Lec13: Graph Algorithms III

Algorithm I
COMP319-003
Spring 2023

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Kyungpook National University **(KNU)**

Last time

- Graph representation
- depth-first search (DFS)
- Plus, applications!
  - Topological sorting
  - Strongly Connected Components

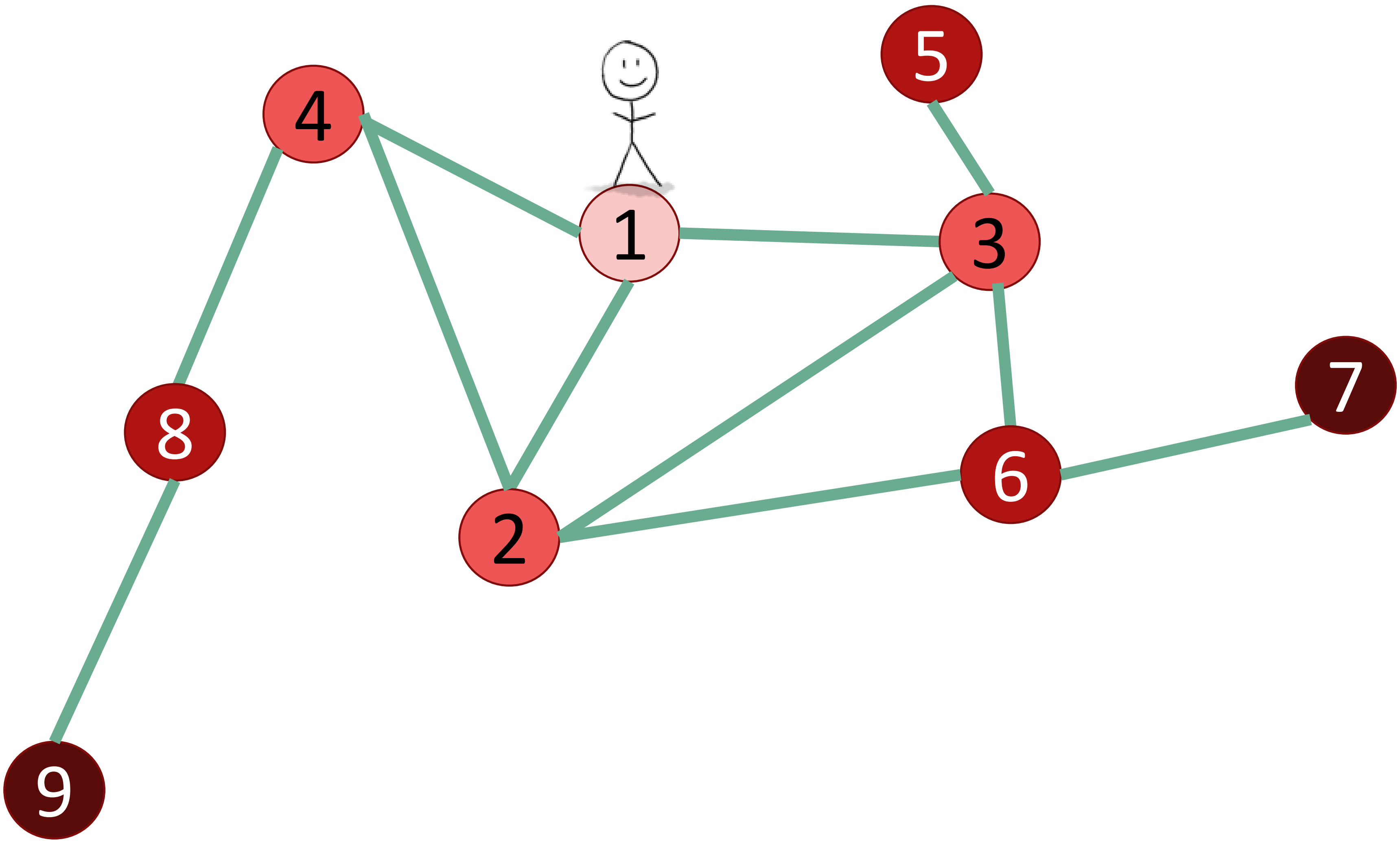
Outline

1. Graph Representations
2. DFS (Depth First Search)
   - Topological Ordering
   - Strongly Connected Components
3. **BFS (Breadth-First Search)**
   - **Dijkstra's Algorithm**

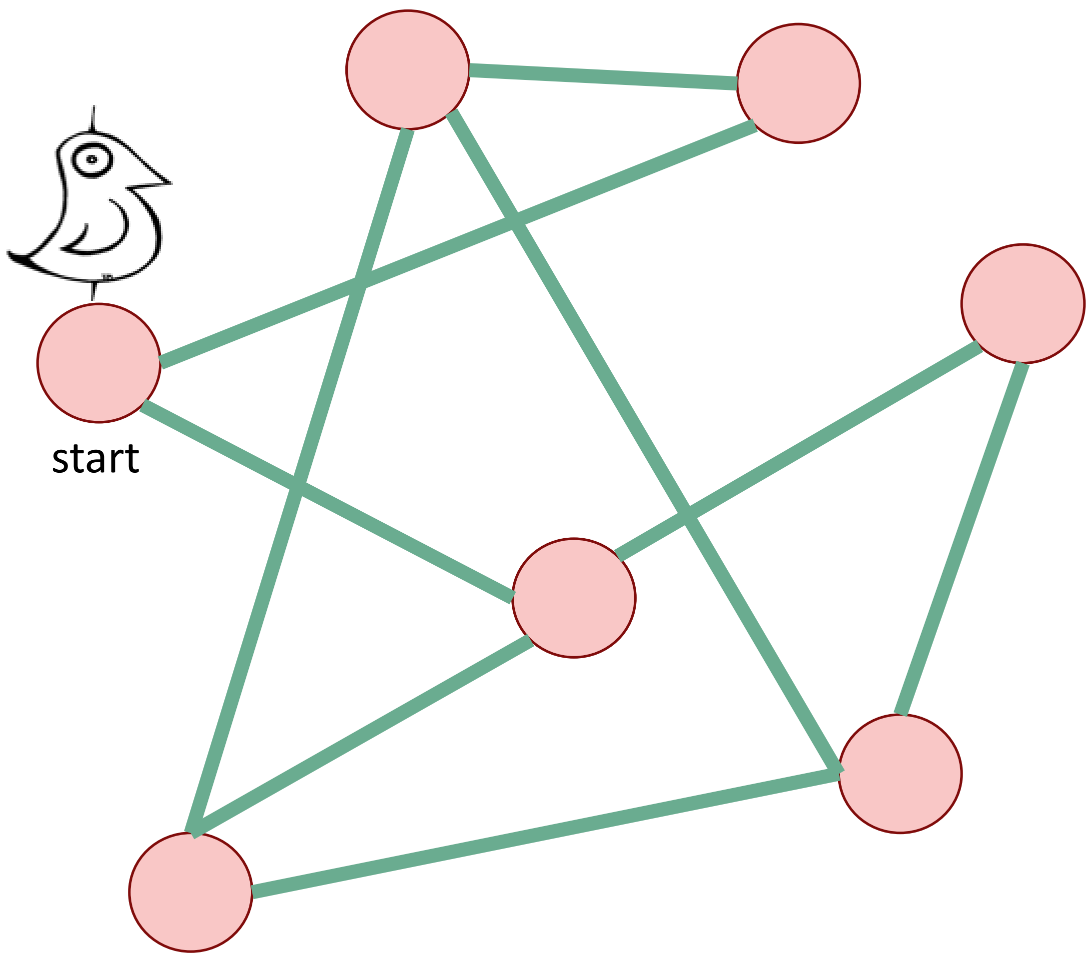
- *Reading: CLRS 22.1 – 22.4*

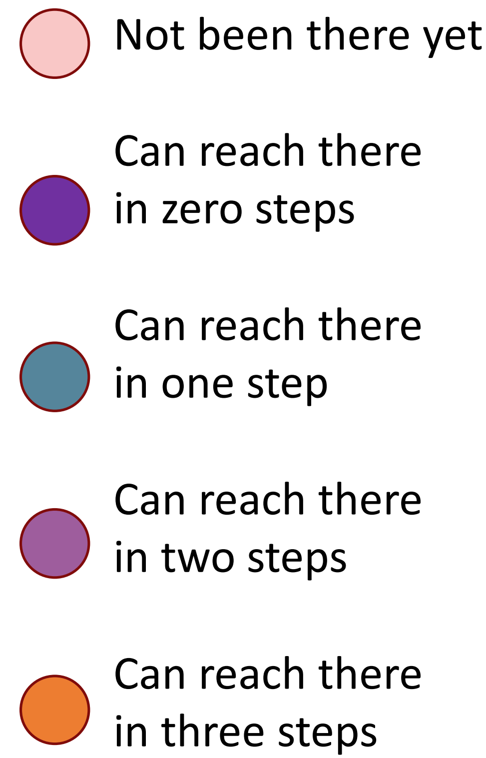
Breadth-First Search

How do we explore a graph?



Breadth-First Search





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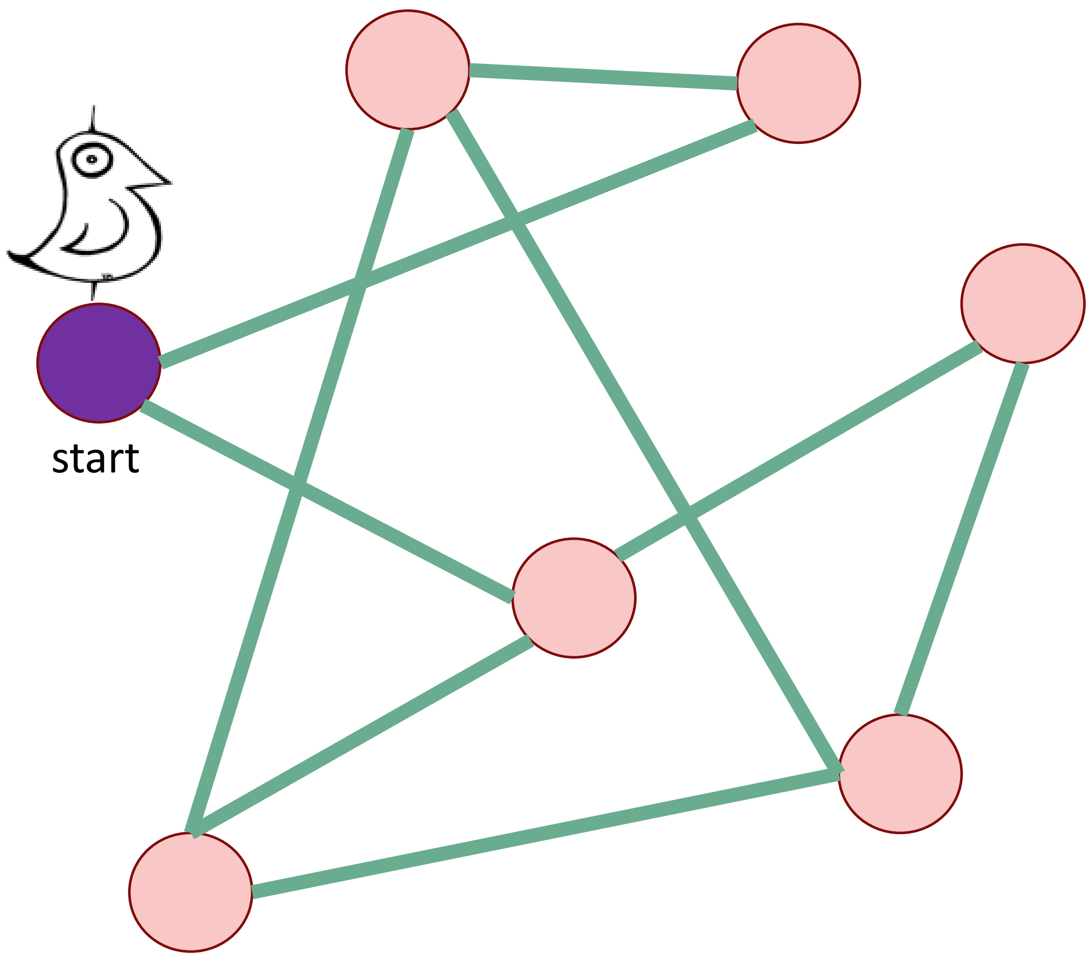


Not been there yet

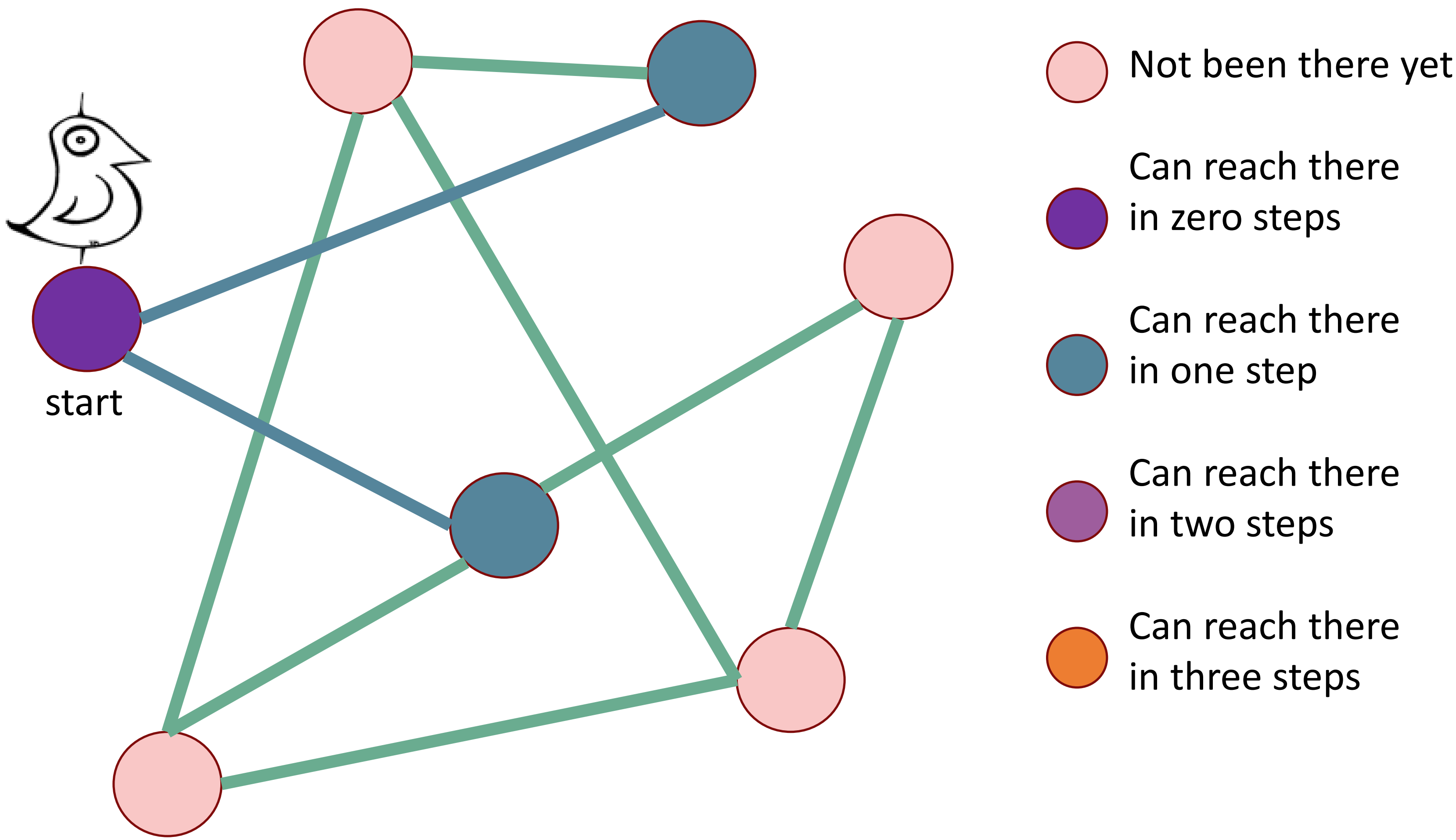
Can reach there in zero steps

Can reach there in one step

Can reach there in two steps

Can reach there in three steps

Breadth-First Search



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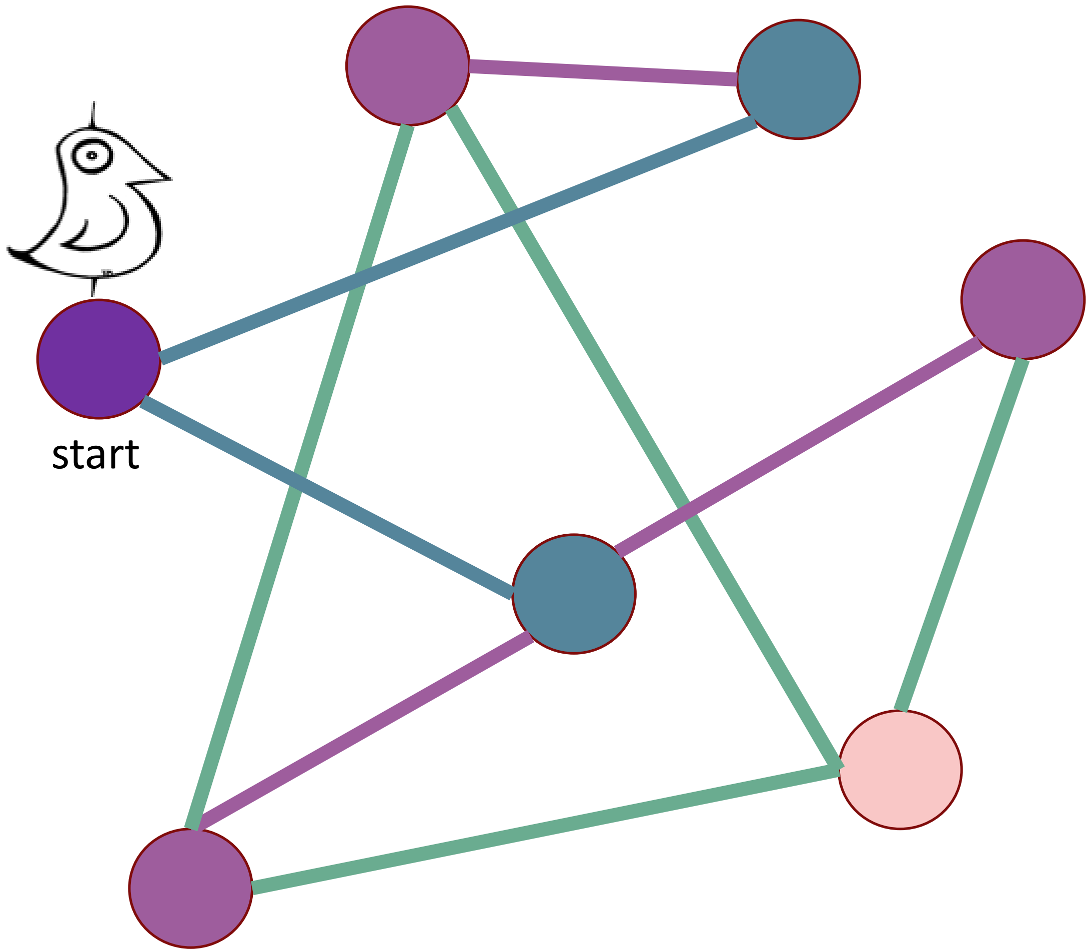

