Session 12

- The Turing Machine

The Turing Machine

Problems That Computer Cannot Solve

Example C Programs that print “hello, world”.

```
main()
{
    printf("hello, world\n");
}
```

This program prints `hello, world` and terminates. There are other programs that also print `hello, world`; yet the fact that they do so is far from obvious.

```
main()
{
    int n, total, x, y, z;
    scanf("%d", &n);
    tatal=3;
    while (1){
        for (x=1; x<=total-2; x++)
            for (y=1; y<=total-x-1; y++)
            {
                z=total-x-y;
                if (power(x,n)+power(y,n)==power(z,n))
                      printf("hello, world\n");
            }
        total++;}
}
```

This program searches every triple of positive integers (x, y, z) in some order, and tests to see if $x^n + y^n = z^n$. If so, the program prints `hello, world`, and if not, it prints nothing.

Fermat's Last Theorem

If $n > 2$, there are no integer solutions to the equation $x^n + y^n = z^n$.

Fermat's last theorem was made by Fermat 300 years ago, but no proof was found until quite recently.

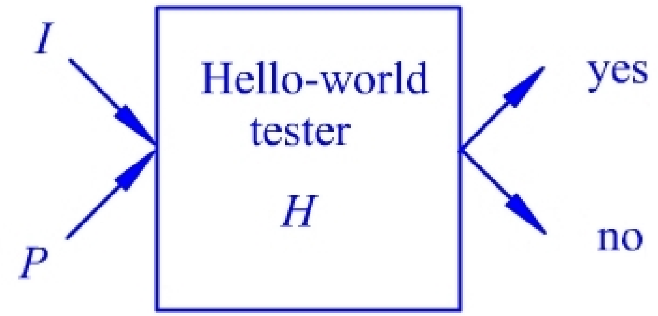
Hello-world Problem

Determine whether a given C program, with a given input, prints `hello, world` *(as the first 12 characters that it prints).*

It seems likely that, if it takes mathematicians 300 years to resolve a question about a single program, then the general problem must be hard indeed.

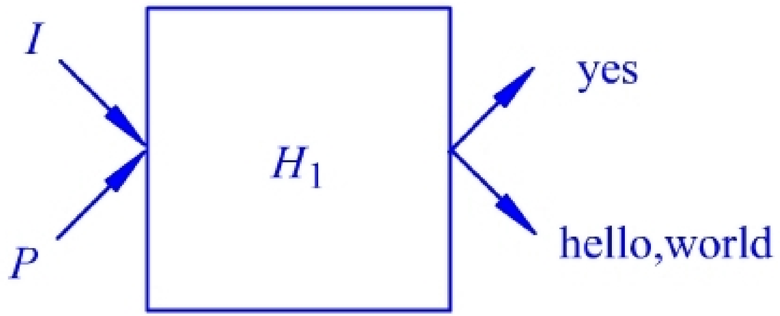
We shall prove that no program or algorithm exists to resolve "Hello-world Problem". That means, computer cannot solve this problem.

Assume there is a program H that takes as input a program P and an input I, and tells whether P with input I prints `hello, world`.



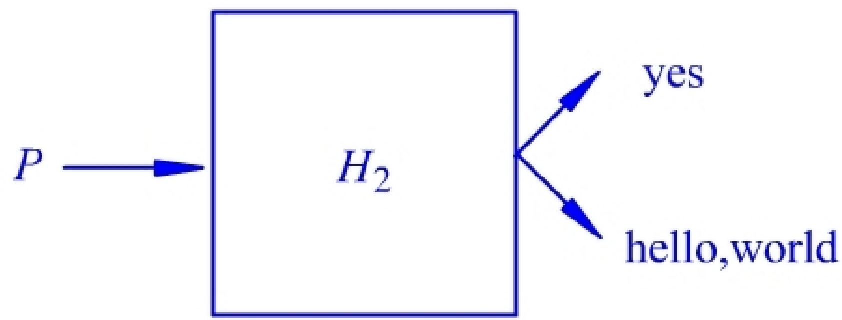
We will prove that H doesn't exist by contradiction.

First, we make a slightly modification to H. Change the output `no` of H to `hello, world`. The new program is called H_1.



