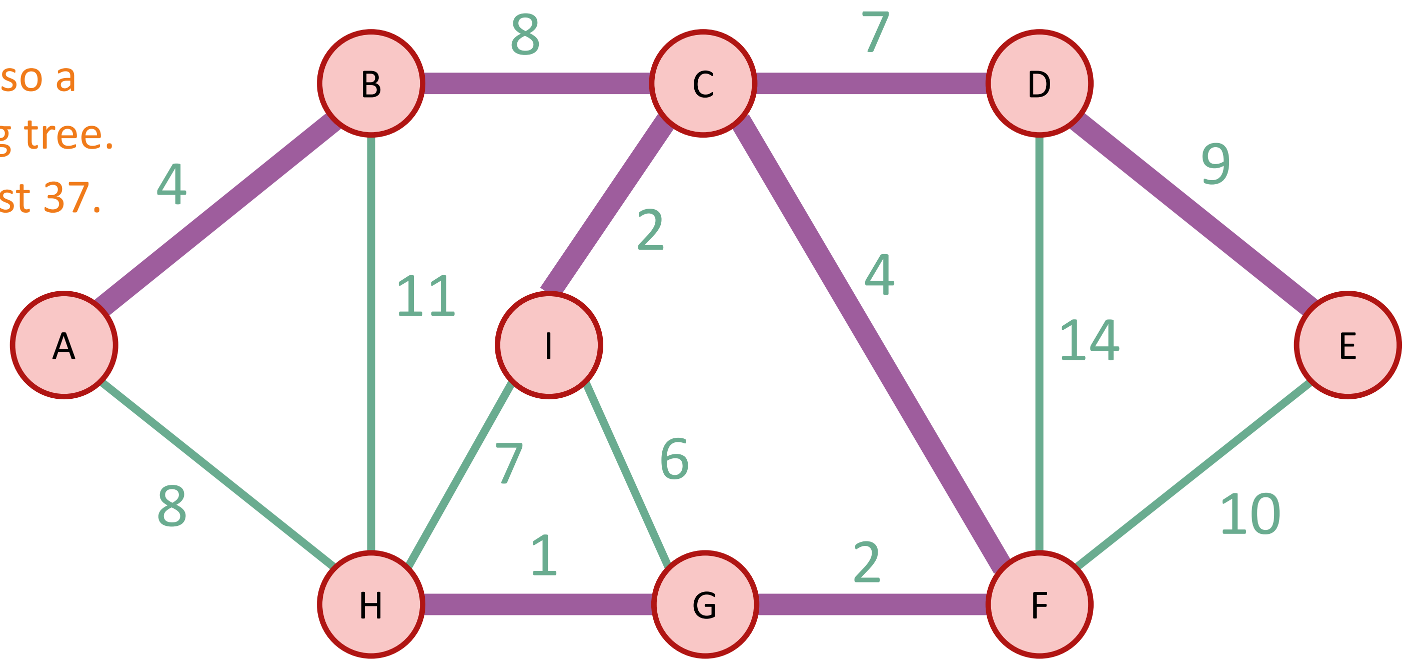
Minimum Spanning Tree

- Say we have an undirected weighted graph.
- The **cost** of a spanning tree is the sum of the weights on the edges.

This is also a spanning tree.
It has cost 37.



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Minimum Spanning Tree

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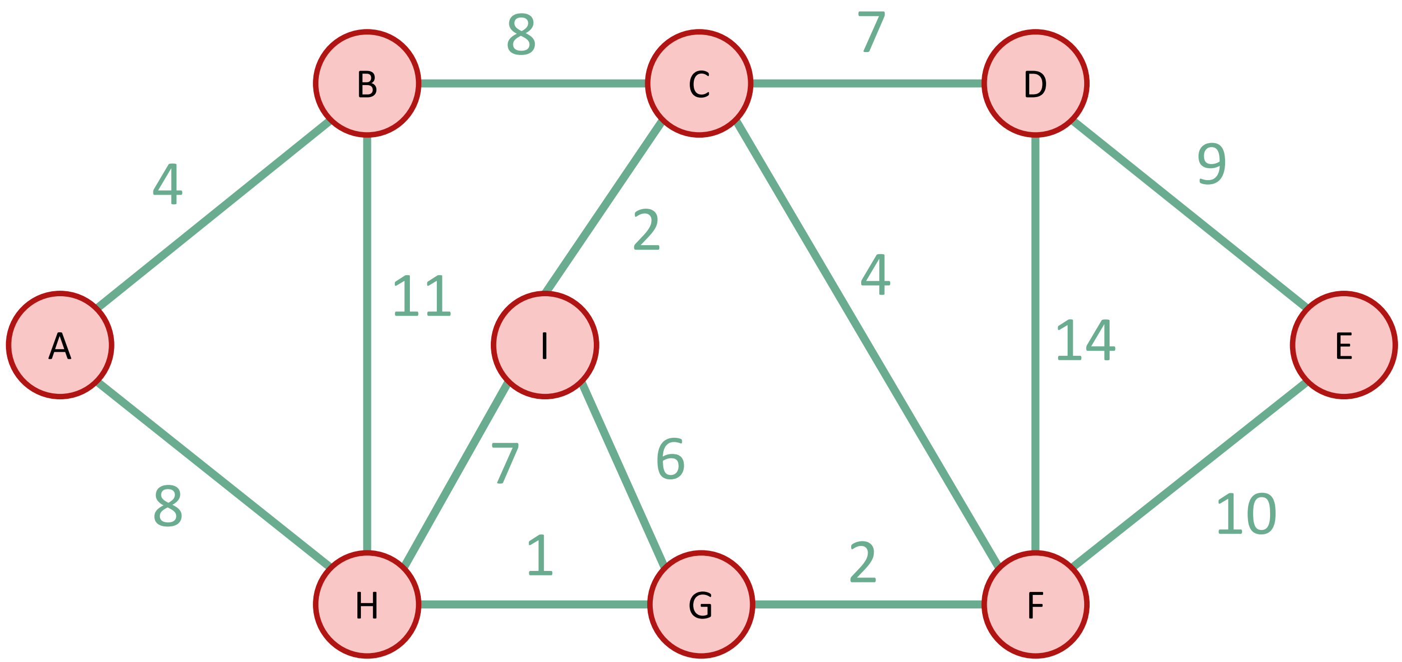

minimum

of minimal cost

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Minimum Spanning Tree

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This is a **minimum spanning tree.**
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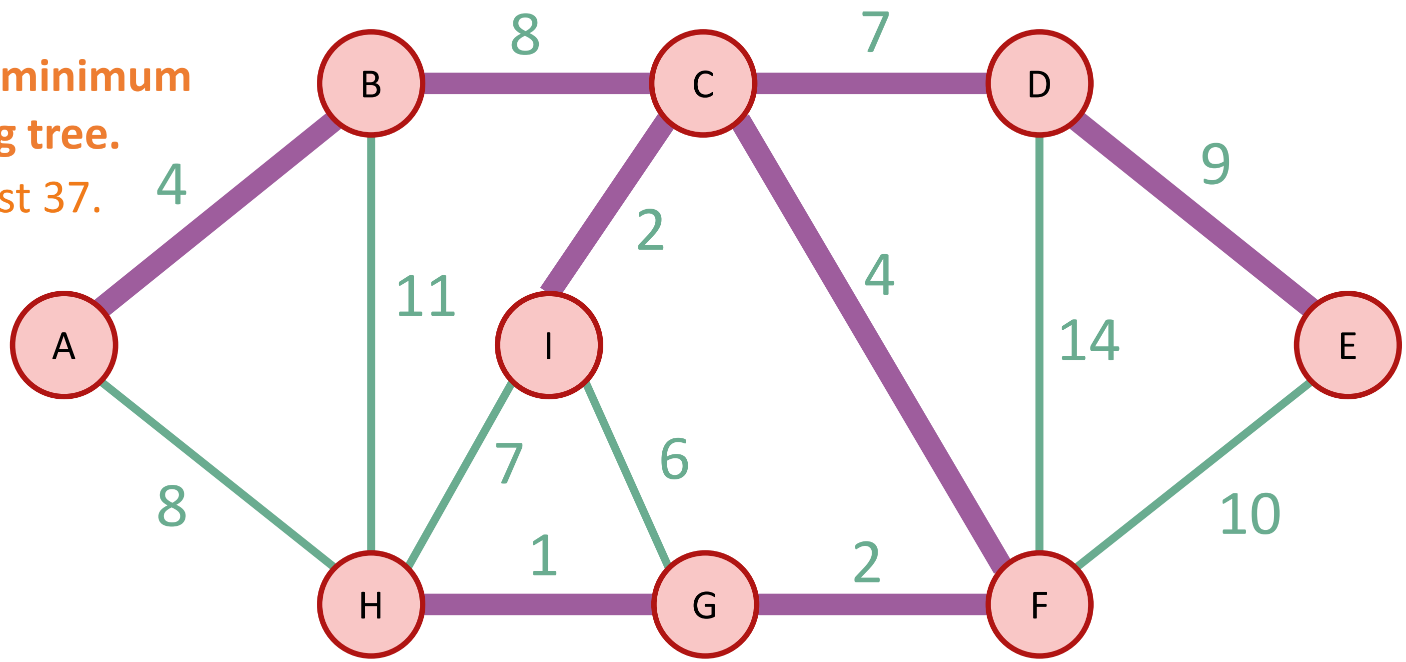


*A **tree** is a connected graph with no cycles!

minimum **of minimal cost**

- A **spanning tree** is a **tree** that connects all of the vertices.

Why MSTs?

- Network design
  - Connecting cities with roads/electricity/telephone/…
- Cluster analysis
  - e.g., genetic distance
- Image processing
  - e.g., image segmentation
- Useful primitive
  - For other graph algorithms

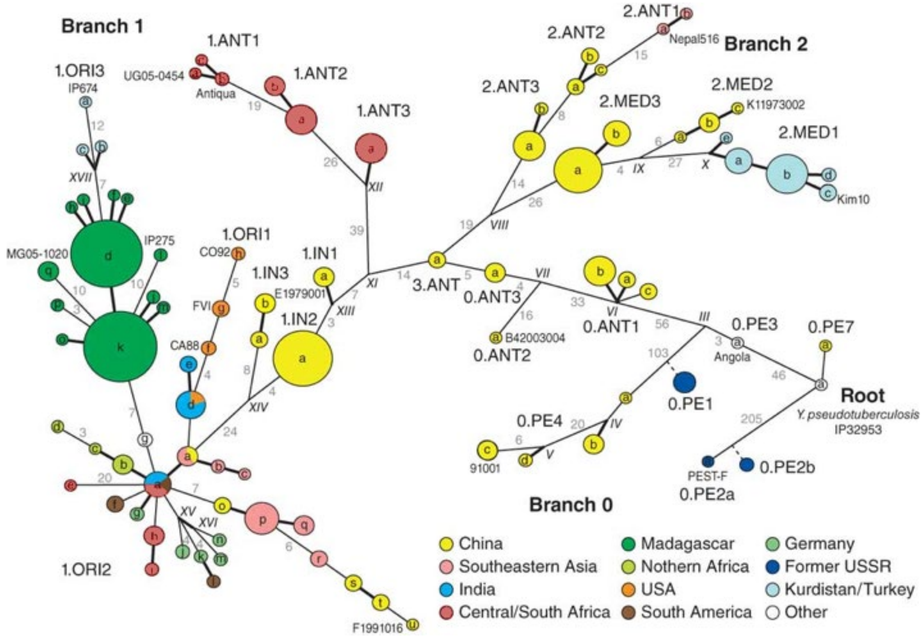


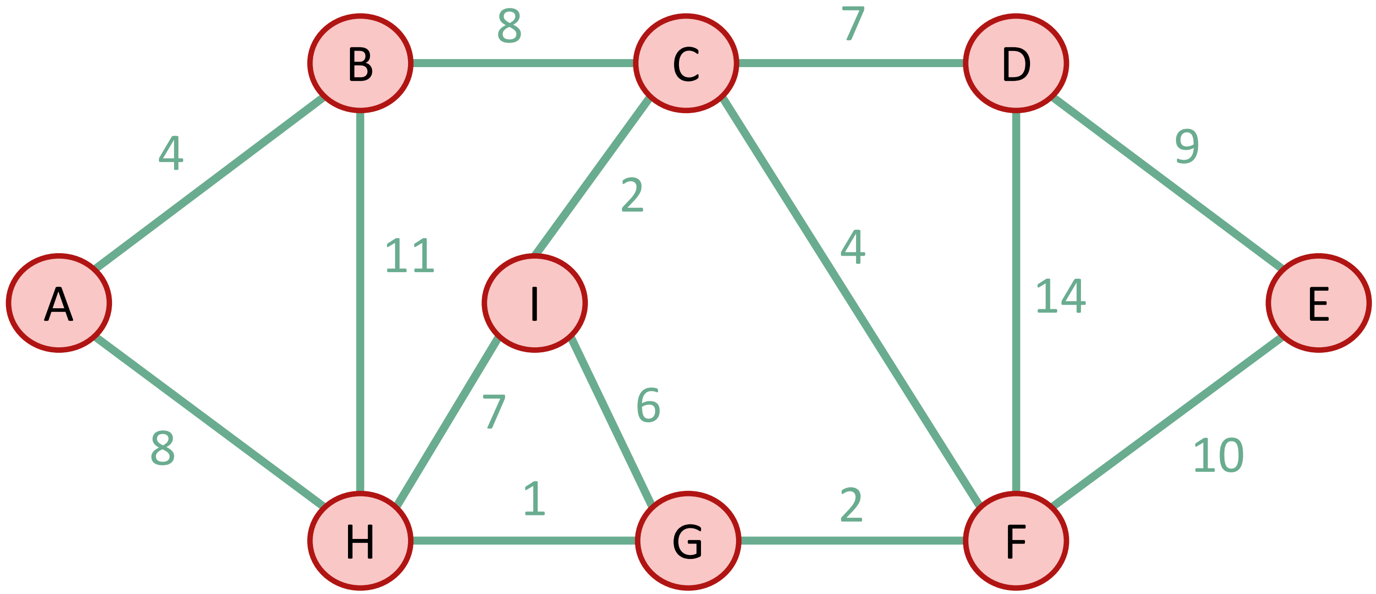
Figure 2: Fully parsimonious minimal spanning tree of 933 SNPs for 282 isolates of *Y. pestis* colored by location. Morelli et al. Nature genetics 2010

How to find an MST?

- Today, we'll see two greedy algorithms.
- In order to prove that these greedy algorithms work, we'll show something like:
  - *Suppose that our choices so far are consistent with an MST.*
  - *Then the next greedy choice that we make is still consistent with an MST.*

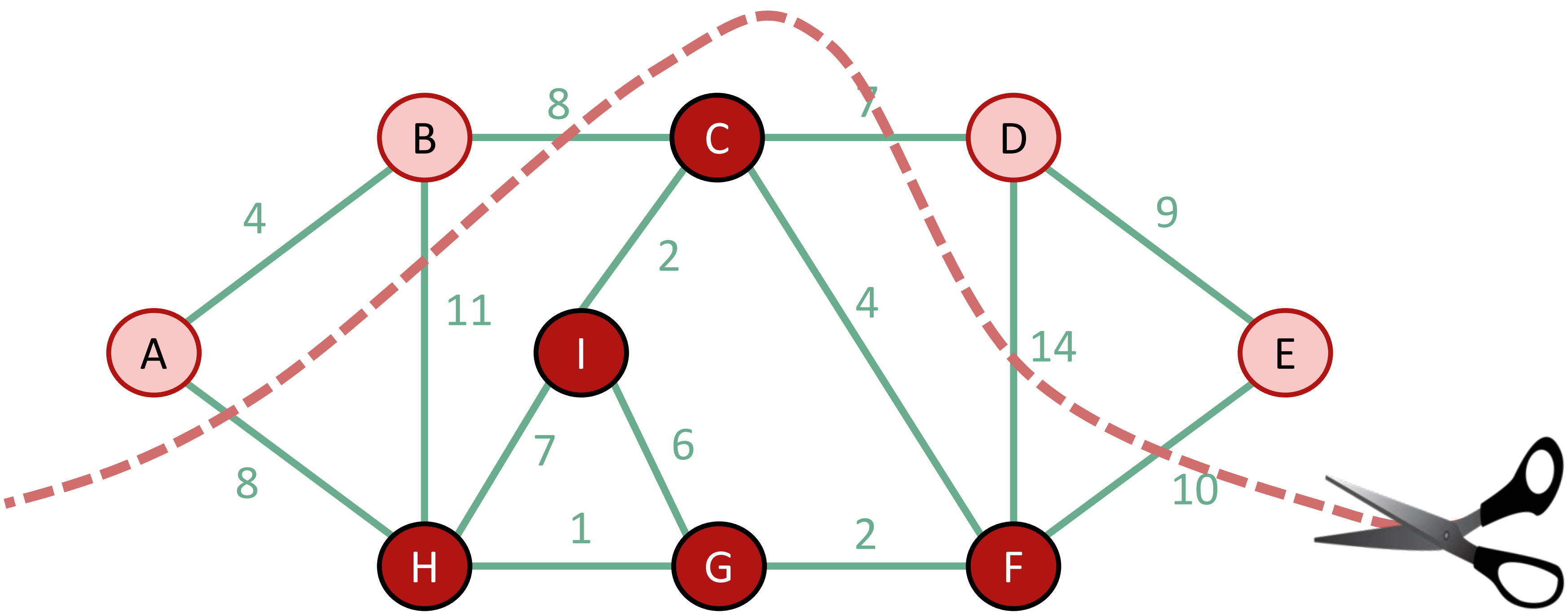
- Again, this is not the only way to prove that these algorithms work.

Let's brainstorm some greedy algorithms!



Brief aside: Cuts in graphs

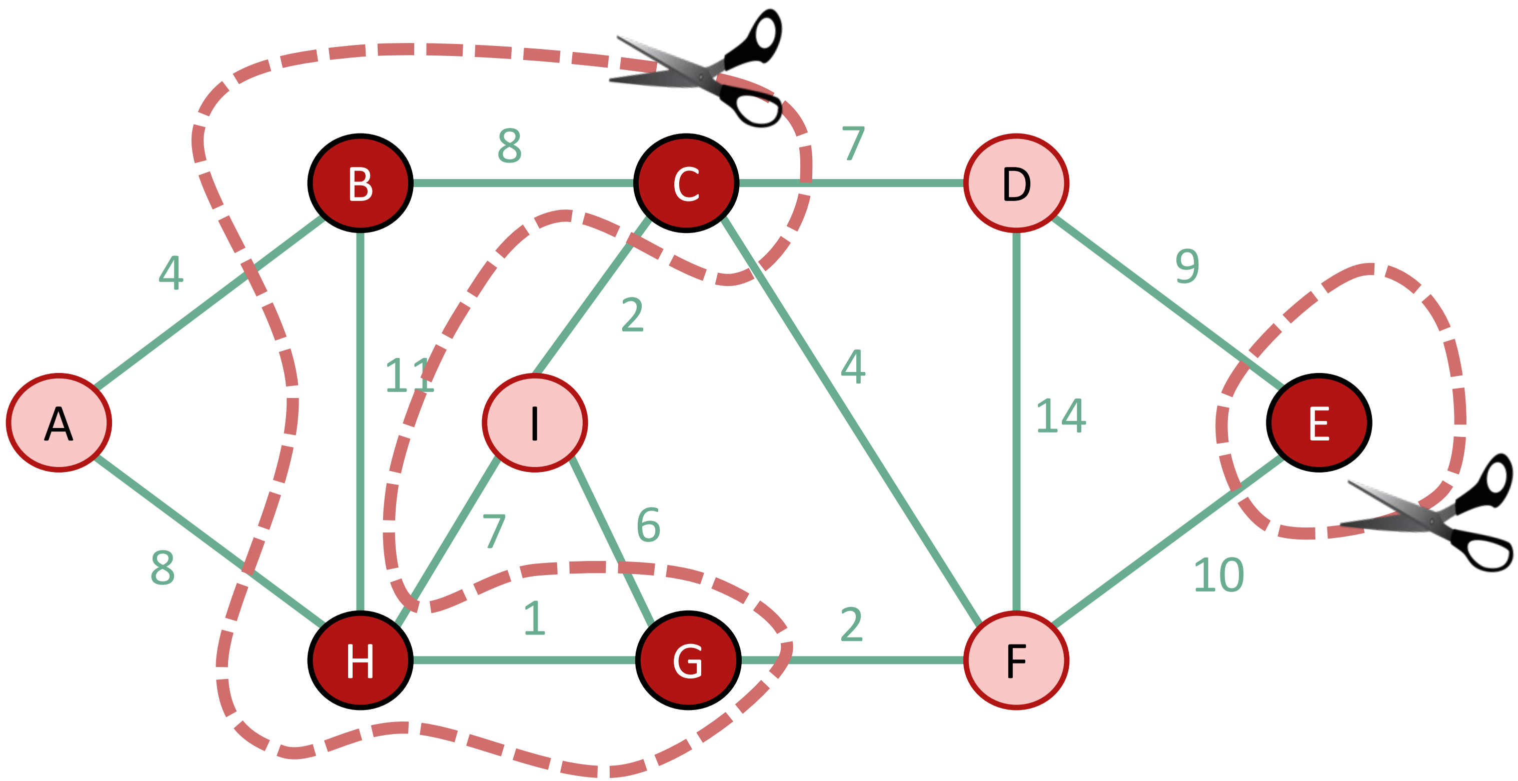
- A cut is a partition of the vertices into two parts:



This is the cut **"{A,B,D,E} and {C,I,H,G,F}"**

Cuts in graphs

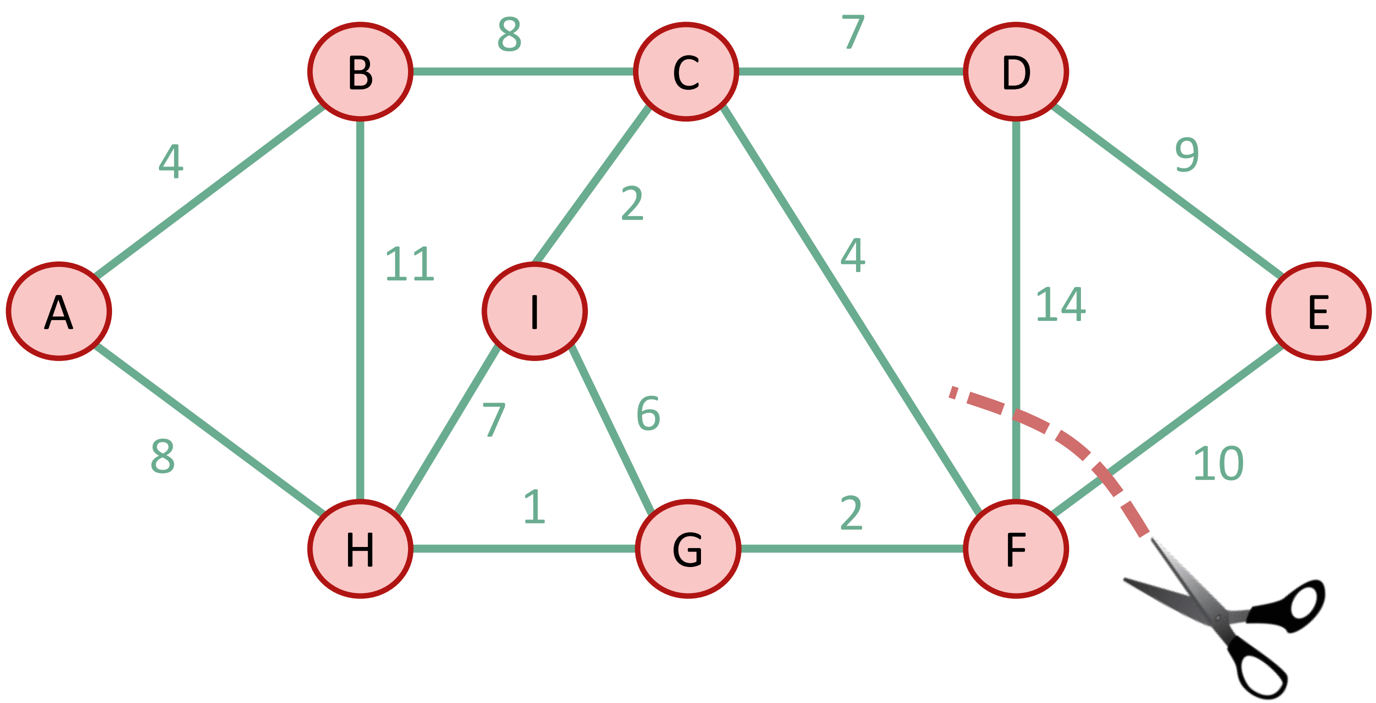
- One or both of the two parts might be disconnected.



This is the cut **"{B,C,E,G,H} and {A,D,I,F}"**

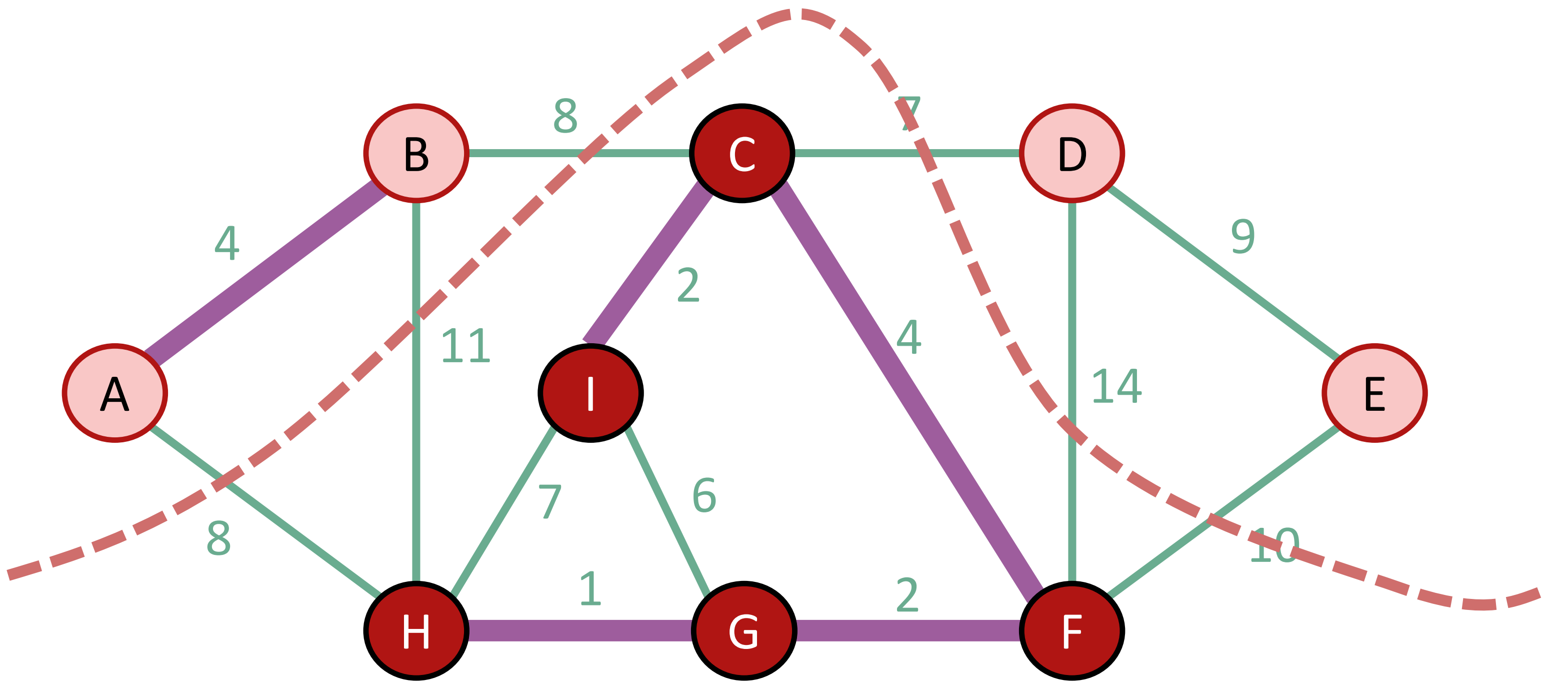
Cuts in graphs

- This is ***not*** a cut. Cuts are partitions of vertices.



Let S be a set of edges in G

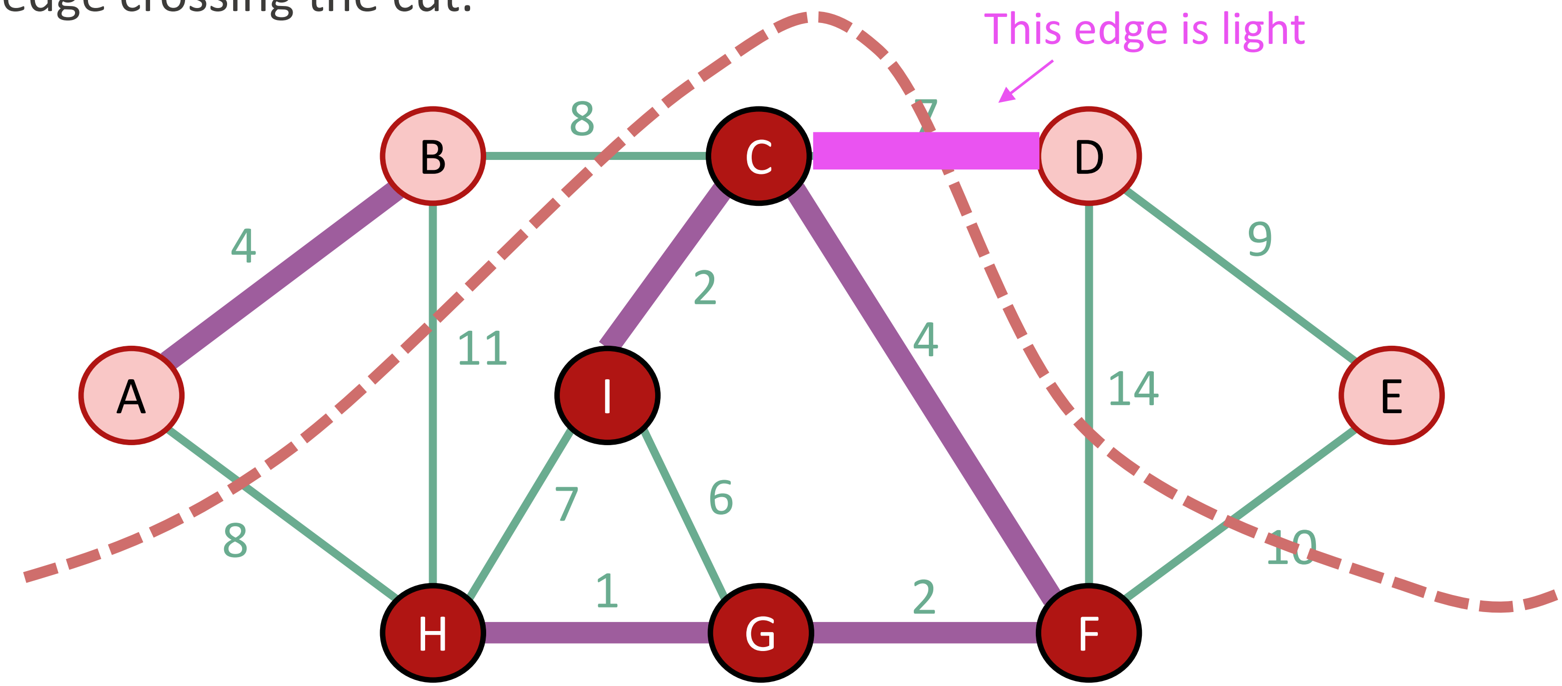
- We say a cut **respects S** if no edges in S cross the cut.



S is the set of **thick purple** edges

Let S be a set of edges in G

- We say a cut **respects S** if no edges in S cross the cut.
- An edge crossing a cut is called **light** if it has the smallest weight of any edge crossing the cut.



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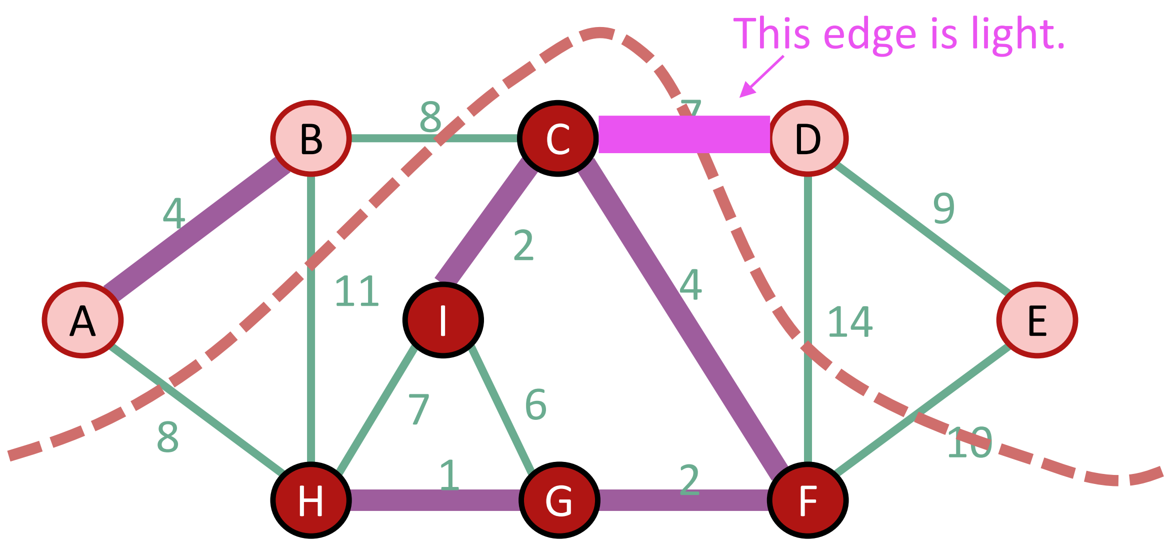
Lemma

- Let S be a set of edges, and consider a cut that respects S.
- Suppose there is an MST containing S.
- Let {u, v} be a light edge.
- Then there is an MST containing S ∪ {{u, v}}



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- Suppose there is an MST containing S.
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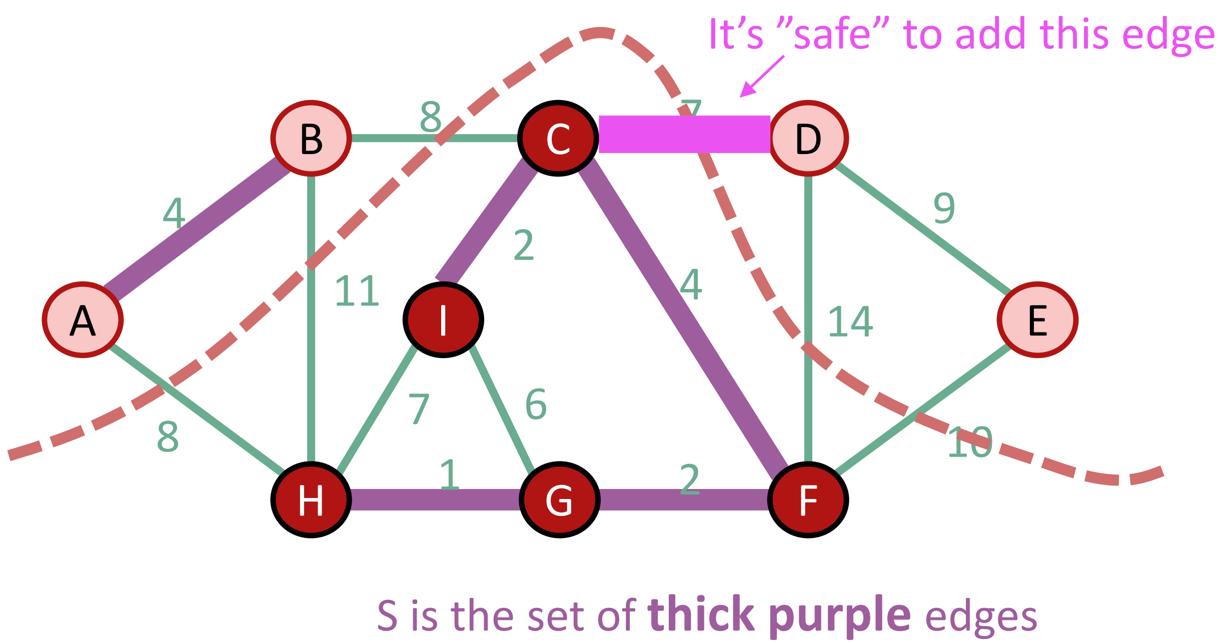
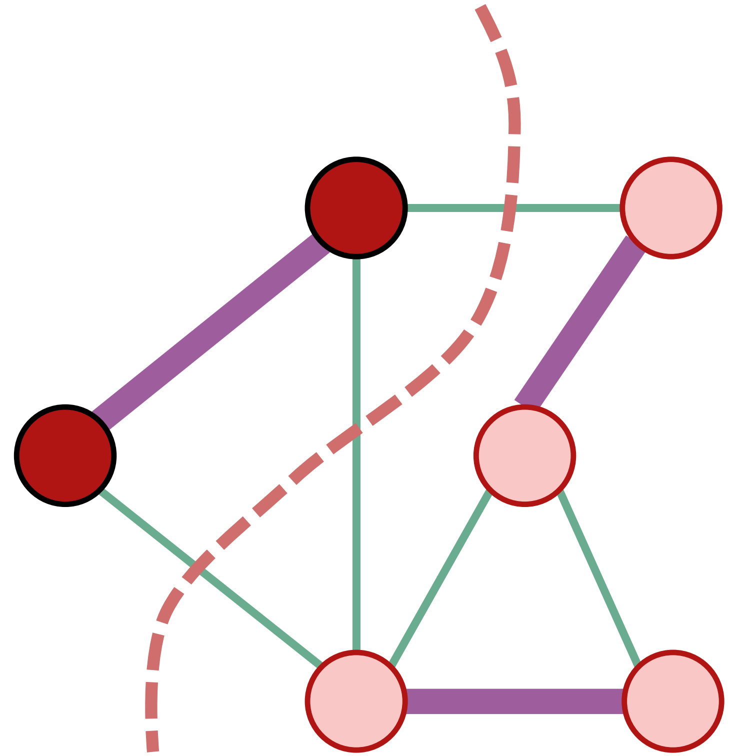
Aka:
If we haven't ruled out the possibility of success so far, then adding a light edge still won't rule it out.