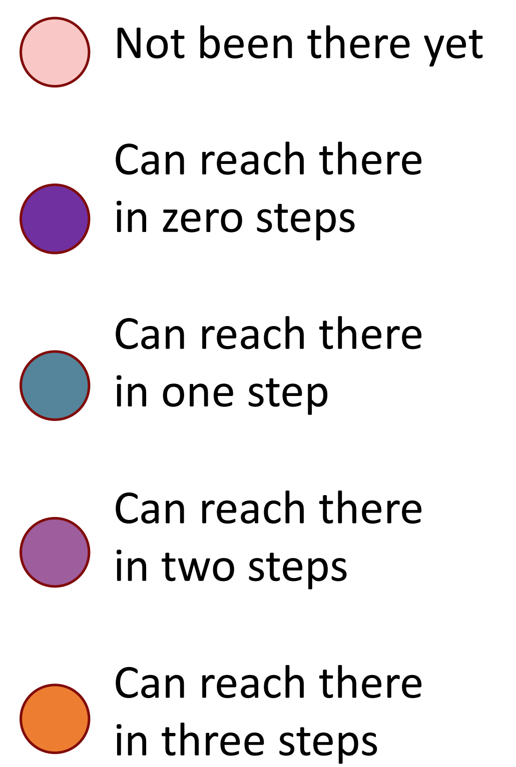

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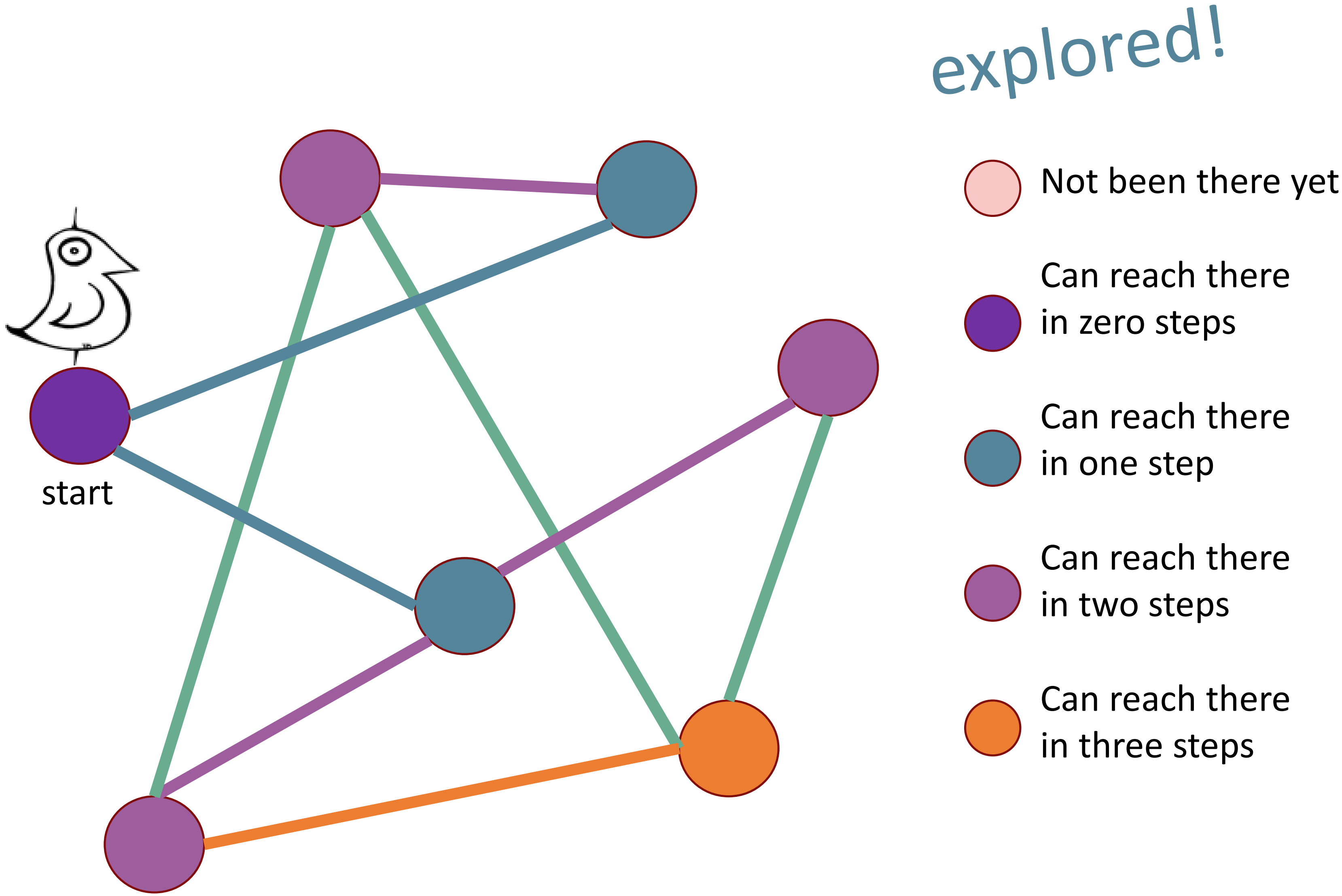
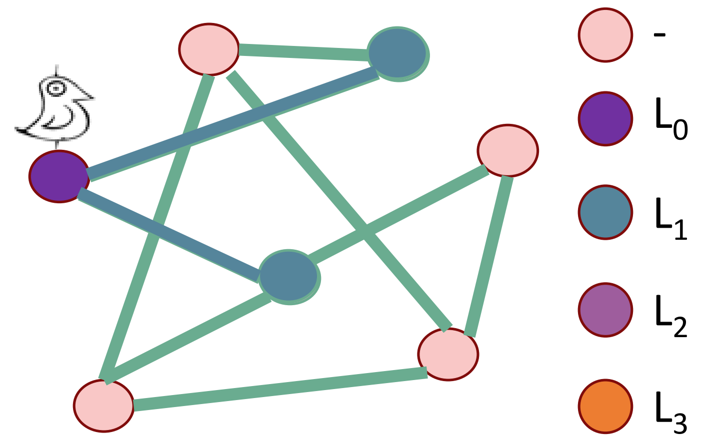


Breadth-First Search

- Set L_i = [] for i=1,…,n
- L_0 = [w], where w is the start node
- Mark w as visited
- **For** i = 0, …, n-1:
  - **For** u in L_i:
    - **For** each v which is a neighbor of u:
      - **If** v isn't yet visited:
        - mark v as visited, and put it in L_{i+1}



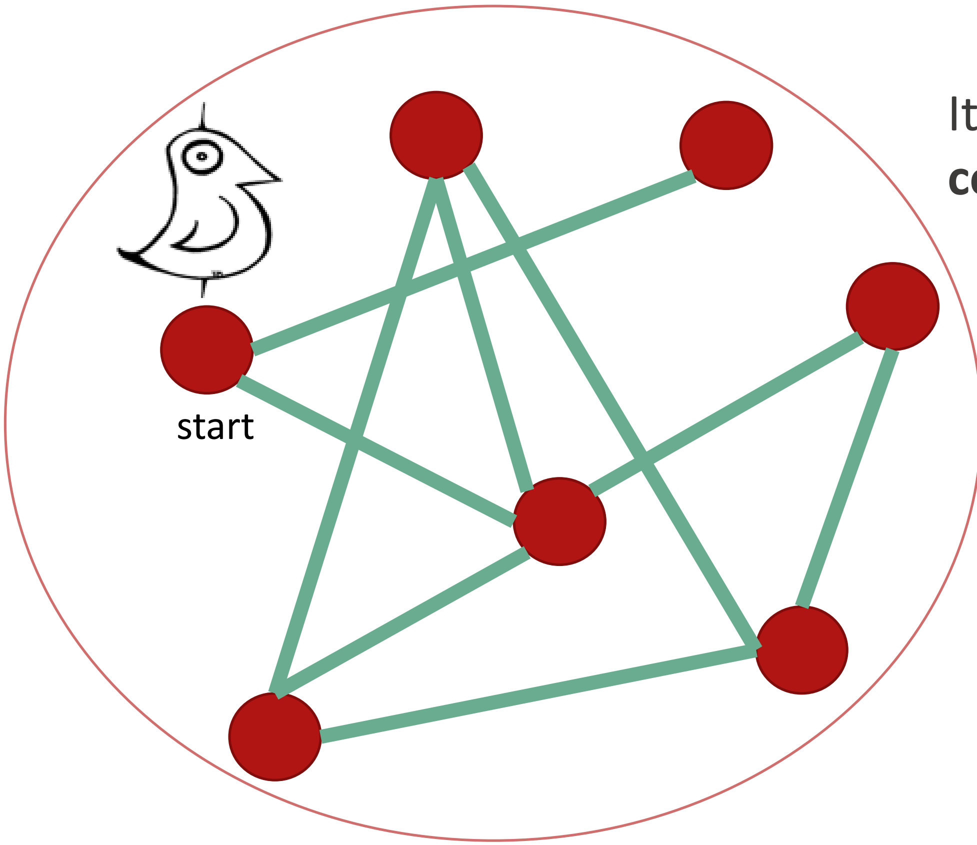
L_i is the set of nodes we can reach in i steps from w

Go through all the nodes in L_i and add their unvisited neighbors to L_{i+1}

- **Runtime?**
  - Same argument as DFS: BFS running time is **O(n + m)**

Breadth-First Search

- BFS also finds all the nodes reachable from the starting point

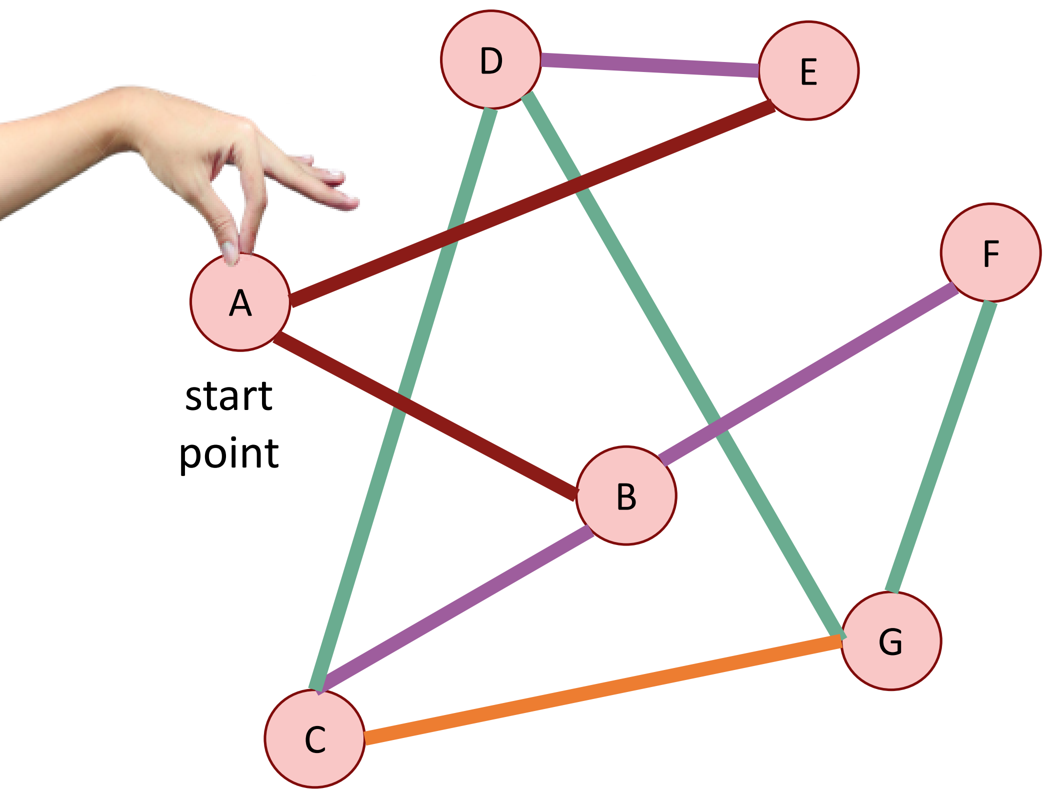


It is a good way to find all the **connected components.**

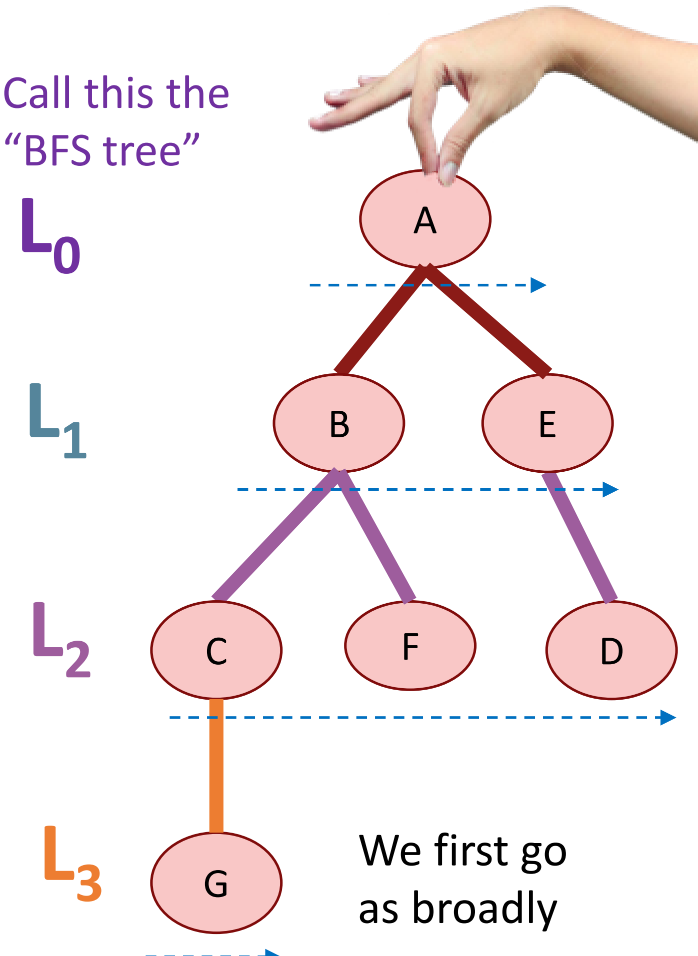
Like DFS, BFS also works fine on directed graphs.

Why is it called breadth-first?

- We are implicitly building a tree:



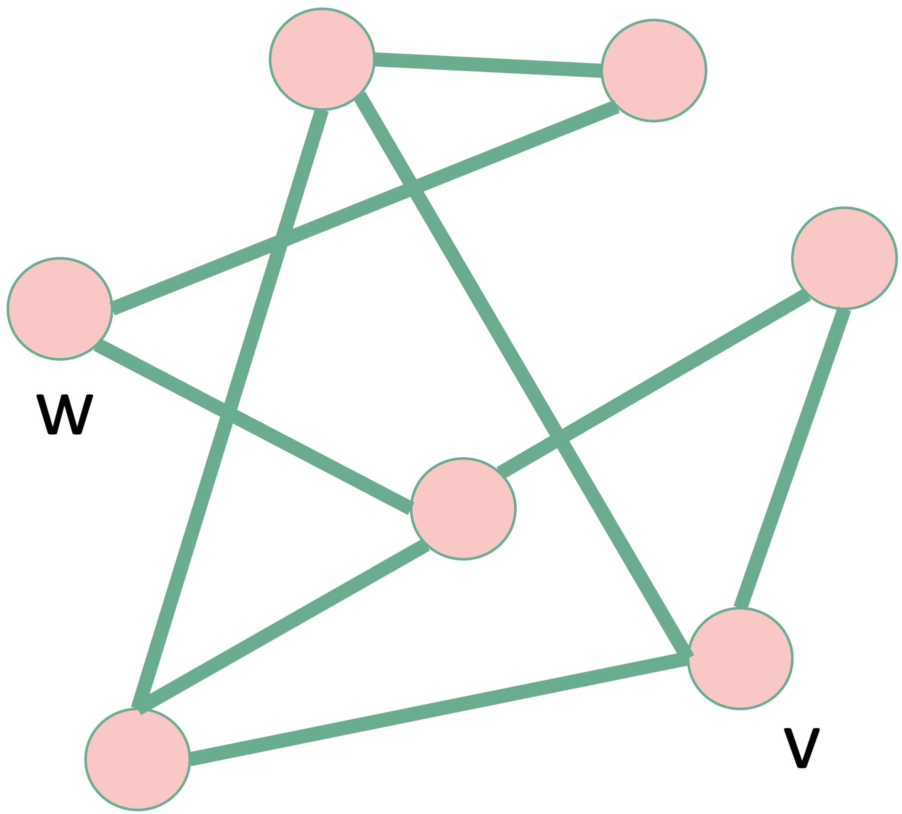
Call this the "BFS tree"



We first go as broadly as we can.