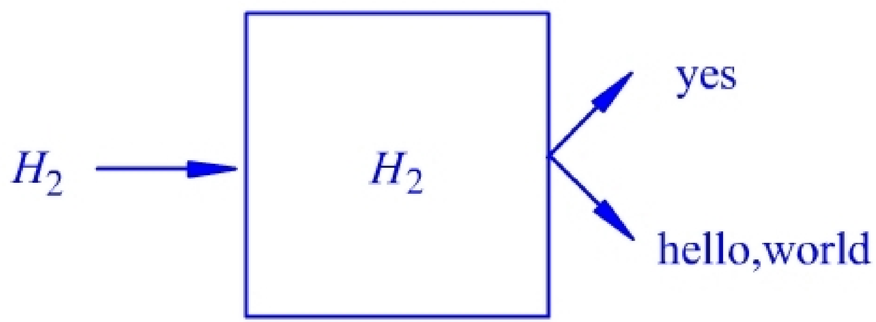
H_1 behaves like H except it prints `hello, world` exactly when H print `no`.

The next modification we perform on H_1 to produce the program H_2, whose input is the program P with its own code as its input.



H_2 behaves like H_1, but uses its input P as both P and I.

Now we prove that H_2 cannot exist. Thus, H_1 does not exist, and likewise, H does not exist. The heart of the argument is: *what H_2 does when given itself as input*.

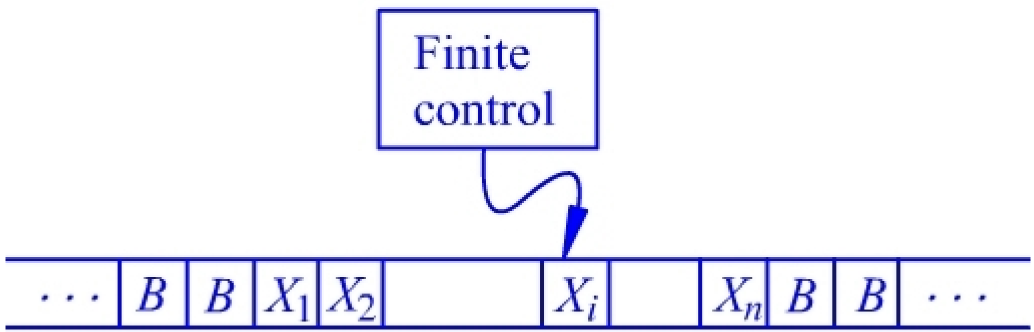


The situation is paradoxical, and we conclude that H_2 cannot exist.

A problem that cannot be solved by computer is called *undecidable*.

Notation for the Turing Machine

We need tools that will allow us to prove problem undecidable or intractable. The theory of undecidability / intractability are both based on a very simple model of a computer, called the Turing machine.



The Turing machine consists of a **finite control**, a **tape** divided into cells, and a **tape head** positioned at one of the tape cells.

- Initially, the *input* (finite-length string of symbols) is placed on the tape.
- All other tape cells, extending infinitely to the left and right, initially hold *blank* symbol.
- In one *move*, the Turing machine change state, write a tape symbol in the cell scanned, and move the tape head left or right.

A Turing machine (TM) is a 7-tuple $(Q, \Sigma, \Gamma, \delta, q_0, B, F)$.

- Q is a finite set of *states* of the finite control.
- Σ is a finite set of *input symbols*.
- Γ is a complete set of *tape symbols*, $\Sigma \subset \Gamma$.
- δ is a *transition function* from $Q \times \Gamma$ to $Q \times \Gamma \times \{L, R\}$.
- q_0 is a *start state*, a member of Q.
- B is *blank* symbol bing in Γ but not in Σ.
- F is a set of *final* or *accepting* states, a subset of Q.

☞ The value of $\delta(q, X)$, *if it is defined*, is a triple (p, Y, D), where:

1. p is the next state, in Q.
2. Y is the symbol, in Γ, written in the cell being scanned, replacing whatever symbol was there.
3. D is a direction, either L or R, standing for “left” or “right”, respectively, and telling us the direction in which the head moves.

The transition function δ gives the principle by which a Turing machine operates, and we often call it the “program” of the machine.

☞ In general, δ is a partial function on $Q \times \Gamma$. $\delta(q, X)$ may be undefined for some $q \in Q$ and $X \in \Gamma$.