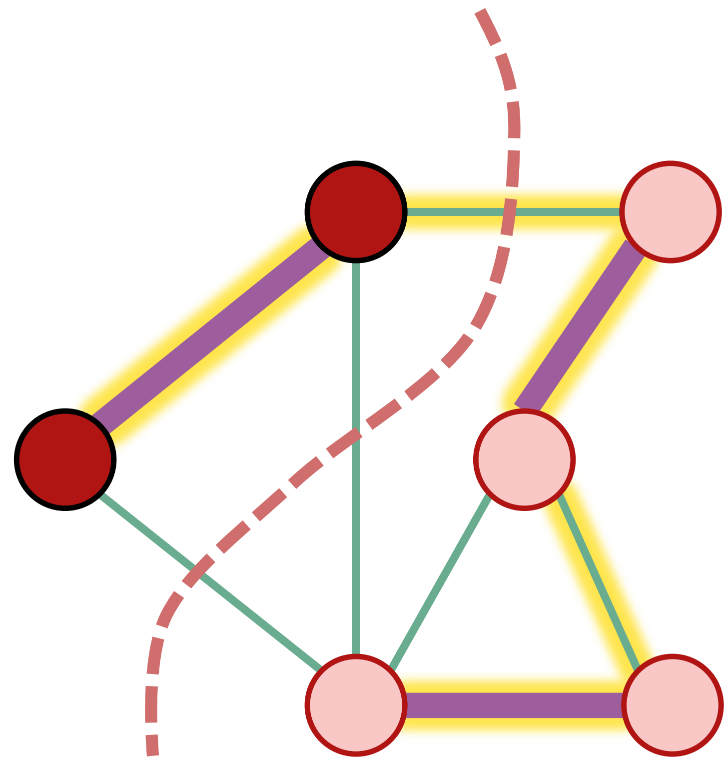
Proof of Lemma

- Assume that we have:
  - a **cut** that **respects S**



Proof of Lemma

- Assume that we have:
  - a **cut** that respects **S**
  - **S** is part of some **MST T**.
- Say that **{u, v}** is light.
  - lowest cost crossing the cut



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  - a **cut** that respects **S**
  - **S** is part of some **MST T**.
- Say that **{u, v}** is light.
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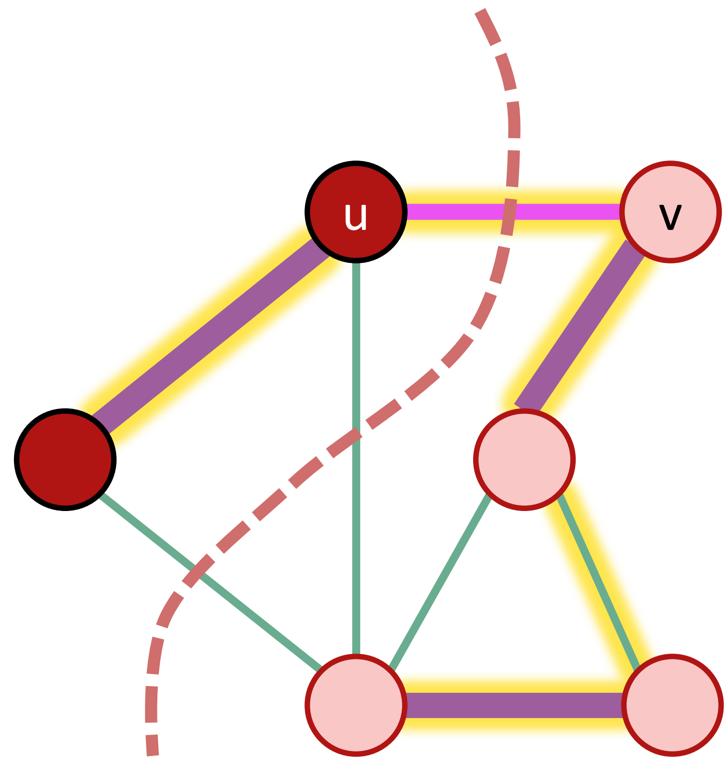
- If **{u, v}** is in T, we are done.
  - T is an MST containing

    both {u, v} and S.



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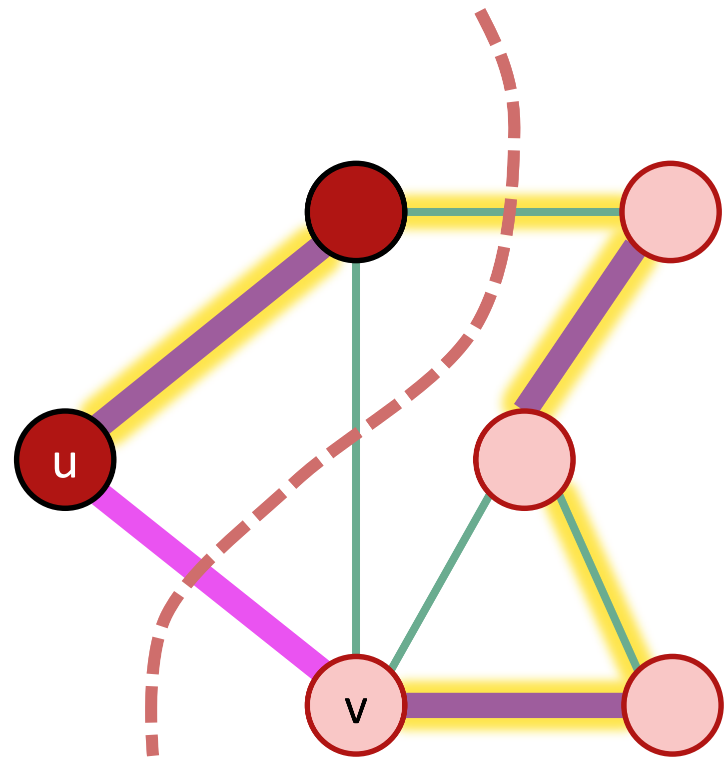
  - **S** is part of some **MST T**.
- Say that **{u, v}** is light.
  - lowest cost crossing the cut
- Say **{u, v}** is not in **T**.
- Note that adding

  {u, v} to **T** will make a cycle.

Claim: Adding any additional edge to a spanning tree will create a cycle.
Proof: Both endpoints are already in the tree and connected to each other.



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- Say that **{u, v}** is light.
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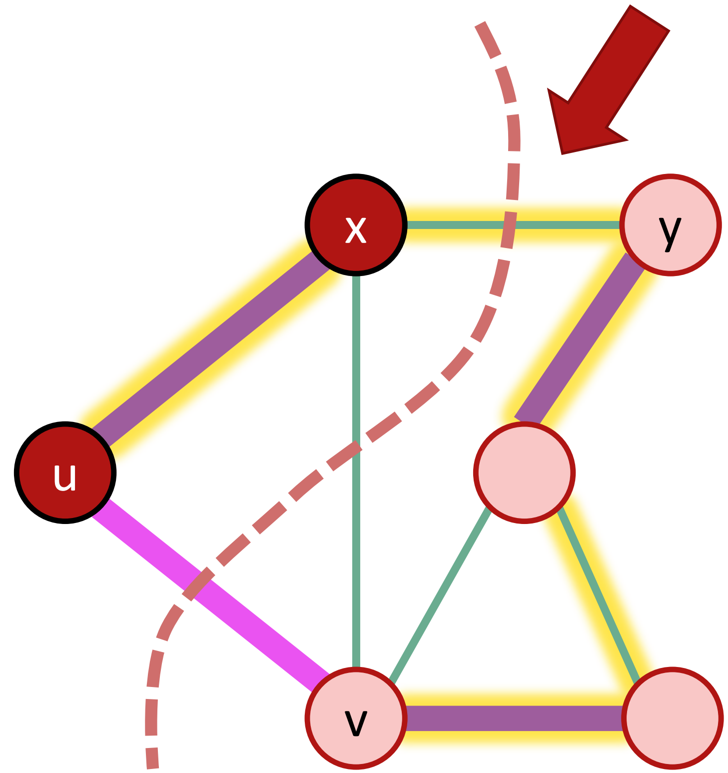
- Say **{u, v}** is not in **T**.
- Note that adding

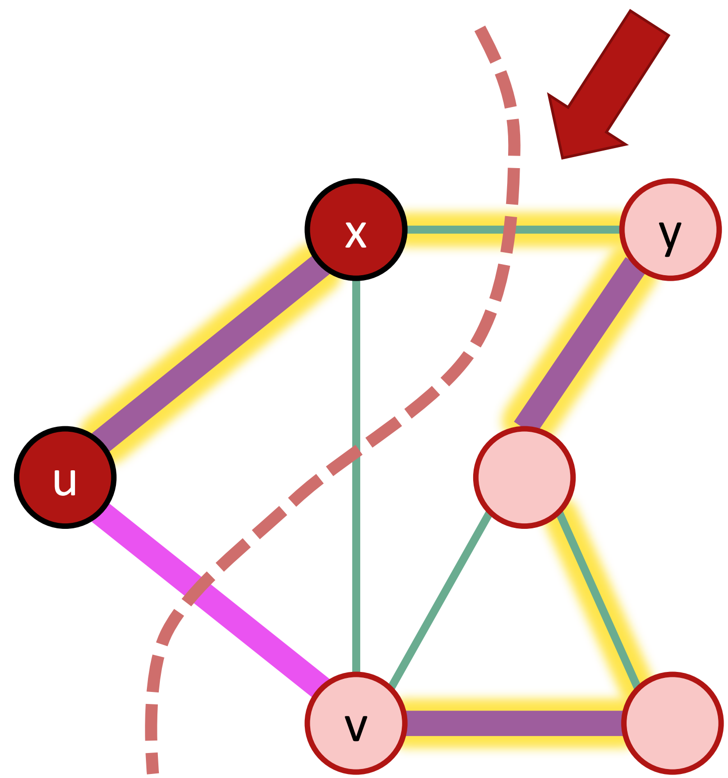
  {u, v} to **T** will make a cycle.
- There is at least one other edge, **{x, y}**, in this cycle crossing the cut.

Claim: Adding any additional edge to a spanning tree will create a cycle.
Proof: Both endpoints are already in the tree and connected to each other.



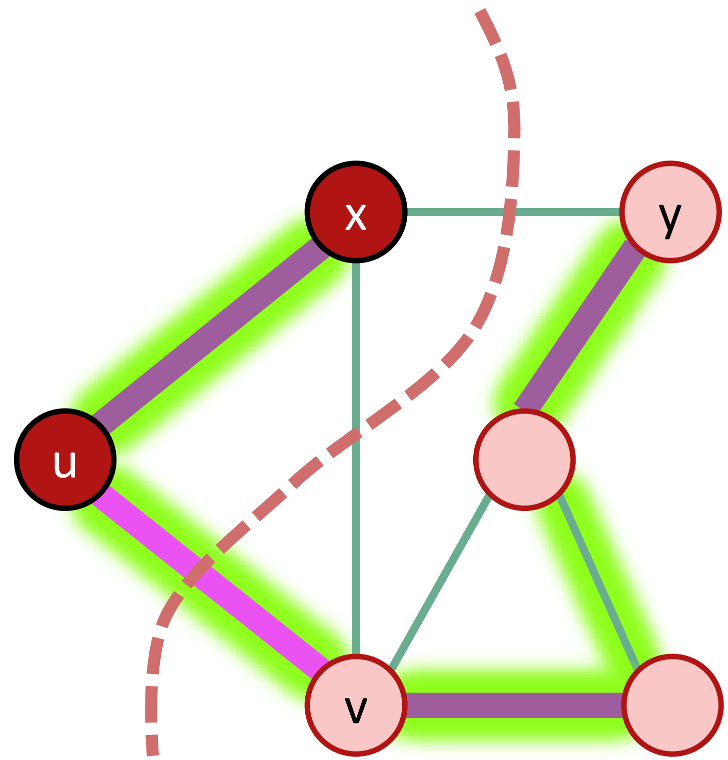
Proof of Lemma

- Consider swapping {u, v} for {x, y} in **T**.
  - Call the resulting tree **T'**.



Proof of Lemma

- Consider swapping {u, v} for {x, y} in **T**.
  - Call the resulting tree **T'**.
- **Claim**: **T'** is still an MST.
  - It is still a spanning tree (why?)
    - It has cost at most that of **T**
    - because {u, v} was light.
    - **T** had minimal cost.
    - So **T'** does too.
- So **T'** is an MST containing S and {u, v}.
  - This is what we wanted.



Lemma

- Let S be a set of edges, and consider a cut that respects S.
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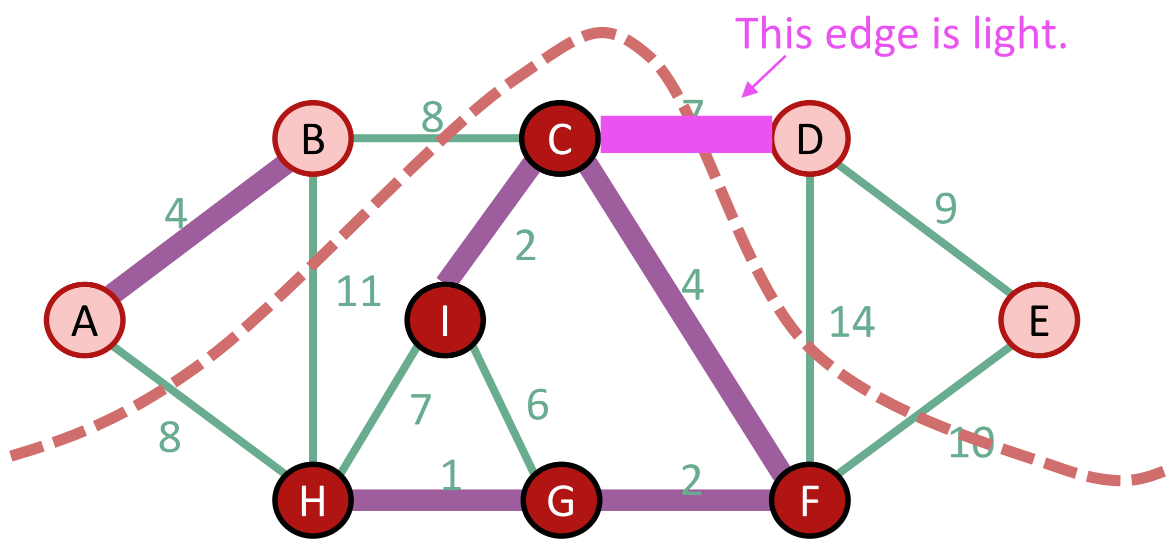


S is the set of **thick purple** edges

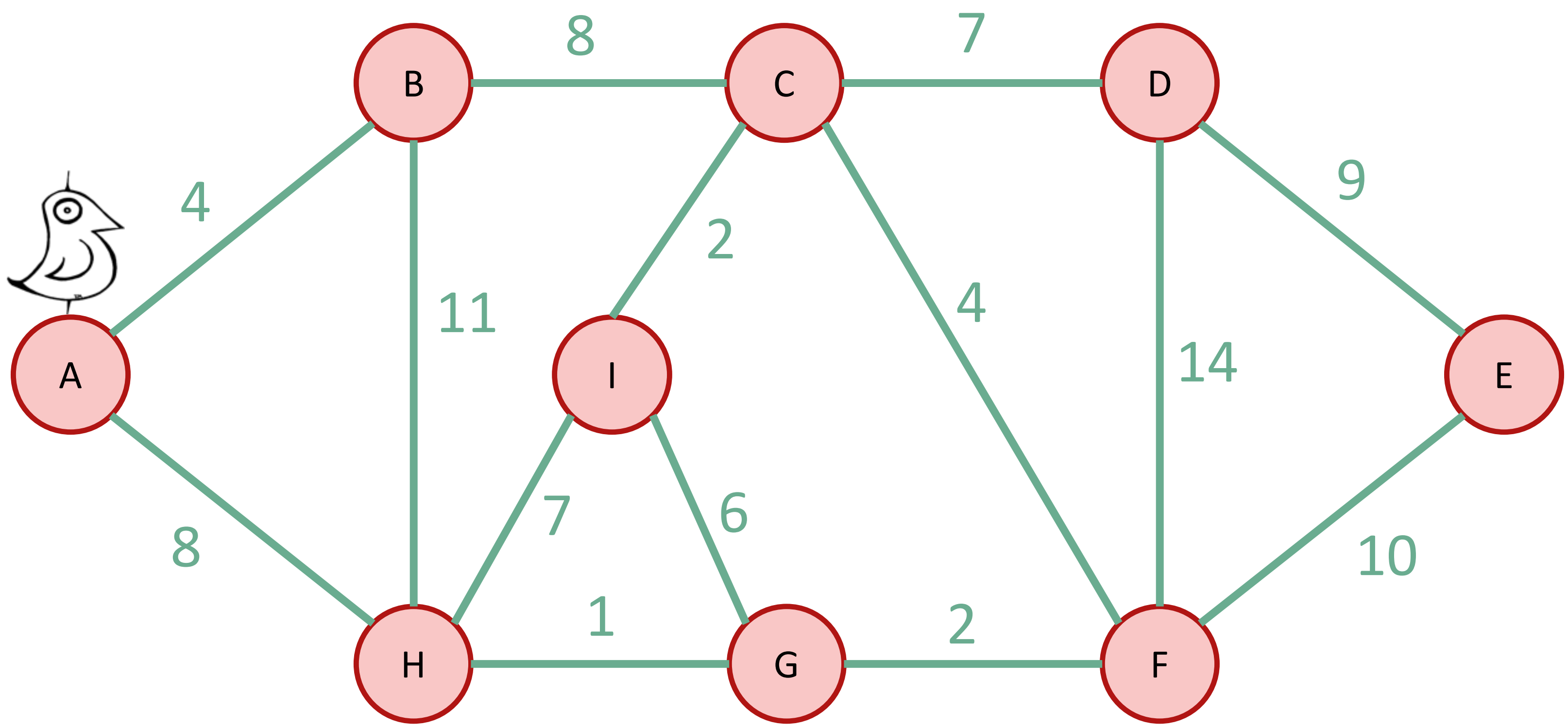
Back to MSTs

- How do we find one?
- Today we'll see two greedy algorithms.

- The strategy:
  - Make a series of choices, adding edges to the tree.
  - Show that each edge we add is safe to add:
    - we do not rule out the possibility of success
    - we will choose light edges crossing cuts and use the Lemma.
  - Keep going until we have an MST.

Idea 1

- Start growing a tree, greedily add the shortest edge we can to grow the tree.



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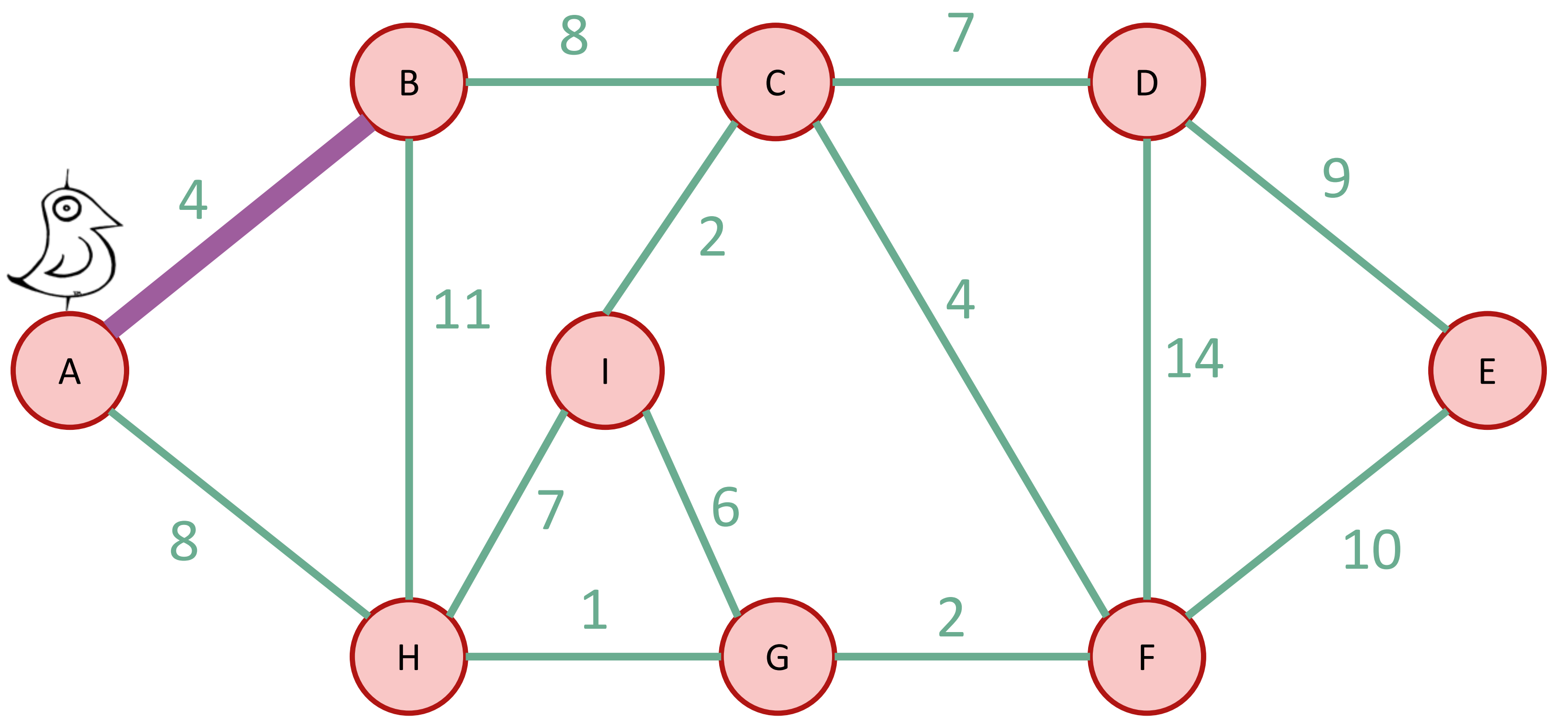

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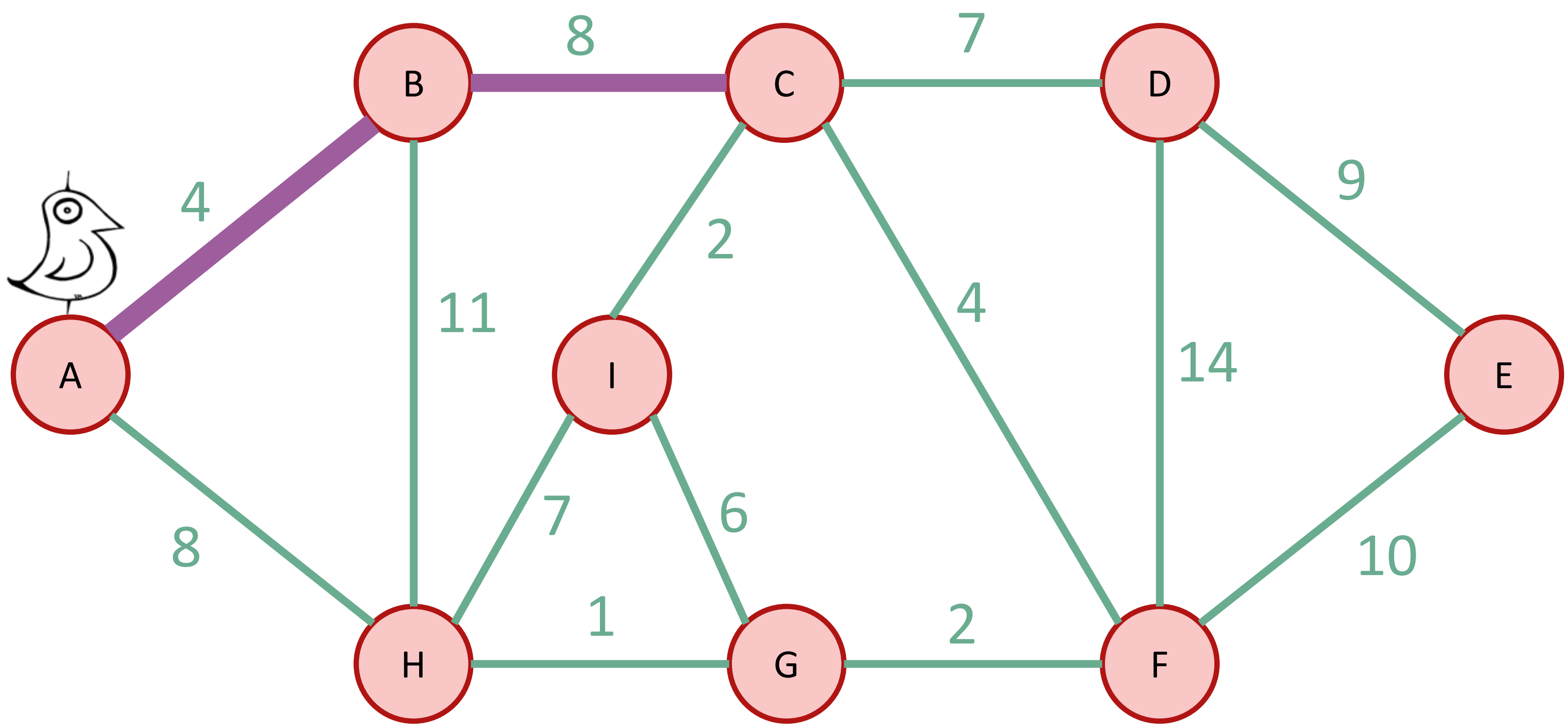

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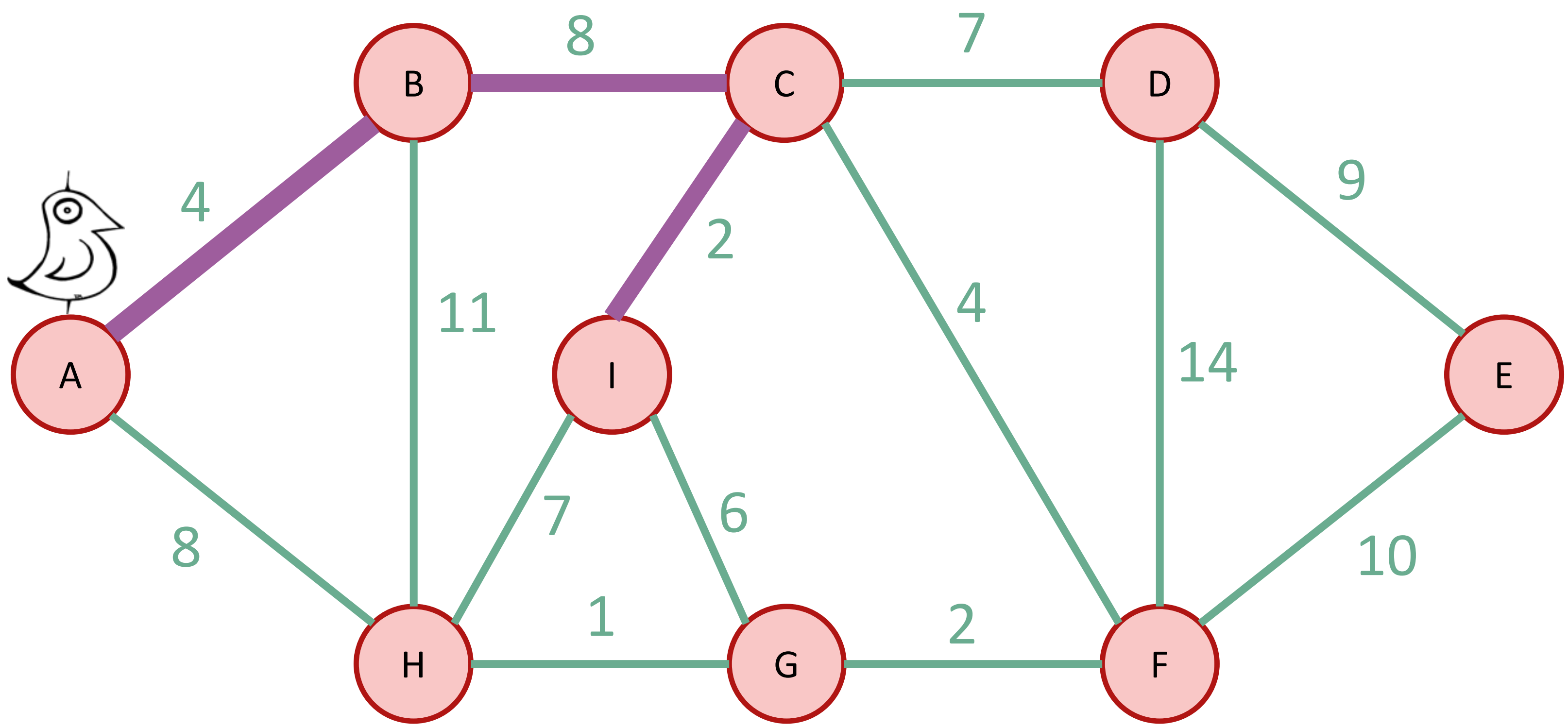

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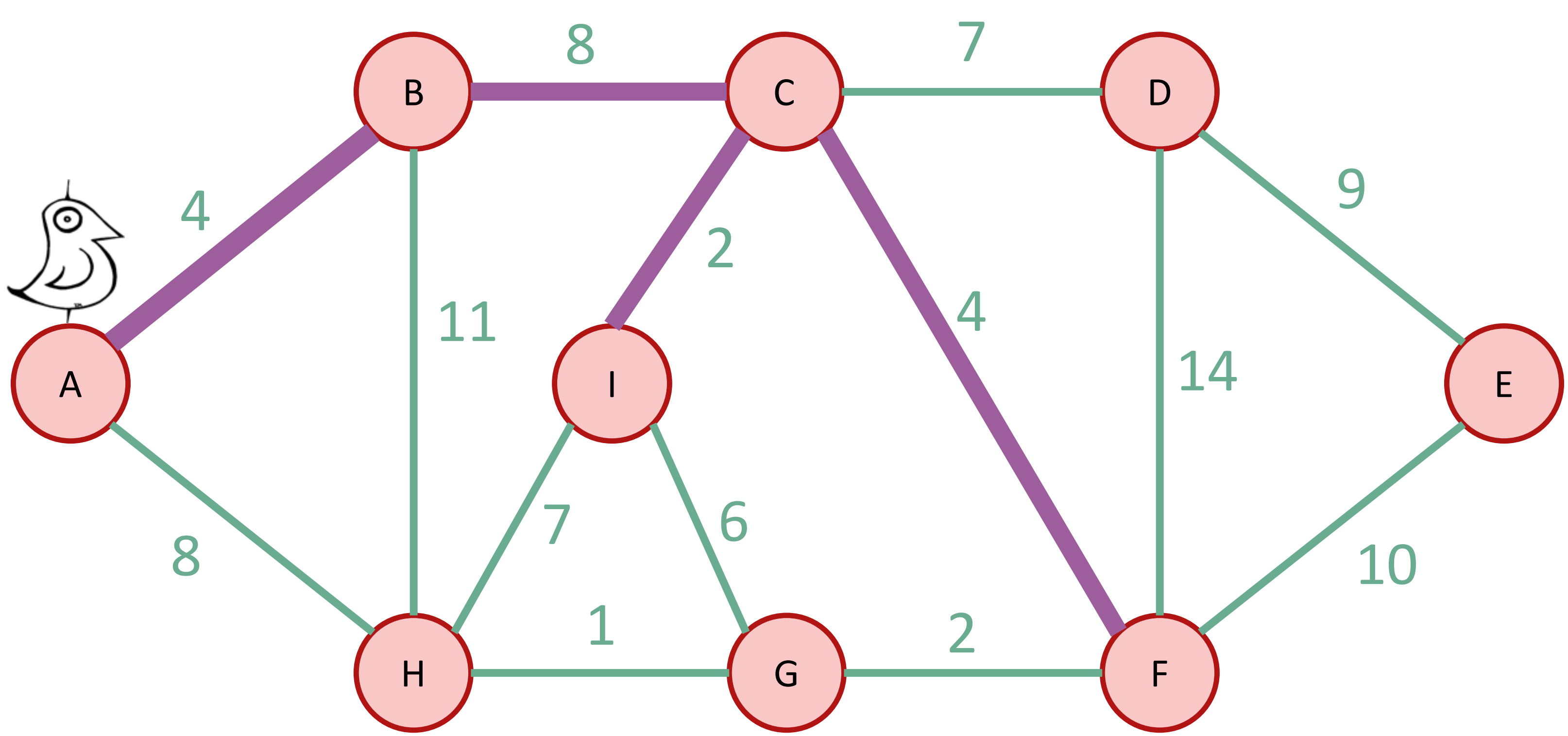