Application of BFS

- How long is the shortest path between w and v?



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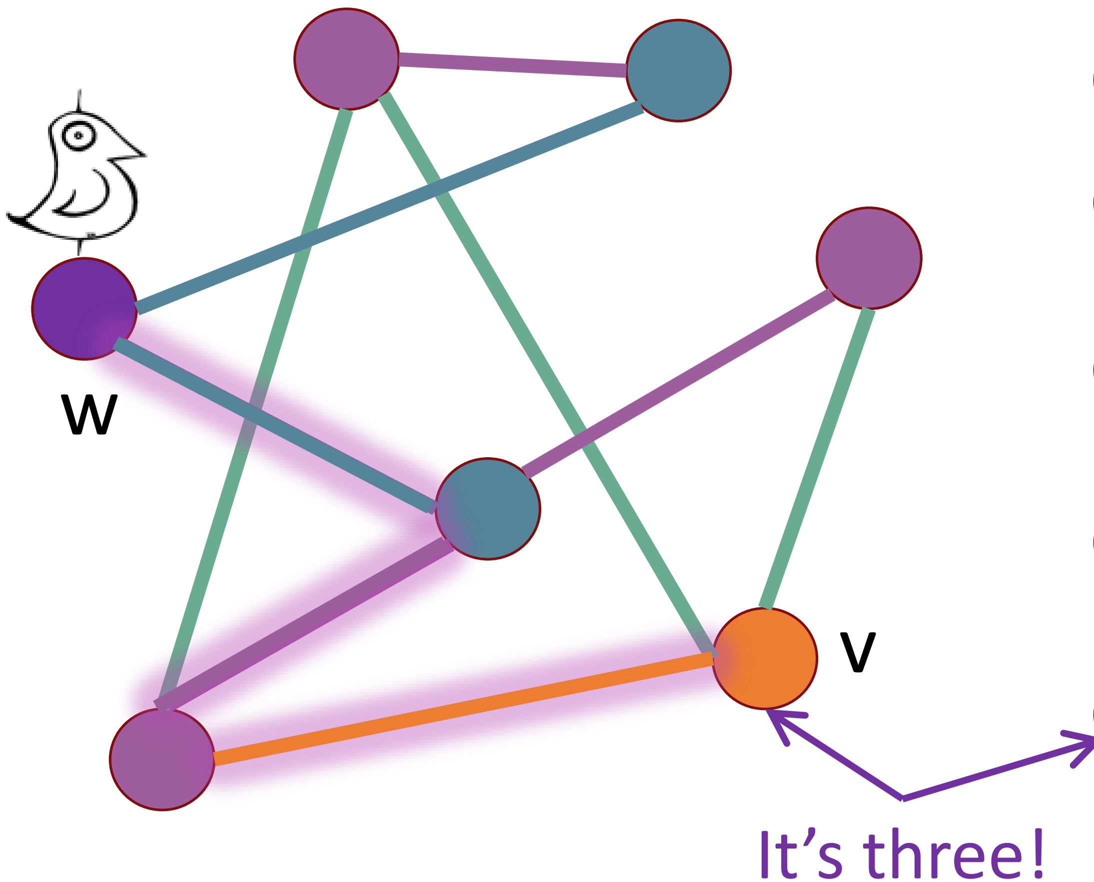

- Not been there yet
- Can reach there in zero steps
- Can reach there in one step
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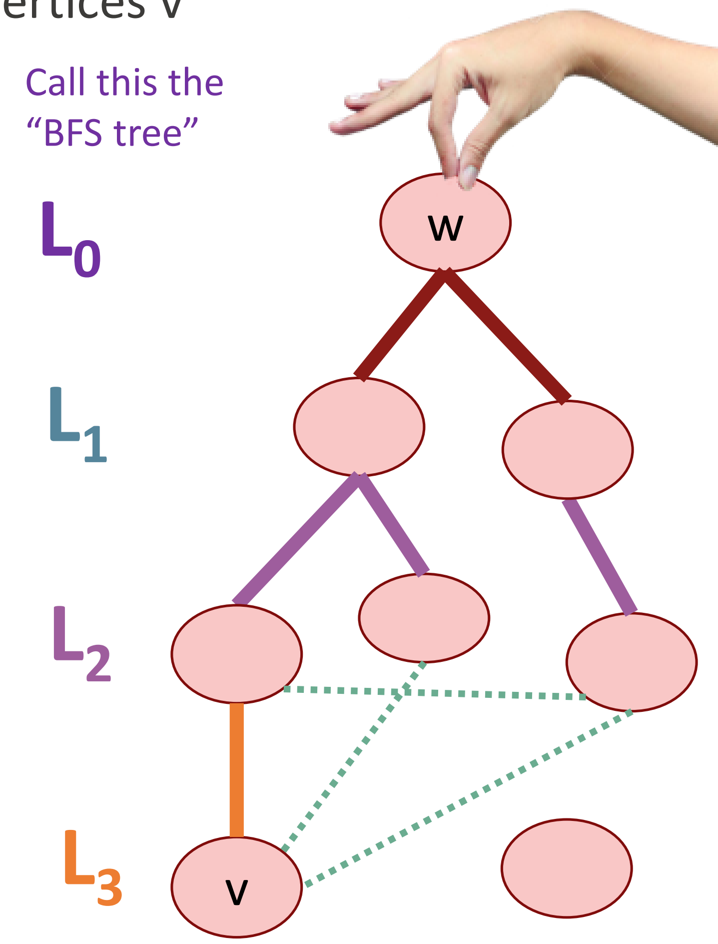
The Shortest Path

- To find the distance between w and all other vertices v
  - Do a **BFS** starting at w
  - For all v in L_i
    - The shortest path between w and v has length i. It is given by the path in the BFS tree.
  - If we never found v, the distance is infinite.



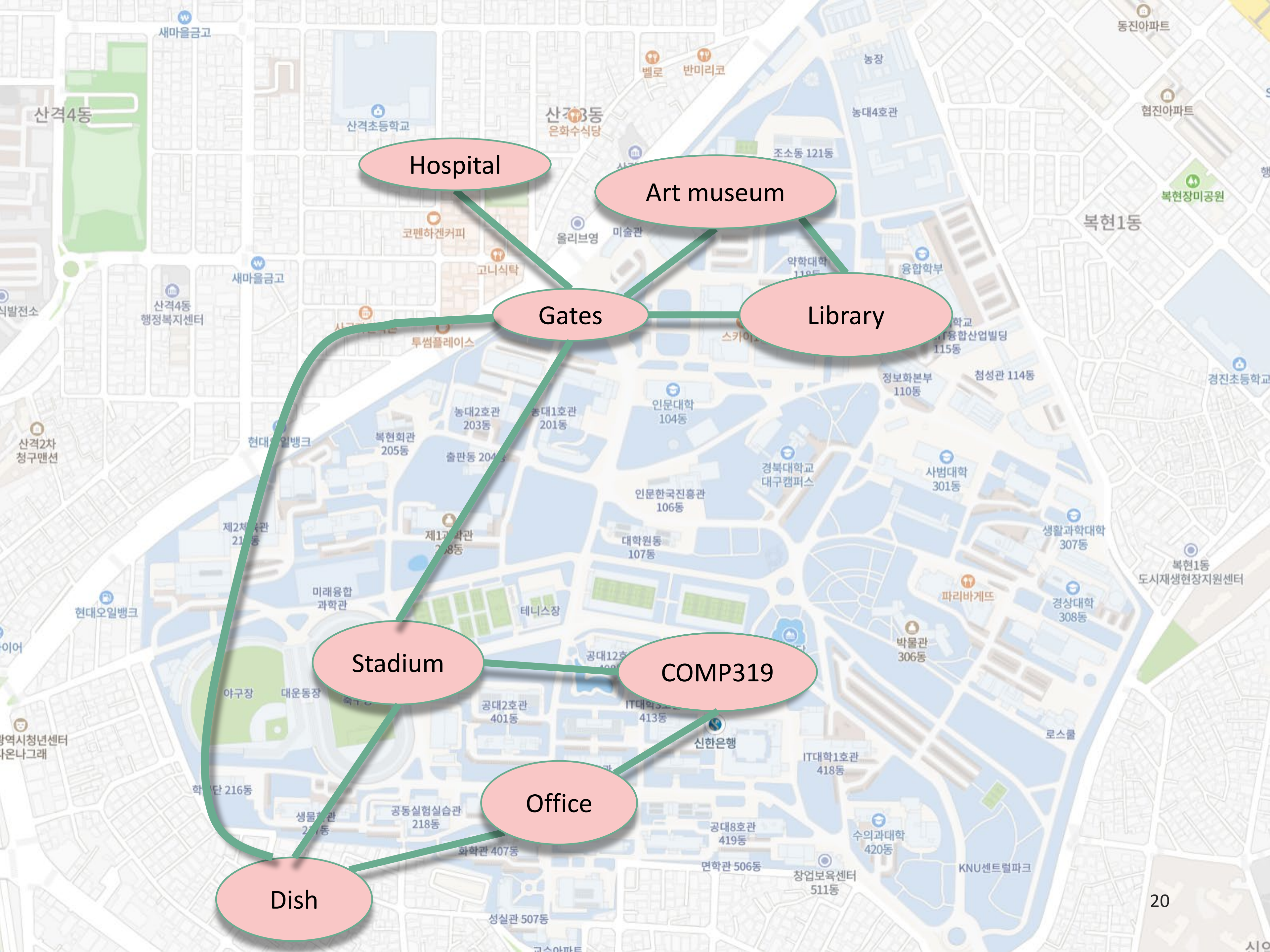
Summary

- The BFS tree is useful for computing distances between pairs of vertices.
- We can find the shortest path between u and v in time **O(n + m)**.

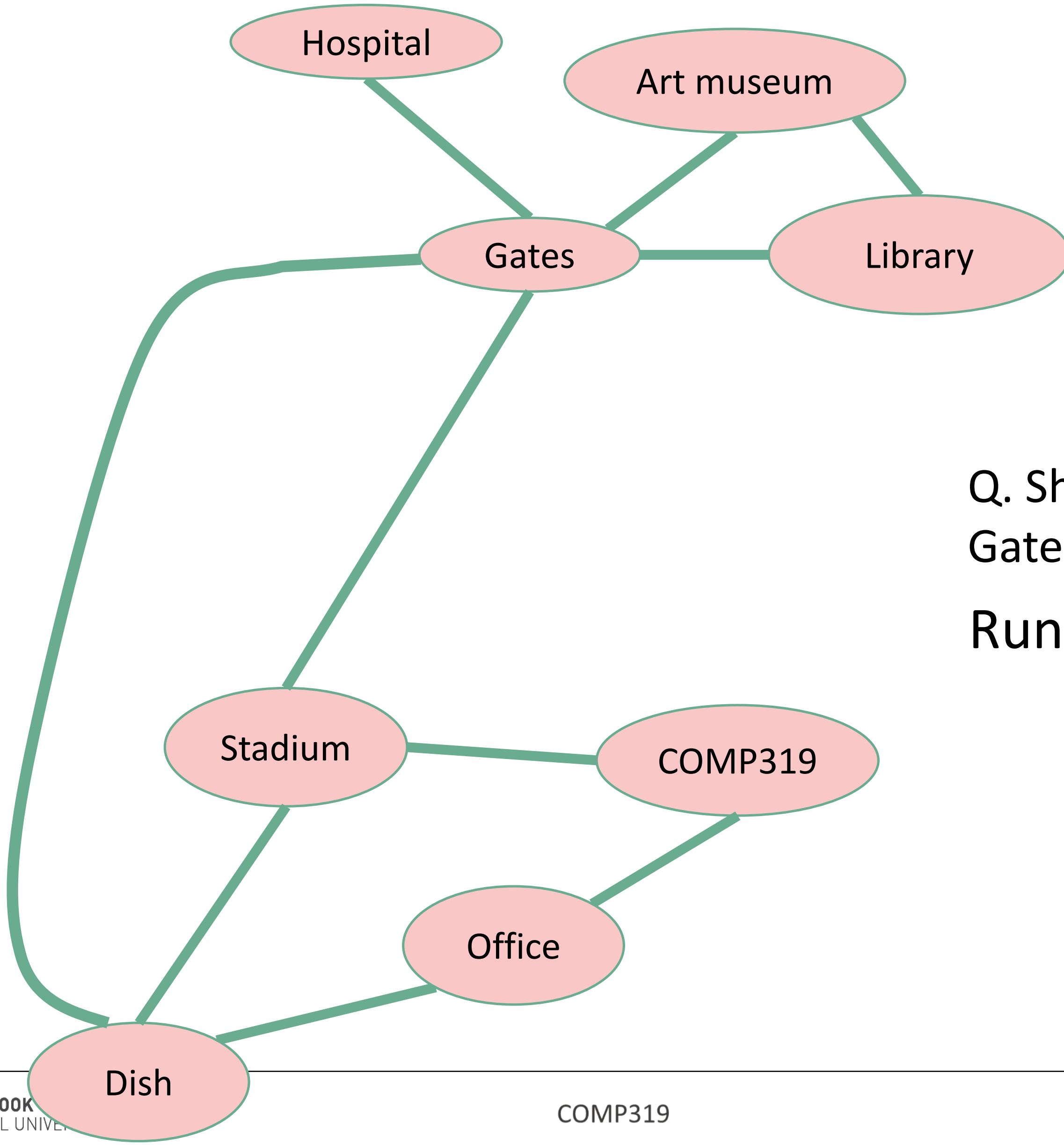
Dijkstra’s Algorithm

What if the graphs are weighted?

- Finding distance on a weighted graph
  - All nonnegative weights: Dijkstra
  - If there are negative weights: Bellman-Ford