A Turing machine is said to *halt* whenever it reaches a configuration for which δ is not defined; this is possible because δ is a partial function.

In general, without otherwise stating so:

- We assume that a TM always halts when it is in an accepting state.

Example Let $M = (\{q_0, q_1, q_2\}, \{0, 1\}, \{0, 1, B\}, \delta, q_0, B, \{q_2\})$, where δ is defined by

$$\delta(q_0, 0) = (q_0, 0, R),\ \ \delta(q_0, 1) = (q_1, 1, R),\ \ \delta(q_1, 0) = (q_1, 0, R),\ \ \delta(q_1, B) = (q_2, B, R).$$

Turing machine M accepts the (0,1)-strings including one and only one 1.

The transition function can also be given by a table

δ	0	1	B
q_0	$(q_0, 0, R)$	$(q_1, 1, R)$	–
q_1	$(q_1, 0, R)$	–	(q_2, B, R)
q_2	–	–	–

Example Let $M = (\{q_0, q_1\}, \{0, 1\}, \{0, 1, B\}, \delta, q_0, B, \{\emptyset\})$, where δ is defined by

$$\begin{array}{lll} \delta(q_0, 0) = (q_1, 0, R), & \delta(q_0, 1) = (q_1, 1, R), & \delta(q_0, B) = (q_1, B, R), \\ \delta(q_1, 0) = (q_0, 0, L), & \delta(q_1, 1) = (q_0, 1, L), & \delta(q_1, B) = (q_0, B, L). \end{array}$$

Suppose that the tape initially contains $01\cdots$, what will happen here?

It is clear from this that the machine will run forever!

This is an instance of a Turing machine that does not halt. As an analogy with programming terminology, we say that the Turing machine is in an *infinite loop*.

Since one can make several different definitions of a Turing machine, it is worthwhile to summarize the main features of our model, which we will call a standard Turing machine:

1. The Turing machine has a tape that is unbounded in both directions.
2. The Turing machine is deterministic in the sense that δ defines at most one move for each configuration.
3. There is no special output device. Whenever the machine halts, some or all of the contents of the tape may be viewed as output.

Instantaneous Descriptions for the Turing Machine

We use the string

$$X_1X_2\cdots X_{i-1}qX_iX_{i+1}\cdots X_n$$

to represent an instantaneous description (ID) in which

- q is the state of the Turing machine.
- The tape head is scanning the ith symbol from the left.
- $X_1X_2\cdots X_n$ is the portion of the tape between the leftmost to the rightmost non-blank.

Now we describe moves of a TM by the $\vdash$ notation.

Suppose $\delta(q, X_i) = (p, Y, L)$, then

$$X_1X_2\cdots X_{i-1}qX_iX_{i+1}\cdots X_n \underset{M}{\vdash} X_1X_2\cdots X_{i-2}pX_{i-1}YX_{i+1}\cdots X_n$$

There are two important exceptions:

1. If $i = 1$, then $\quad qX_1X_2\cdots X_n \underset{M}{\vdash} pBYX_2\cdots X_n$.
2. If $i = n$ and $Y = B$, then $\quad X_1X_2\cdots X_{n-1}qX_n \underset{M}{\vdash} X_1X_2\cdots X_{n-2}pX_{n-1}$.

Now suppose $\delta(q, X_i) = (p, Y, R)$, then

$$X_1X_2 \cdots X_{i-1}qX_iX_{i+1} \cdots X_n \underset{M}{\vdash} X_1X_2 \cdots X_{i-1}YpX_{i+1} \cdots X_n$$

Again, there are two important exceptions:

1. If $i = n$, then $X_1X_2 \cdots X_{n-1}qX_n \underset{M}{\vdash} X_1X_2 \cdots X_{n-1}YpB$.
2. If $i = 1$ and $Y = B$, then $qX_1X_2 \cdots X_n \underset{M}{\vdash} pX_2 \cdots X_n$.

As usual, $\overset{*}{\underset{M}{\vdash}}$, or just $\overset{*}{\vdash}$ when the TM M is understood, will be used to indicate zero, one, or more moves of the TM M.

The TM M is said to halt starting from some initial configuration $\alpha_1 q \alpha_2$ if

$$\alpha_1 q \alpha_2 \overset{*}{\vdash} \beta_1 p X \beta_2$$

for any p and X, for which $\delta(p, X)$ is undefined.