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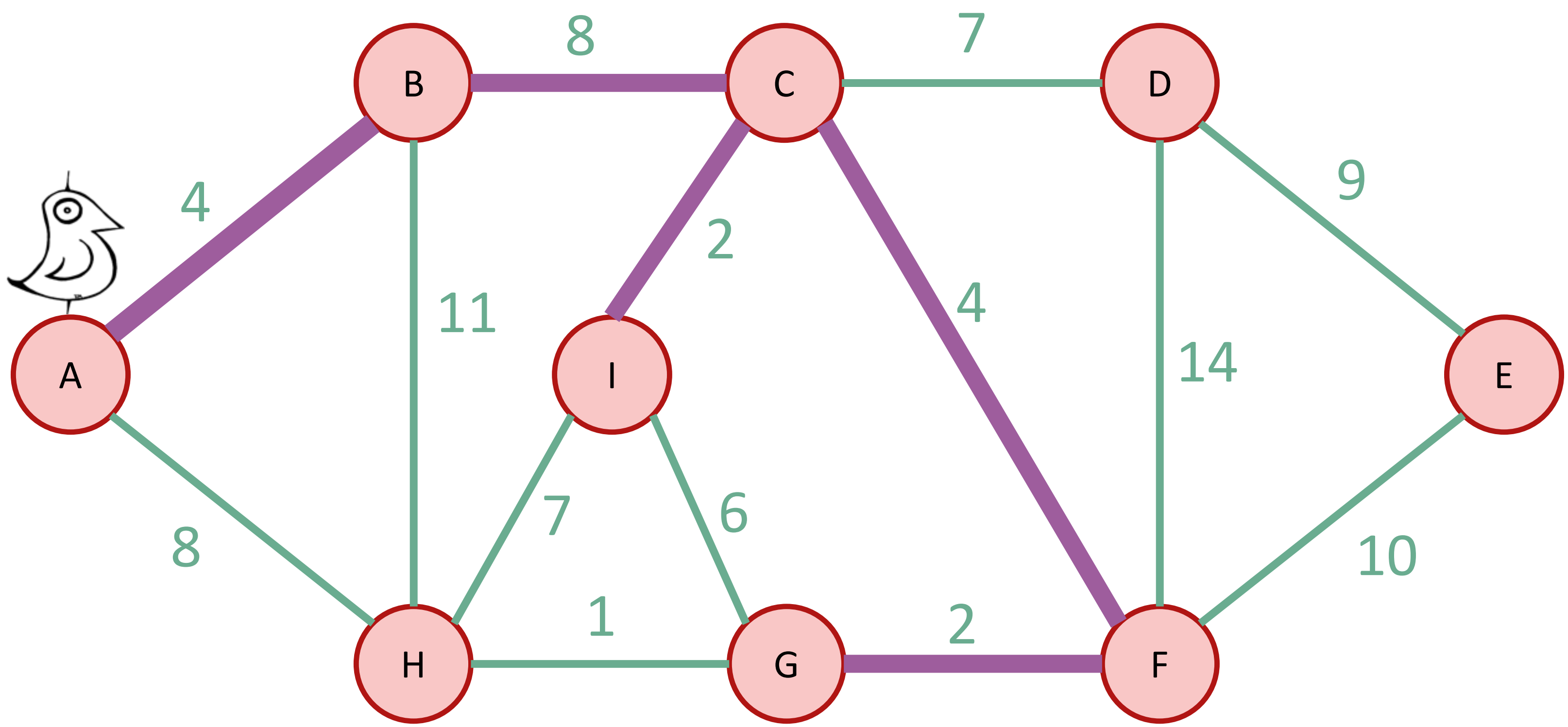

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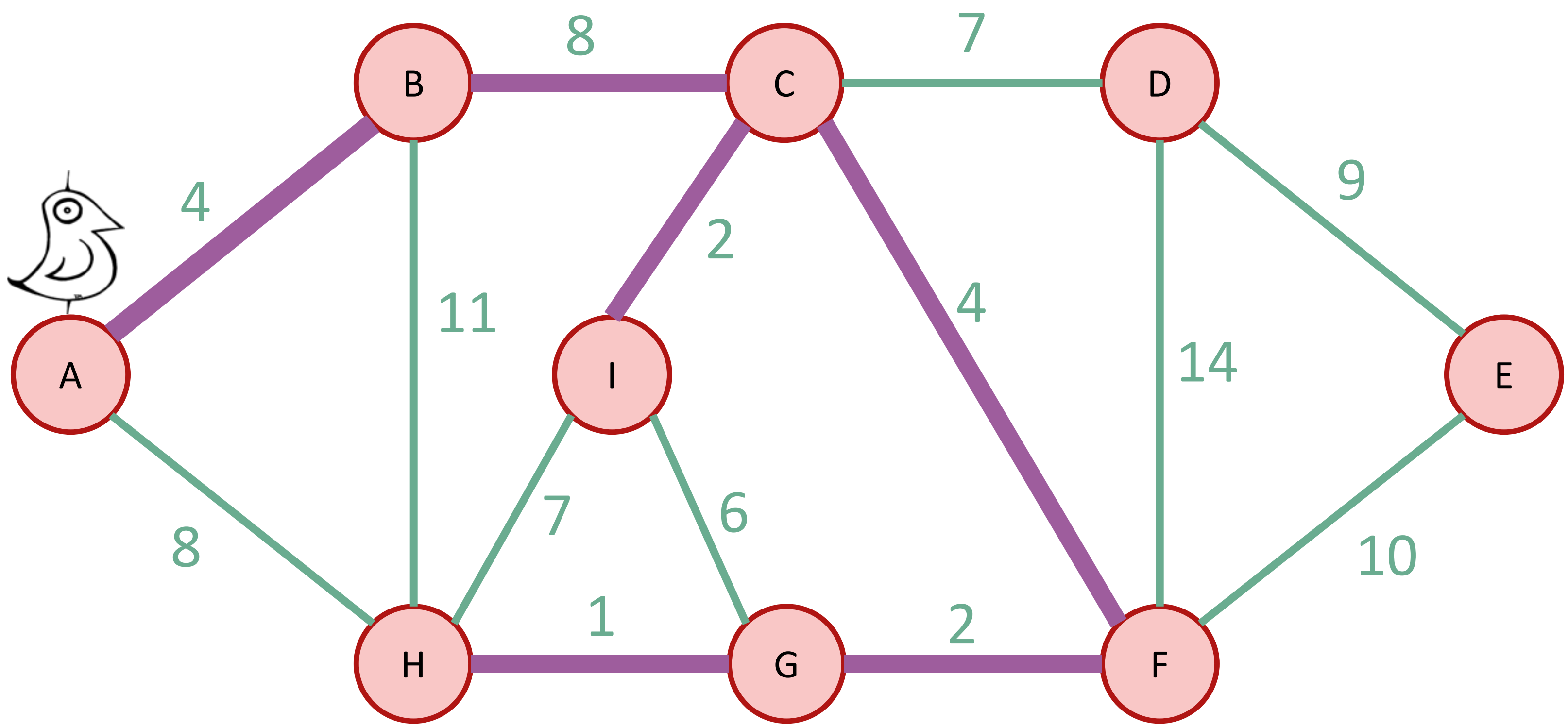

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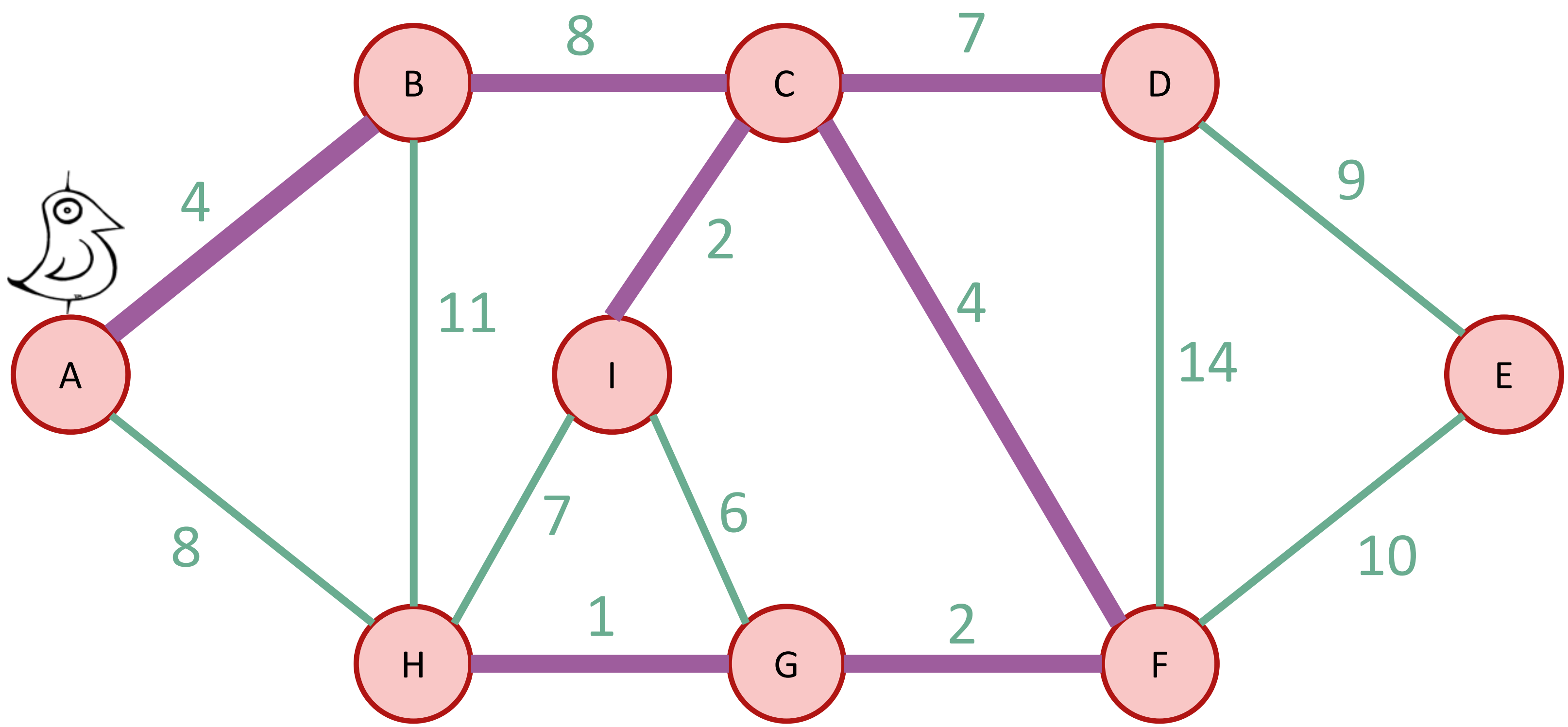

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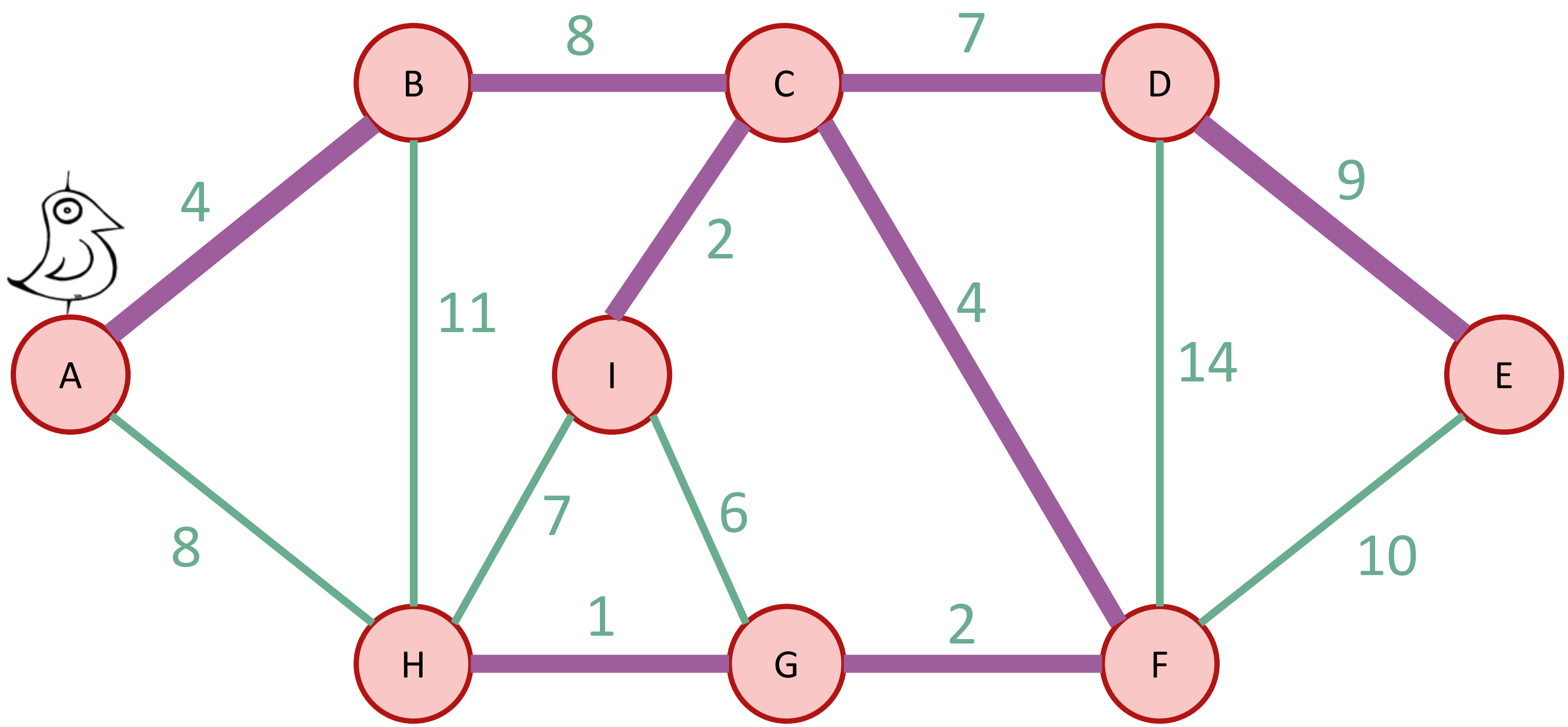


We've discovered Prim's algorithm!

- **slowPrim (G = (V, E), starting vertex s):**
  - Let (s, u) be the lightest edge coming out of s.
  - MST = {(s, u)}
  - verticesVisited = {s, u}
  - **while** |verticesVisited| < |V|:
    - find the lightest edge {x, v} in E so that:
      - x is in verticesVisited
      - v is not in verticesVisited
    - add {x, v} to MST
    - add v to verticesVisited
  - **return** MST

At most n iterations of this while loop.

Time at most m to go through all the edges and find the lightest.

Naively, the running time is O(nm):

- For each of $\leq$n iterations of the while loop:
  - Go through all the edges.

Two questions

1. Does it work?
   - That is, does it actually return a MST?

2. How do we actually implement this?
   - The pseudocode above says “slowPrim”…

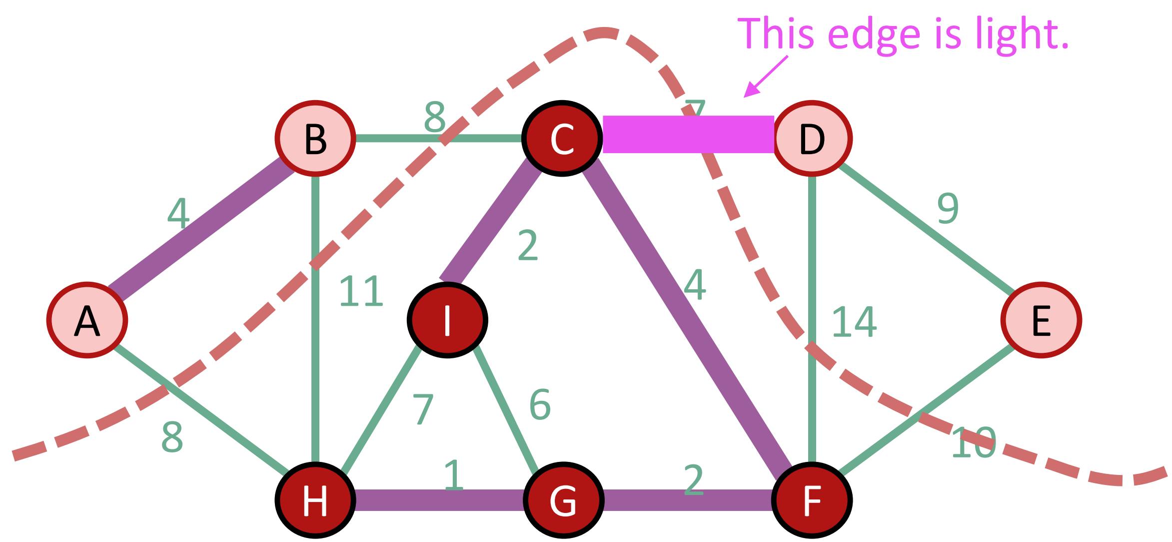
Does it work?

- We need to show that our greedy choices don't rule out success.
- That is, at every step:
  - If there exists an MST that contains all of the edges S we have added so far…
  - …then when we make our next choice {u, v}, there is still an MST containing S and {u, v}.

- Now it is time to use our lemma!

Lemma

- Let S be a set of edges, and consider a cut that respects S.
- Suppose there is an MST containing S.
- Let {u, v} be a light edge.
- Then there is an MST containing S ∪ {{u, v}}



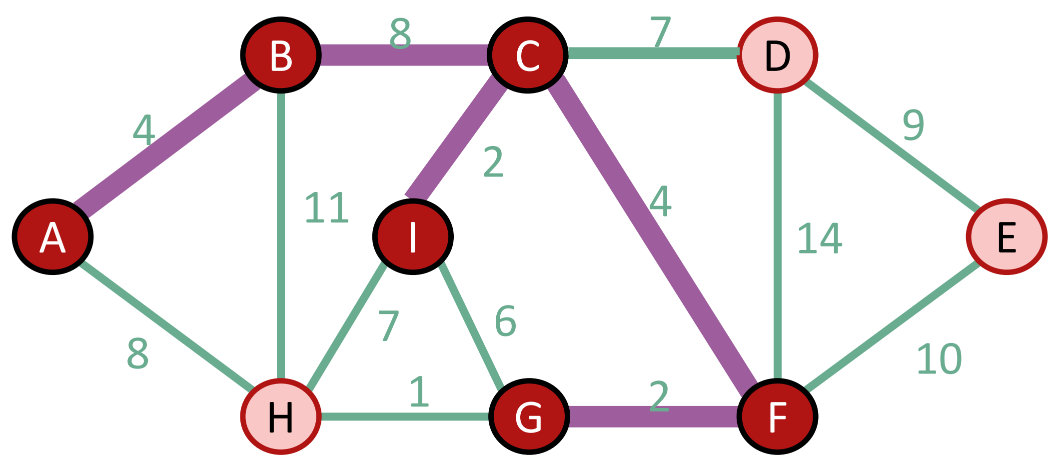
S is the set of **thick purple** edges

Partway through Prim

- Assume that our choices **S** so far don't rule out success
  - There is an MST consistent with those choices

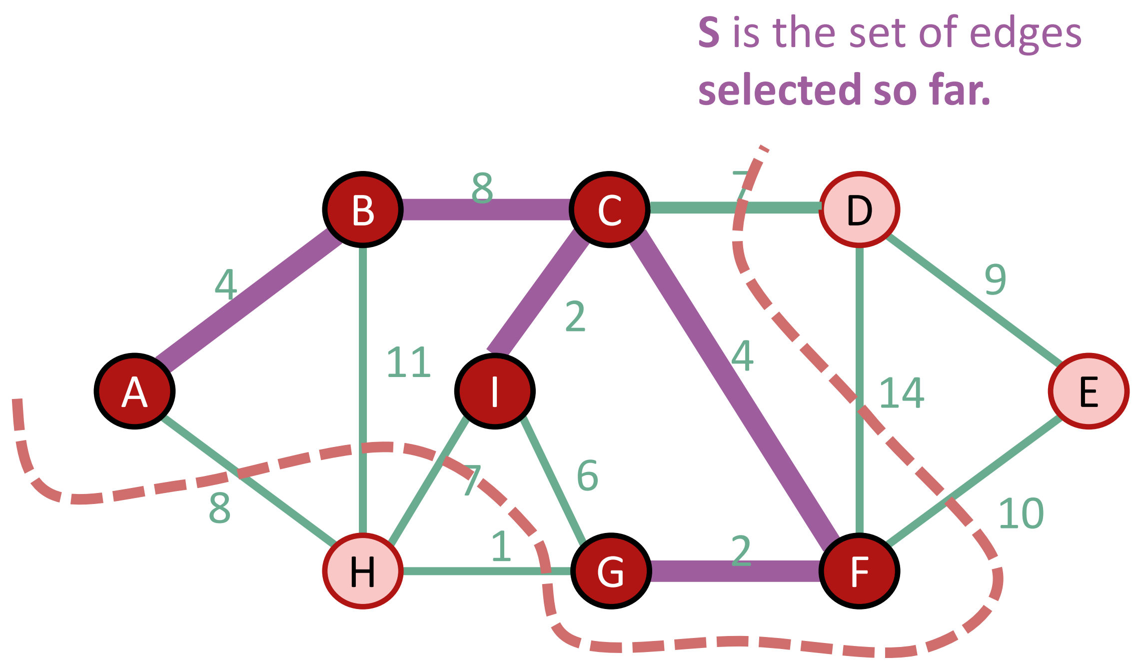
    Q. How can we use our lemma to show that our next choice also does not rule out success?

S is the set of edges **selected so far.**



Partway through Prim

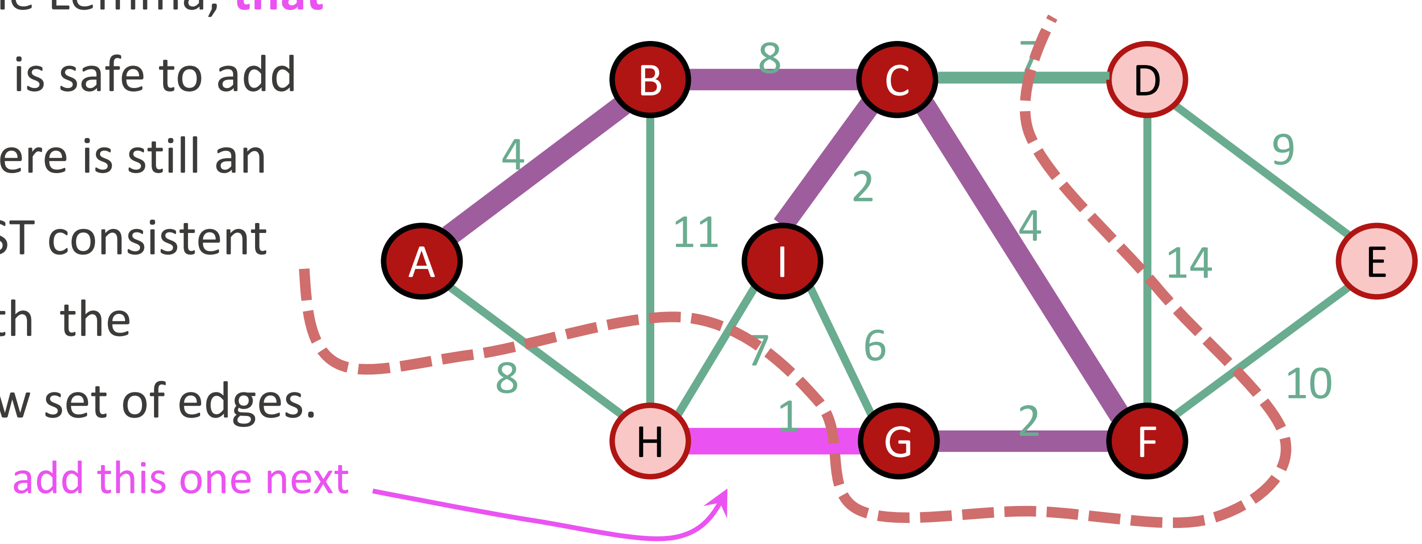
- Assume that our choices **S** so far don't rule out success
  - There is an MST consistent with those choices
- Consider the cut {**visited**, **unvisited**}
  - This cut respects S.



Partway through Prim

- Assume that our choices **S** so far don't rule out success
  - There is an MST consistent with those choices
- Consider the cut {**visited**, **unvisited**}
  - This cut respects S.
- The edge we add next is a **light edge**.
  - Least weight of any edge crossing the cut.
- By the Lemma, **that edge** is safe to add
  - There is still an MST consistent with the new set of edges.

S is the set of edges **selected so far.**



Hooray!

- Our greedy choices **don't rule out success**.
- This is enough (along with an argument by induction) to guarantee correctness of Prim's algorithm.

Formally

- Inductive hypothesis:
  - After adding the t'th edge, there exists an MST with the edges added so far.
- Base case:
  - After adding the 0'th edge, there exists an MST with the edges added so far. **YEP.**
- Inductive step:
  - If the inductive hypothesis holds for t (aka, the choices so far are safe), then it holds for t+1 (aka, the next edge we add is safe).
  - **That's what we just showed.**
- Conclusion:
  - After adding the n-1'st edge, there exists an MST with the edges added so far.
  - At this point we have a spanning tree, so it better be minimal.

Two questions

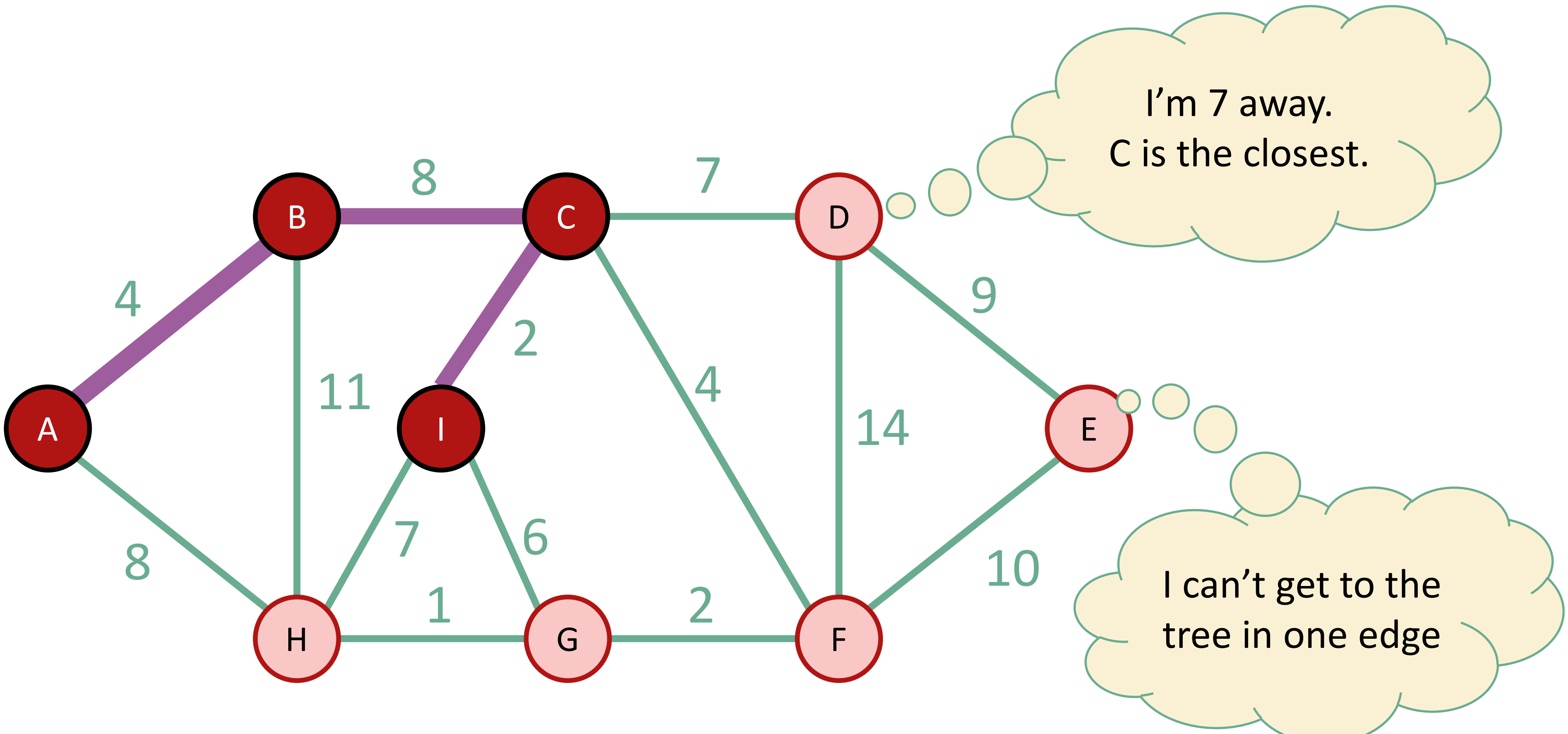
1. Does it work?
   - That is, does it actually return a MST?
     - **Yes!**

2. How do we actually implement this?
   - The pseudocode above says “slowPrim”…

How do we actually implement this?

- Each vertex keeps:
  - the **distance** from itself to the **growing spanning tree**
  - **how to get there**.

if you can get there in one edge.



How do we actually implement this?

- Each vertex keeps:
  - the **distance** from itself to the **growing spanning tree**
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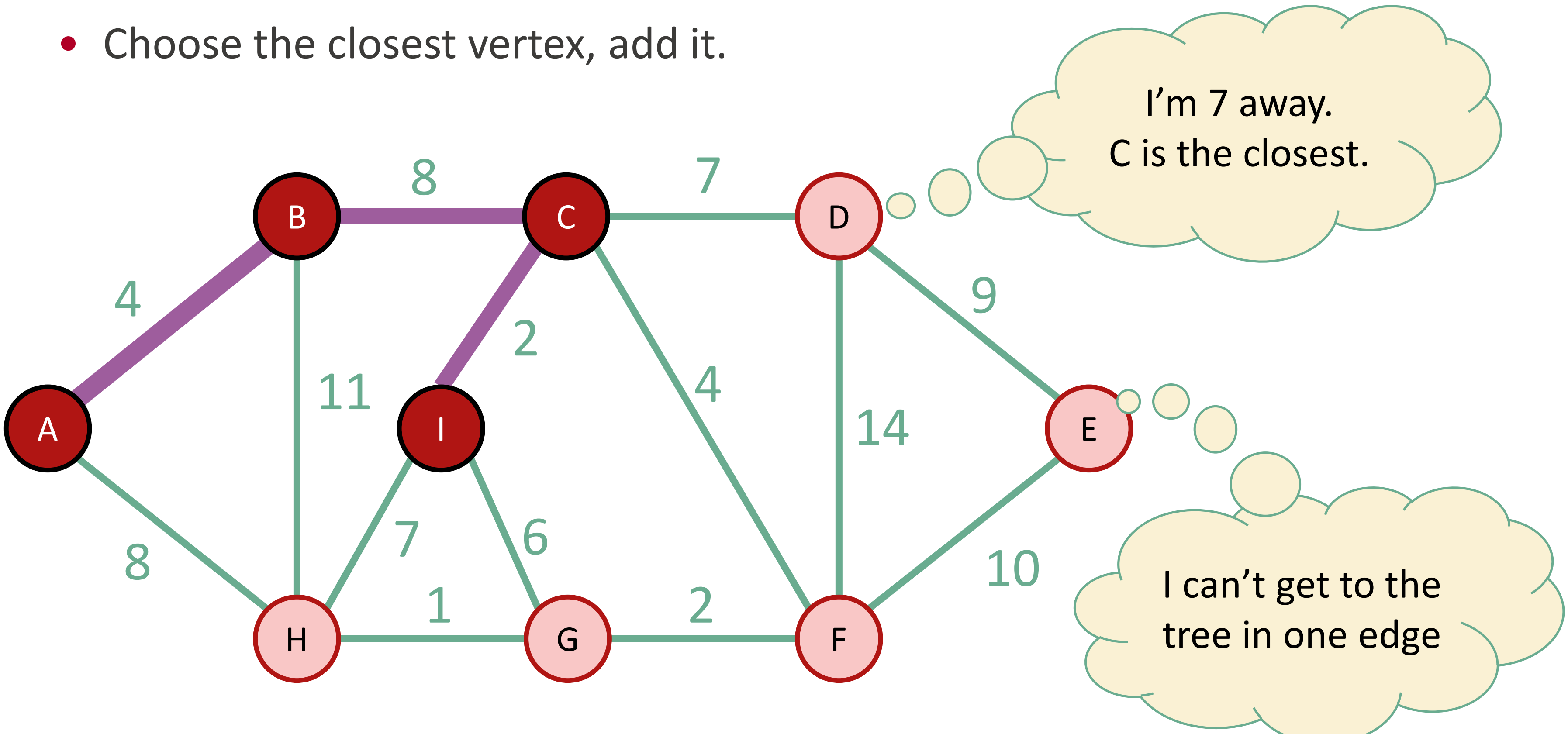
if you can get there in one edge.

- Choose the closest vertex, add it.