Hospital
Art museum
Gates
Library
Stadium
COMP319
Office
Dish

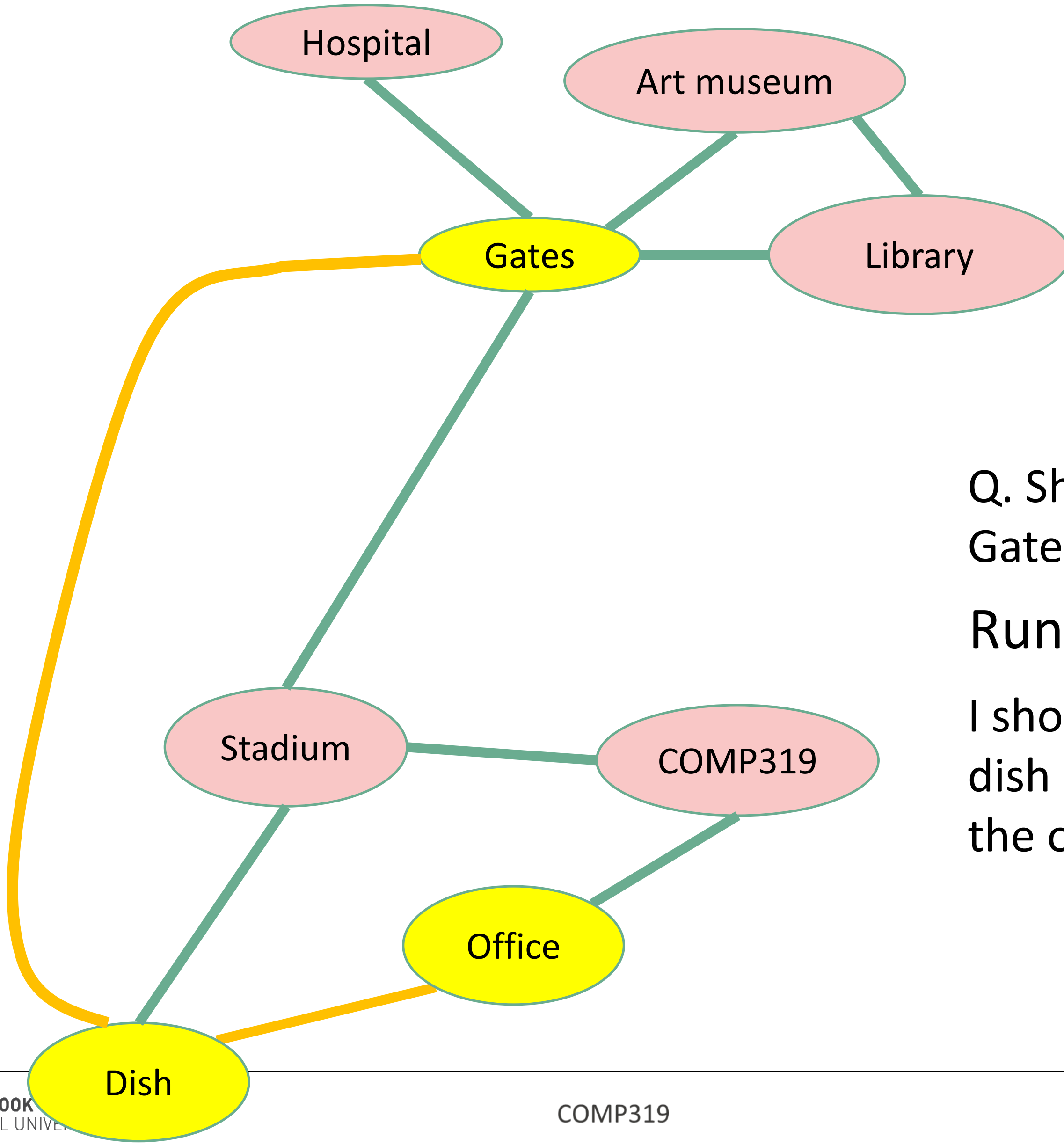
Just the graph



Q. Shortest path from Gates to the Office?

Run BFS …

Just the graph

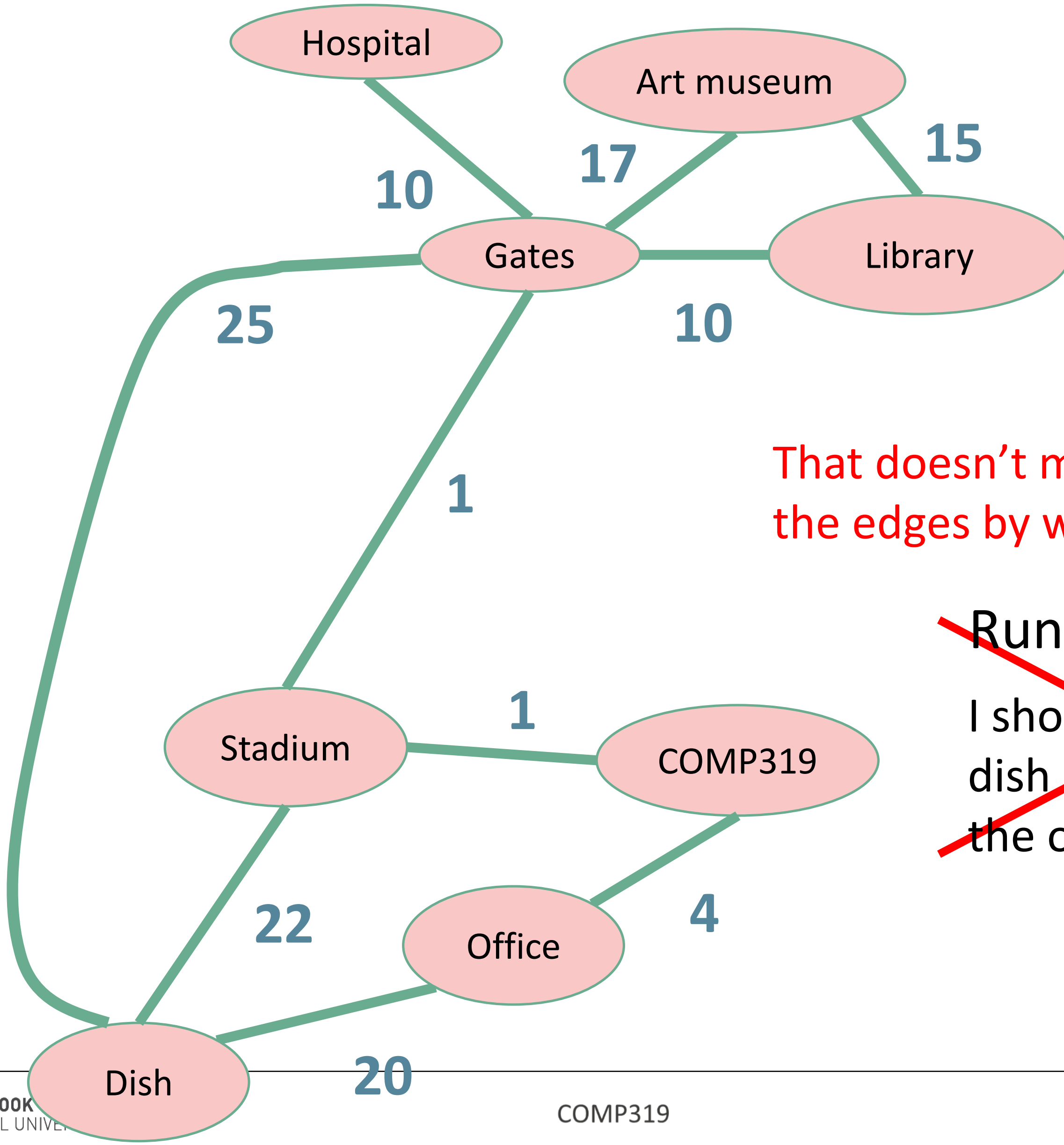


Q. Shortest path from Gates to the Office?

Run BFS …

I should go to the dish and then back to the office!

What if the graph has weights?



That doesn't make sense if I label the edges by walking time.

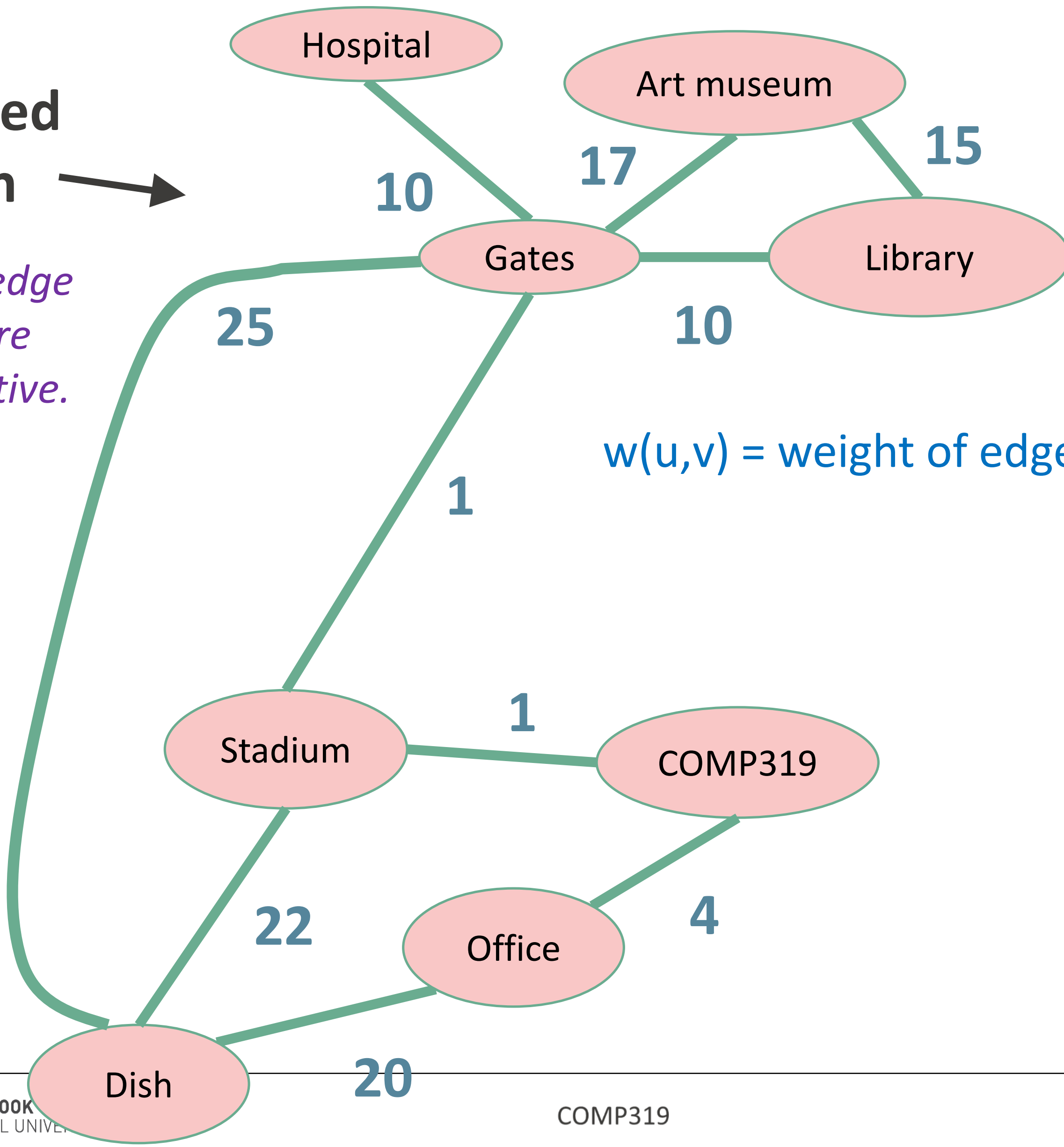
~~Run BFS …~~

~~I should go to the dish and then back to the office!~~

What if the graph has weights?

weighted graph

For now, edge weights are non-negative.



w(u,v) = weight of edge between u and v.

What if the graph has weights?

weighted graph

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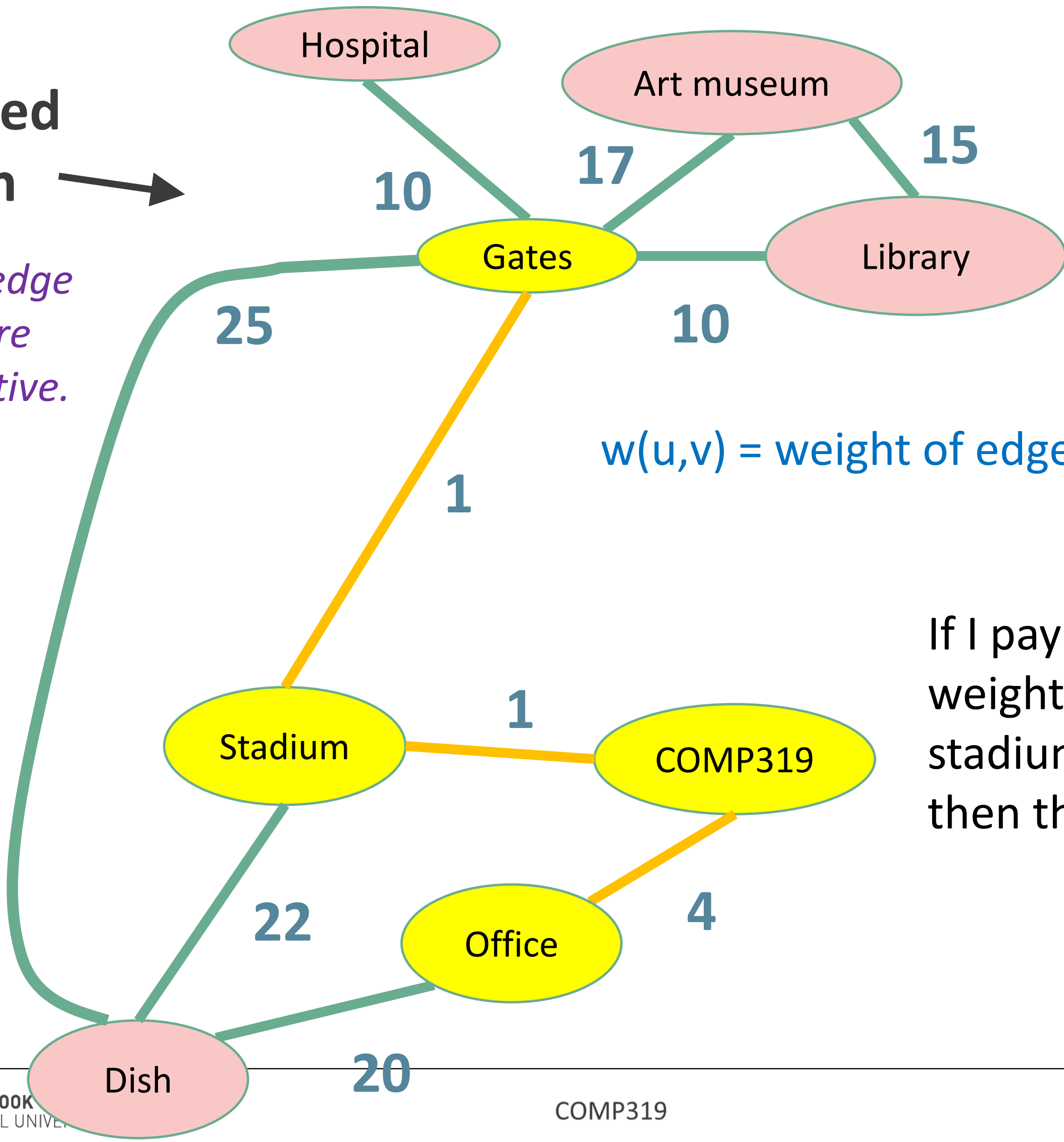
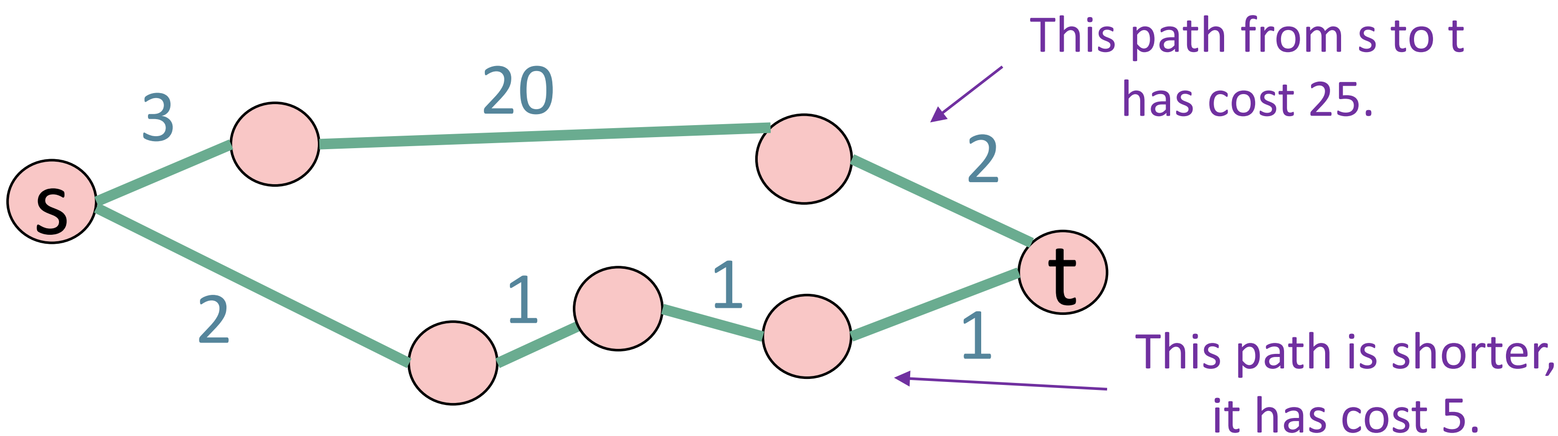


w(u,v) = weight of edge between u and v.

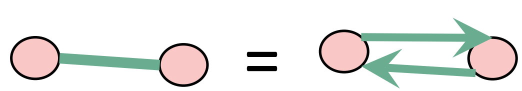
If I pay attention to the weights, I should go to stadium, then COMP319, then the Office.

Shortest path problem

- What is the **shortest path** between u and v in a weighted graph?
  - the **cost** of a path is the sum of the weights along that path.
  - The **shortest path** is the one with the minimum cost.



  - The **distance d(u,v)** between two vertices u and v is the cost of the shortest path between u and v.

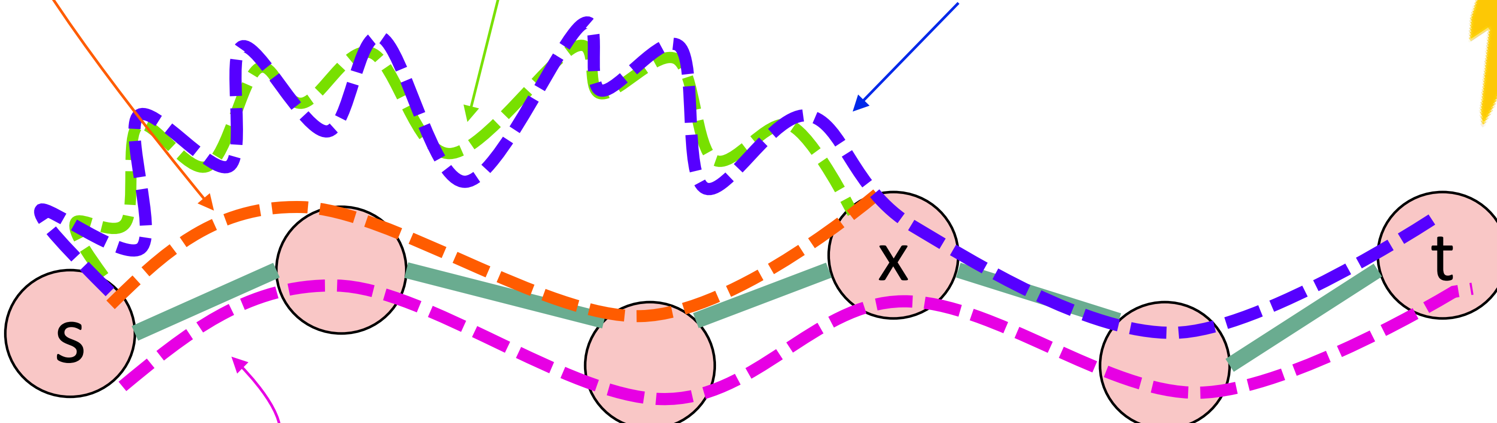


Note: For this lecture **all graphs are directed**, but to save on notation I'm just going to draw undirected edges.

Warm-up

- A sub-path of a shortest path is also a shortest path.

- Say **this** is a shortest path from s to t.
- Claim: **this** is a shortest path from s to x.
  - Suppose not, **this** one is a shorter path from s to x.
  - But then that gives an **even shorter path** from s to t!



Single-source shortest-path problem

- I want to know the shortest path from one vertex (Gates) to all other vertices.