The sequence of configuration leading to a halt state will called a *computation*.

If a Turing machine never halts, we will represent it by

$$\alpha_1 q \alpha_2 \overset{*}{\vdash} \infty.$$

Example Consider $M = (\{q_0, q_1, q_2\}, \{0, 1\}, \{0, 1, B\}, \delta, q_0, B, \{q_2\})$, where δ is defined by

δ	0	1	B
q_0	$(q_0, 0, R)$	$(q_1, 1, R)$	–
q_1	$(q_1, 0, R)$	–	(q_2, B, R)
q_2	–	–	–

Let's see the moves of M on input 00100

$$q_0 00100 \vdash 0q_0 0100 \vdash 00q_0 100 \vdash 001q_1 00 \vdash 0010q_1 0 \vdash 00100q_1 B \vdash 00100Bq_2 B$$

We find that this is an accepting computation.

Here is the case of non-accepting computations by M.

- The ID sequence of moves of M on 00000

$$q_0 00000 \vdash 0q_0 0000 \vdash 00q_0 000 \vdash 000q_0 00 \vdash 0000q_0 0 \vdash 00000q_0 B \quad \text{dies}$$

- The ID sequence of moves of M on 00101

$$q_0 00101 \vdash 0q_0 0101 \vdash 00q_0 101 \vdash 001q_1 01 \vdash 0010q_1 1 \quad \text{dies}$$

The Language of a Turing Machine

The Turing machine is started in the initial state q_0 with the head positioned on the leftmost symbol of input string w. If, after a sequence of moves, the machine enters a final state and halts, the w is considered to be accepted.

Let $M = (Q, \Sigma, \Gamma, \delta, q_0, B, F)$ be a Turing machine. The language of M is defined by

$$L(M) = \{w \mid w \in \Sigma^*, \;\; q_0 w \overset{*}{\vdash} \alpha p \beta \;\; \text{for some } p \in F \text{ and } \alpha, \beta \in \Gamma^*\}$$

$L(M)$ is often called the recursively enumerable languages or RE languages.

☞ Note that there is a important difference between TM and FA/PDA. The TM can decide whether it accepts the input before scanned all symbols in the input string!

☞ The above definition tells what must happen when $w \in L(M)$. It says nothing about the outcome for any other input.

When w is not in $L(M)$, one of two things can happen: the machine can halt in a nonfinal state or it can enter an infinite loop and never halt. Any sting for which M does not halt by definition not in $L(M)$.

Example For $\Sigma = \{0, 1\}$, design a Turing machine that accepts the language denoted by the regular expression $(\mathbf{01})^*\mathbf{0}$.

This is an easy exercise in Turing machine programming.

$$M = (\{q_0, q_1, q_2\}, \{0, 1\}, \{0, 1, B\}, \delta, q_0, B, \{q_2\})$$

where δ is given by

δ	0	1	B
q_0	$(q_1, 0, R)$	–	–
q_1	–	$(q_0, 1, R)$	(q_2, B, R)
q_2	–	–	–

Example Let $M = (\{q_0, q_1, q_2, q_3\}, \{0, 1\}, \{0, 1, B\}, \delta, q_0, B, \{q_3\})$ where δ is given by

δ	0	1	B
q_0	$(q_0, 0, R)$	$(q_1, 1, R)$	–
q_1	$(q_1, 0, R)$	$(q_2, 1, R)$	–
q_2	$(q_2, 0, R)$	$(q_3, 1, R)$	–
q_3	–	–	–

Analyzing the moves of M, we can see

$$L(M) = \{w \mid w \in \{0, 1\}^*,\ w \text{ including at least three } 1's\}$$

For instance, $q_0 100110010010 \overset{*}{\vdash} 10011 q_3 0010010$

The recognition of more complicated languages is more difficult.