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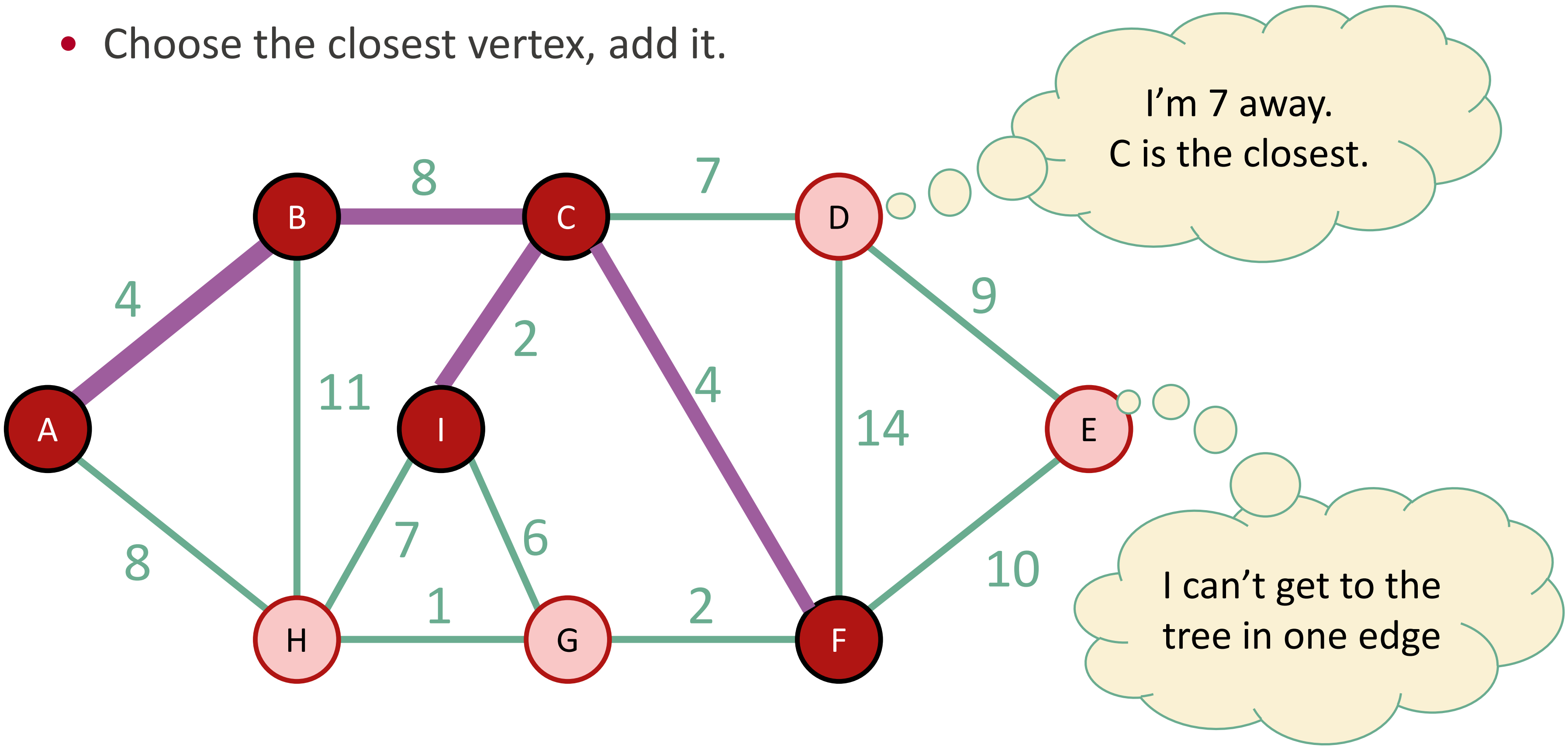
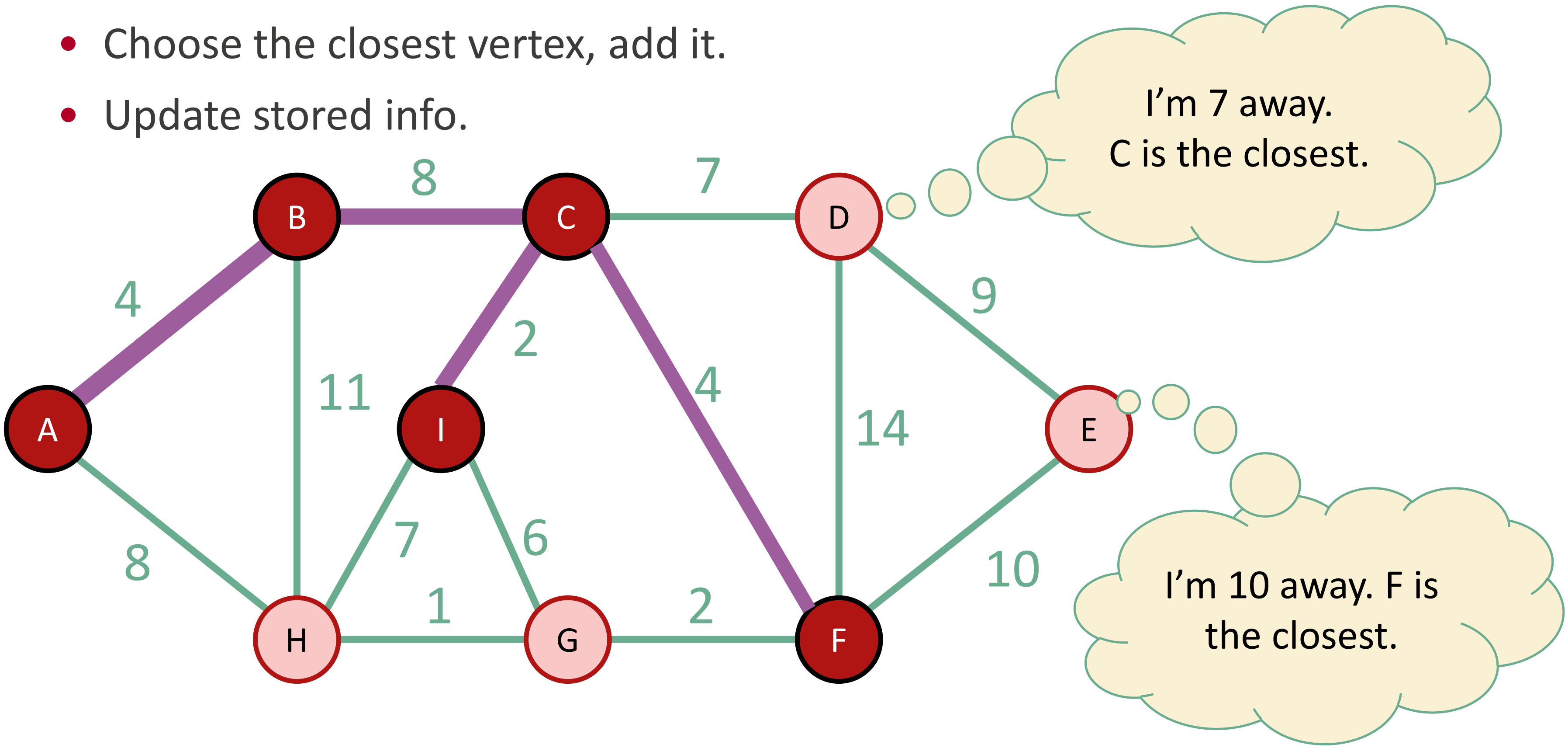


How do we actually implement this?

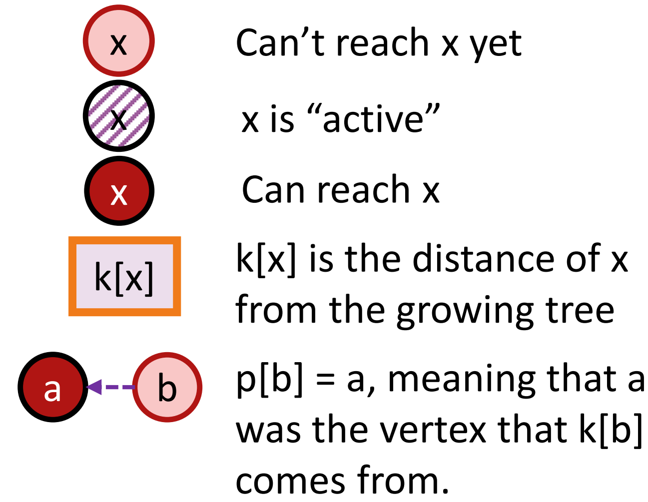
- Each vertex keeps:
  - the **distance** from itself to the **growing spanning tree**
  - **how to get there**. if you can get there in one edge.
- Choose the closest vertex, add it.
- Update stored info.

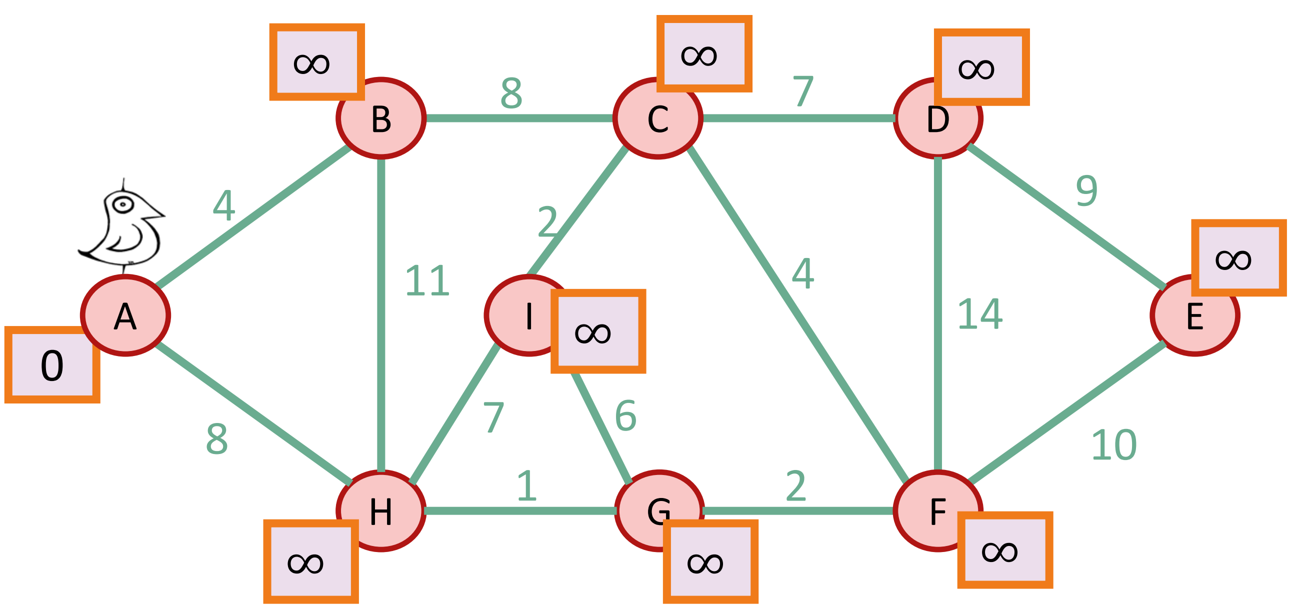


Efficient implementation

Every vertex has a key and a parent

Until all the vertices are **reached:**



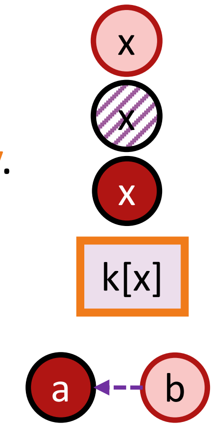


Efficient implementation

Every vertex has a key and a parent

Until all the vertices are **reached:**

Activate the **unreached** vertex u with the smallest key.



Can't reach x yet

x is "active"

Can reach x

k[x] is the distance of x from the growing tree

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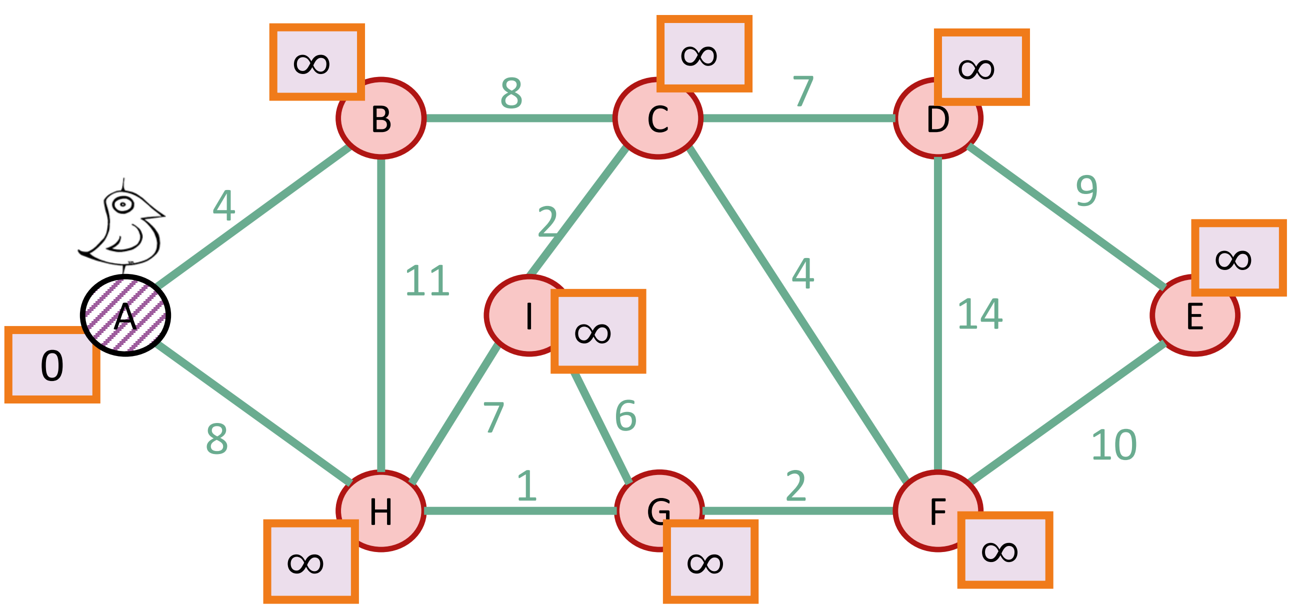

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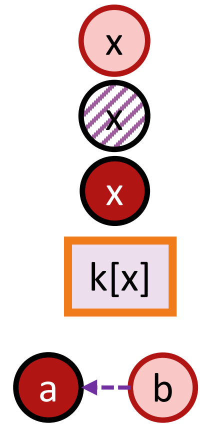

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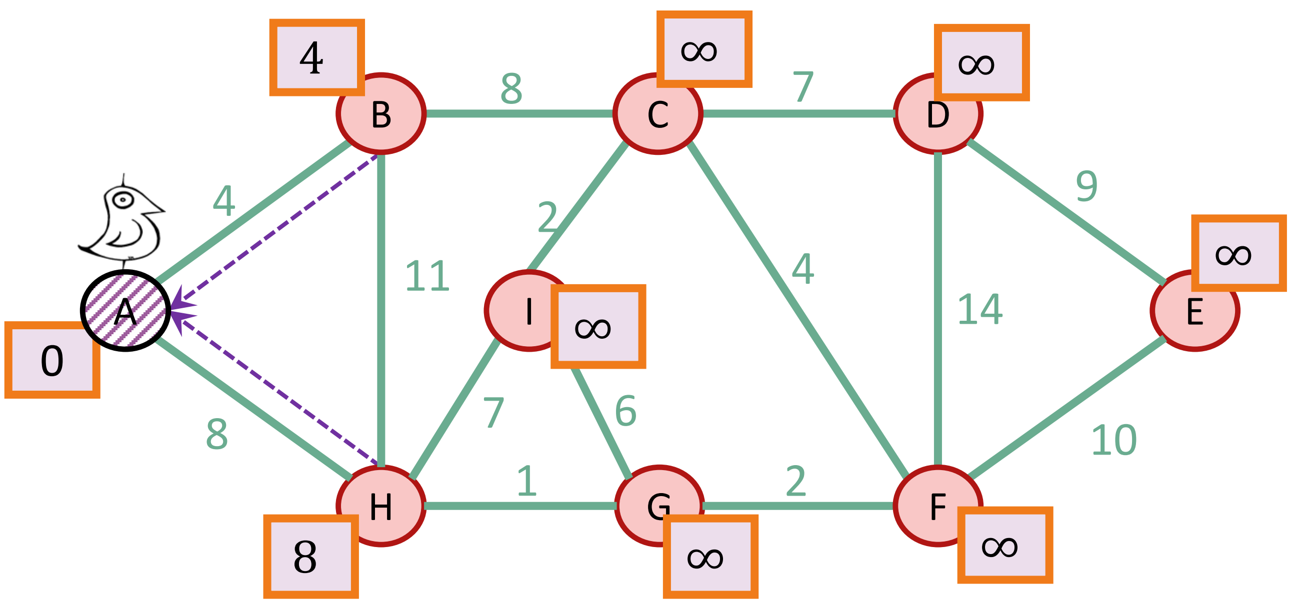

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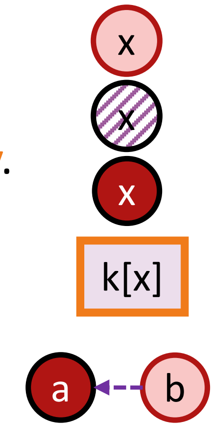

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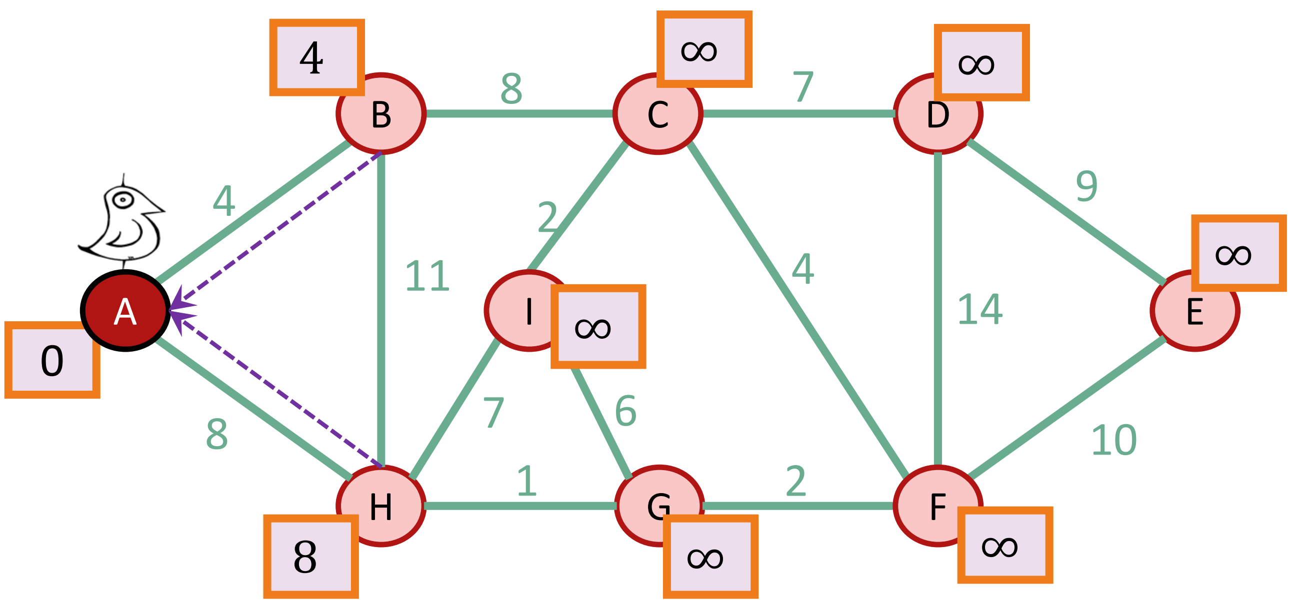

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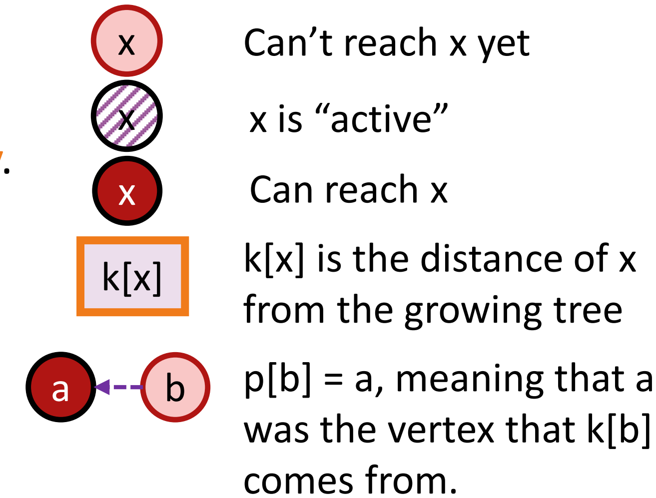

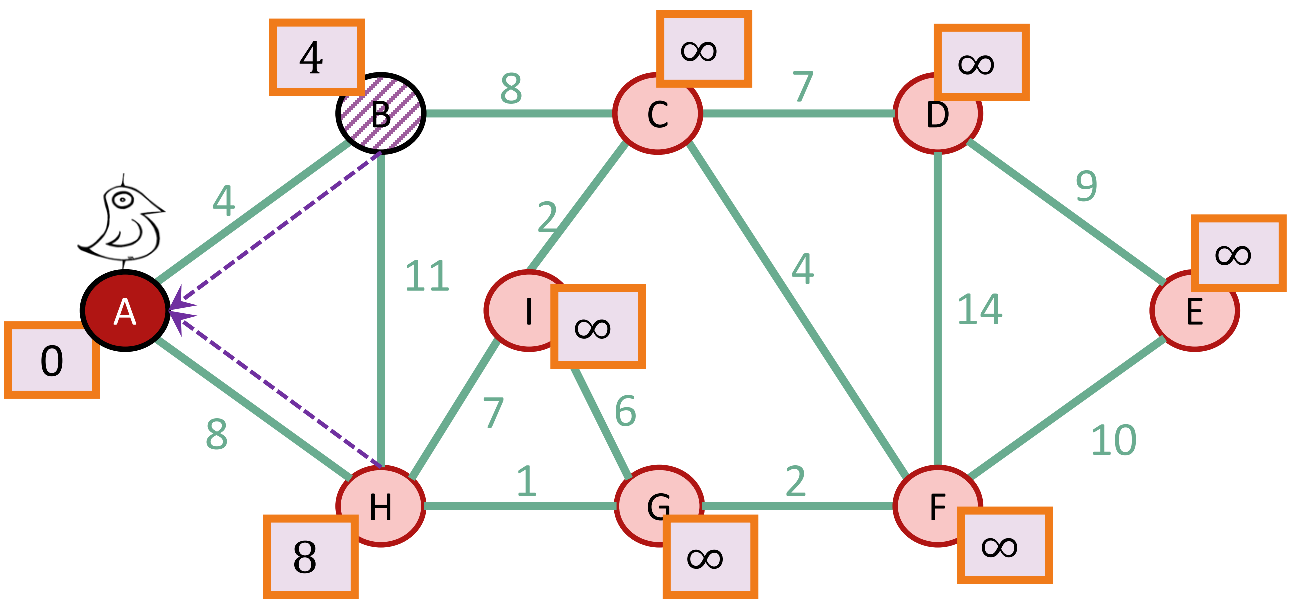

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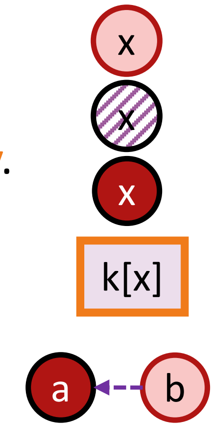

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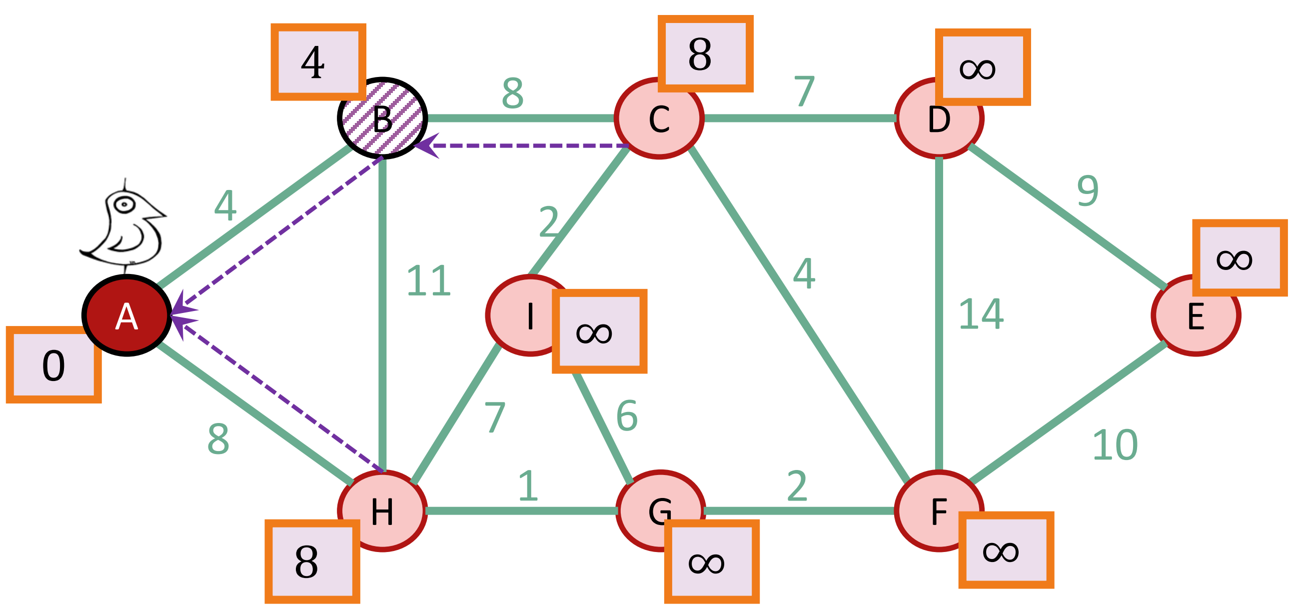

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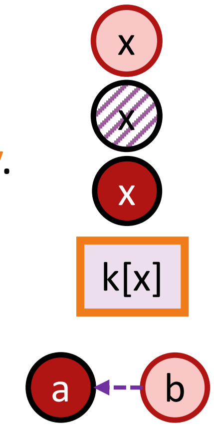

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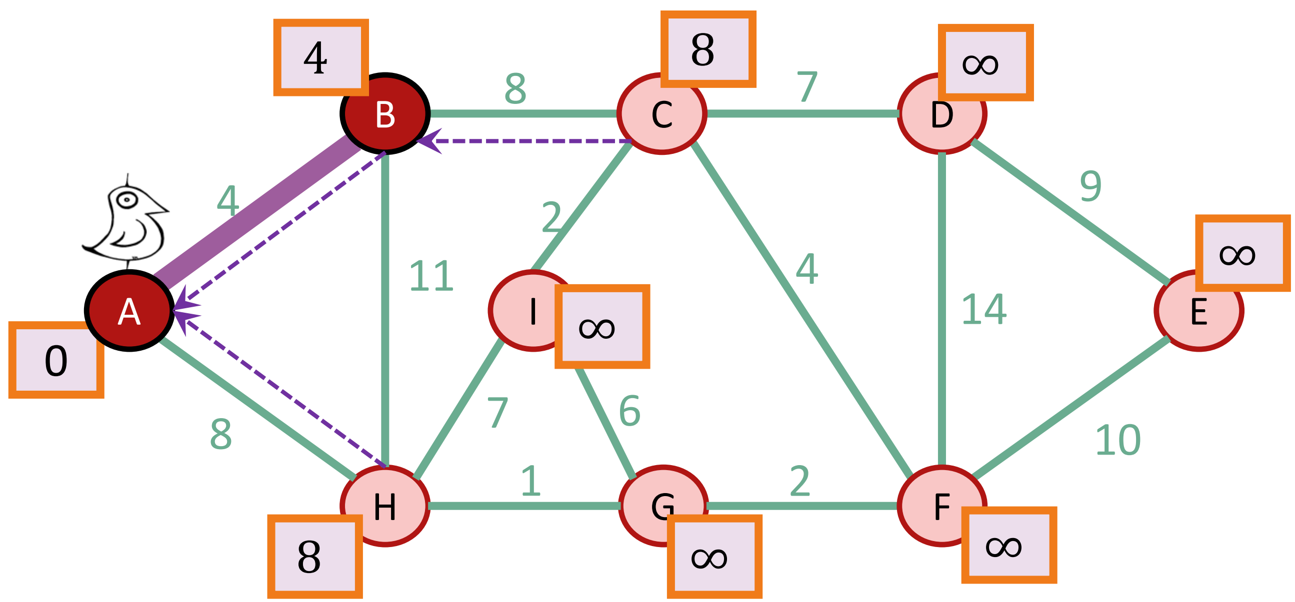

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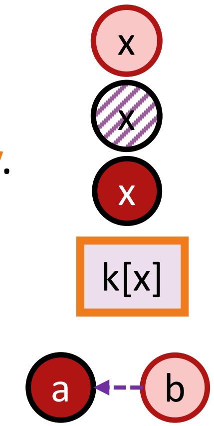

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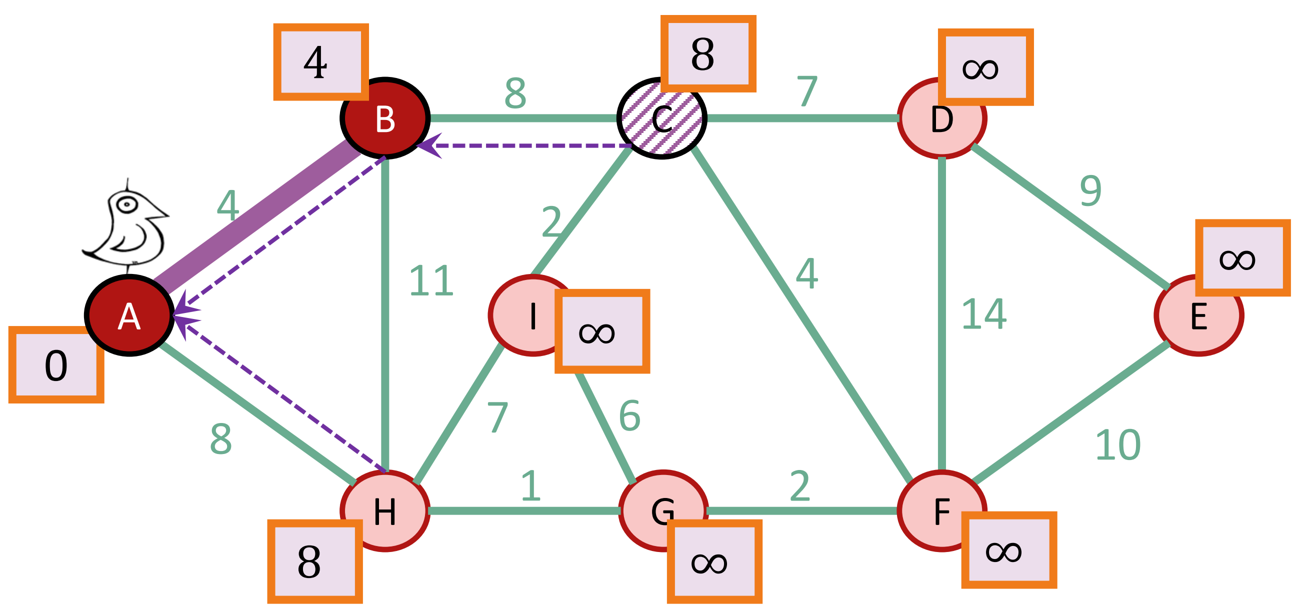

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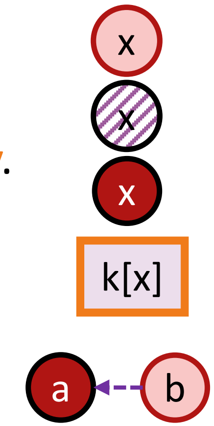

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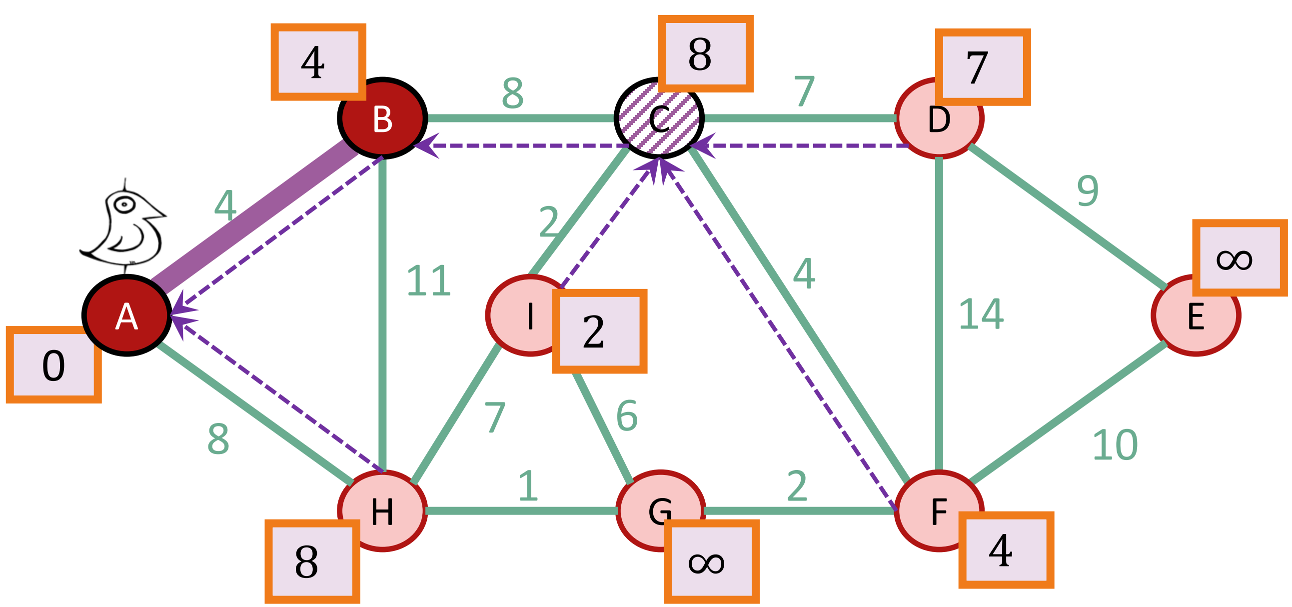

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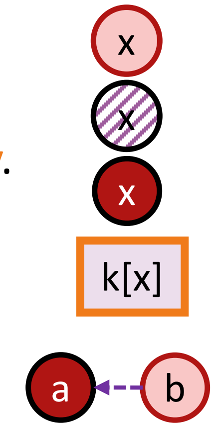

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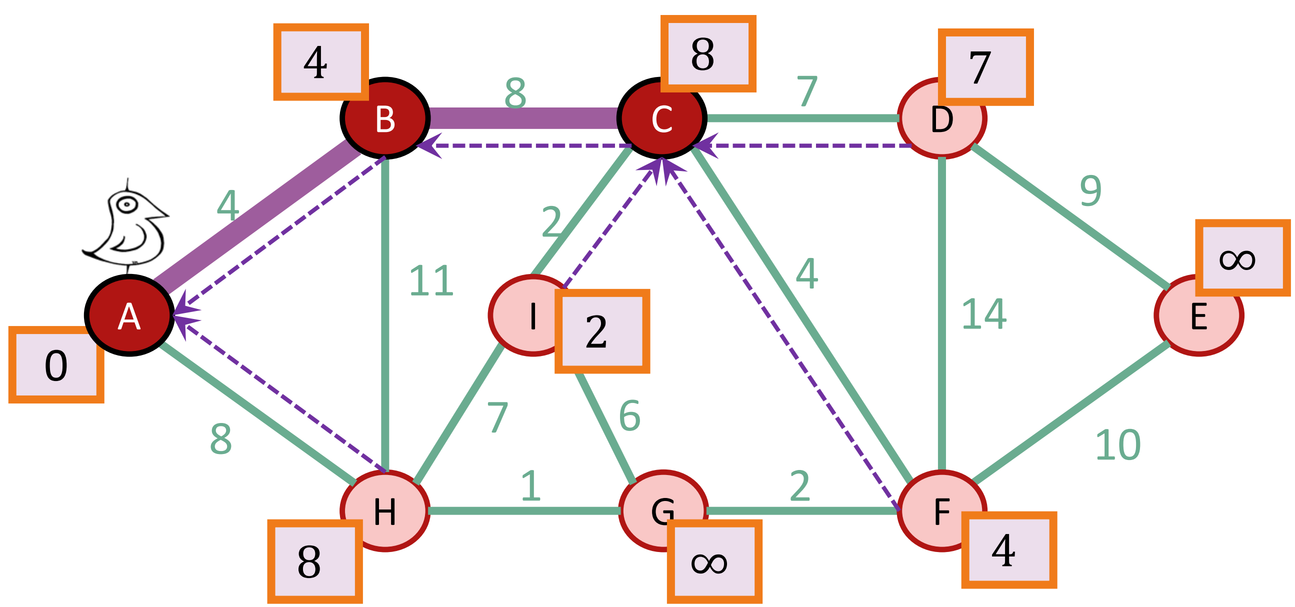

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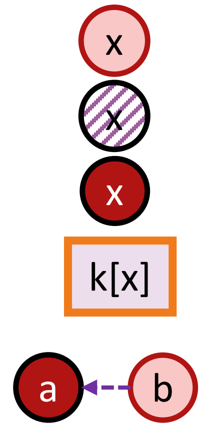