Destination	Cost	To get there
Stadium	1	Gates-Stadium
COMP319	2	Gates-Stadium-COMP319
Hospital	10	Gates-Hospital
Art museum	17	Gates-Art museum
Office	6	Gates-Stadium-COMP319-Office
Library	10	Gates-Stadium
Dish	23	Gates-Stadium-Dish

Example

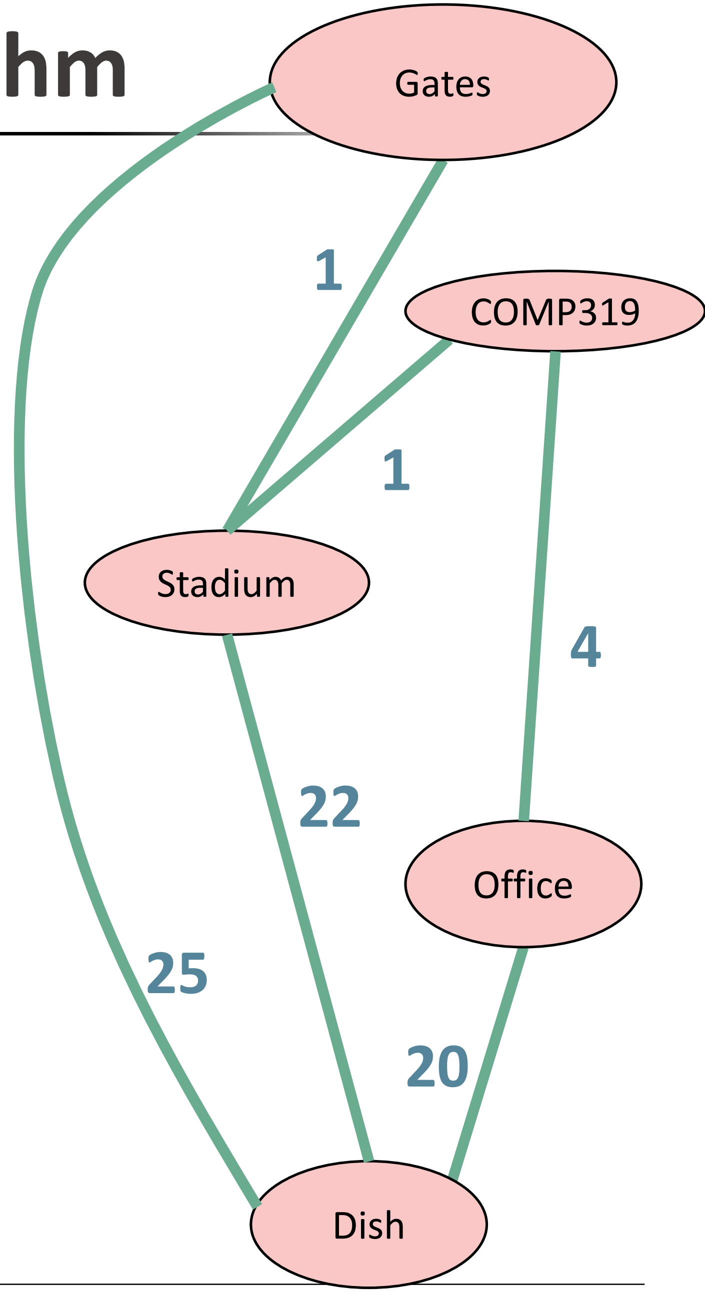
- I regularly have to solve **“what is the shortest path from home to [anywhere else]”** using Car, Bus, KTX, Flight, Taxi, Bike, Walking.
  - Edge weights have something to do with time, money, traffic, convenience.
- **Network routing**
  - I send information over the internet, from my computer to all over the world.
  - Each path has a cost which depends on link length, traffic, other costs, etc..

Back to this example

Dijkstra's algorithm

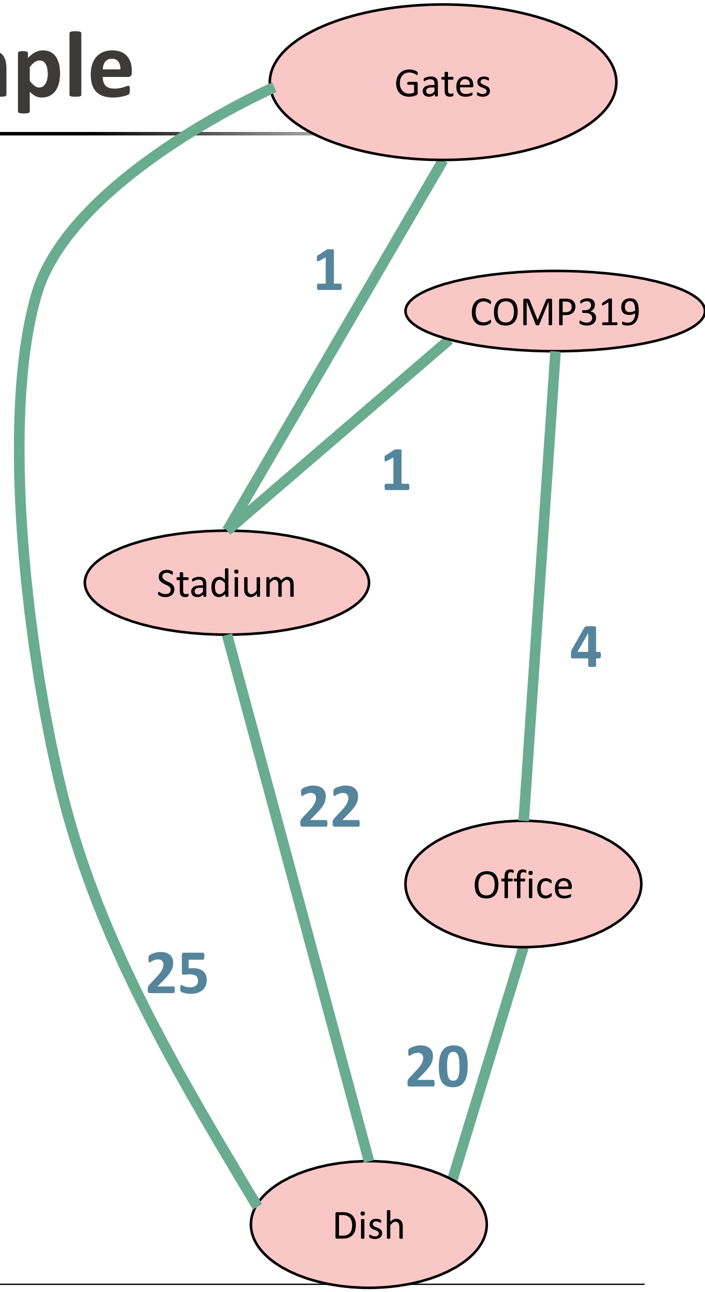
- Finds shortest paths from Gates to everywhere else.

- Dijkstra is used in practice
  - e.g., OSPF (Open Shortest Path First), a routing protocol for IP networks, uses Dijkstra.



Dijkstra by example

- How far is a node from Gates?



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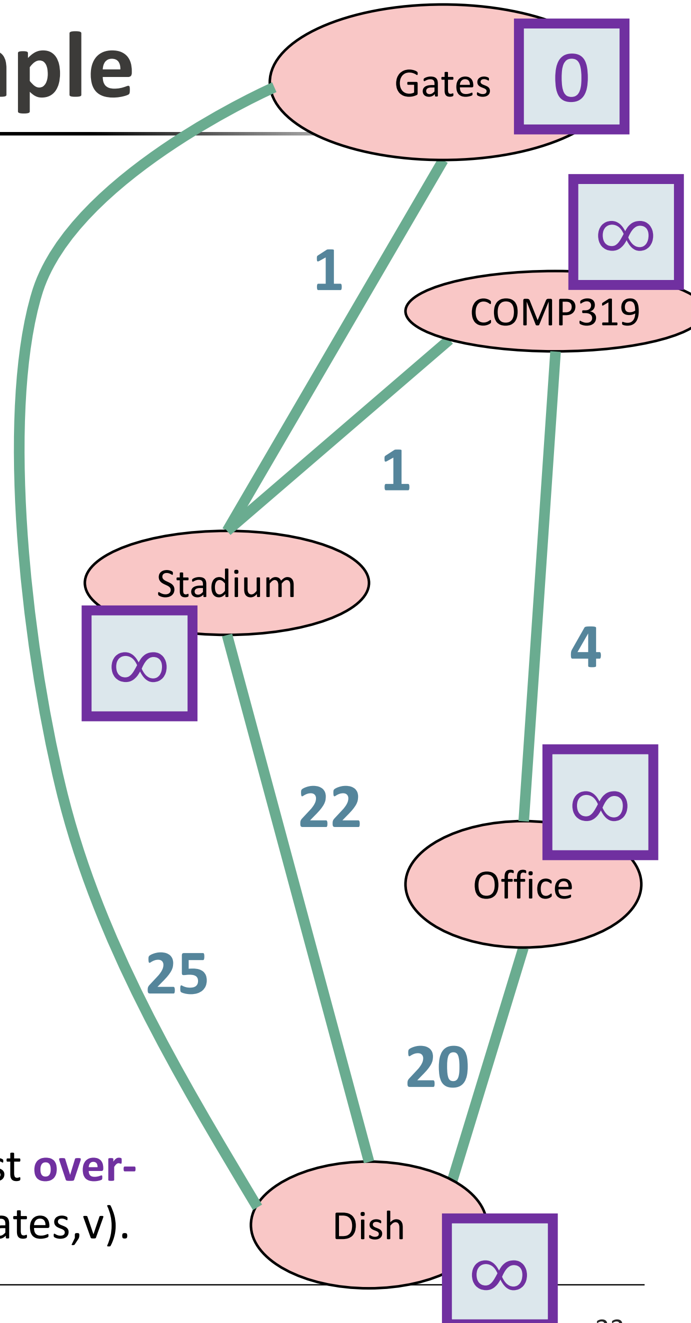
- **How far is a node from Gates?**
  - Initialize d[v] = ∞ for all non-starting vertices v, and d[Gates] = 0



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x **x = d[v]** is my best **over-estimate** for d(Gates,v).

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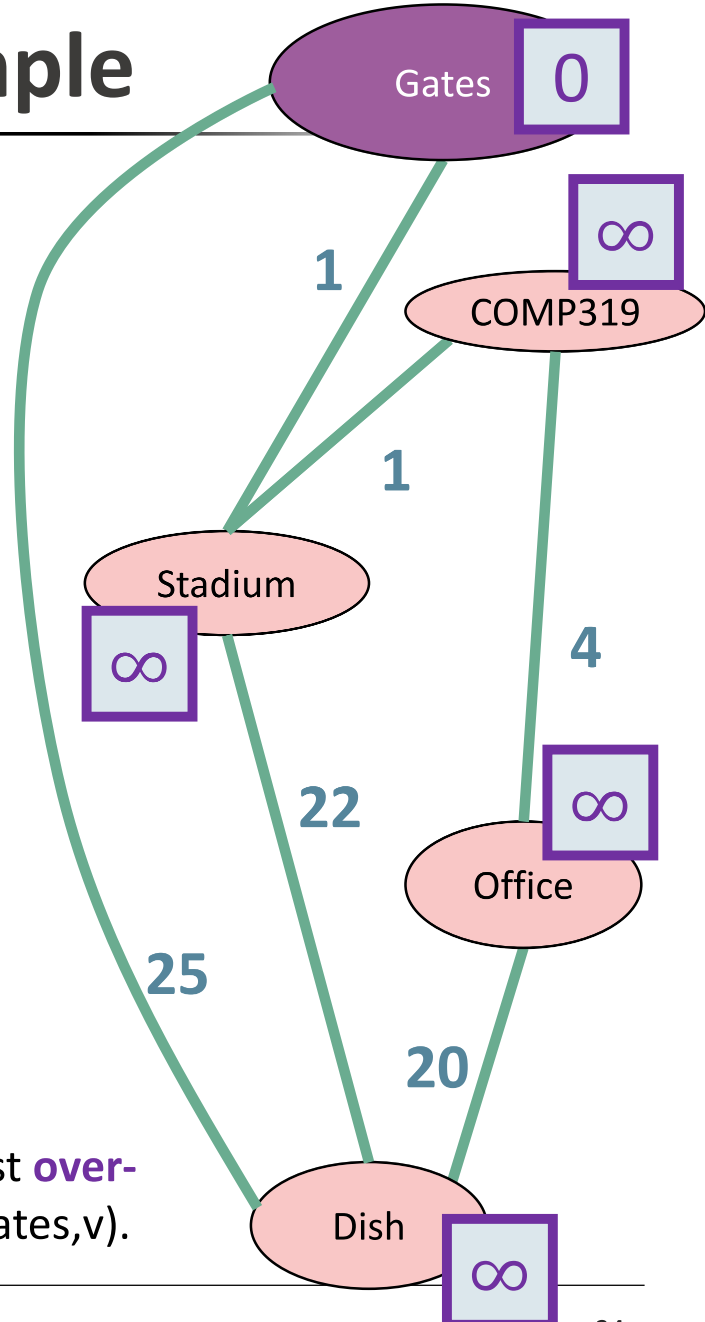

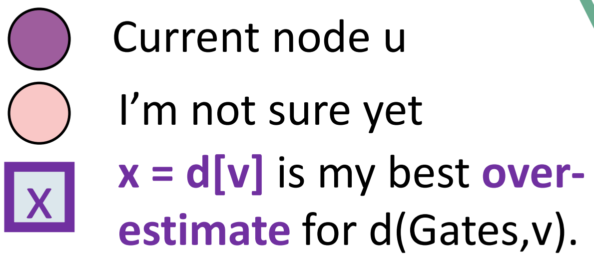

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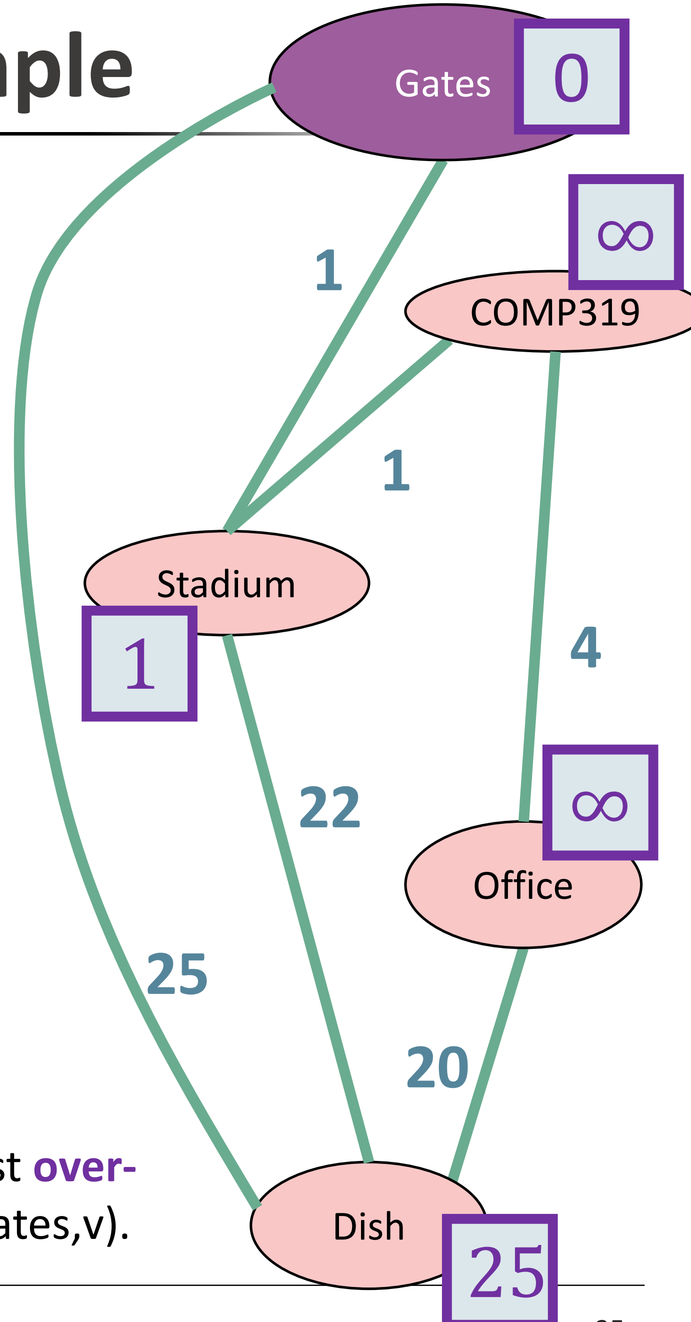

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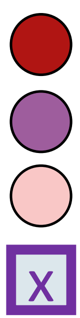