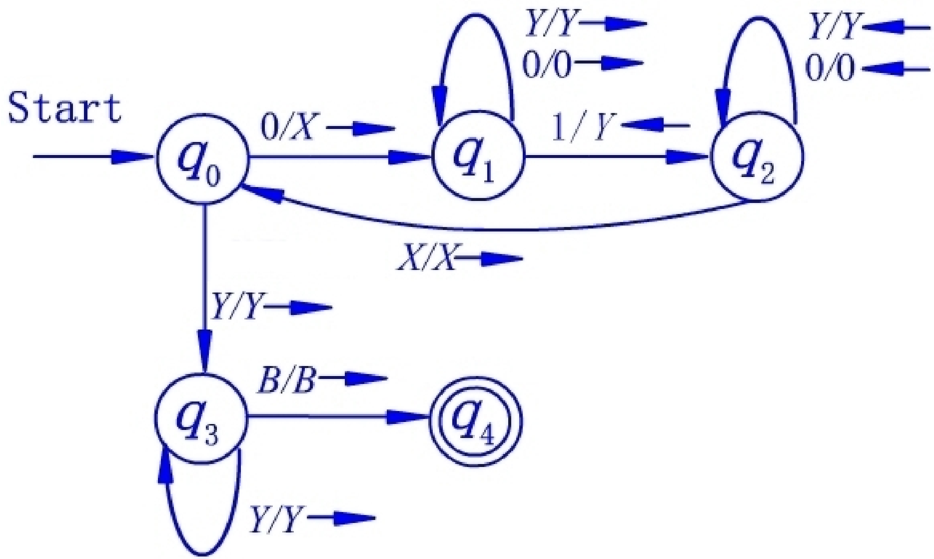
Example Design a Turing machine M accepting the language $\{0^n1^n \mid n \geq 1\}$.

$$M = (\{q_0, q_1, q_2, q_3, q_4\}, \{0, 1\}, \{0, 1, X, Y, B\}, \delta, q_0, B, \{q_4\})$$

where δ is given by

δ	0	1	X	Y	B
q_0	(q_1, X, R)	–	–	(q_3, Y, R)	–
q_1	$(q_1, 0, R)$	(q_2, Y, L)	–	(q_1, Y, R)	–
q_2	$(q_2, 0, L)$	–	(q_0, X, R)	(q_2, Y, L)	–
q_3	–	–	–	(q_3, Y, R)	(q_4, B, R)
q_4	–	–	–	–	–

We can represent the transition of this TM pictorially, much as we did for the PDA.



Here is an example of an accepting computation by M. Its input is 0011.

$q_0 0011 \vdash Xq_1 011 \vdash X0q_1 11 \vdash Xq_2 0Y1 \vdash q_2 X0Y1 \vdash Xq_0 0Y1 \vdash XXq_1 Y1 \vdash XXYq_1 1 \vdash XXq_2 YY \vdash Xq_2 XYY \vdash XXq_0 YY \vdash XXYq_3 Y \vdash XXYYq_3 B \vdash XXYYBq_4 B$

For another example, consider what M does on the input 0010.

$q_0 0010 \vdash Xq_1 010 \vdash X0q_1 10 \vdash Xq_2 0Y0 \vdash q_2 X0Y0 \vdash Xq_0 0Y0 \vdash XXq_1 Y0 \vdash XXYq_1 0 \vdash XXY0q_1 B$

M dies and does not accept its input.

Example Design a Turing machine M accepting the language $\{0^n 1^n 2^n \mid n \geq 1\}$.

The ideas used of the previous example are easily carried over to this case. Although conceptually a simple extension, writing the actual program is tedious. We leave it as a somewhat lengthy, but straightforward exercise.

One conclusion we can draw from this example is that a Turing machine can recognize some languages that are not context-free, a first indication that *Turing machines are more powerful than pushdown automata*.


Break for 15 minutes

Turing Machine as Transducers

Turing machines are not only interesting as recognizers of languages, they provide us with a simple abstract model for digital computers in general. Since the primary purpose of a computer is to transform input into output, it acts as a transducer.

We can view a Turing machine a transducer as an implementation of a function f defined by $\hat{w} = f(w)$, provided that $q_0 w \underset{M}{\vdash} \hat{w} p B$, for some final state p.

A function f with domain D is said to be Turing-computable or just computable if there exists some Turing machine $M = (Q, \Sigma, \Gamma, \delta, q_0, B, F)$ such that

$$q_0 w \underset{M}{\vdash} f(w)pB, \qquad p \in F,$$

for all $w \in D$.

Note that the position of the tape head after computation is not important. We also ask the machine $q_0 w \underset{M}{\vdash} pf(w)$.