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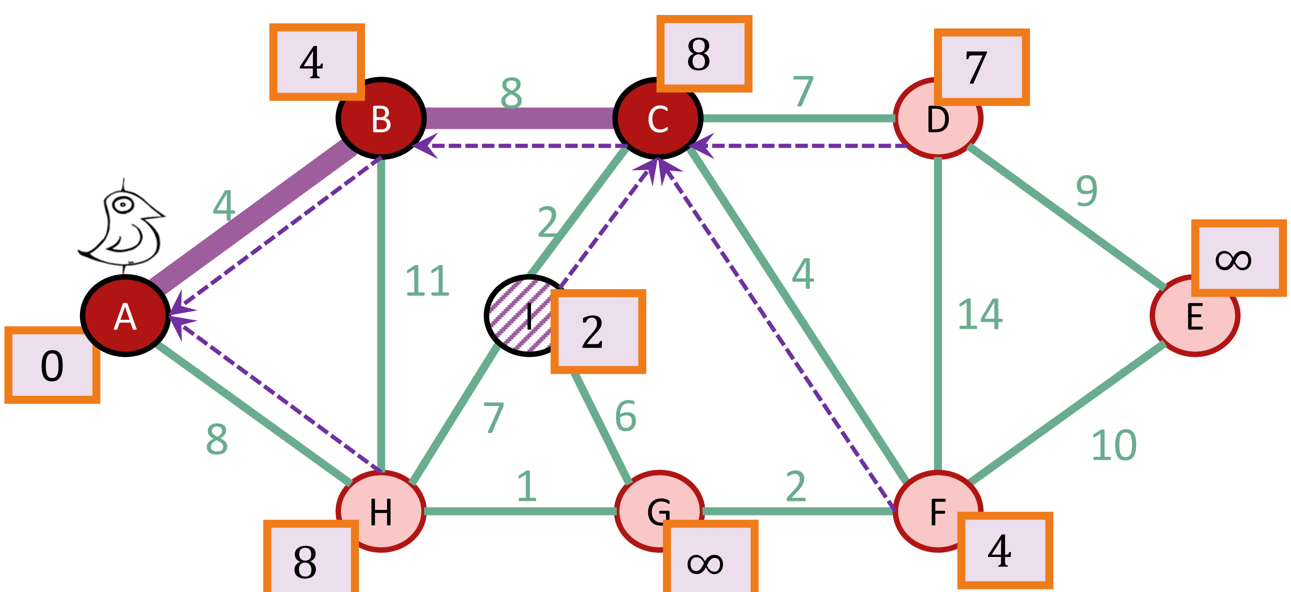

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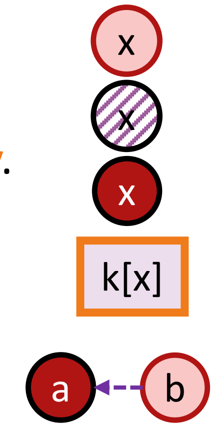

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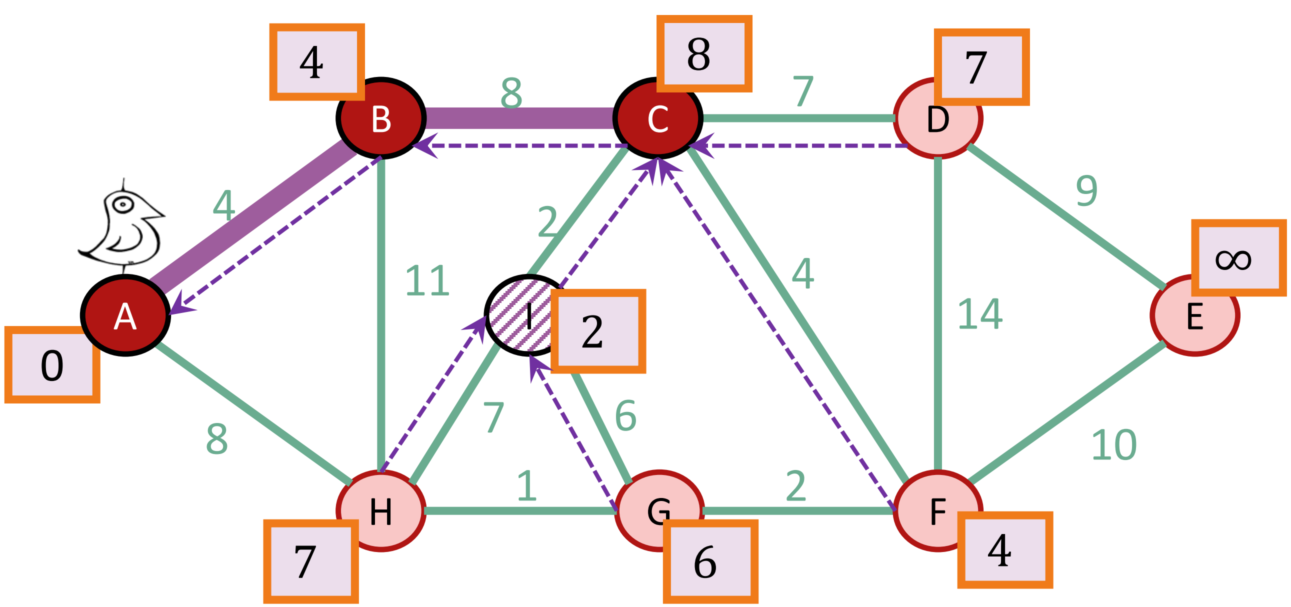

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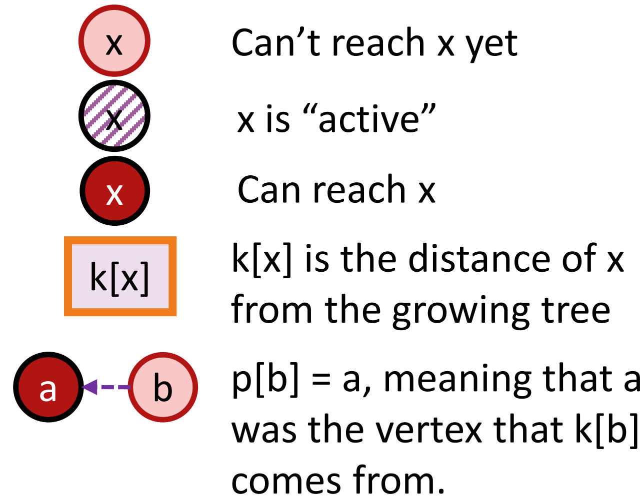

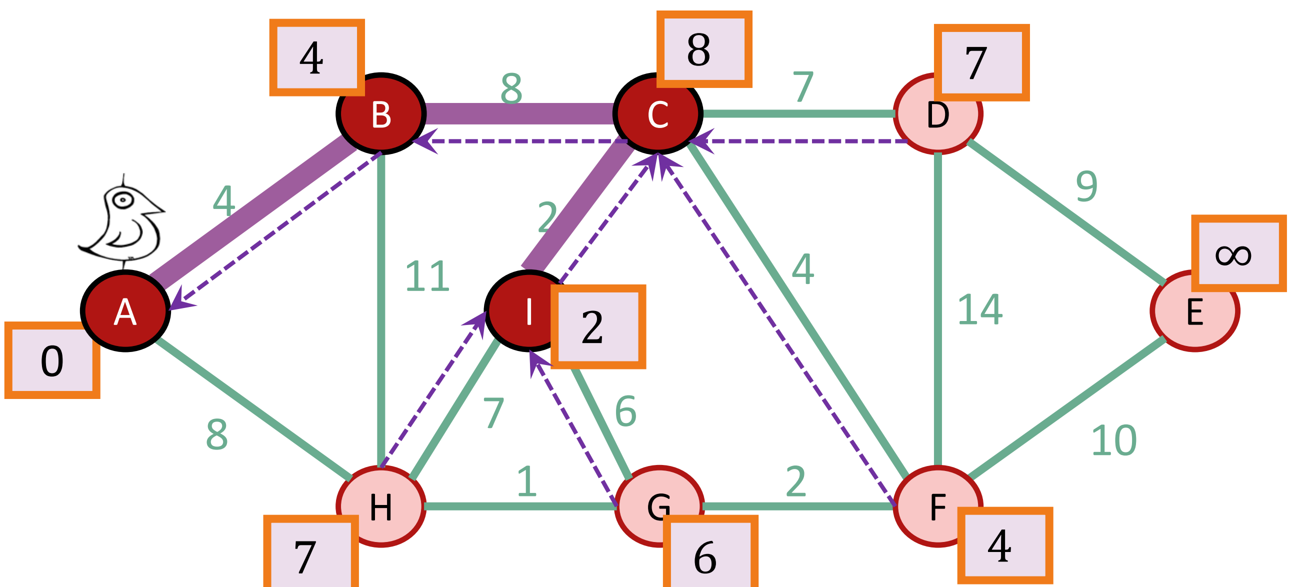

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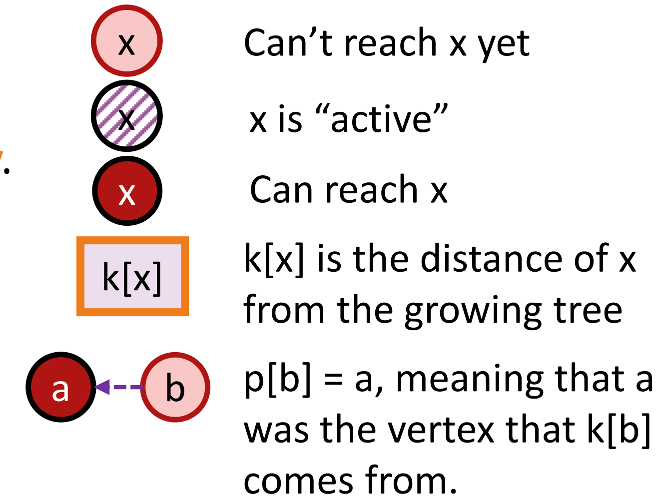

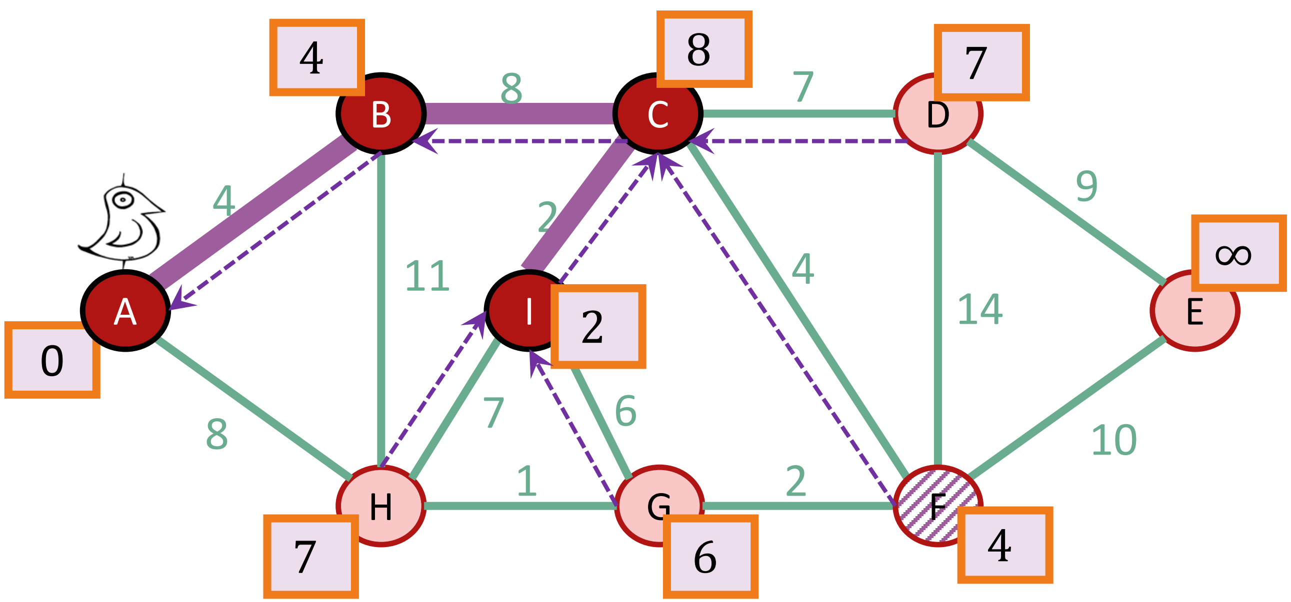

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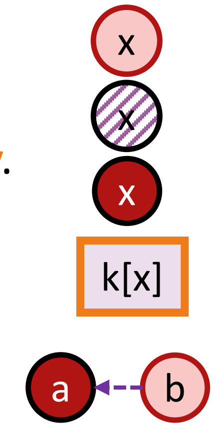

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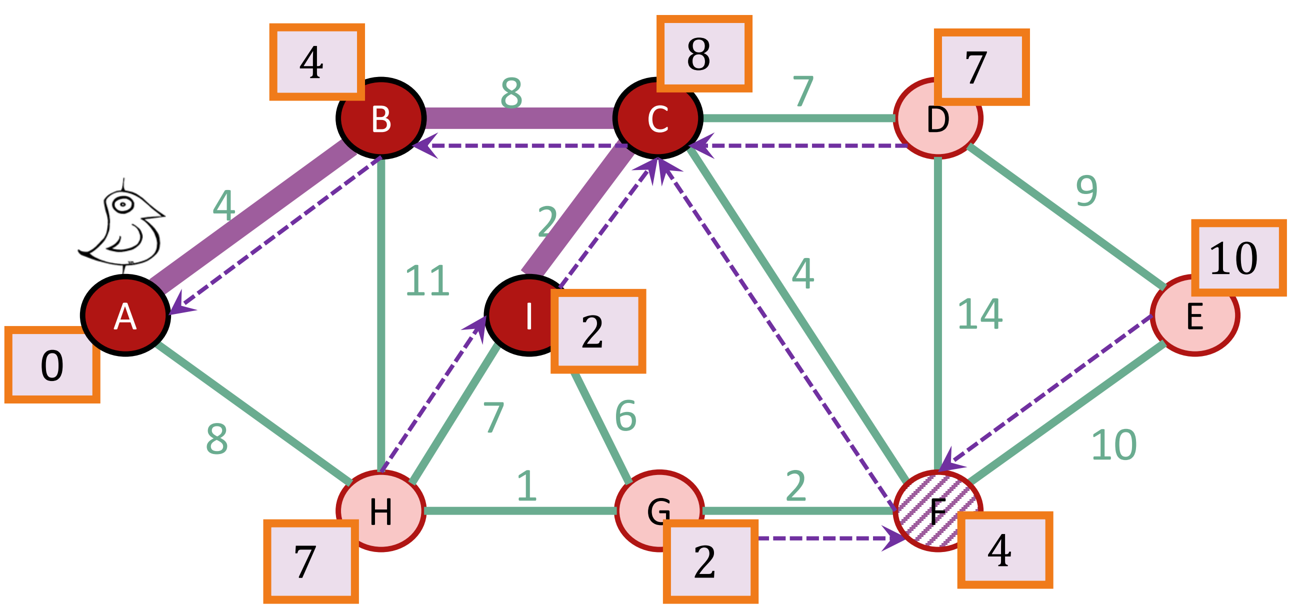

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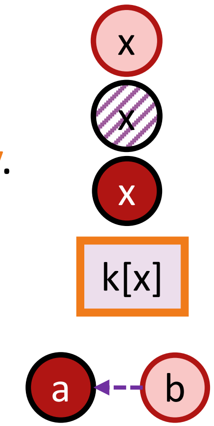

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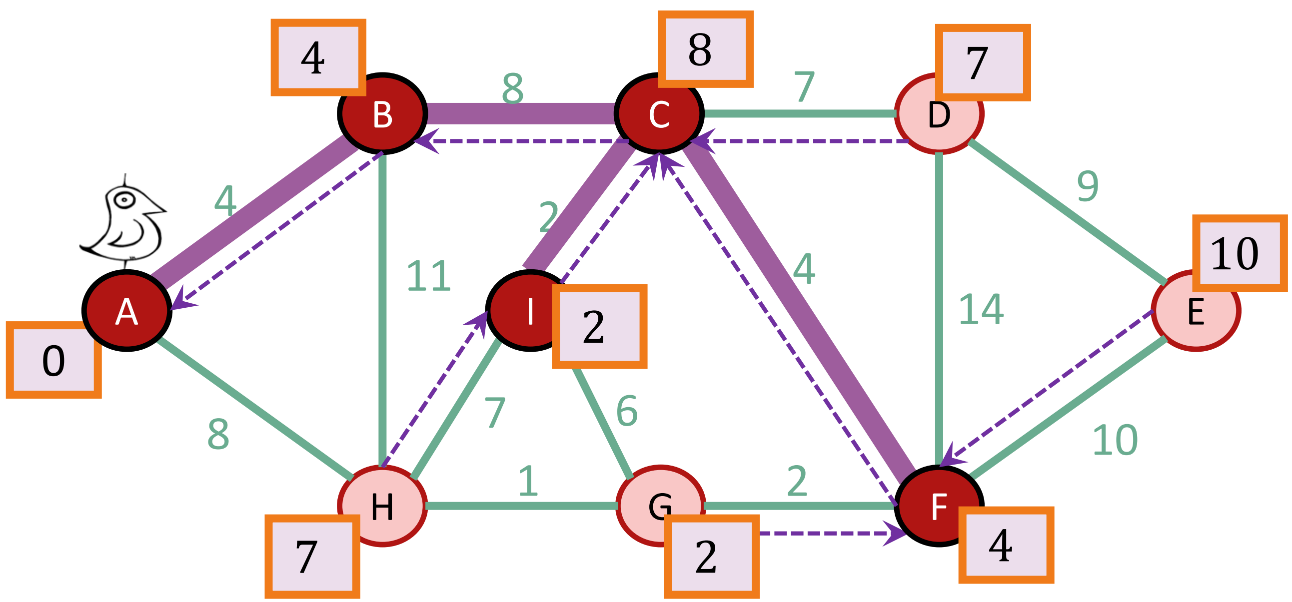

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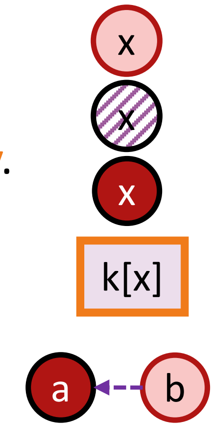

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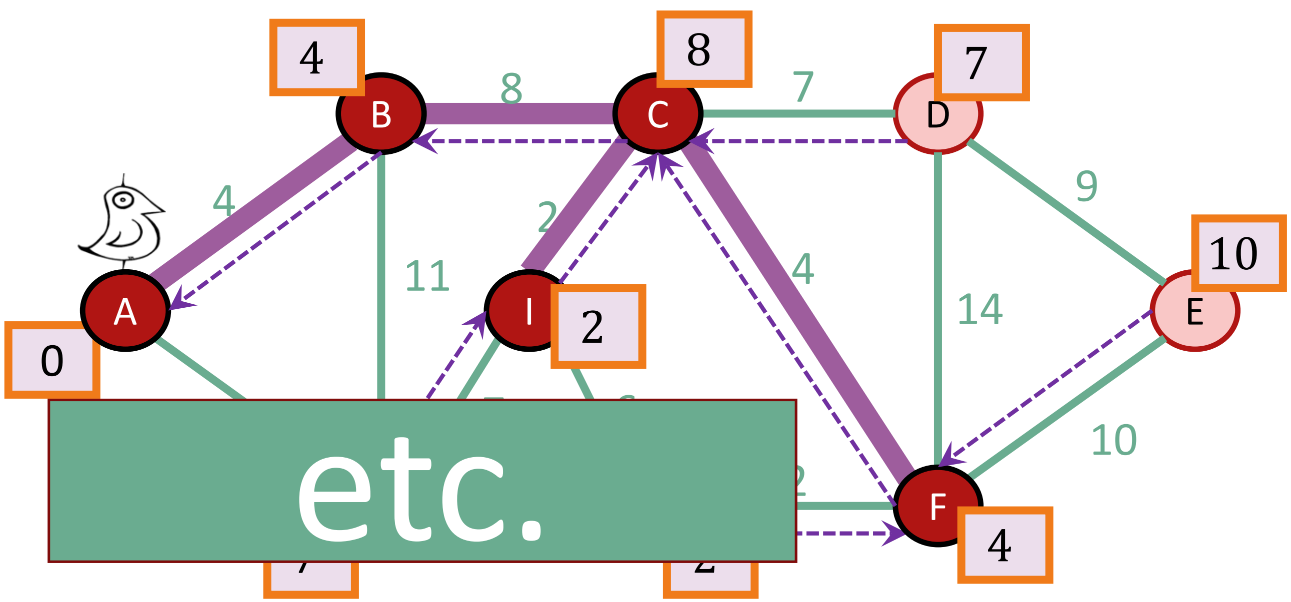


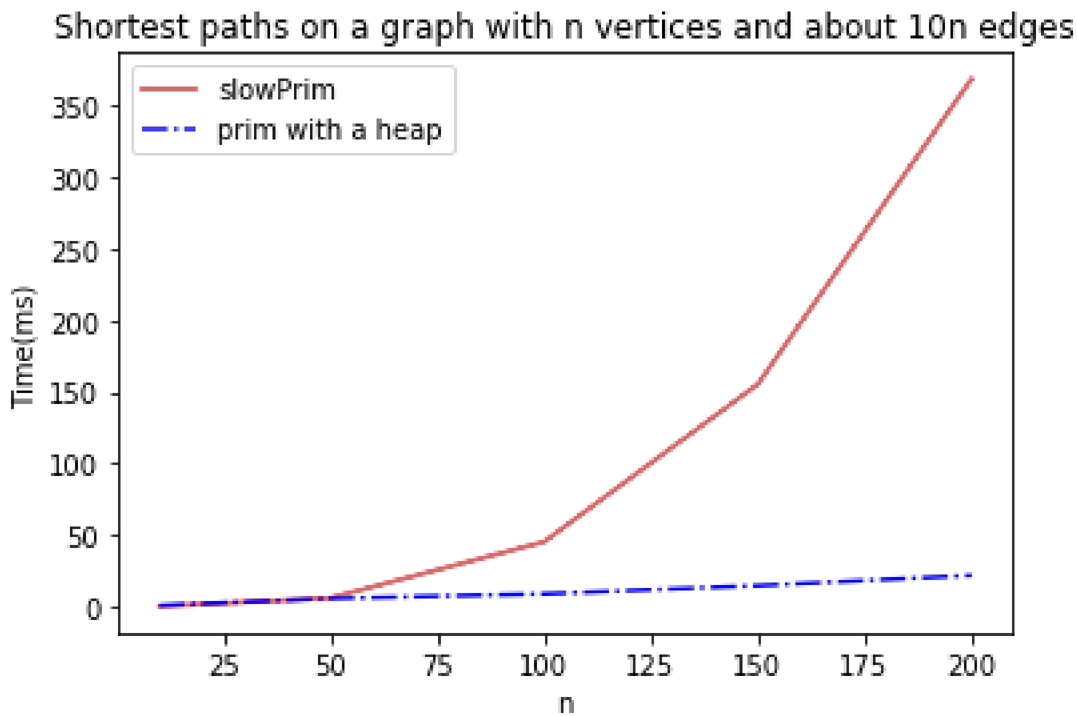
This should look pretty familiar

- Very similar to Dijkstra's algorithm!
- **Differences:**
  1. Keep track of p[v] in order to return a tree at the end
  2. Instead of d[v] which we update by
     - d[v] = min(d[v], d[u] + w(u, v))

     we keep k[v] which we update by
     - k[v] = min(k[v], w(u, v))

Running time

- Exactly the same as Dijkstra:
  - **O(mlog(n))** using a Red-Black tree as a priority queue.
  - **O(m + nlog(n))** time if we use a Fibonacci Heap.



Two questions

1. Does it work?
   - That is, does it actually return a MST?
     - Yes!

2. How do we actually implement this?
   - The pseudocode above says “slowPrim”…
     - Implement it basically the same way we’d implement Dijkstra.

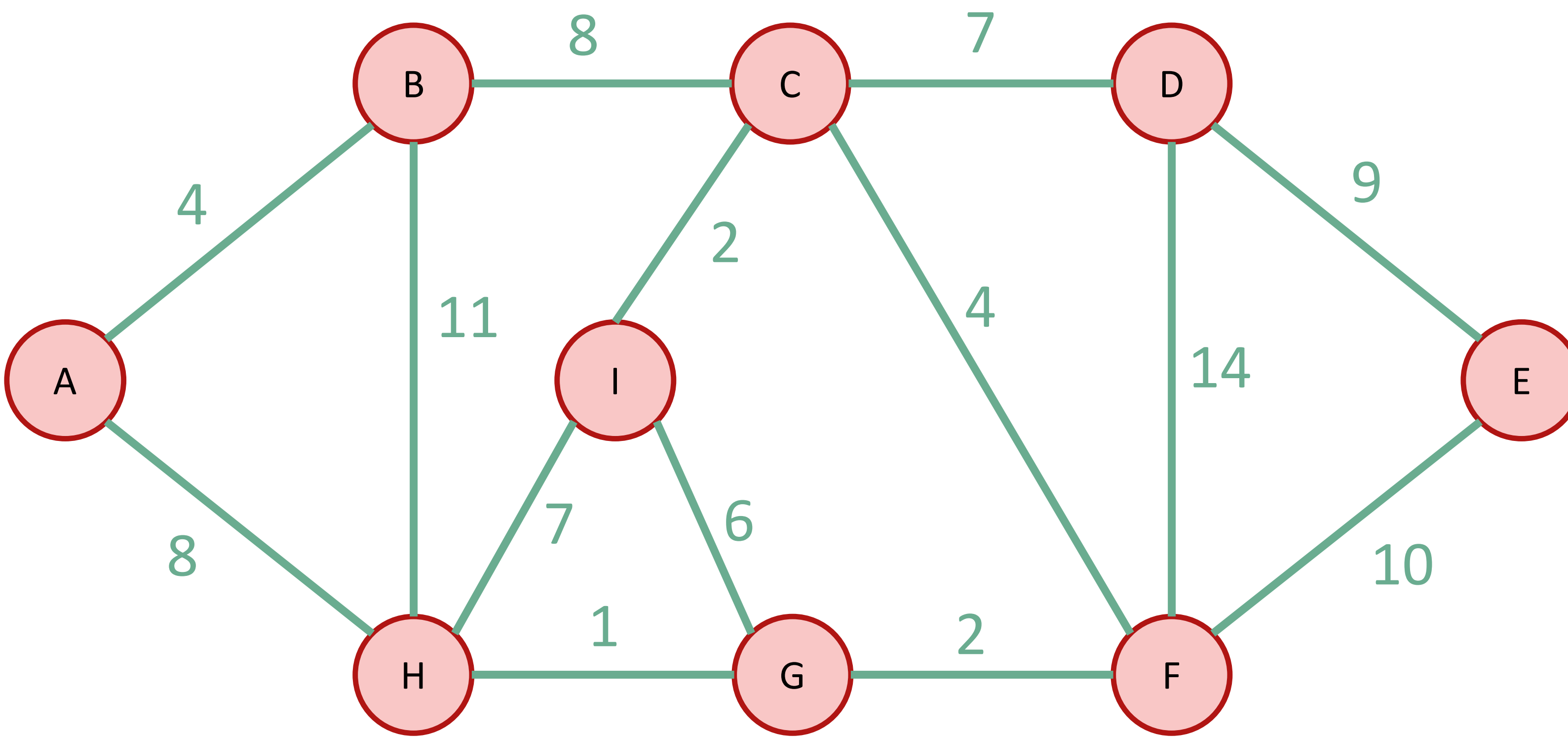
What have we learned?

- Prim's algorithm greedily grows a tree
  - similar to Dijkstra's algorithm
- It finds a Minimum Spanning Tree!
  - in time $O(m\log(n))$ if we implement it with a Red-Black Tree.
  - In amortized time $O(m+n\log(n))$ with a Fibonacci heap.
- To prove it worked, we followed the same recipe for greedy algorithms we saw last time.
  - Show that, at every step, we **don't rule out success.**

That’s not the only greedy algorithm for MST!

Other strategy

- what if we just always take the cheapest edge whether or not it's connected to what we have so far?



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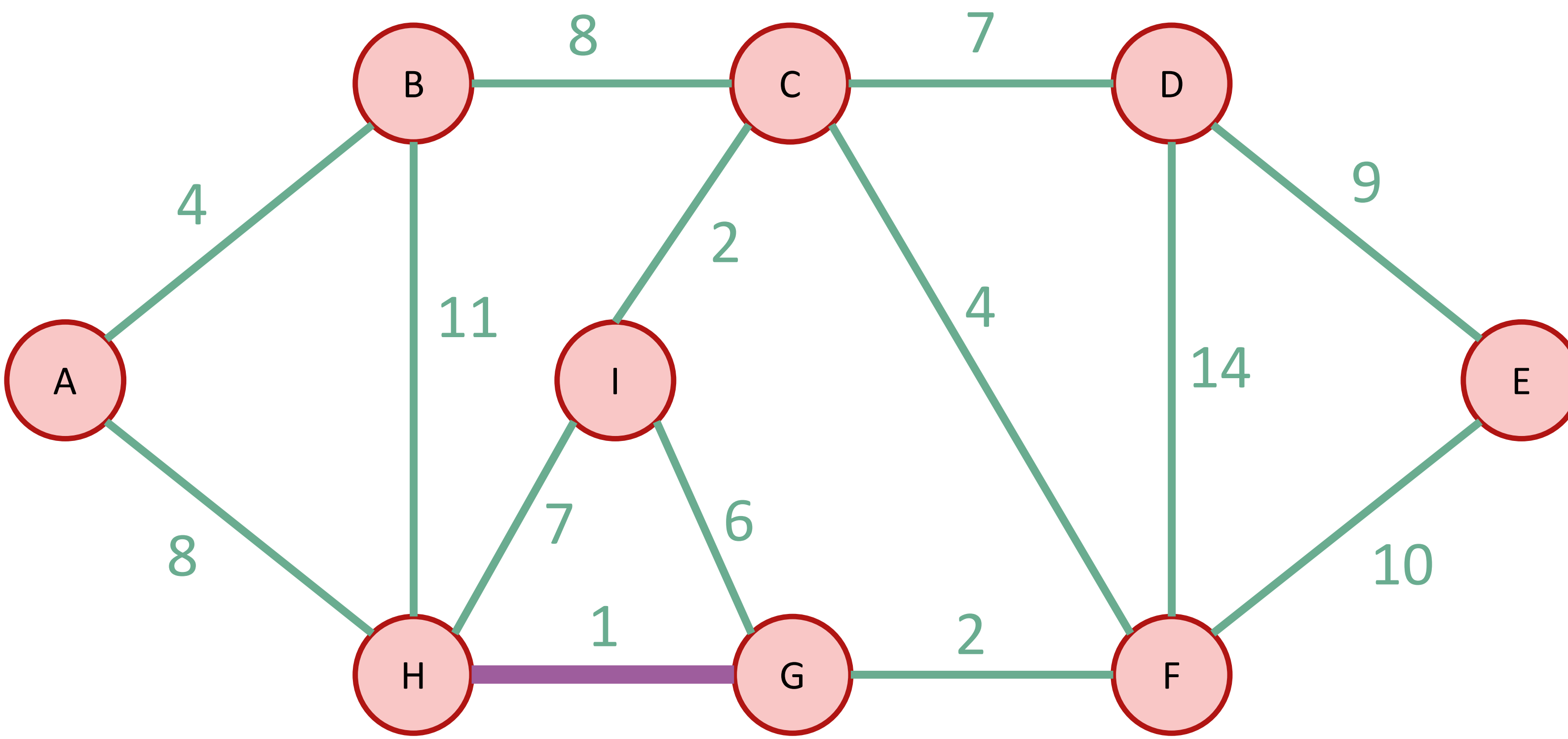

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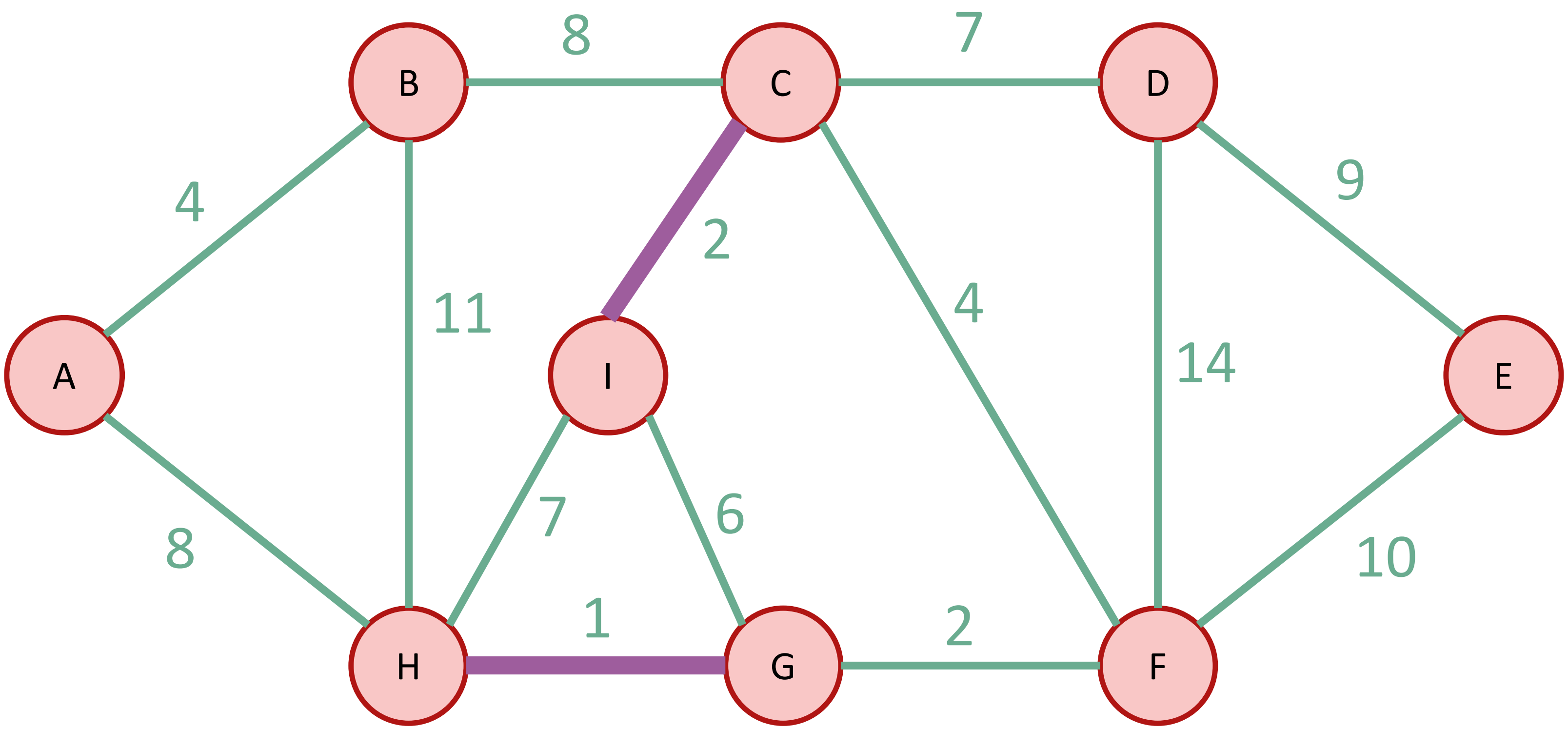