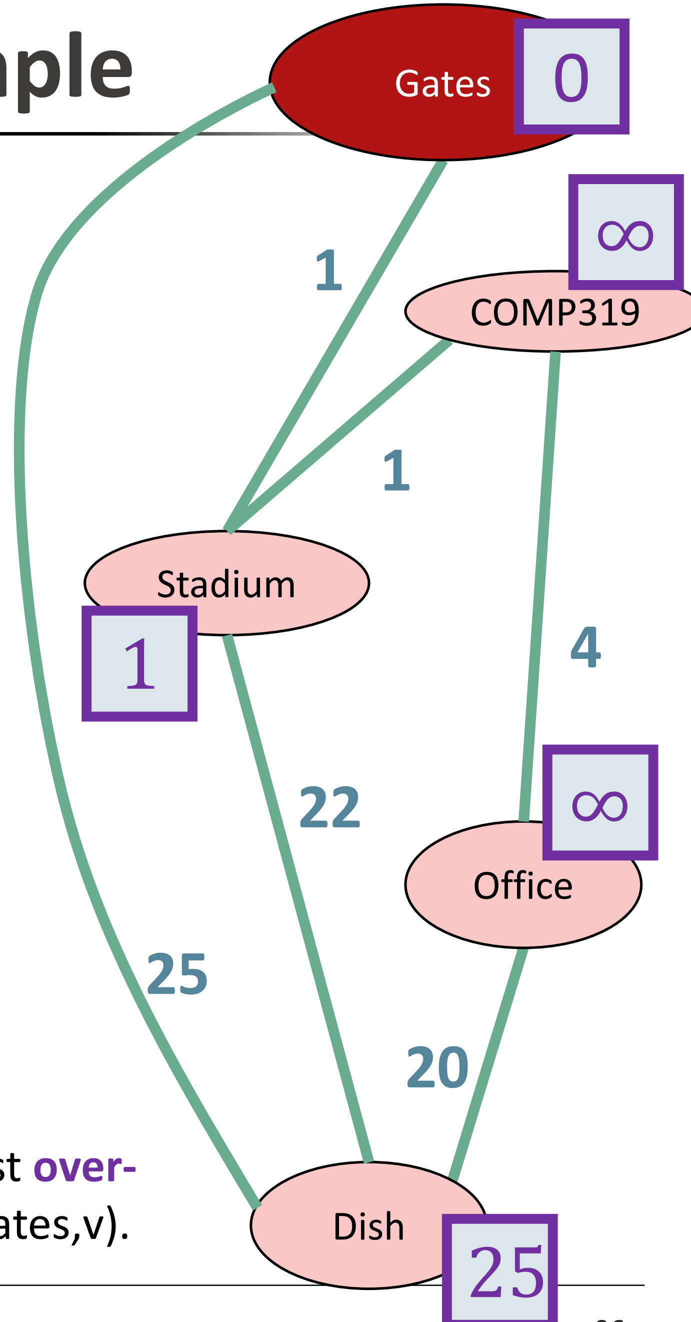

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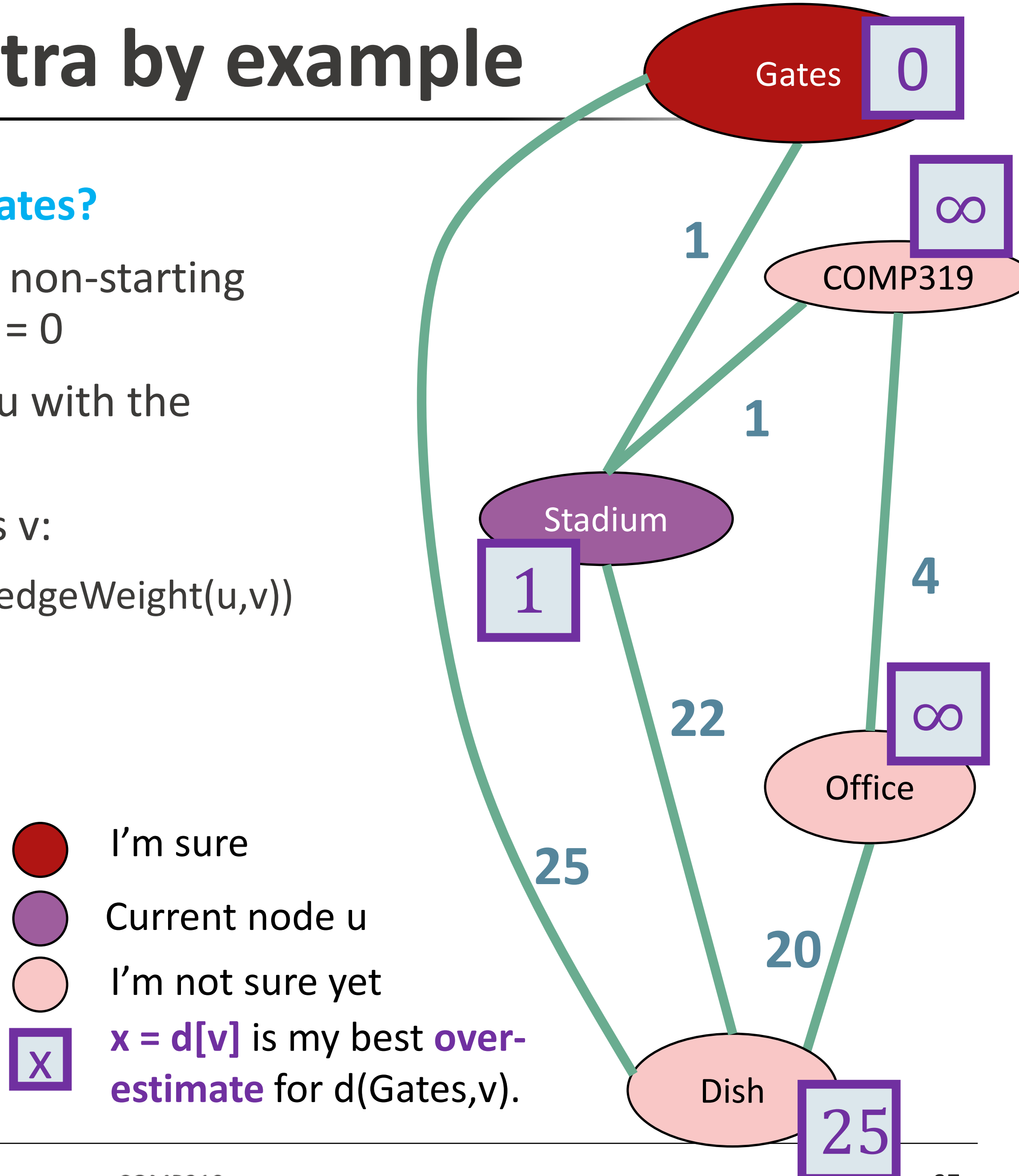

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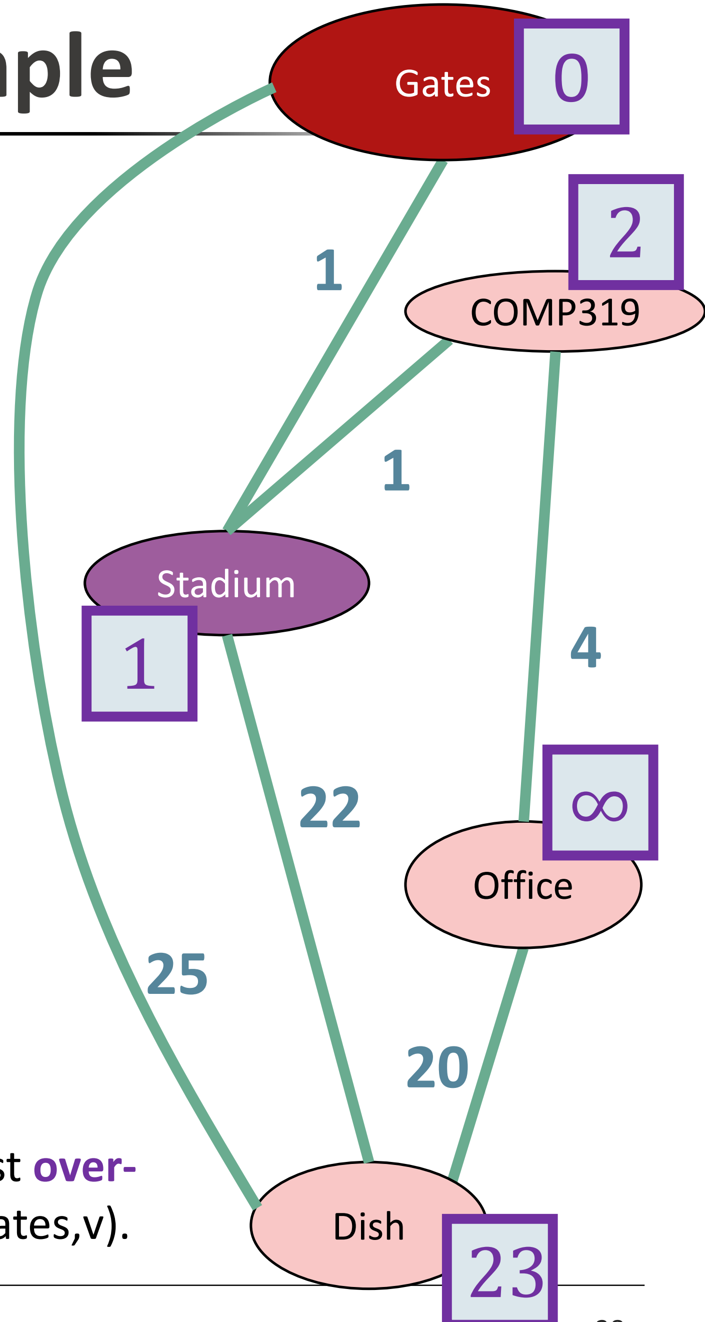

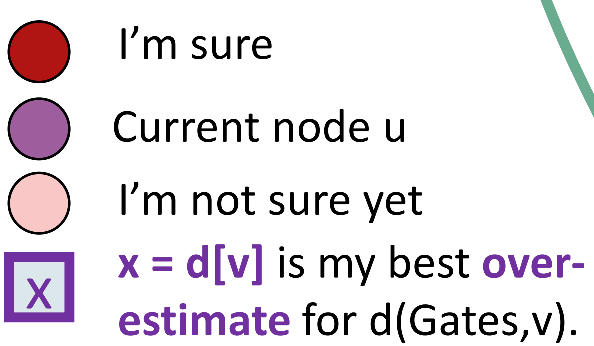

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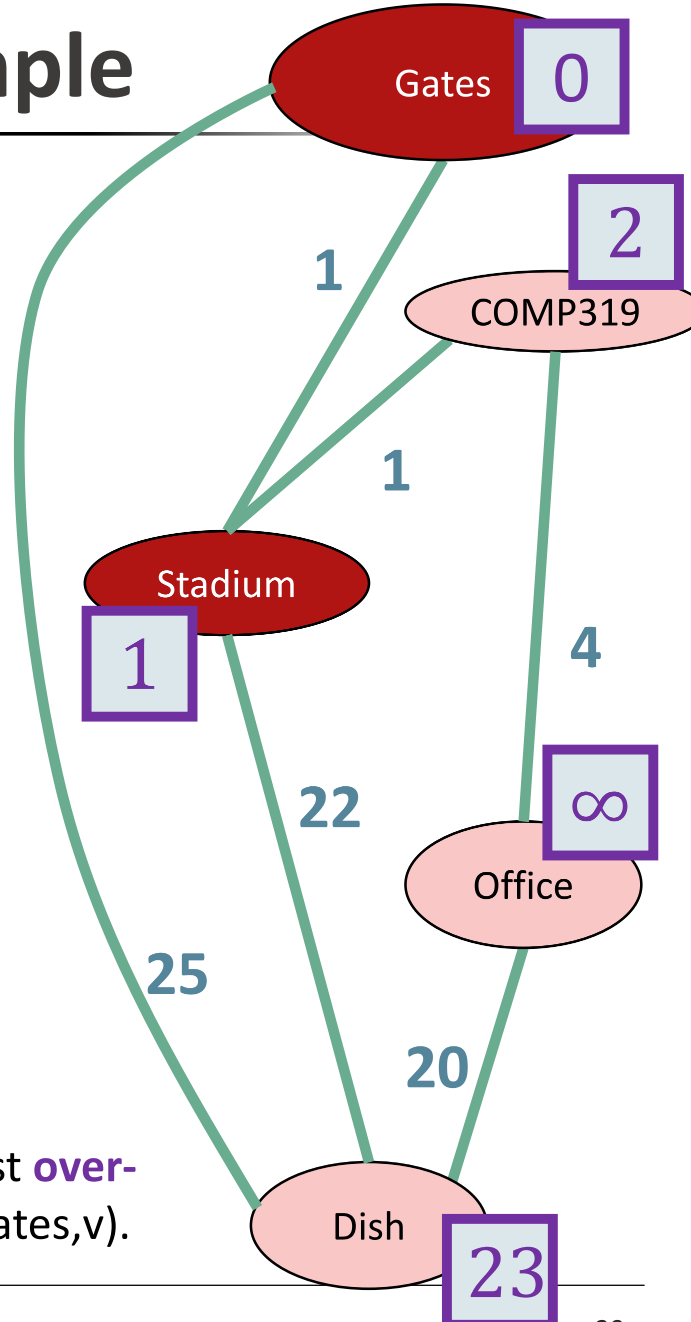

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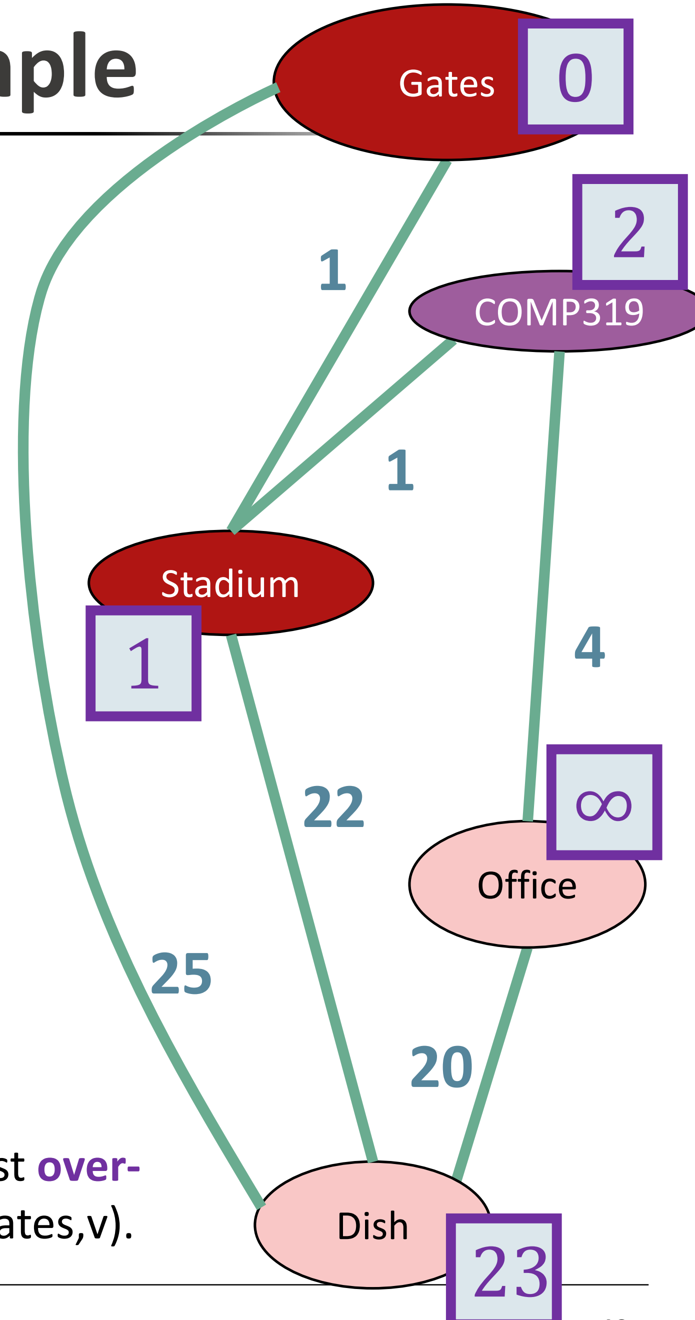

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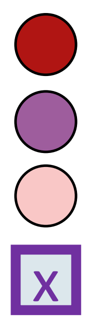

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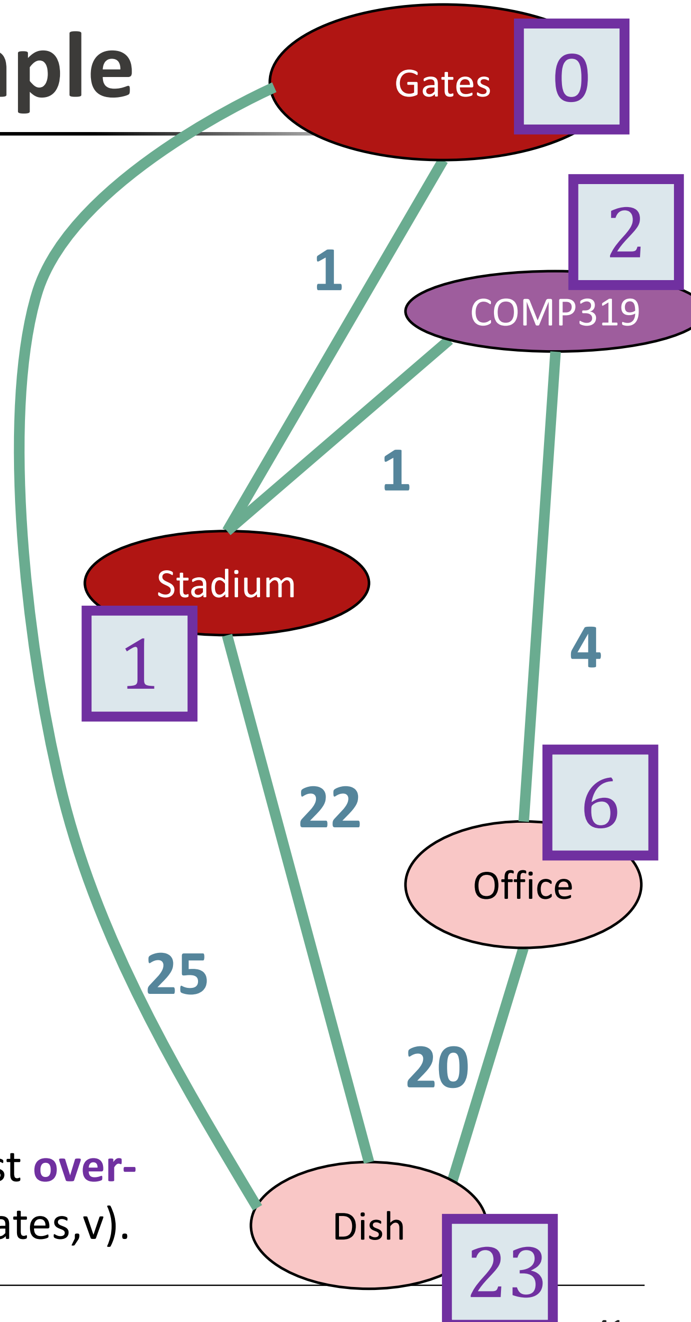