As we will shortly claim, all the common mathematical functions, no matter how complicated, are Turing-computable.

Let's first consider the integer-valued functions. In Turing's scheme, integers are represented in unary. We use 0^n represent any nonnegative integer n.

For a integer-valued function $f(n_1, n_2, \cdots, n_k)$, we use string $0^{n_1}10^{n_2}1\cdots10^{n_k}$ represent value of its variables $n_1, n_2, \cdots, n_k$.

Turing machine might compute the function $f(n_1, n_2, \cdots, n_k)$. It will start with a tape consisting of $0^{n_1}10^{n_2}1\cdots10^{n_k}$ surrounded by blanks. If $f(n_1, n_2, \cdots, n_k) = m$, Turing machine halts with 0^m on its tape, surrounded by blanks.

Example Design a Turing machine M, compute $n+m$ for any nonnegative integers n and m.

The input of M is 0^n10^m, the output should be 0^{n+m} when M halts.

- M scans symbols in the input string. After finding 1, replaces 1 by 0, then searches right until meets blank. M then return left, replaces the last 0 by a blank.
- There is an exception. When $n = 0$, what M does is just change 1 to blank.

Now we give a formal description for desired Turing machine.

$$M = (\{q_0, q_1, q_2, q_3\}, \{0, 1\}, \{0, 1, B\}, \delta, q_0, B, \{q_3\})$$

where δ is given by

δ	0	1	B
q_0	$(q_1, 0, R)$	(q_3, B, R)	–
q_1	$(q_1, 0, R)$	$(q_1, 0, R)$	(q_2, B, L)
q_2	(q_3, B, R)	–	–
q_3	–	–	–

e.g. $q_0 000100 \vdash 0q_1 00100 \vdash 00q_1 0100 \vdash 000q_1 100 \vdash 0000q_1 00 \vdash 00000q_1 0 \vdash 000000q_1 B \vdash 00000q_2 0B \vdash 00000Bq_3 B$

Example Design a Turing machine M, compute $m \dot{-} n = \max(m - n, 0)$ for any nonnegative integers m and n.

$$M = (\{q_0, q_1, q_2, q_3, q_4, q_5, q_6\}, \{0, 1\}, \{0, 1, B\}, \delta, q_0, B, \{q_6\})$$

where δ is given by

δ	0	1	B
q_0	(q_1, B, R)	(q_5, B, R)	–
q_1	$(q_1, 0, R)$	$(q_2, 1, R)$	–
q_2	$(q_3, 1, L)$	$(q_2, 1, R)$	(q_4, B, L)
q_3	$(q_3, 0, L)$	$(q_3, 1, L)$	(q_0, B, R)
q_4	$(q_4, 0, L)$	(q_4, B, L)	$(q_6, 0, R)$
q_5	(q_5, B, R)	(q_5, B, R)	(q_6, B, R)
q_6	–	–	–