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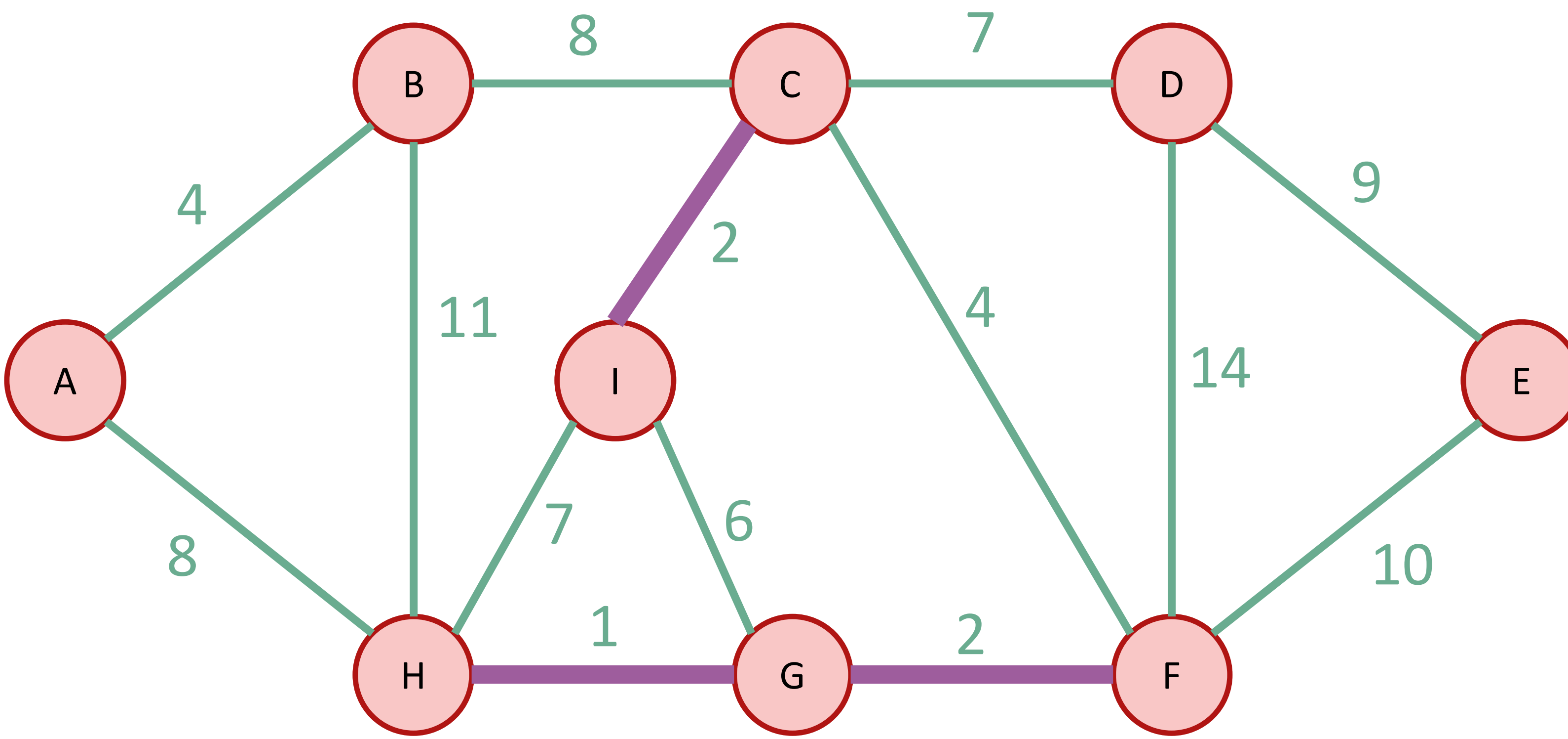

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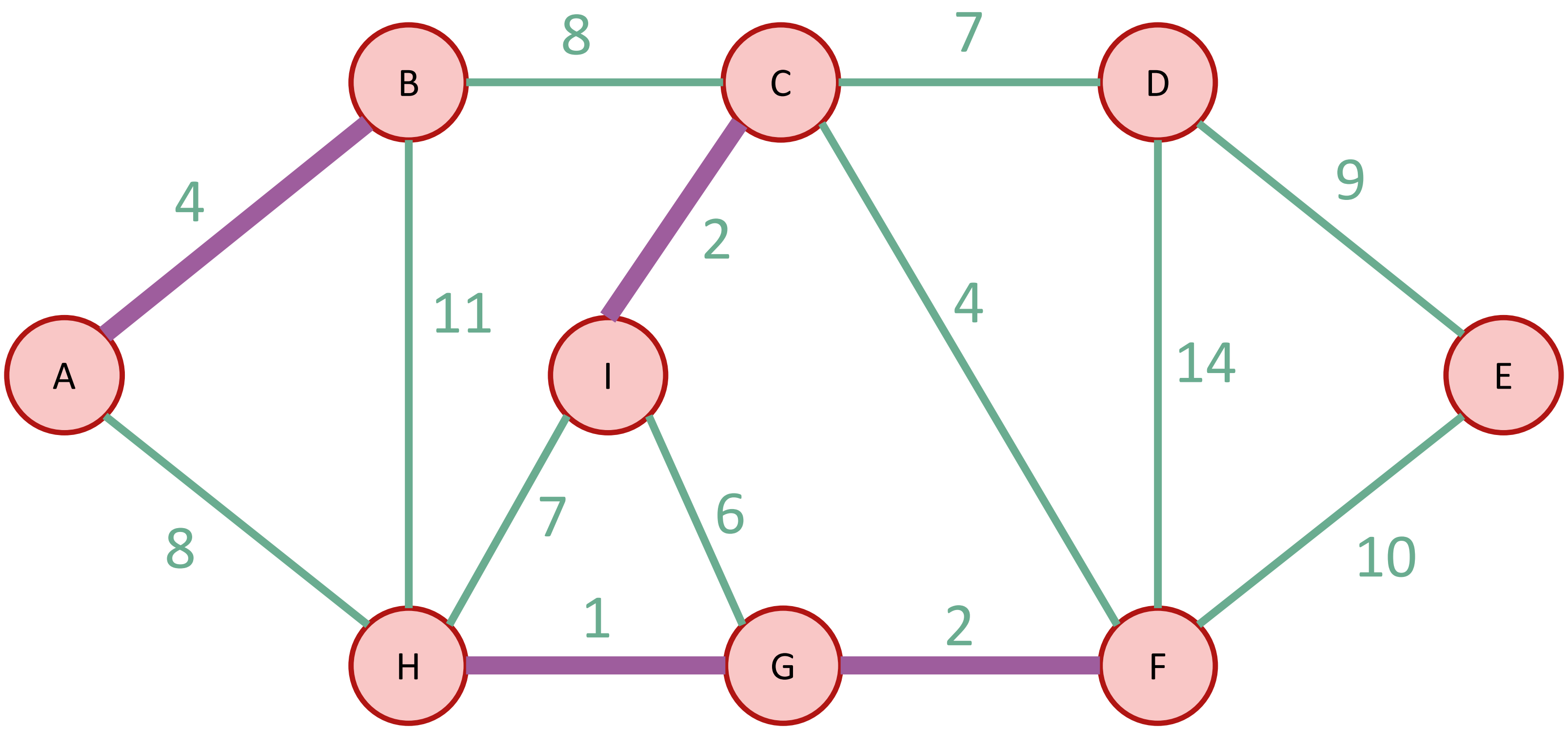

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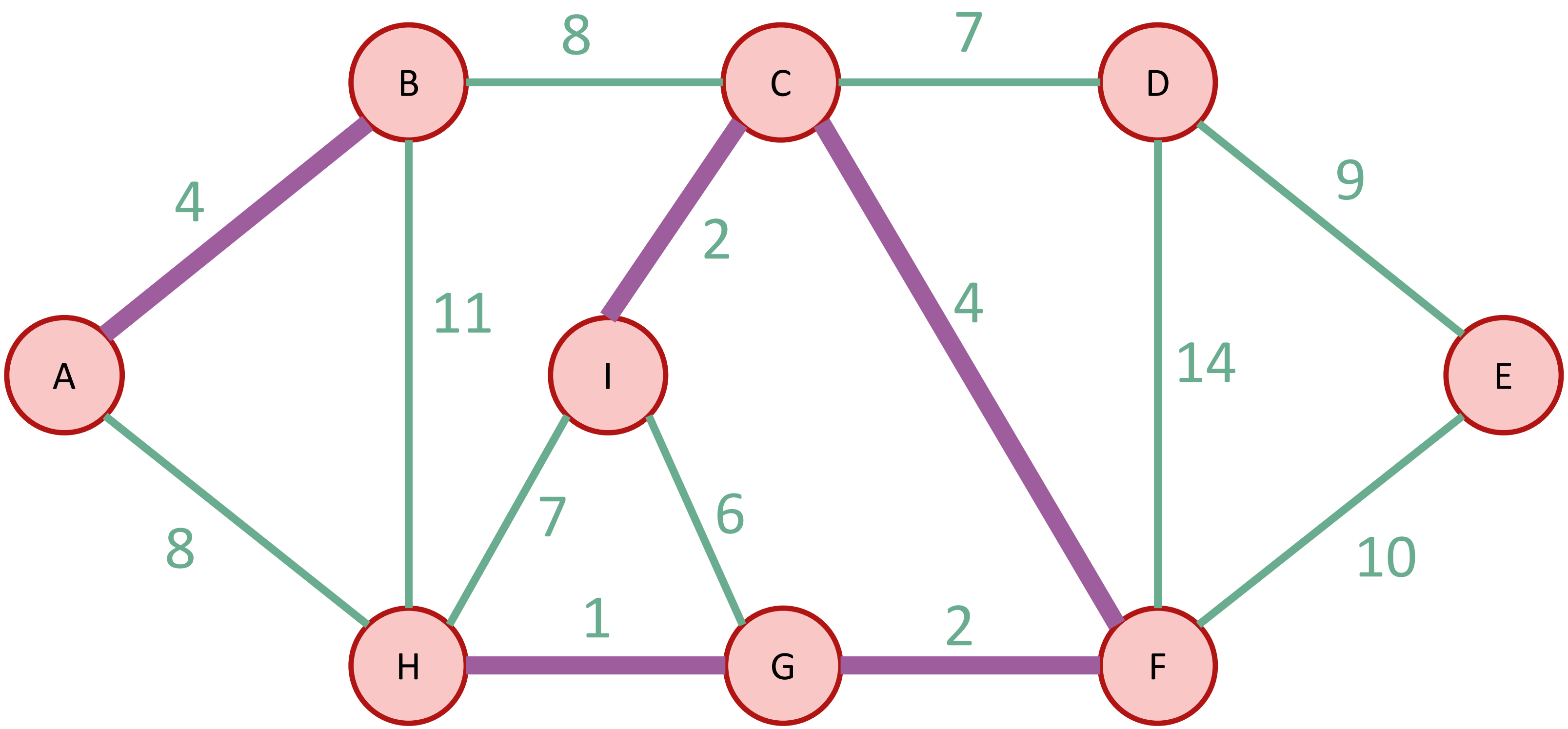

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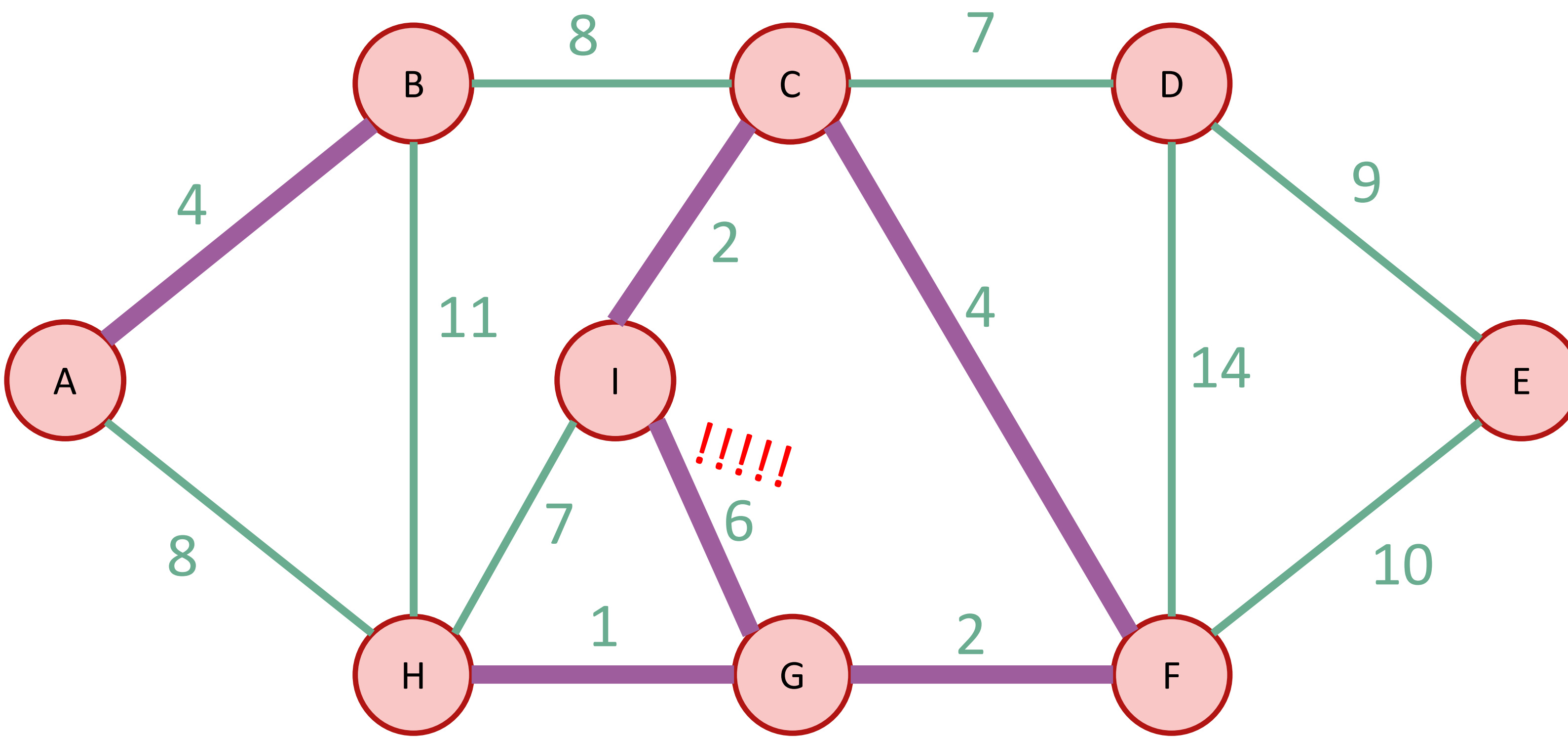
that won't cause a cycle



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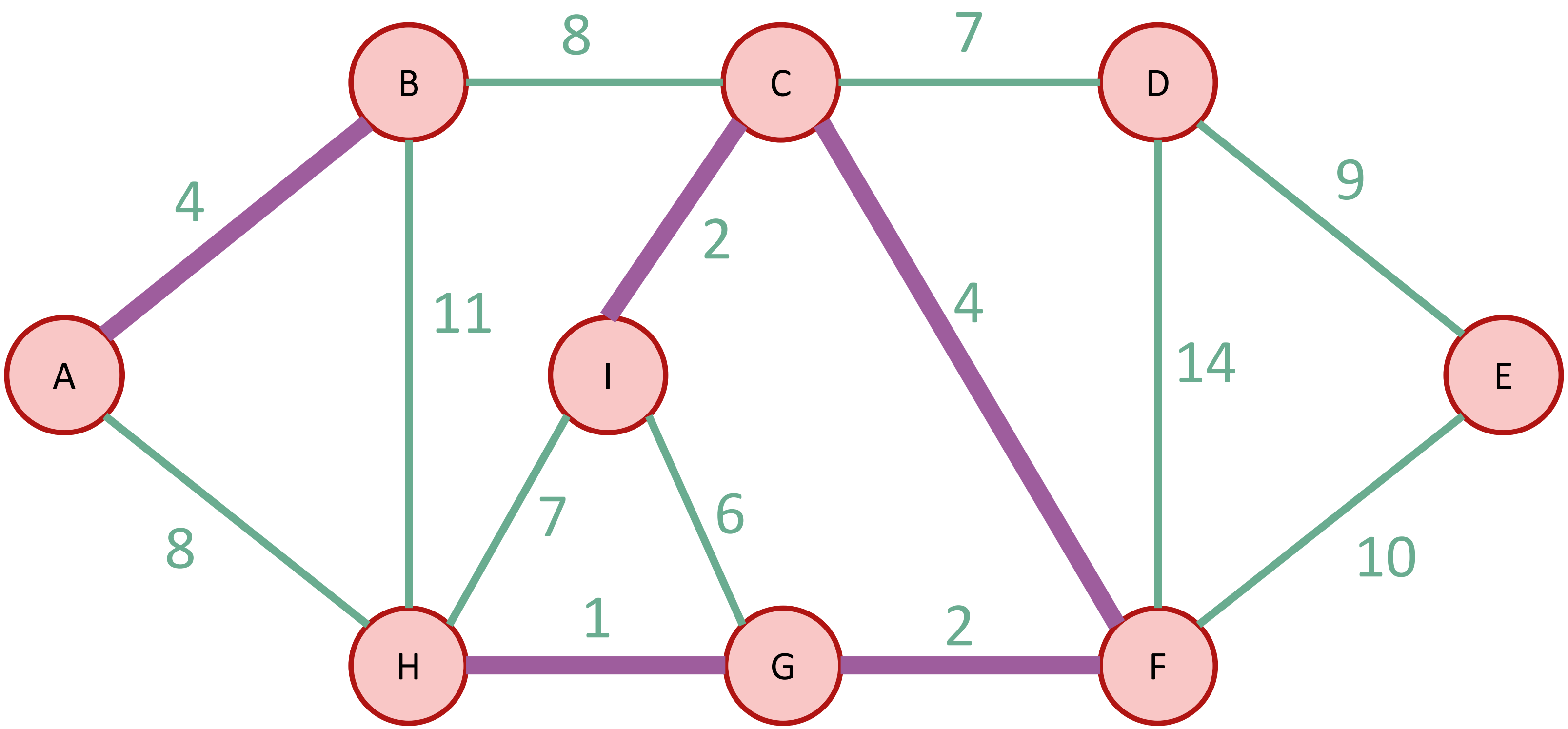

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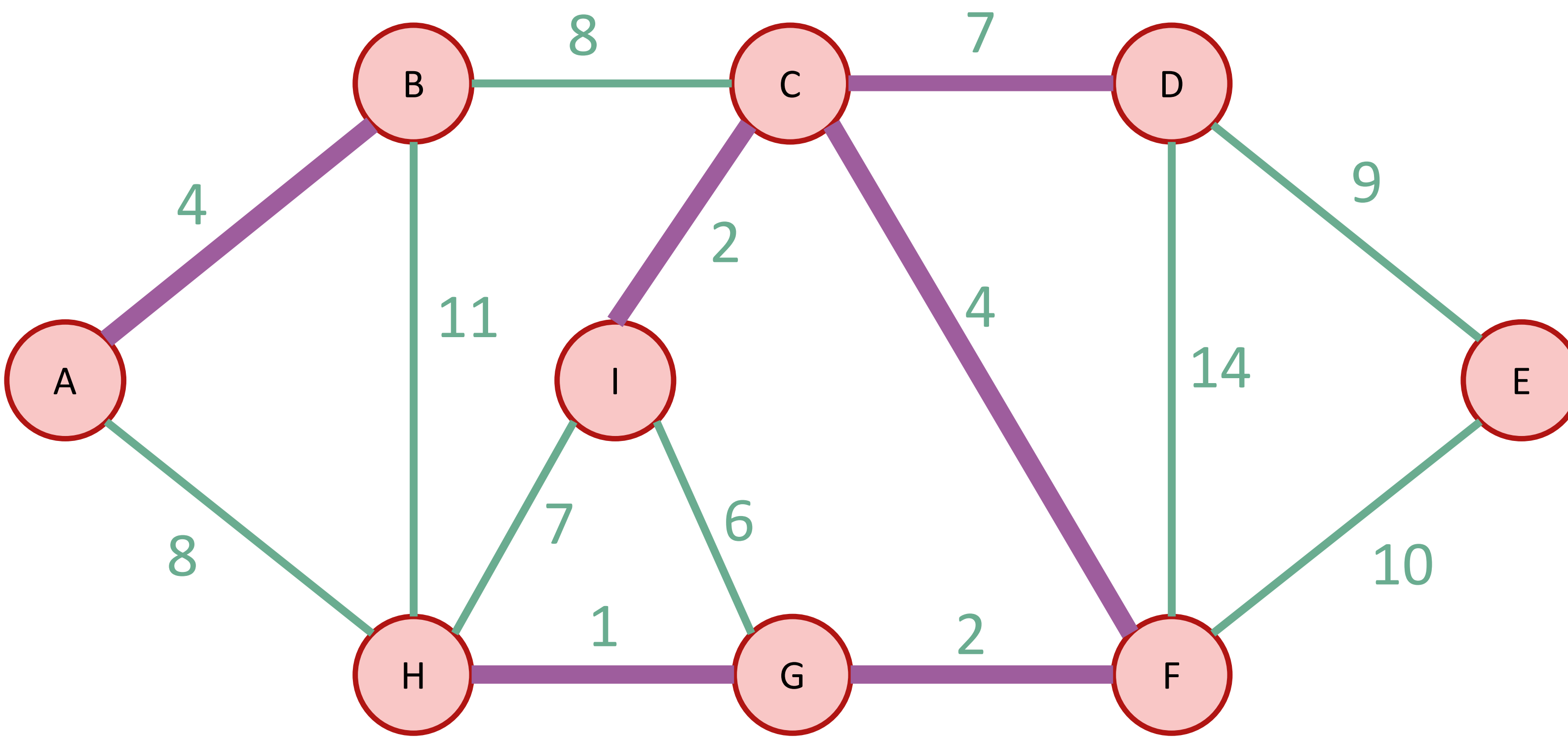

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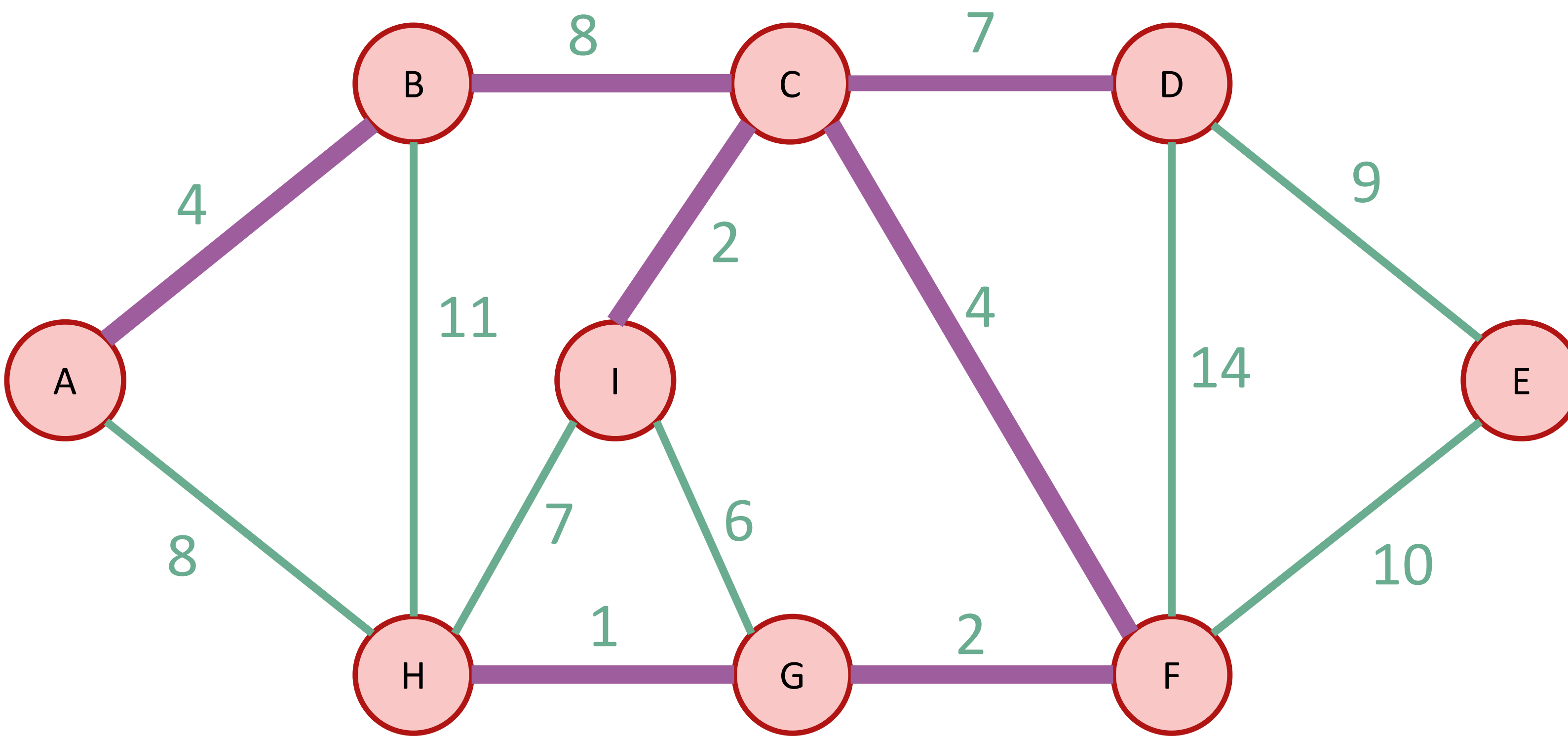

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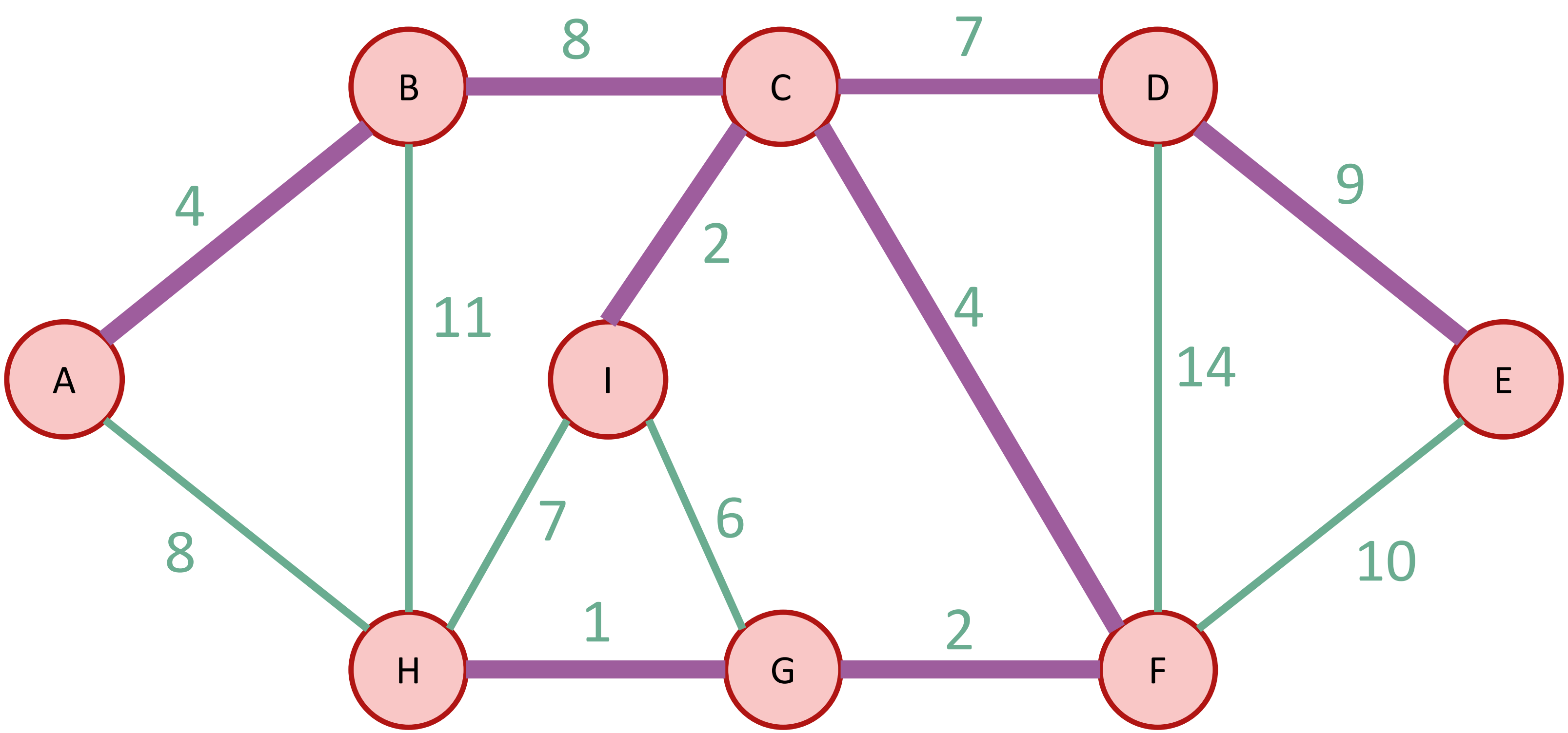


We’ve discovered Kruskal’s algorithm!

- **slowKruskal** (G = (V,E)):
  - Sort the edges in E by non-decreasing weight.
  - MST = {}
  - **for** e in E (in sorted order):
    - **if** adding e to MST won’t cause a cycle:
      - add e to MST.
  - **return** MST

m iterations through this loop

How do we check this?

How **would** you figure out if added e would make a cycle in this algorithm?

<u>Naively, the running time is ???:</u>
- For each of m iterations of the for loop:
  - Check if adding e would cause a cycle…

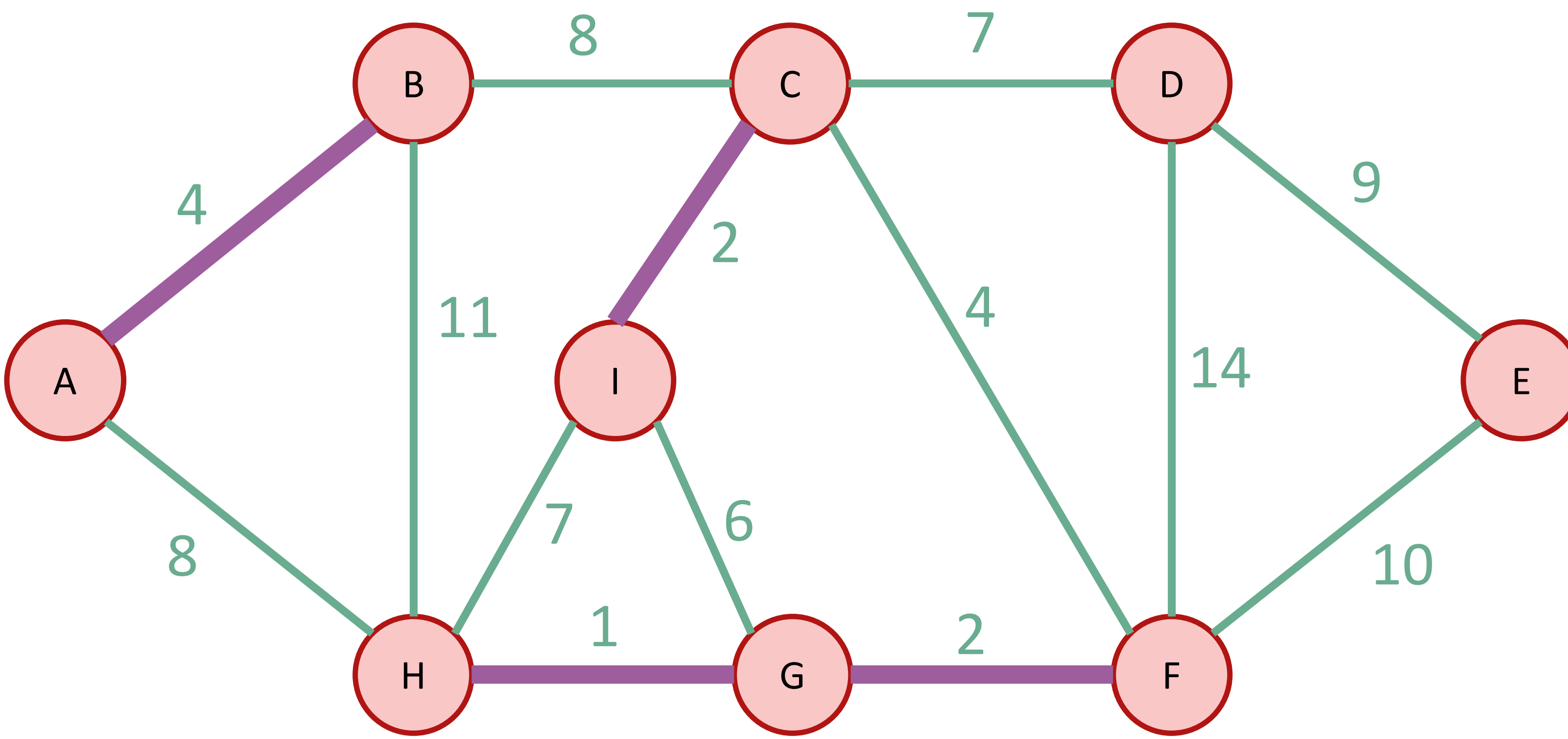
Two questions

1. Does it work?
   - That is, does it actually return a MST?

2. How do we actually implement this?
   - The pseudocode above says “slowKruskal”…

At each step, maintaining a forest.