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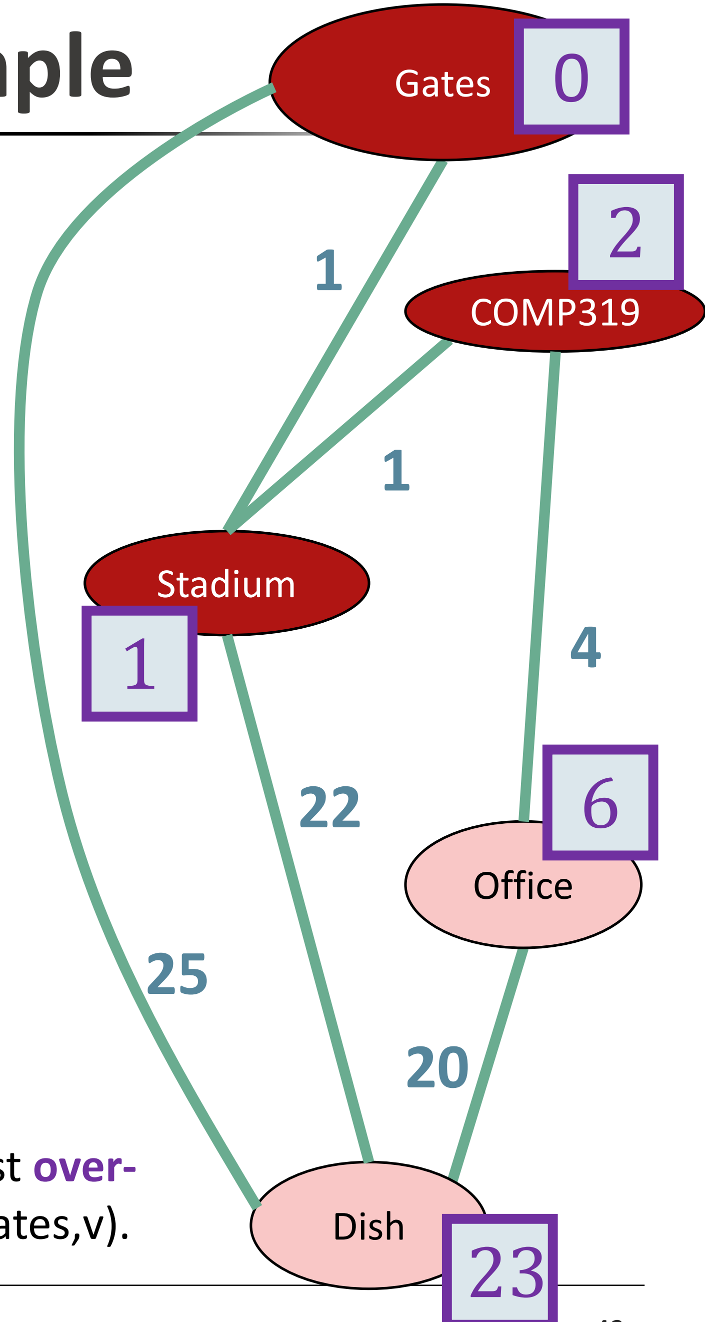

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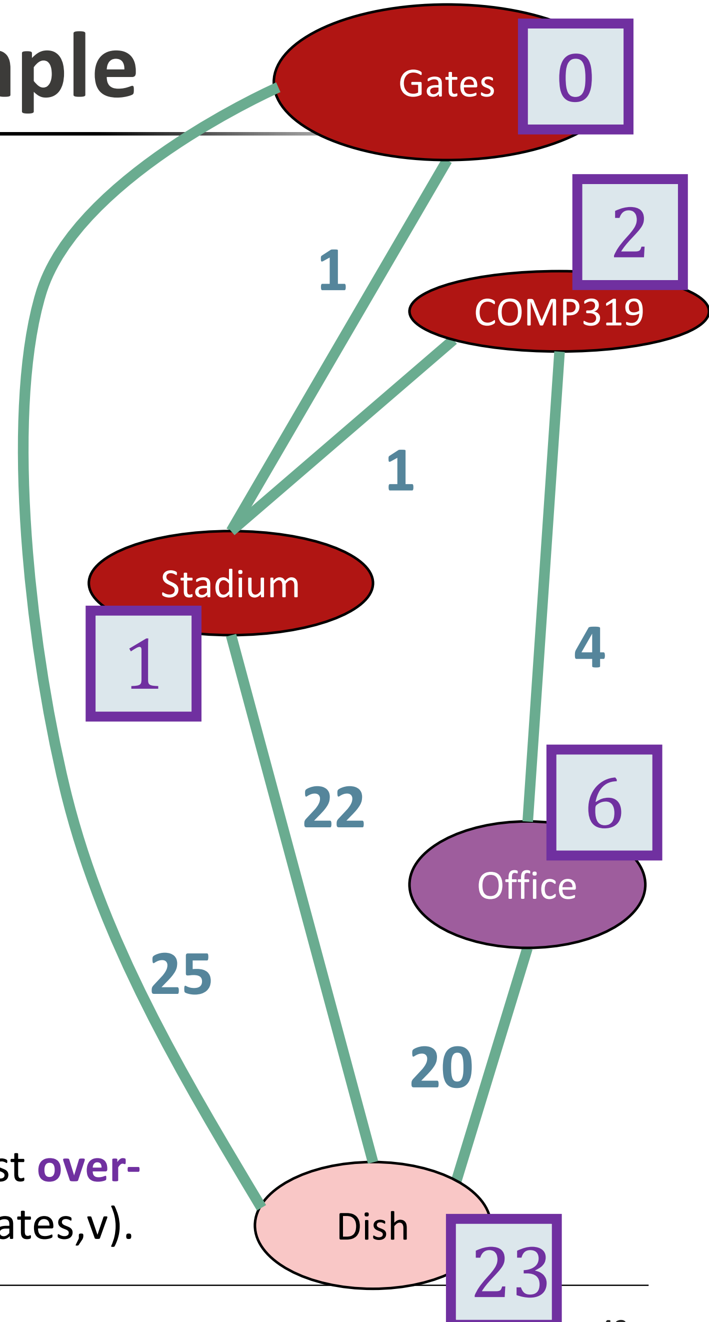

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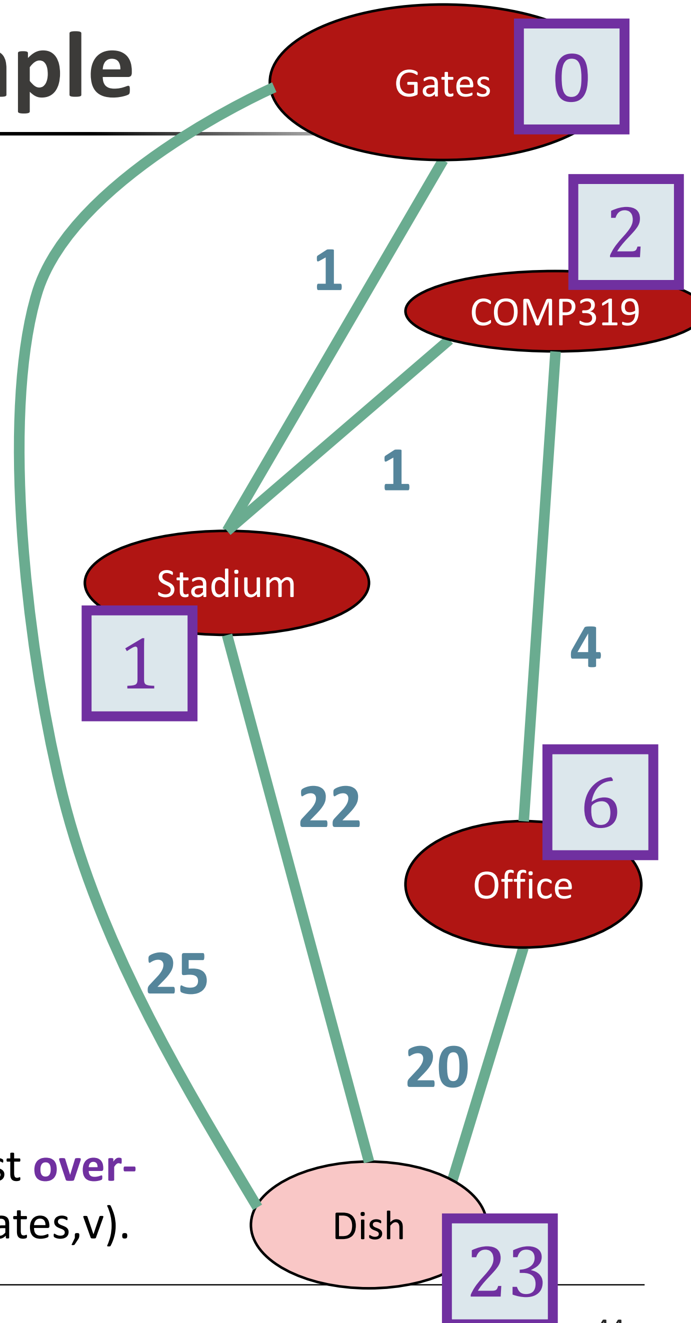

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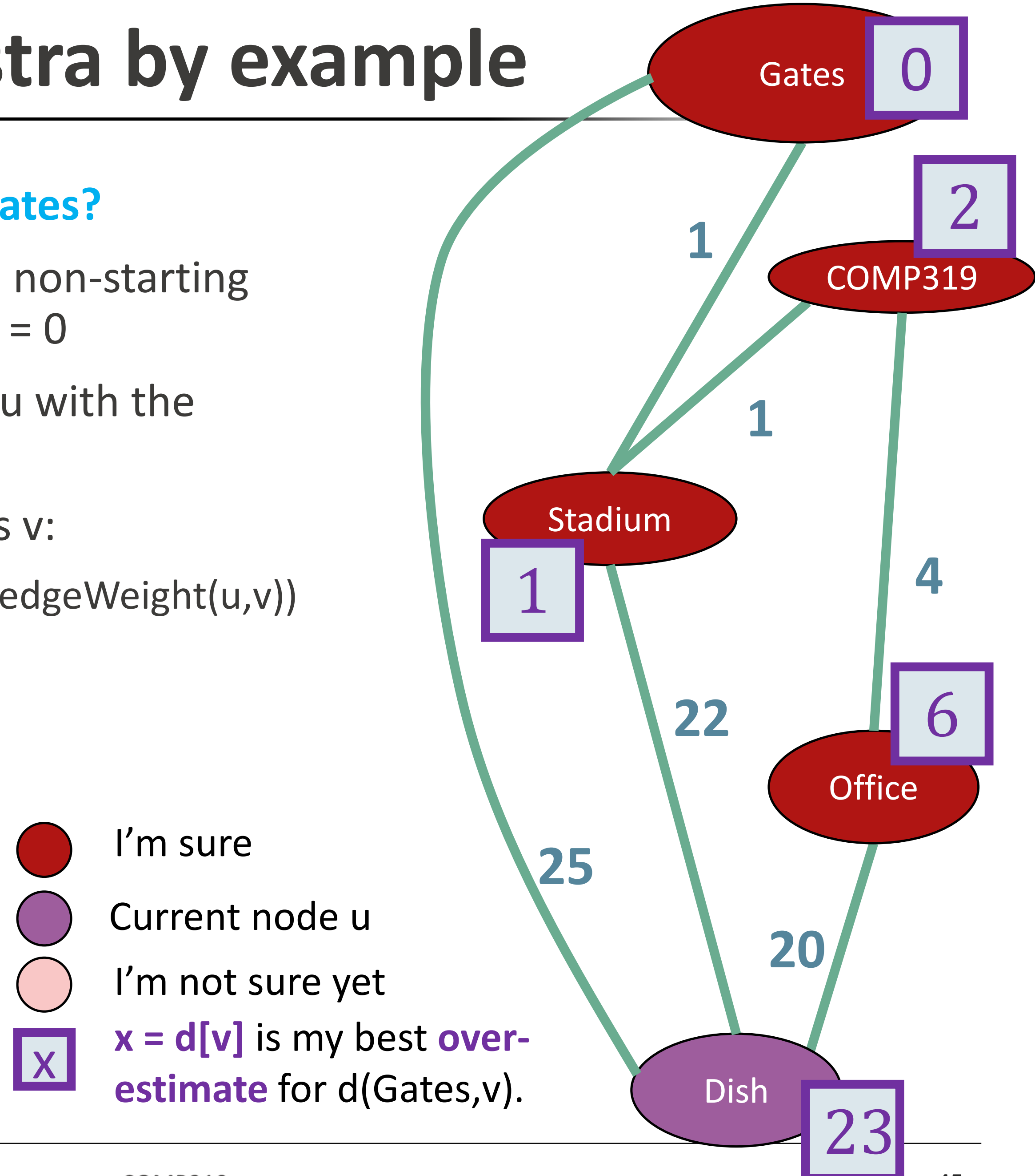

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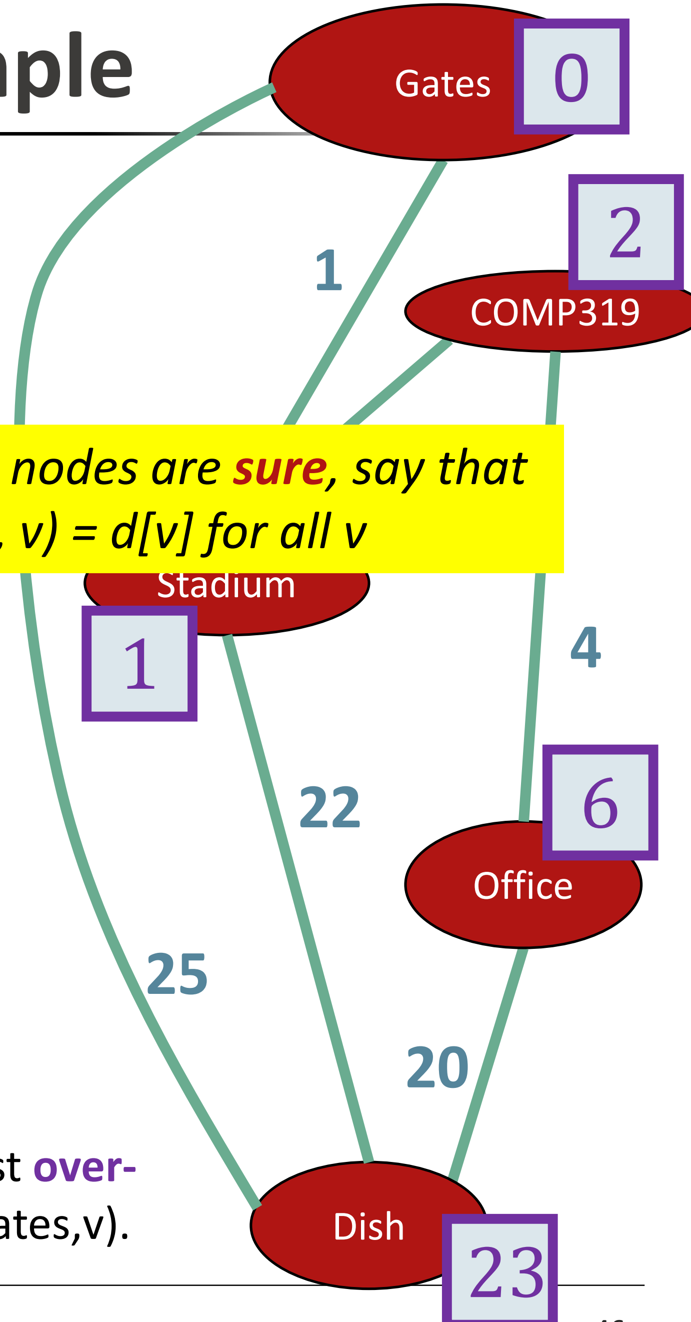
  - Repeat

*After all nodes are **sure**, say that d(Gates, v) = d[v] for all v*



I'm sure

Current node u

I'm not sure yet

x **x = d[v]** is my best **over-estimate** for d(Gates,v).

Dijkstra's algorithm

- Dijkstra(G,s):

> - Set all vertices to **not-sure**
> - d[v] = ∞ for all v in V
> - d[s] = 0
> - **While** there are **not-sure** nodes:
>   - Pick the **not-sure** node u with the smallest estimate **d[u]**
>   - **For** v in u.neighbors:
>   - d[v] ← min(d[v], d[u] + edgeWeight(u,v))
>   - Mark u as **sure**
> - Now d(s, v) = d[v]

- As usual, we want to check:
  - Does it work?
  - Is it fast?

Several useful properties

- **Corollary 1**. If there is no path from s to v, then we have:

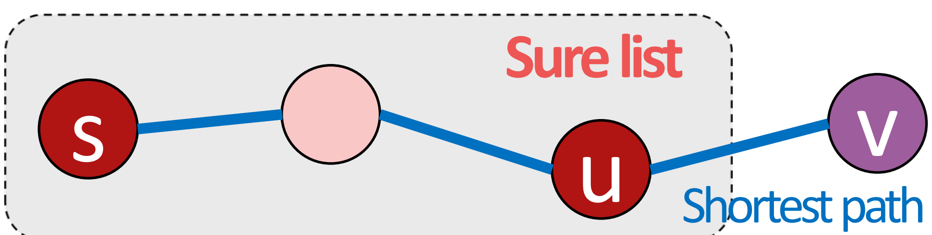
$$d[v] = d(s, v) = \infty$$

- **Lemma 1.** We always have $d[v] \geq d(s, v)$ for all v.

$$d[v] \leftarrow \min(\ \underbrace{d[v]}_{\text{Whatever path we had in mind before}},\ \underbrace{d[u] + \text{edgeWeight}(u,v)}_{\text{The shortest path to u, and then the edge from u to v.}}\ )$$

d[v] = length of the path we have in mind ≥ length of shortest path = d(s,v)

- **Lemma 2.** If s -> u -> v is a shortest path in G for some u, v in V, and if d[u] = d(s, u) at any time prior to update w(u, v),



Then,
$d[v] = d(s, u) + w(u, v) = d(s, v)$
at all time afterward.