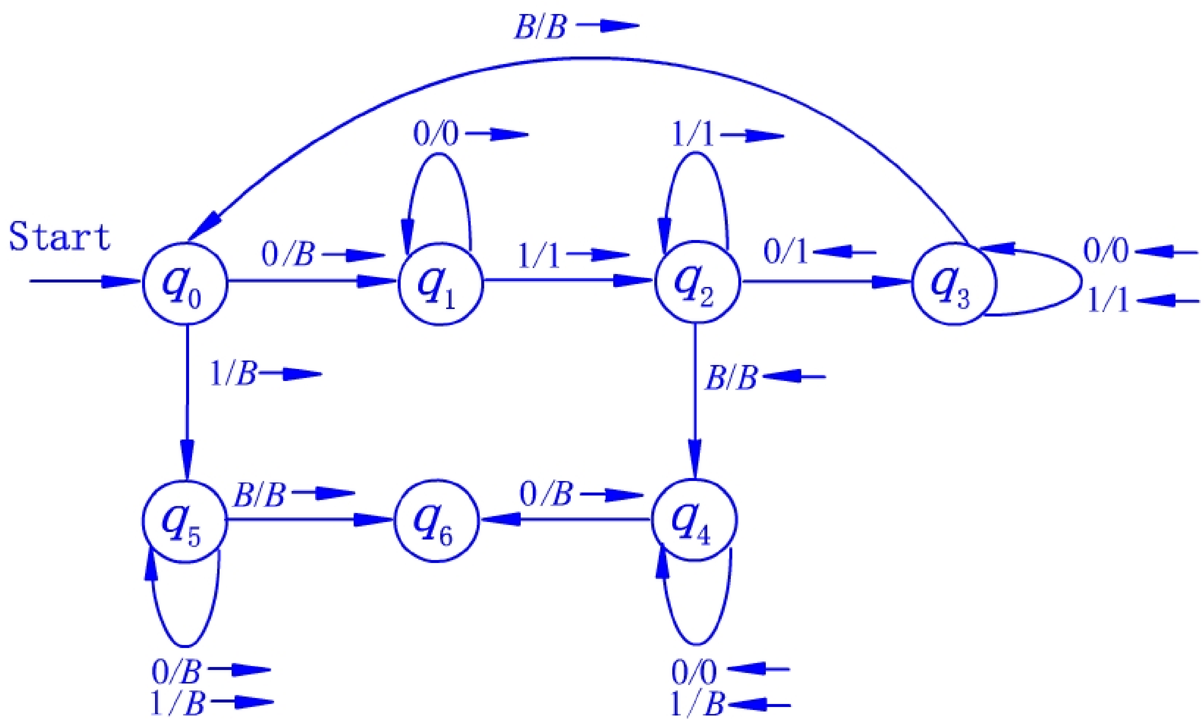
The Turing machine M will start with a tape consisting of 0^m10^n surrounded by blanks. M halts with $0^{m \dot{-} n}$ on its tape, surrounded by blanks.

M repeatedly finds its leftmost remaining 0 and replaces it by a blank. It then searches right, looking for a 1. After finding a 1, it continues right, until it comes to a 0, which it replaces by a 1.

M then returns left, seeking the leftmost 0, which it identifiers when it first meets a blank and then moves one cell to the right.

The repetition ends if either:

- Searching right for a 0, M encounters a blank. Then the n 0's in 0^m10^n have all been changed to 1's, and $n + 1$ of the m 0's have been changed to B. M replaces the $n + 1$ 1's by $n + 1$ B's, and moves to left, replaces first B by one 0, leaving $m - n$ 0's on the tape. Since $m \geq n$ in the case, $m - n = m \dot{-} n$.
- Beginning the cycle, M cannot find a 0 to change to a blank, because the first m 0's already have been changed to B. Then $m \leq n$, so $m \dot{-} n = 0$. M replaces all remaining 1's and 0's by B and ends with a completely blank tape.


B/B →
0/0 →
1/1 →
Start
0/B →
1/1 →
0/1 ←
0/0 ←
1/1 ←
q_0
q_1
q_2
q_3
1/B →
B/B ←
B/B →
0/B →
q_5
q_6
q_4
0/B →
1/B →
0/0 ←
1/B ←

Other basic operations, *e.g.* copying string can also be done on a Turing machine.

Example Design a Turing machine that copies string of 1's. More precisely, find a machine that performs the computation $q_0 w \overset{*}{\vdash} q_f ww$, for any $w \in \{1\}^+$.

To solve the problem, we implement the following process:

1. Replace every 1 by an X.
2. Find the rightmost X and replace it with 1.
3. Travel to the right end of the current nonblank region and create a 1 there.
4. Repeat Step 2 and 3 until there are no more X's.

A Turing machine version of this is

$$M = (\{q_0, q_1, q_2, q_3\}, \{0, 1\}, \{0, 1, X, B\}, \delta, q_0, B, \{q_3\})$$

where δ is given by

δ	0	1	X	B
q_0	–	(q_0, X, R)	–	(q_1, B, L)
q_1	–	$(q_1, 1, L)$	$(q_2, 1, R)$	(q_3, B, R)
q_2	–	$(q_2, 1, R)$	–	$(q_1, 1, L)$
q_3	–	–	–	–

e.g. $q_0 11 \vdash Xq_0 1 \vdash XXq_0 B \vdash Xq_1 X \vdash X1q_2 B \vdash Xq_1 11 \vdash q_1 X11 \vdash 1q_2 11 \vdash 11q_2 1 \vdash 111q_2 B \vdash 11q_1 11 \vdash 1q_1 111 \vdash q_1 1111 \vdash q_1 B1111 \vdash q_3 1111$

Thank you!