- A **forest** is a collection of disjoint trees



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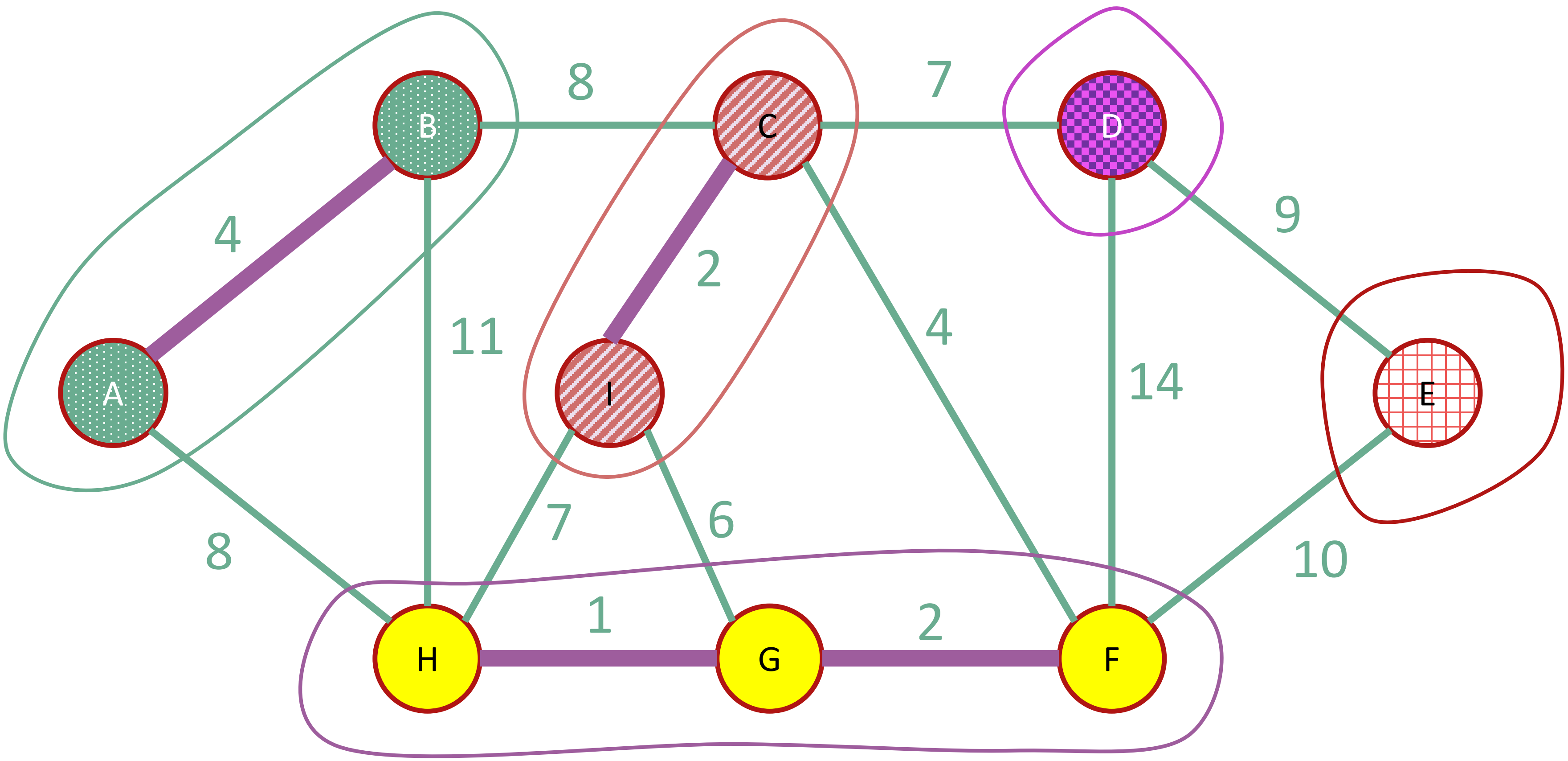

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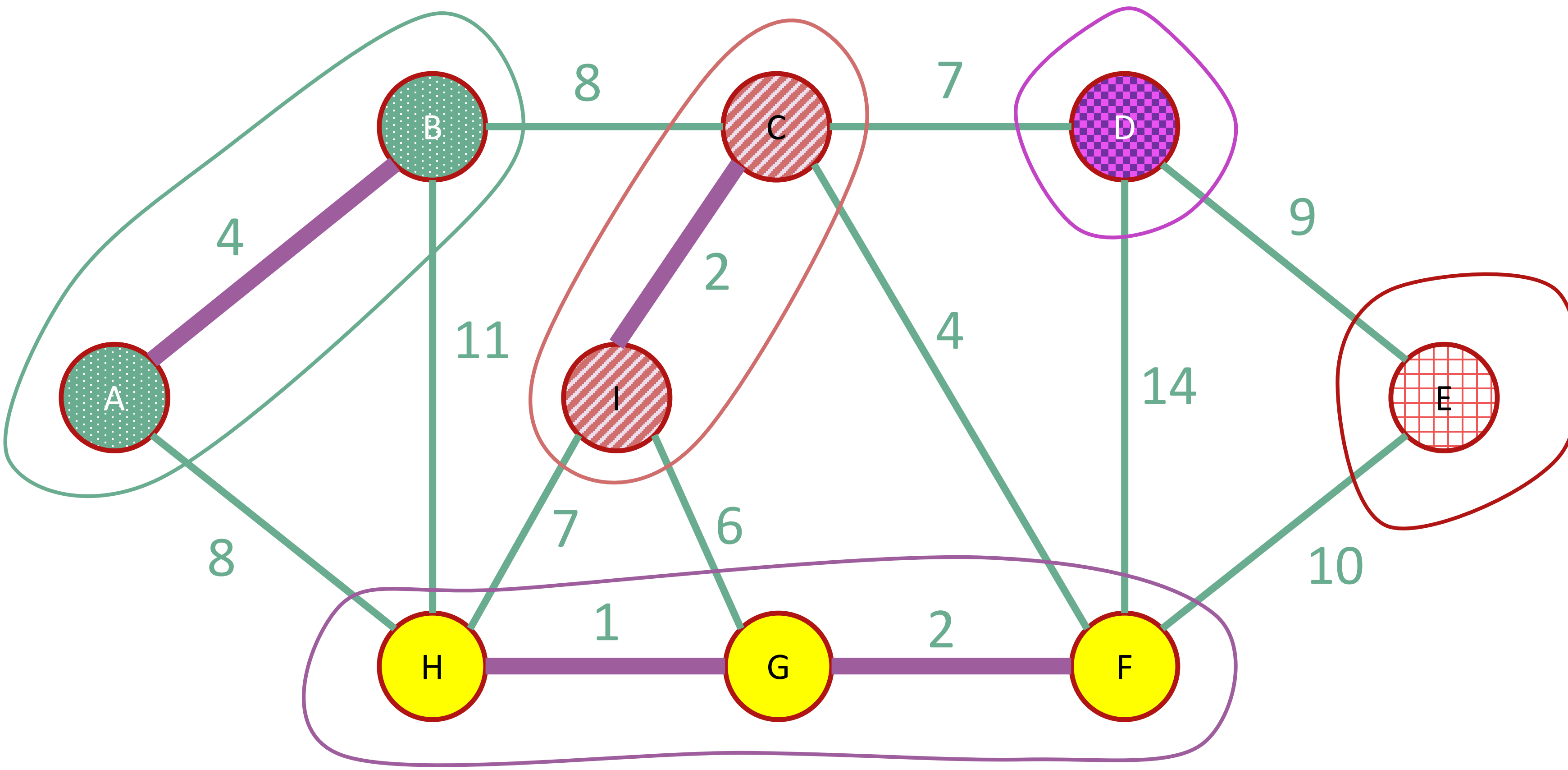

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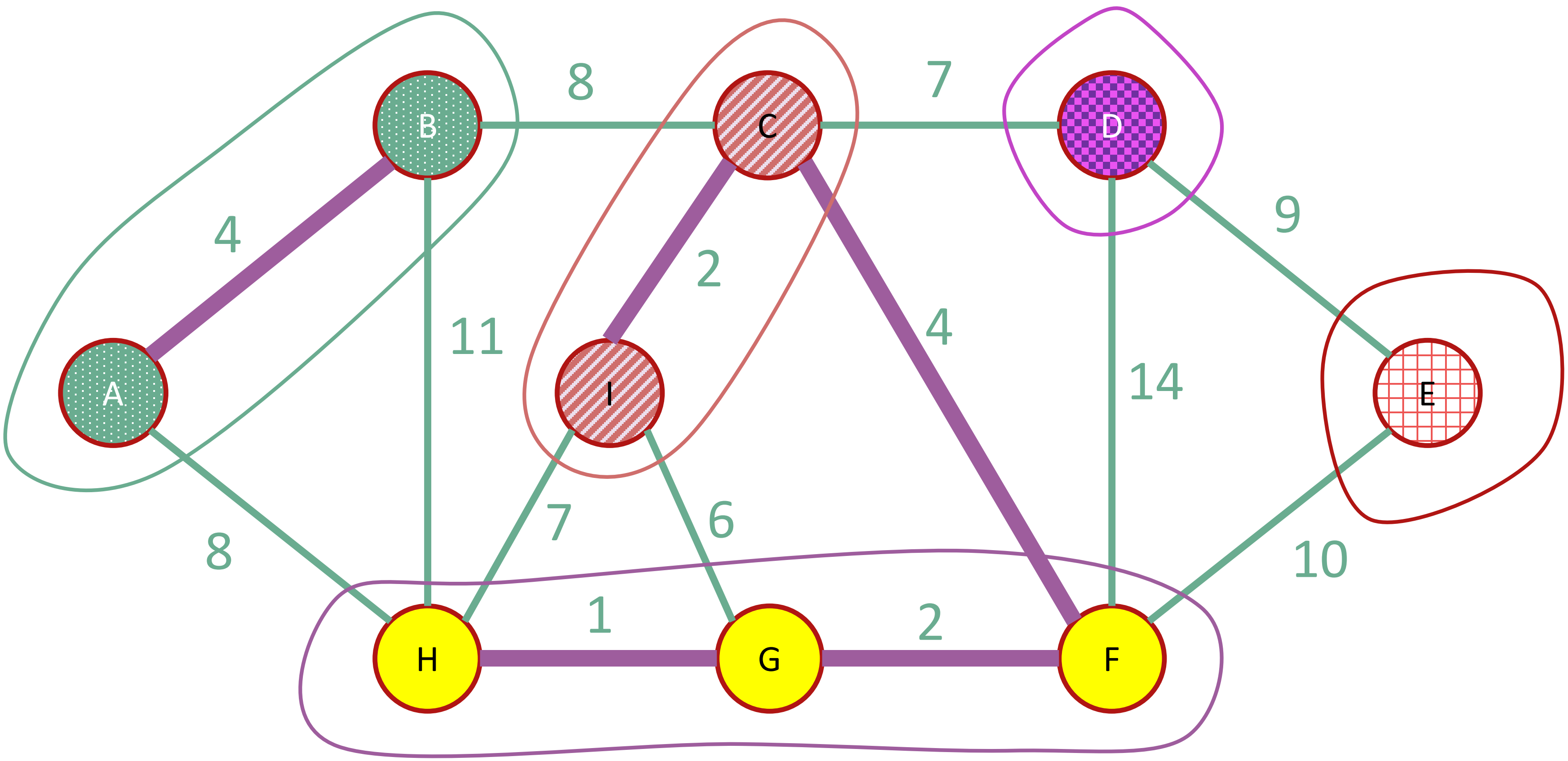
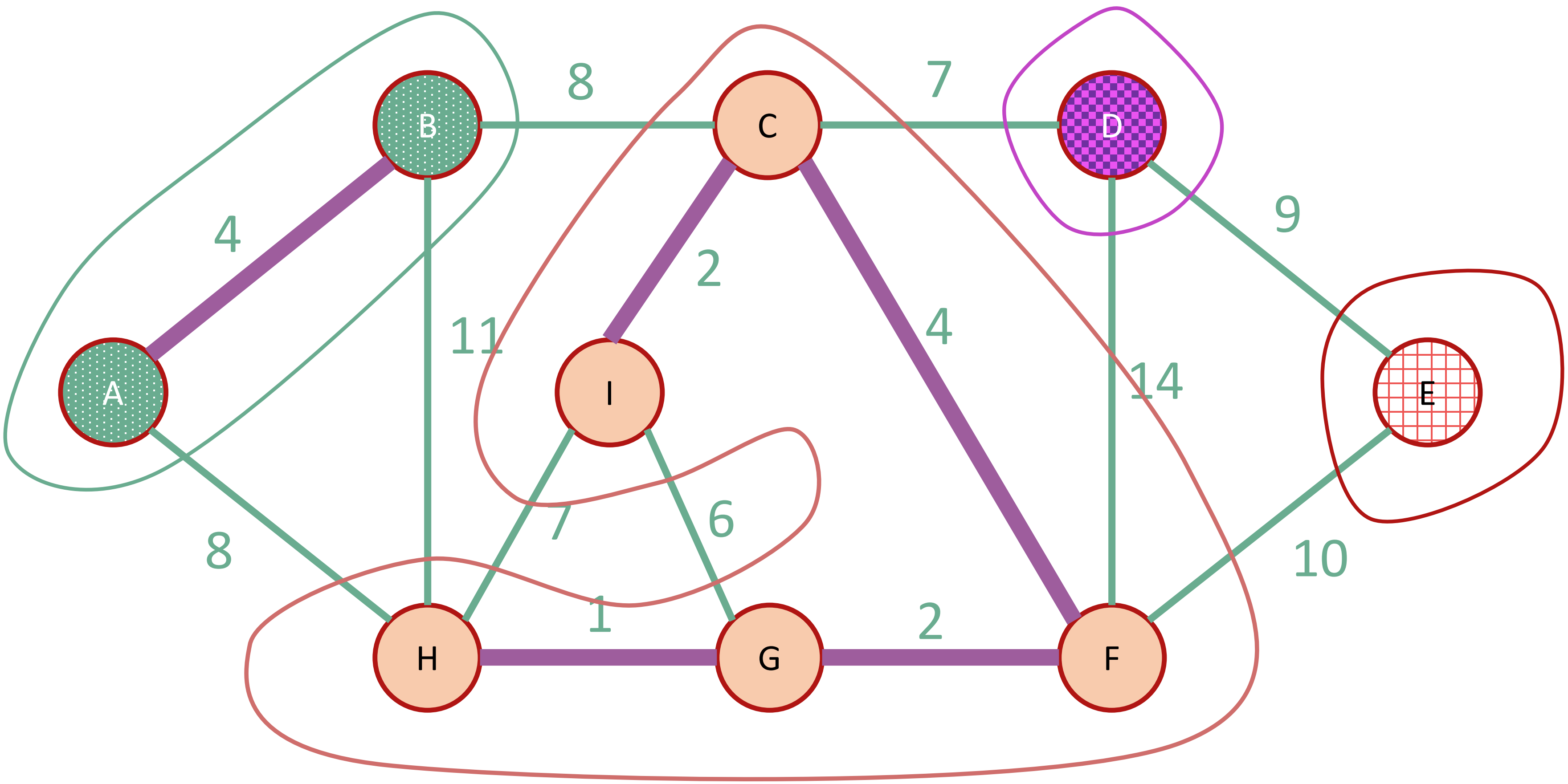


At each step, maintaining a forest.

- A **forest** is a collection of disjoint trees
- When we add an edge, we merge two trees:

We never add an edge within a tree since that would create a cycle.

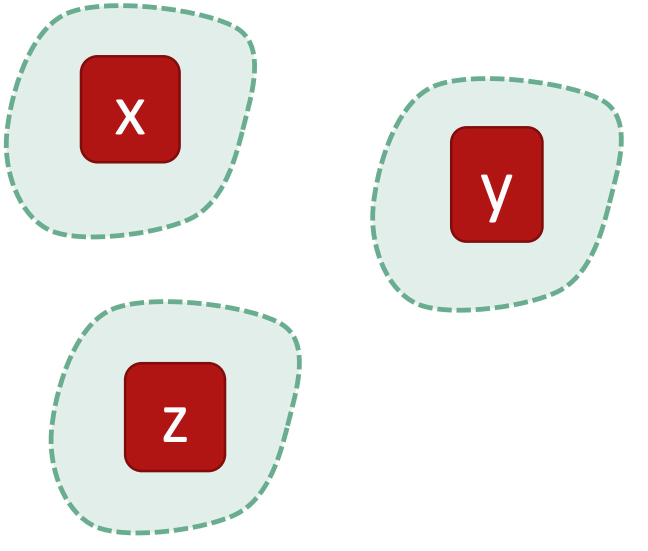


Keep the trees in a special data structure

- **Union-find** data structure also called disjoint-set data structure.
- Used for storing collections of sets
- Supports:
  - **`makeSet(u)`:** create a set {u}
  - **`find(u)`:** return the set that u is in
  - **`union(u,v)`:** merge the set that u is in with the set that v is in.

```
makeSet(x)
makeSet(y)
makeSet(z)

union(x,y)
```



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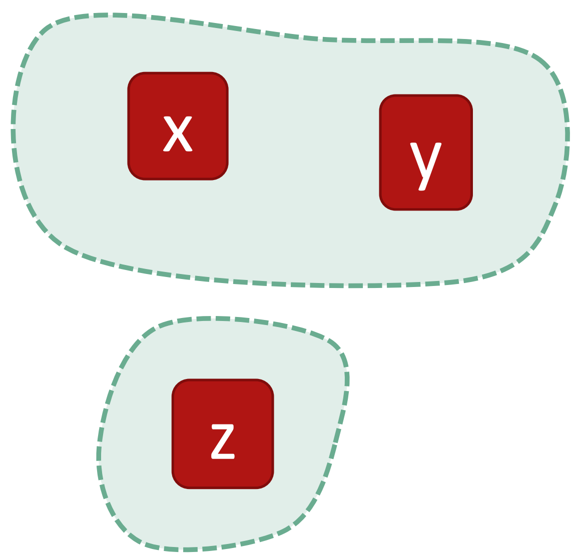

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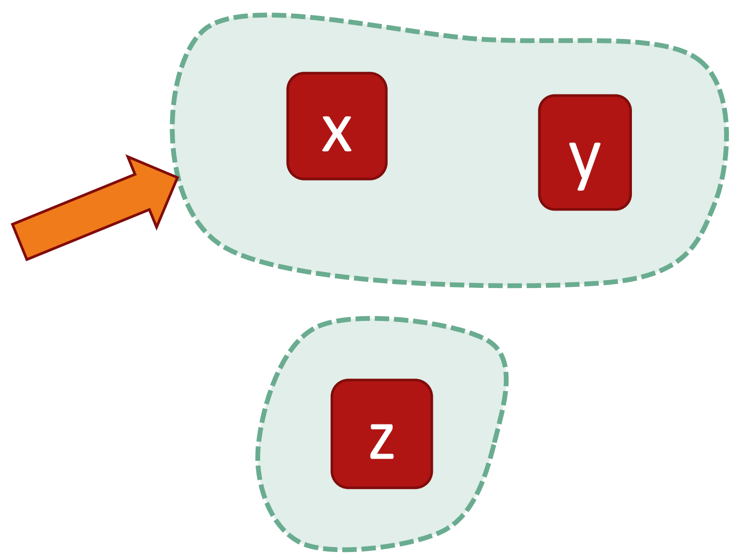
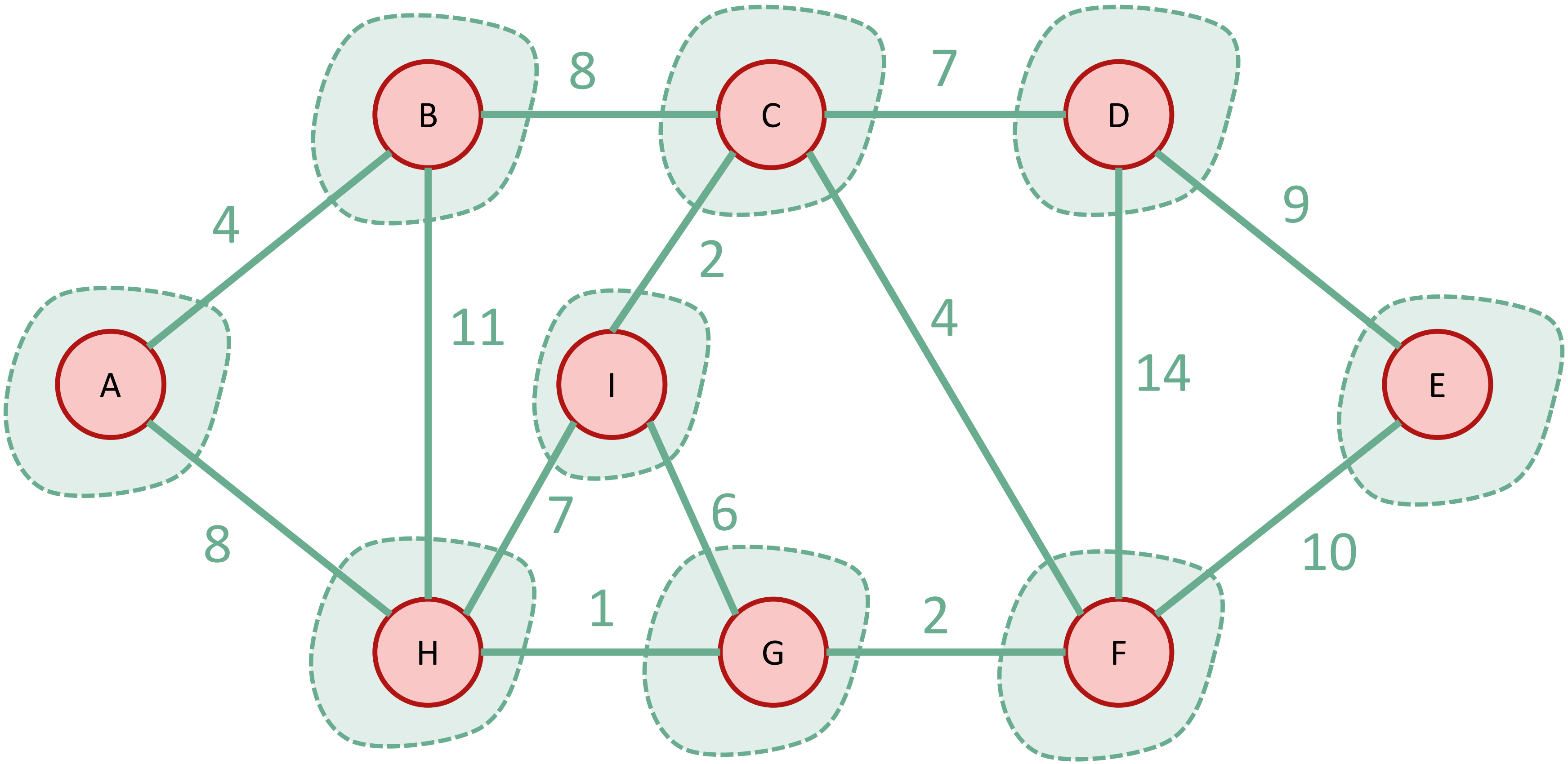


Kruskal pseudo-code

- **kruskal**(G = (V,E)):
  - Sort E by weight in non-decreasing order
  - MST = {} /* initialize an empty tree */
  - **for** v in V:
    - **makeSet**(v) /* put each vertex in its own tree in the forest */
  - **for** (u,v) in E: /* go through the edges in sorted order */
    - **if find**(u) != **find**(v): /* if u and v are not in the same tree */
      - add (u,v) to MST
      - **union**(u,v) /* merge u's tree with v's tree */
  - **return** MST

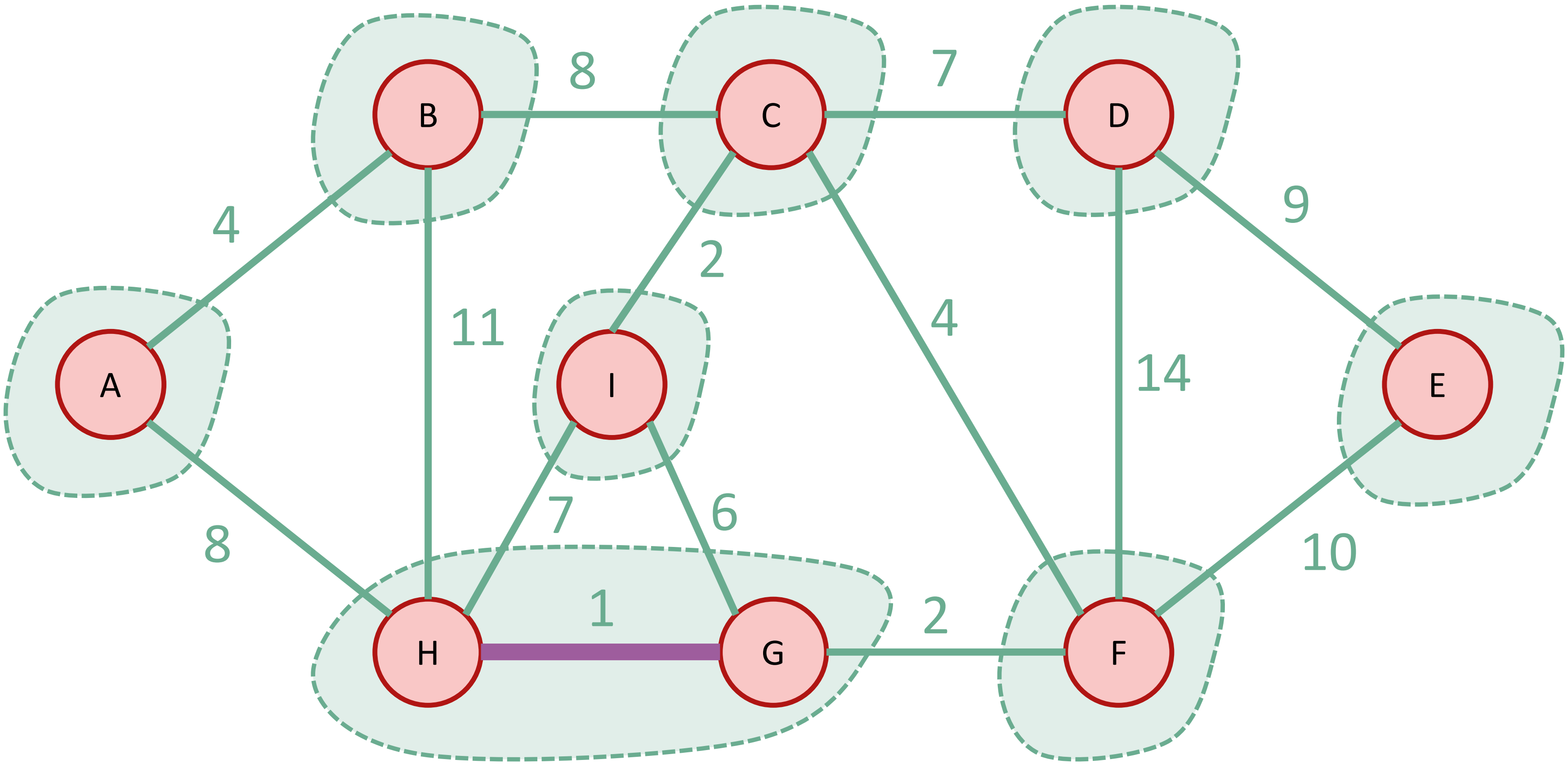
Once more…

- To start, every vertex is in its own tree.



Once more...

- Then start merging.



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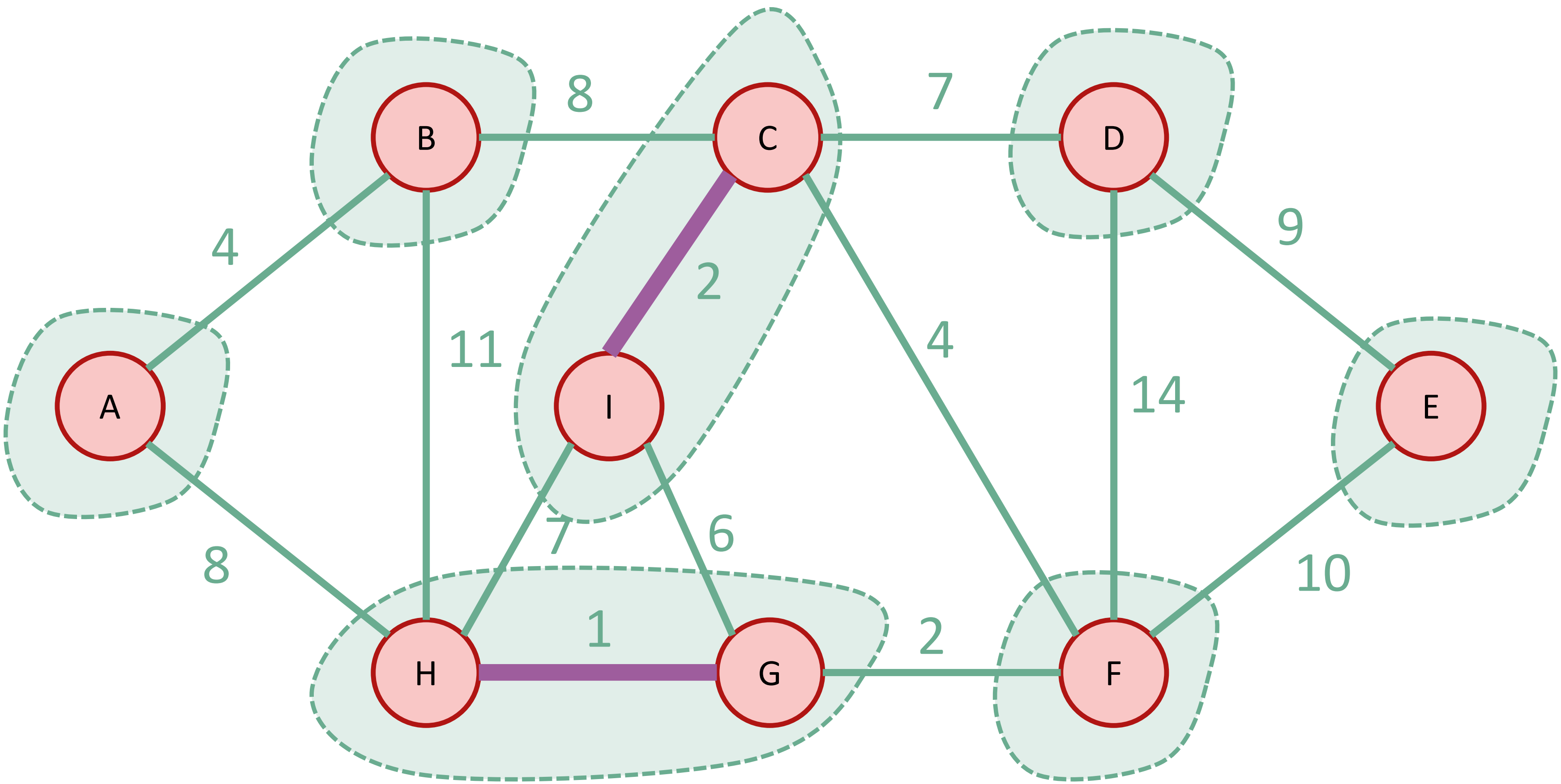

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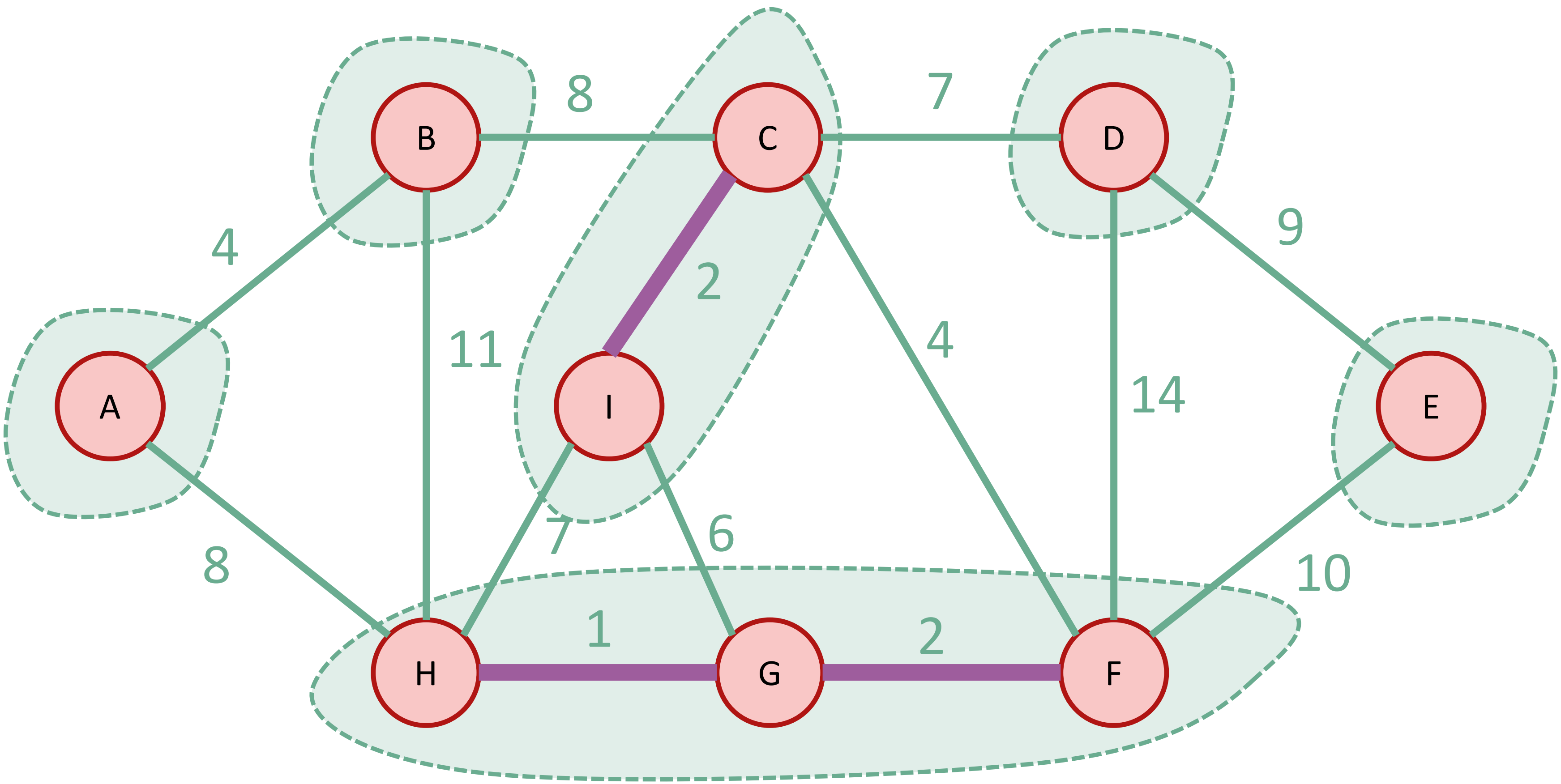

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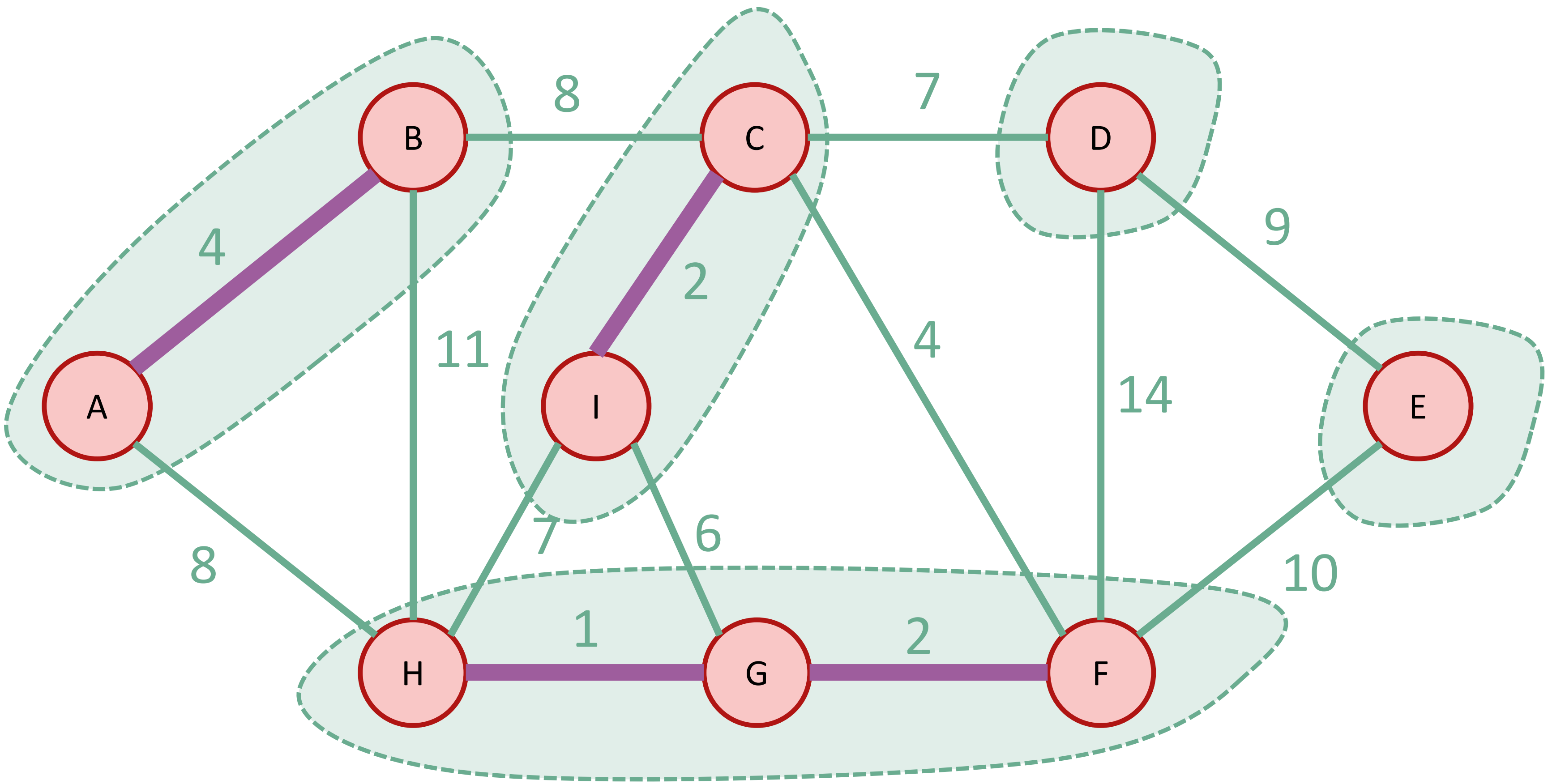