Why does this work?

- **Theorem:**
  - Suppose we run Dijkstra on G = (V,E), starting from s.
  - At the end of the algorithm, the estimate d[v] is equal to the shortest-path weight d(s, v) for all v.

- **Proof.** We use the following loop invariant:

> At the start of each iteration of the **while** loop, d[v] = d(s, v) for all v in **the sure list.**

  - **Initialization:** Initially, there is no v in **the sure list**, so the invariant is trivially true.

Proof cont'd – Maintenance

- Suppose that we are about to add u to **the sure list**.
- That is, we picked u in the first line here:

Recall:

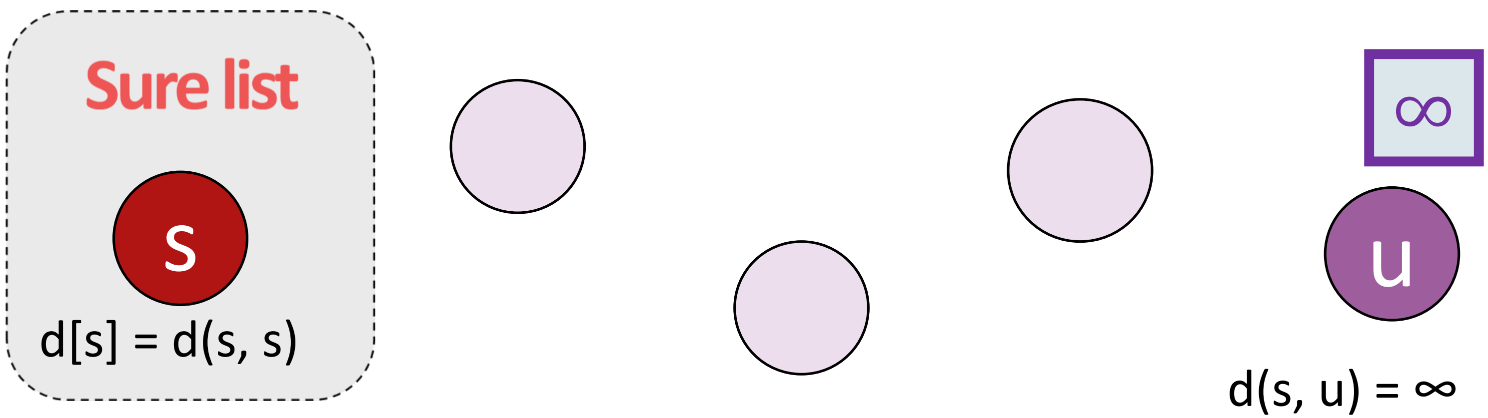
- Pick the **not-sure** node u with the smallest estimate **d[u]**
- Update all u's neighbors v:
  - d[v] ← min(d[v] , d[u] + edgeWeight(u,v))
- Mark u as **sure**
- Repeat

- Also suppose u is the first vertex that marked sure with d[u] != d(s, u)
  - (This is the way of contradiction.)

Proof cont'd – Maintenance

- s is the first vertex that is marked as sure,
  - At this time, d[s] = d(s, s) = 0. Thus, s ≠ u.

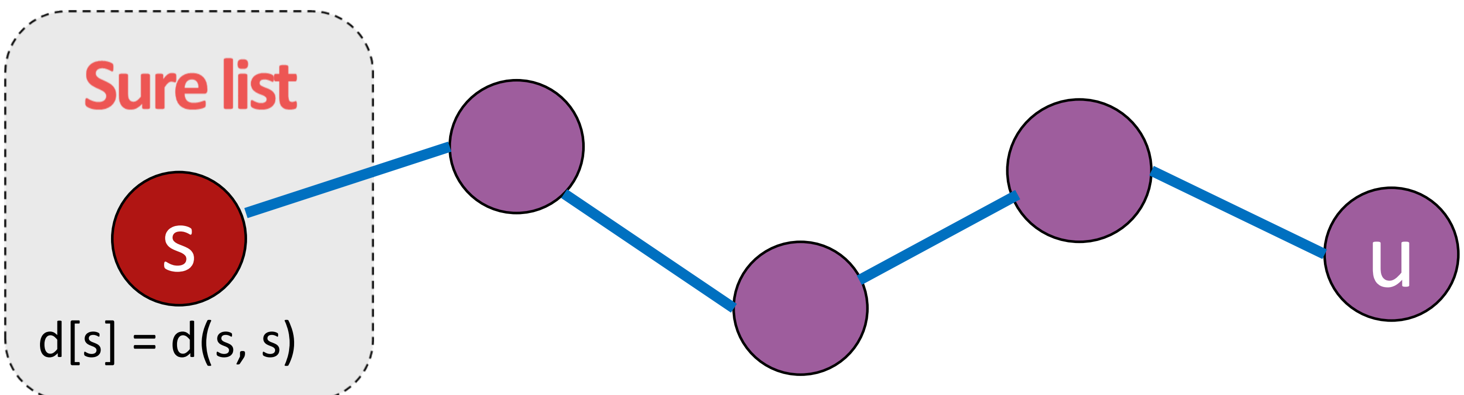
- If there is no path,
  - then, d[u] = d(s, u) = ∞, (by corollary 1) <-- violate the assumption!



Proof cont'd – Maintenance

- s is the first vertex that is marked as sure,
  - At this time, d[s] = d(s, s) = 0. Thus, s ≠ u.

- If there is no path,
  - then, d[u] = d(s, u) = ∞, (by corollary 1) <-- violate the assumption!
  - Therefore, there must exist one path and the shortest path too.

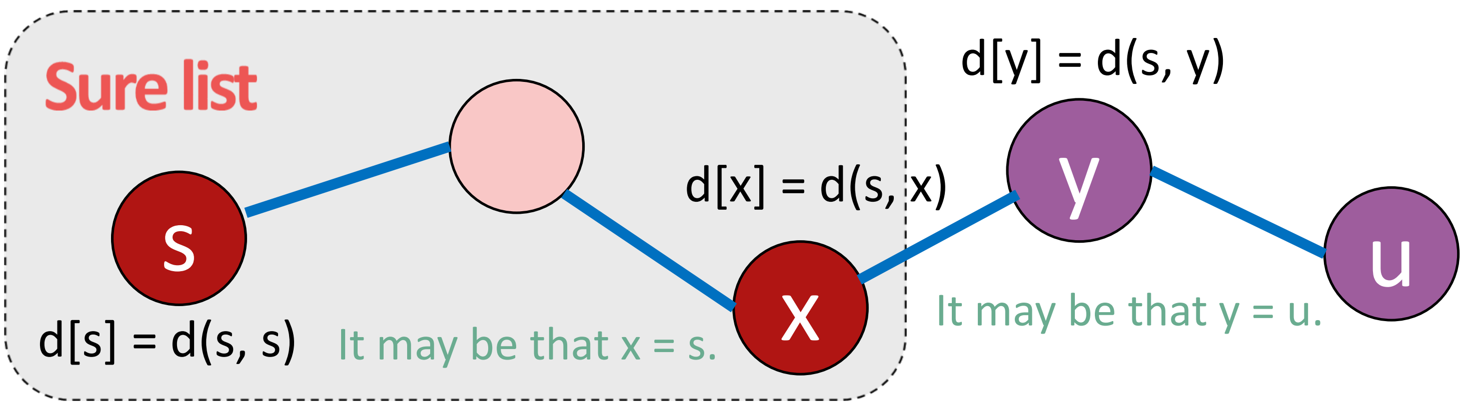


Consider this the shortest path *p*.

Proof cont'd – Maintenance

- let x, and y:
  - y be the first vertex along *p* such that y is in **the not-sure list**.
  - x be y's predecessor along *p*.
- Then,
  - d[x] = d(s, x). (By the hypothesis)
  - Also, d[y] = d(s, y). (By the Lemma 2)

u is the first vertex that marked sure with d[u] ≠ d(s, u))



Consider this the shortest path *p*.

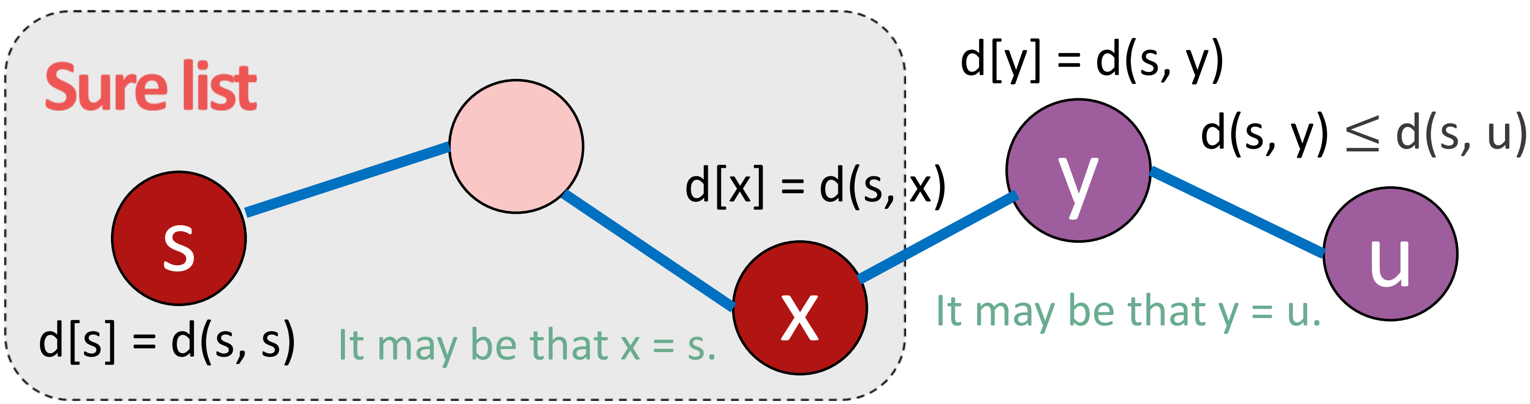
Proof cont'd – Maintenance

- Then, d[y] = d(s, y) ≤ d(s, u) ≤ d[u]

  (by Lemma 1)

  - But because both y, and u were in **the not-sure list**, when u was chosen, we have d[u] ≤ d[y].
  - Consequently, everything is equal. Thus d[u] = d(s, u). **CONTRADICTION!!**

- We conclude that d[u] = d(s, u) when u is added to **the sure list.**



Consider this the shortest path *p*.

Proof cont'd – Termination

- **Termination:**
  - At termination, **the not-sure list** = Ø which, along with our earlier invariant implies that **the sure list** is equal to V.
  - Thus, d[u] = d(s, u) for all u in V.

As usual

- Does it work?
  - Yes.

- **Is it fast?**

Running time?

- Dijkstra(G,s):

  - Set all vertices to **not-sure**
  - d[v] = ∞ for all v in V
  - d[s] = 0
  - **While** there are **not-sure** nodes:
    - Pick the **not-sure** node u with the smallest estimate **d[u]**
    - **For** v in u.neighbors:
      - d[v] ← min(d[v], d[u] + edgeWeight(u,v))
    - Mark u as **sure**
  - Now d(s, v) = d[v]

  - Operate n iterations (one per vertex)

- **How long does one iteration take?**
  - Depends on how you implement it..

We need a data structure that:

- find u with minimum d[u]
  - `findMin()`
- update (decrease) d[v]
  - `updateKey(v,d)`
- remove that u
  - `removeMin(u)`

Just the inner loop:

- Pick the **not-sure** node u with the smallest estimate **d[u]**
- Update all u's neighbors v:
  - d[v] ← min(d[v] , d[u] + edgeWeight(u,v))
- Mark u as **sure**

Total running time is big-oh of:

$$\sum_{u \in V} \left(T(\text{findMin}) + \left(\sum_{v \in u.neighbors} T(\text{updateKey}) \right) + T(\text{removeMin}) \right)$$

= n (T(`findMin`) + m T(`updateKey`) + T(`removeMin`))

If we use an array

- T(`findMin`) = O(n)
- T(`removeMin`) = O(n)
- T(`updateKey`) = O(1)

- Running time of Dijkstra
  = O(n(T(`findMin`) + T(`removeMin`)) + m T(`updateKey`))
  = $O(n^2) + O(m)$
  = $O(n^2)$

If we use a red-black tree

- T(`findMin`) = O(log(n))
- T(`removeMin`) = O(log(n))
- T(`updateKey`) = O(log(n))

- Running time of Dijkstra

  = O(n(T(`findMin`) + T(`removeMin`)) + m T(`updateKey`))

  = O(nlog(n)) + O(mlog(n))

  = O((n + m)log(n))

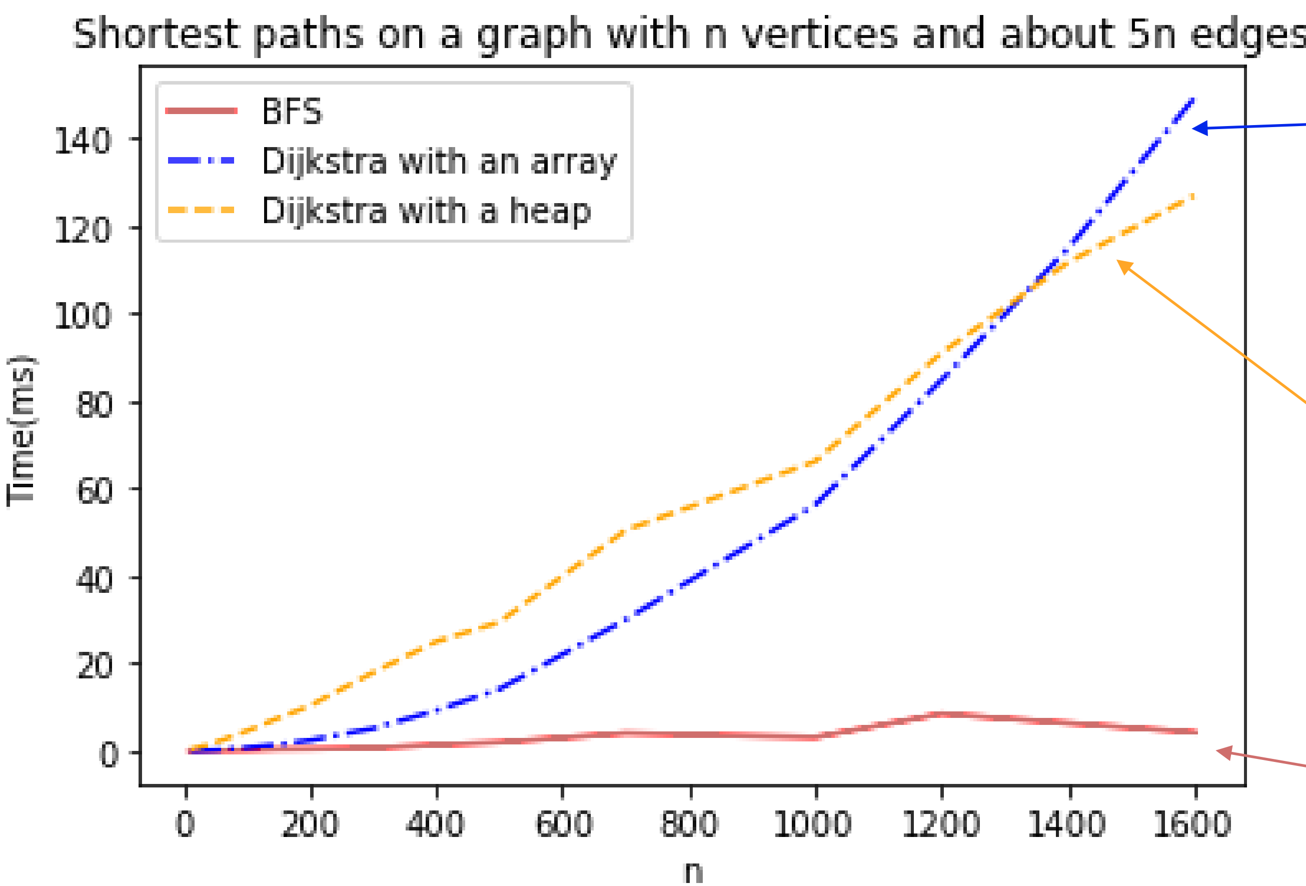
Better than an array if the graph is sparse!
aka if m is much smaller than n^2

Is a hash table a good idea here?

- **Not really**:
  - `Search(v)` is fast (in expectation)
  - But `findMin()` will still take time O(n) without more structure.

$$O(n(\,T(\texttt{findMin}) + T(\texttt{removeMin})\,) + m\,T(\texttt{updateKey}))$$

In practice



Dijkstra using a Python list to keep track of vertices has quadratic runtime.

Dijkstra using a heap looks a bit more linear

BFS is really fast by comparison! But it doesn't work on weighted graphs.

Dijkstra Drawbacks

- Needs non-negative edge weights.
- If the weights change, we need to re-run the whole thing.
  - In OSPF, a vertex broadcasts any changes to the network, and then every vertex re-runs Dijkstra's algorithm from scratch.

What have we learned?

- Dijkstra's algorithm finds shortest paths in weighted graphs with non-negative edge weights.

- Along the way, it constructs a nice tree.
  - We could post this tree in Gates!
  - Then people would know how to get places quickly.


Any Question?