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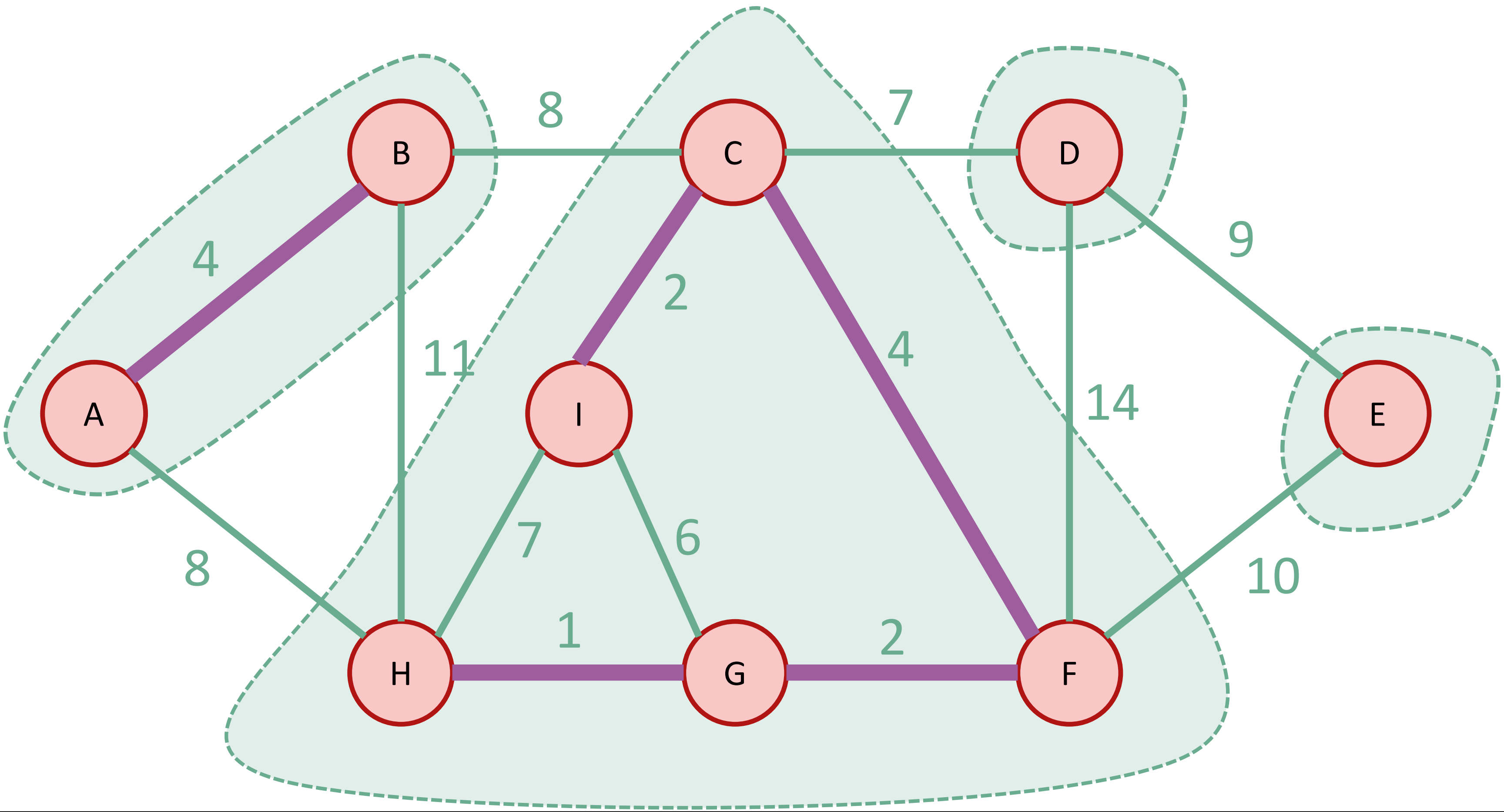

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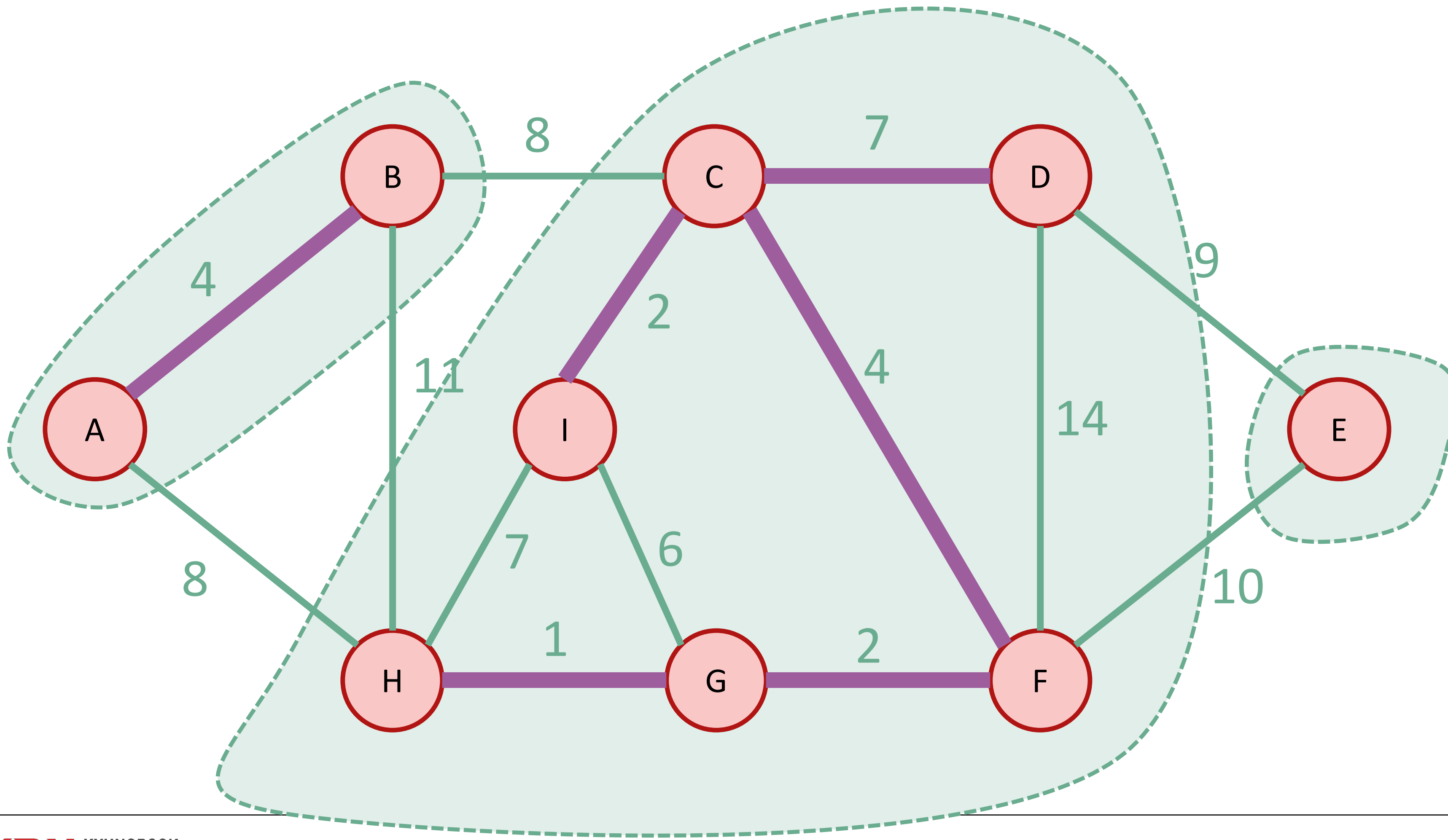

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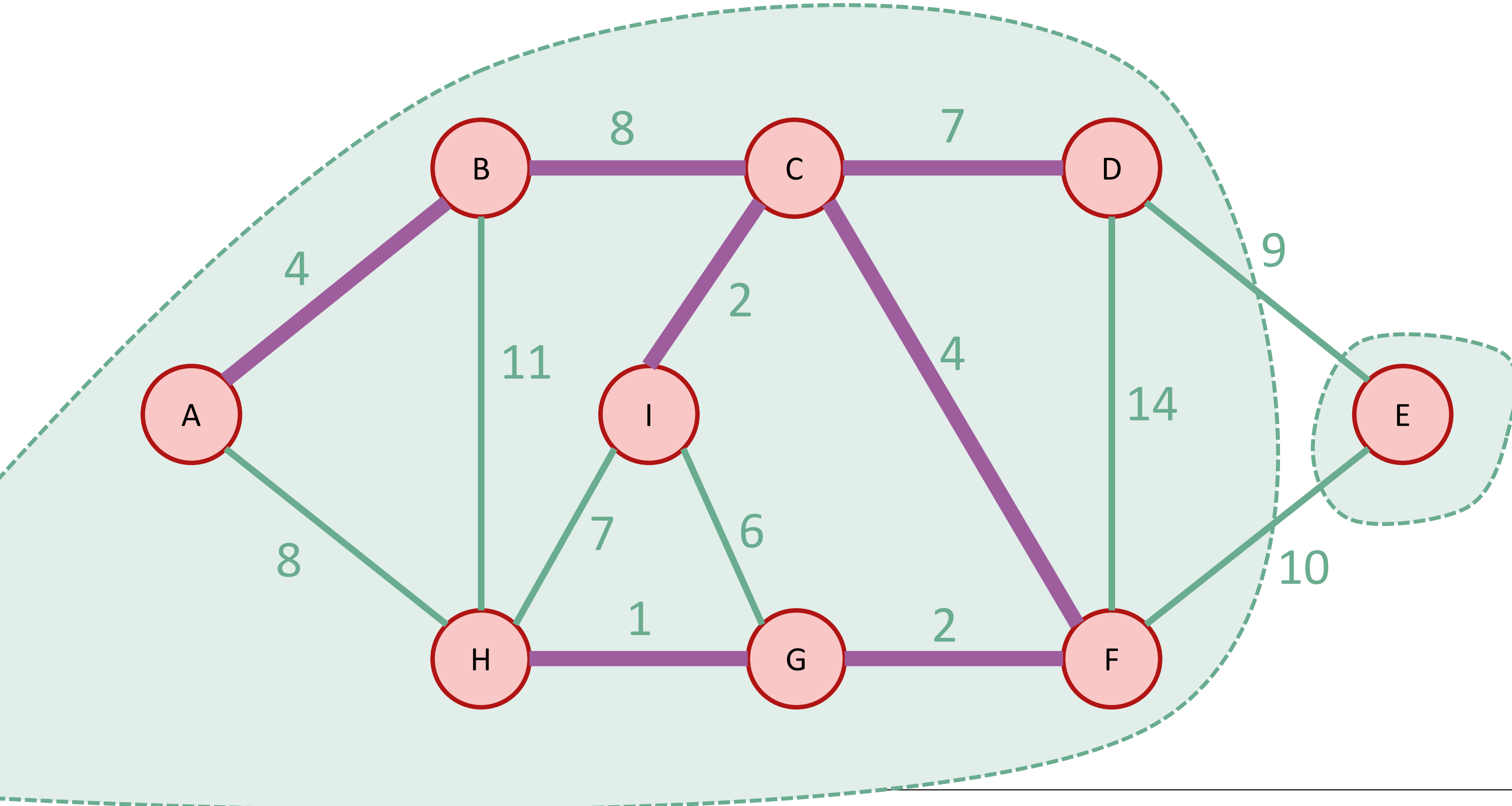
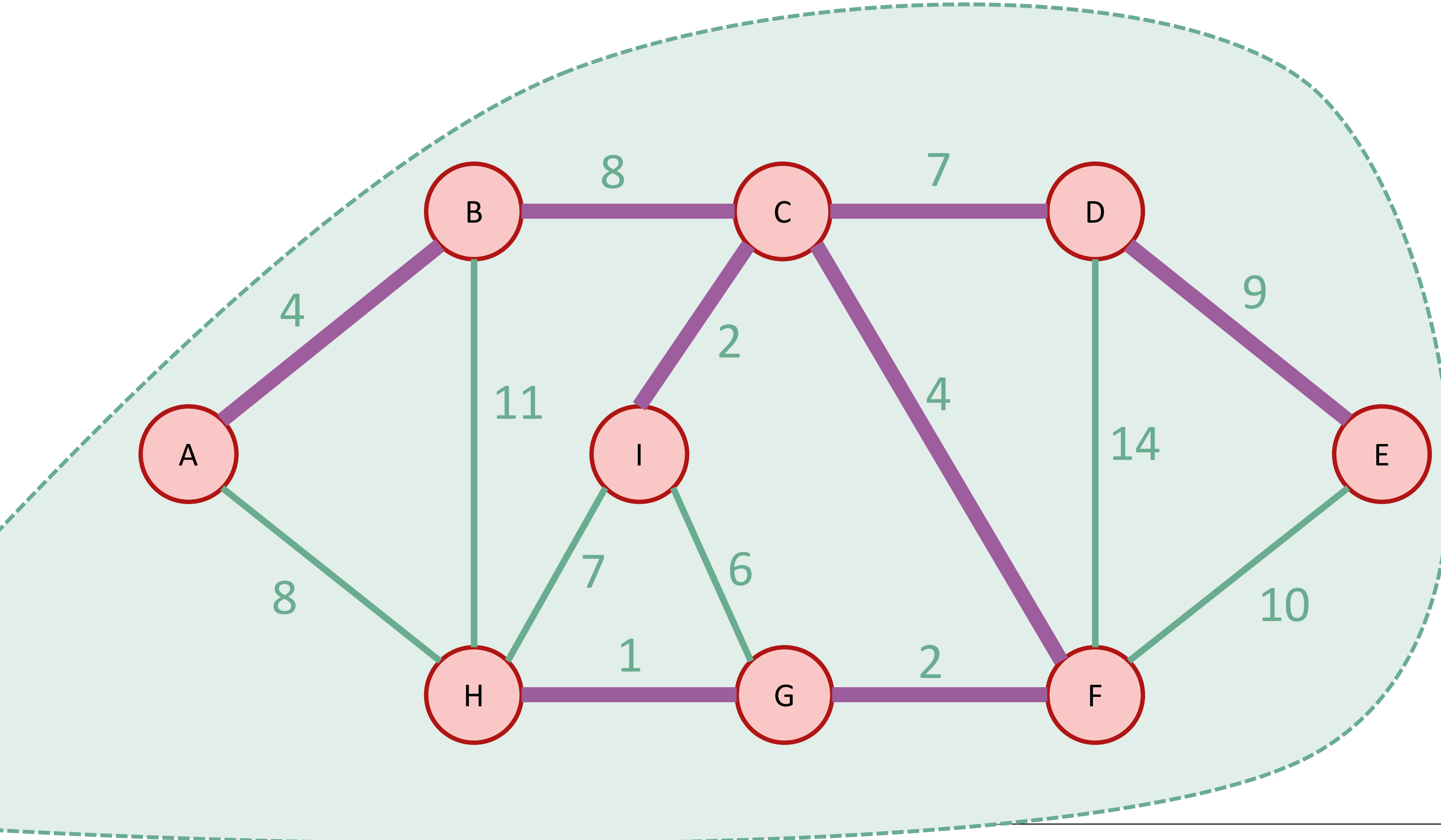


Once more…

- Then start merging.

Stop when we have one big tree!



Running time

- Sorting edges takes O(mlog(n))
  - In practice, if the weights are small integers we can use radixSort and take time O(m)
- For the rest:
  - n calls to **`makeSet`** (put each vertex in its own set)
  - 2m calls to **`find`** (for each edge, **`find`** its endpoints)
  - n-1 calls to **`union`**
    - (we will never add more than n-1 edges to the tree, so we will never call **`union`** more than n-1 times.)
- Total running time:
  - Worst-case O(mlog(n)), just like Prim with a RBtree.
  - Closer to O(m) if you can do radixSort

Two questions

1. Does it work?
   - That is, does it actually return a MST?

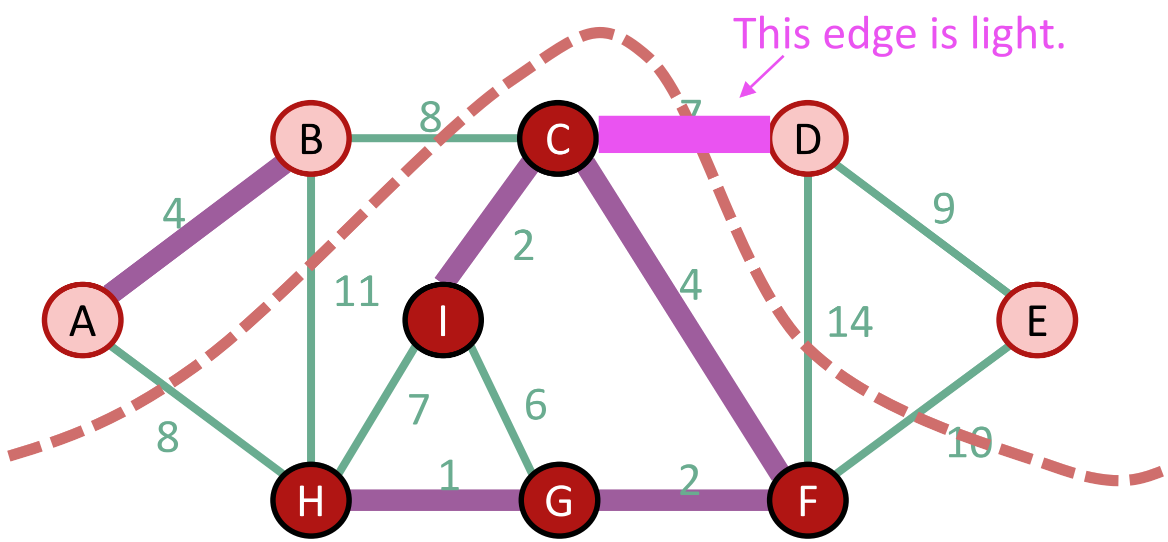
2. How do we actually implement this?
   - The pseudocode above says “slowKruskal”…
     - Worst-case running time O(mlog(n)) using a union-find data structure.

Does it work?

- We need to show that our greedy choices **don't rule out success.**
- That is, at every step:
  - There exists an MST that contains all of the edges we have added so far.
- Now it is time to use our lemma again!

Lemma

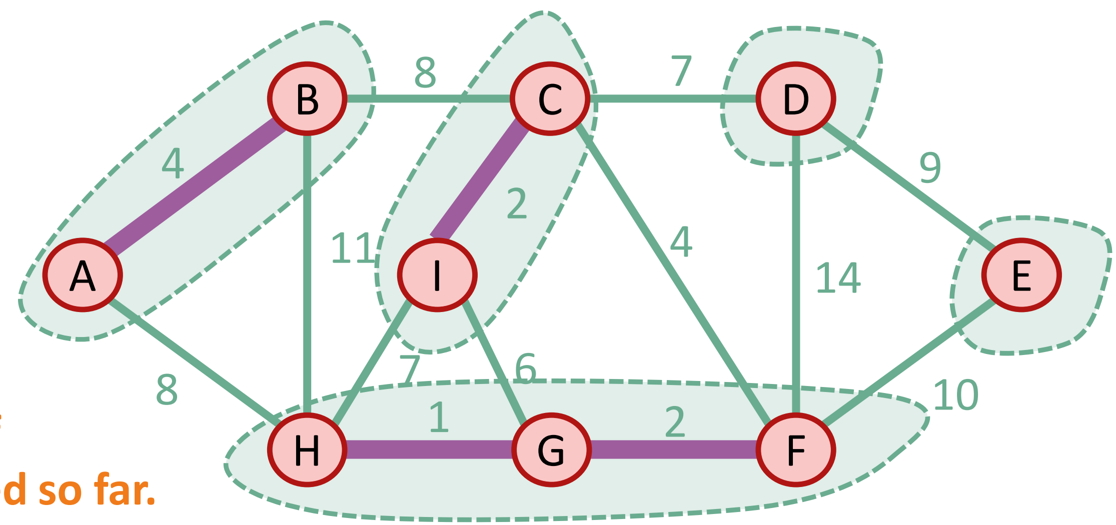
- Let S be a set of edges, and consider a cut that respects S.
- Suppose there is an MST containing S.
- Let {u, v} be a light edge.
- Then there is an MST containing S ∪ {{u, v}}



S is the set of **thick purple** edges

Partway through Kruskal

- Assume that our choices **S** so far don't rule out success.
  - There is an MST extending them
- The **next edge** we add will merge two trees, **T1, T2**



S is the set of edges selected so far.

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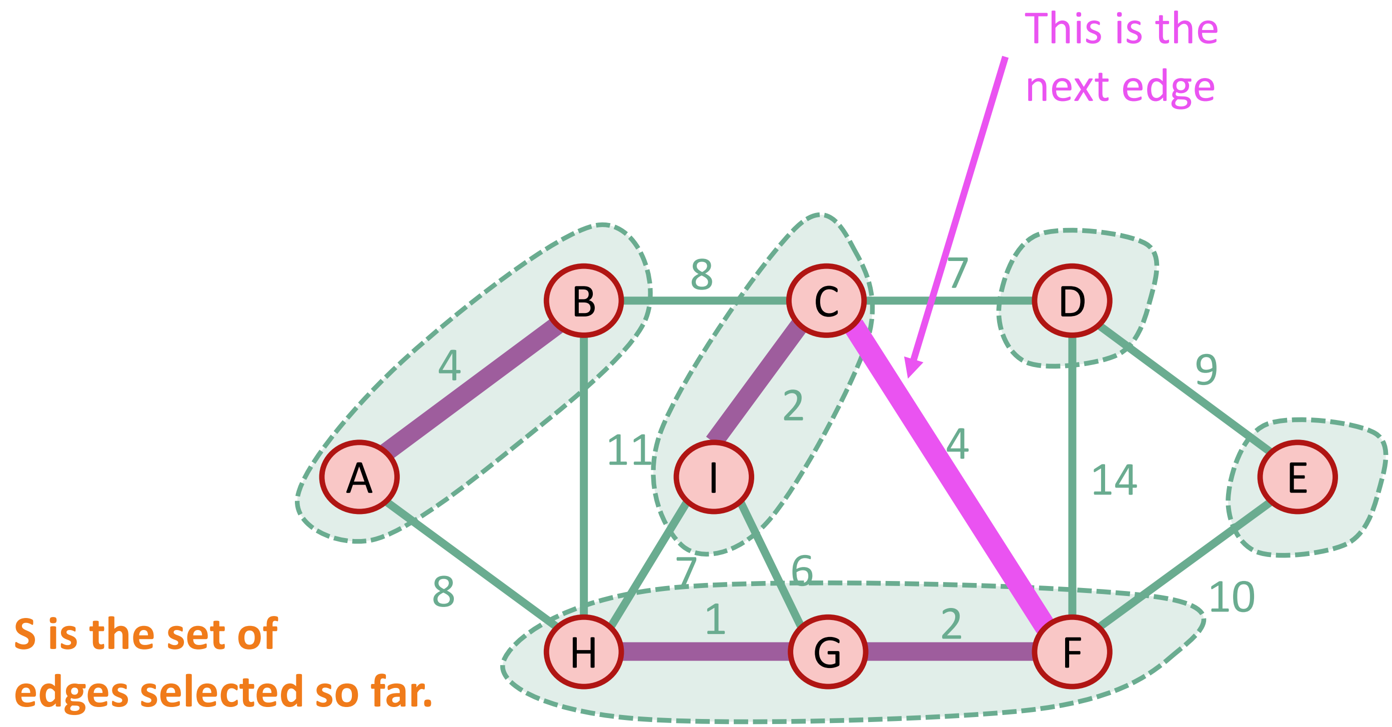

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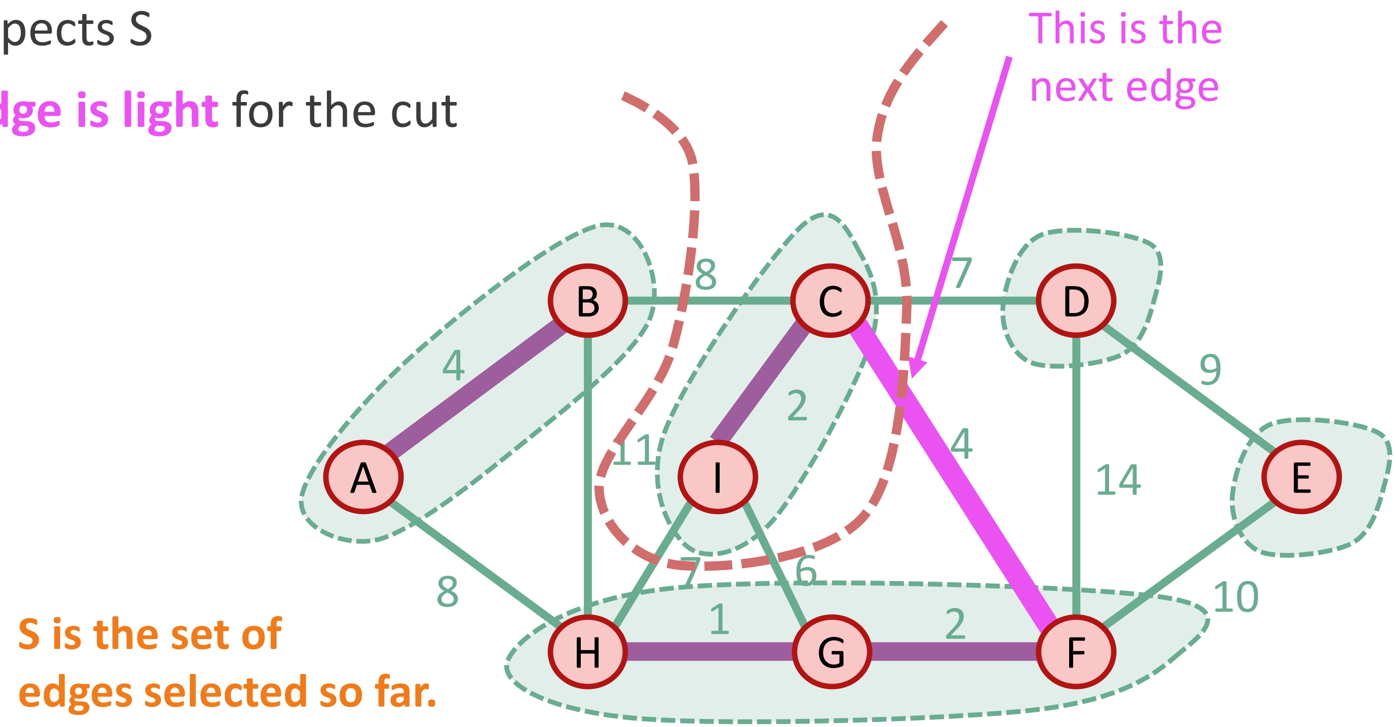
- Consider the cut **{T1, V – T1}.**
  - This cut respects S
  - Our **new edge is light** for the cut



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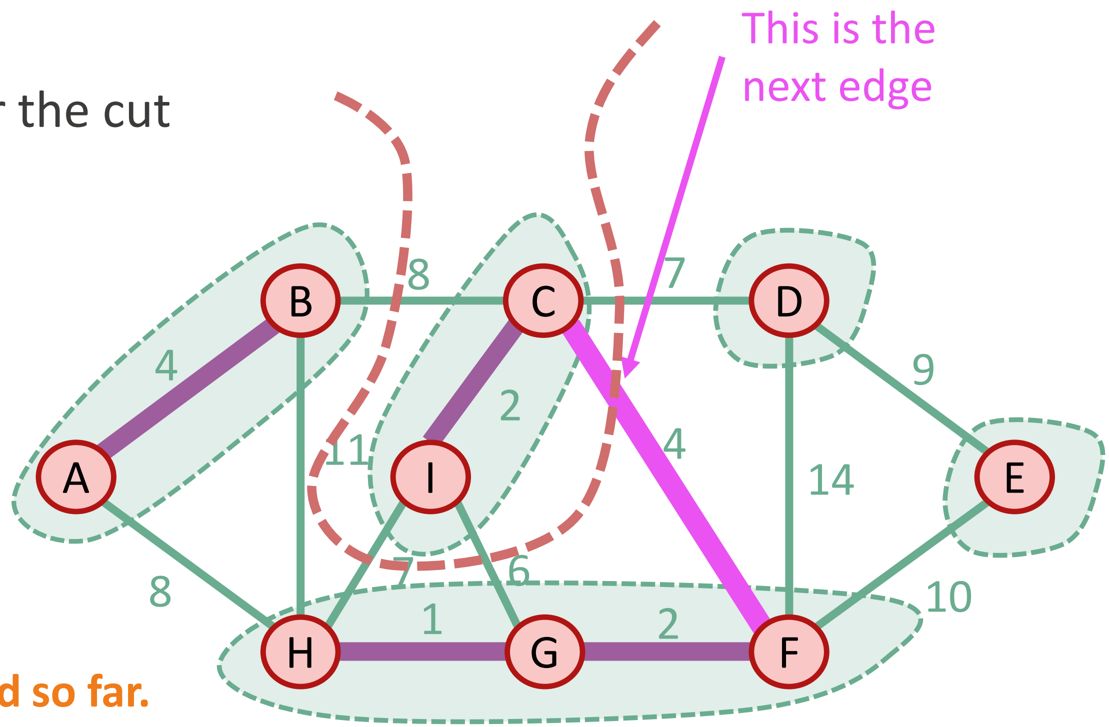
  - There is an MST extending them
- The **next edge** we add will merge two trees, **T1, T2**
- Consider the cut **{T1, V – T1}.**
  - This cut respects S
  - Our **new edge is light** for the cut
- By the Lemma, **that edge** is safe to add.
  - There is still an MST extending the new set



S is the set of edges selected so far.

Hooray!

- Our greedy choices **don’t rule out success**.

- This is enough (along with an argument by induction) to guarantee correctness of Kruskal’s algorithm.

Formally

- Inductive hypothesis:
  - After adding the t'th edge, there exists an MST with the edges added so far.
- Base case:
  - After adding the 0'th edge, there exists an MST with the edges added so far. **YEP.**
- Inductive step:
  - If the inductive hypothesis holds for t (aka, the choices so far are safe), then it holds for t+1 (aka, the next edge we add is safe).
  - **That's what we just showed.**
- Conclusion:
  - After adding the n-1'st edge, there exists an MST with the edges added so far.
  - At this point we have a spanning tree, so it better be minimal.

Two questions

1. Does it work?
   - That is, does it actually return a MST?
     - Yes!

2. How do we actually implement this?
   - The pseudocode above says “slowKruskal”…
     - Using a union-find data structure!

What have we learned?

- Kruskal's algorithm greedily grows a forest
- It finds a Minimum Spanning Tree in time O(mlog(n))
  - if we implement it with a Union-Find data structure
  - if the edge weights are reasonably-sized integers and we ignore the inverse Ackerman function, basically O(m) in practice.

- To prove it worked, we followed the same recipe for greedy algorithms we saw last time.
  - Show that, at every step, we **don't rule out success.**

Compare and contrast

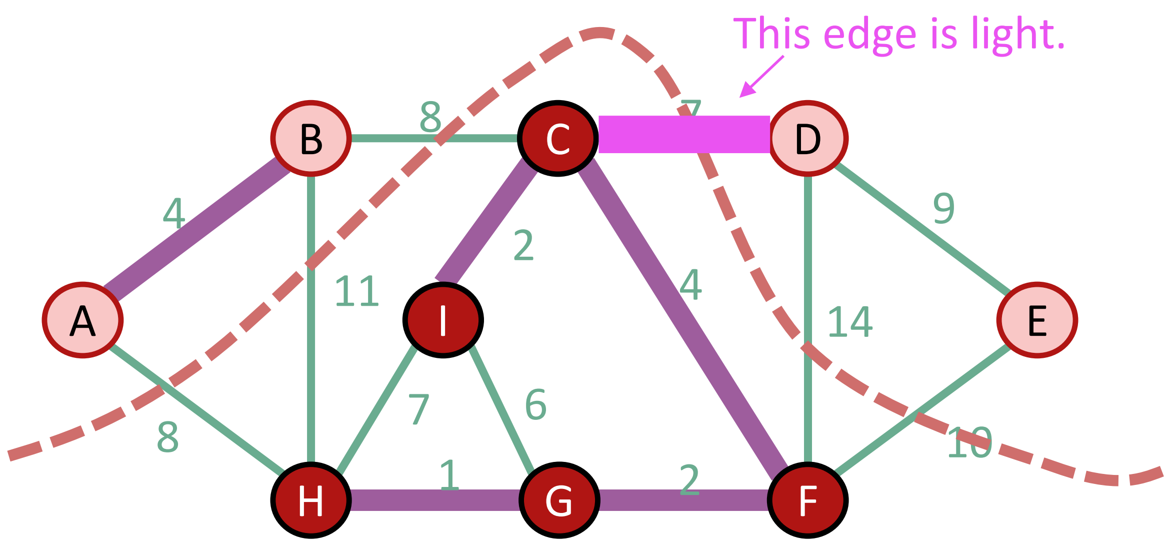
- **Prim:**
  - Grows a tree.
  - Time O(mlog(n)) with a red-black tree
  - Time O(m + nlog(n)) with a Fibonacci heap
- **Kruskal**:
  - Grows a forest.
  - Time O(mlog(n)) with a union-find data structure
  - If you can do radixSort on the edge weights, morally O(m)

Prim might be a better idea on dense graphs if you can't radixSort edge weights

Kruskal might be a better idea on sparse graphs if you can radixSort edge weights

Both Prim and Kruskal

- Greedy algorithms for MST.
- Similar reasoning:
  - Optimal substructure: subgraphs generated by cuts.
  - The way to make safe choices is to choose light edges crossing the cut.



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Recap

- Two algorithms for Minimum Spanning Tree
  - Prim's algorithm
  - Kruskal's algorithm

- Both are examples of greedy algorithms!
  - Make a series of choices.
  - Show that at each step, your choice does not rule out success.
  - At the end of the day, you haven't ruled out success, so you must be successful.


Any Question?