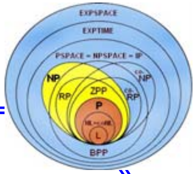
Session 9

- Equivalence of Pushdown Automata and Context-Free Grammars
- Deterministic Pushdown Automata

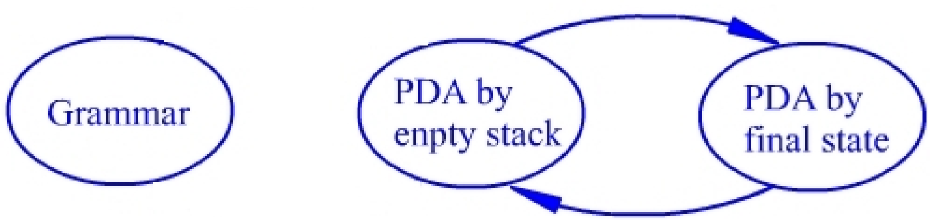


Equivalence of Pushdown Automata and Context-Free Grammars

We'll prove that a language is generated by a CFG

- if and only if it is accepted by a PDA by empty stack,
- if and only if it is accepted by a PDA by final state.

We already know how to go between empty stack and final state.



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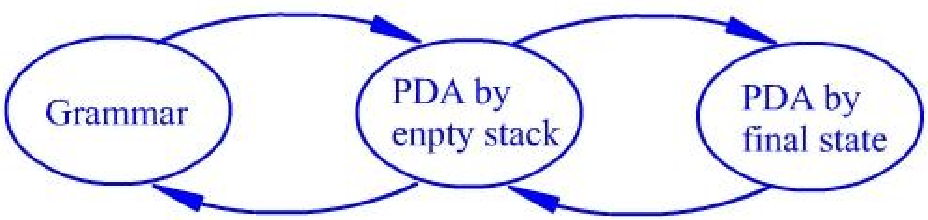


From CFG's to PDA's

Given a CFG G, we construct a PDA that simulates the leftmost derivations $\overset{*}{\underset{lm}{\Rightarrow}}$.

We write left-sentential forms as $xA\alpha$, where A is the leftmost variable in the form, x is whatever terminals, and α is the string of terminals and variables.

For instance,

$$\underbrace{(a+}_{x}\underbrace{\underbrace{E}_{A}\underbrace{)}_{\alpha}}_{\text{tail}}$$

Let $xA\alpha \underset{lm}{\Rightarrow} x\beta\alpha$. This corresponds to the PDA first having consumed x and having $A\alpha$ on the stack, and then on ϵ it pops A and pushes β.

That is, let y such that $w = xy$. Then the PDA goes non-deterministically from configuration $(q, y, A\alpha)$ to $(q, y, \beta\alpha)$.

Now $(q, y, \beta\alpha)$ may not be a representation of the next left-sentential form, because β may have a prefix of terminals.

How to go next step?

✷ Note that there is only one state for this PDA.

Remove the terminals appear at the beginning of $\beta\alpha$, and expose the next variable at the top of the stack!

These terminals are compared against the next input symbols, to make sure the guesses at the leftmost derivation of w are correct; if not, this branch dies.

Formally, let $G = (V, T, R, S)$ be a CFG. Define $P_G = (\{q\}, T, V \cup T, \delta, q, S)$, where

- $\delta(q, \epsilon, A) = \{(q, \beta) \mid A \to \beta \in R\}$ for $A \in V$,
- $\delta(q, a, a) = \{(q, \epsilon)\}$ for $a \in T$.

Theorem 6.5 *If PDA P_G constructed from CFG G by the construction above, then $N(P_G) = L(G)$.*

Proof (⊇-direction). Let $w \in L(G)$. Then

$$S = \gamma_1 \underset{lm}{\Rightarrow} \gamma_2 \underset{lm}{\Rightarrow} \cdots \underset{lm}{\Rightarrow} \gamma_n = w.$$

Let $\gamma_i = x_i\alpha_i$. We show by induction on i that if $S \overset{*}{\underset{lm}{\Rightarrow}} \gamma_i$, then $(q, w, S) \overset{*}{\vdash} (q, y_i, \alpha_i)$, where $w = x_i y_i$.

Since $\gamma_n = w$, we have $\alpha_n = \epsilon$, and $y_n = \epsilon$, thus $(q, w, S) \overset{*}{\underset{lm}{\vdash}} (q, \epsilon, \epsilon)$, i.e. $w \in N(P_G)$.

For instance, consider CFG G:

$$E \to I|E+E|E\times E|(E), I \to a|b|Ia|Ib|I0|I1.$$

Since $a\times b \in L(G)$

$$\begin{array}{ccccccccccc} E & \Rightarrow & E\times E & \Rightarrow & I\times E & \Rightarrow & a\times E & \Rightarrow & a\times I & \Rightarrow & a\times b \\ \gamma_1 & & \gamma_2 & & \gamma_3 & & \gamma_4 & & \gamma_5 & & \gamma_6 \end{array}$$

then

$$\begin{array}{ll} (q, a\times b, E) & \gamma_1 \\ \vdash (q, a\times b, E\times E) & \gamma_2 \\ \vdash (q, a\times b, I\times E) & \gamma_3 \\ \vdash (q, a\times b, a\times E) \vdash (q, \times b, \times E) \vdash (q, b, E) & \gamma_4 \\ \vdash (q, b, I) & \gamma_5 \\ \vdash (q, b, b) \vdash (q, \epsilon, \epsilon) & \gamma_6 \end{array}$$

We continue our proof.

Basis step: For $i = 1$, $\gamma_1 = S$. Thus $x_1 = \epsilon$, and $y = w$. Clearly, $(q, w, S) \overset{*}{\vdash} (q, w, S)$.

Inductive step: The induction hypothesis is $(q, w, S) \overset{*}{\vdash} (q, y_i, \alpha_i)$. We have to show that

$$(q, y_i, \alpha_i) \overset{*}{\vdash} (q, y_{i+1}, \alpha_{i+1}).$$

Now α_i begins with a variable A, and we have the form (e.g. suppose $\beta = \nu B\mu$)

$$\underbrace{x_i A\chi}_{\gamma_i} \underset{lm}{\Rightarrow} x_i\beta\chi = \underbrace{x_{i+1}B\mu\chi}_{\gamma_{i+1}}$$

By the induction hypothesis $A\chi$ is on the stack, y_i is unconsumed. From the construction of P_G is follow that we can make the move

$$(q, y_i, A\chi) \vdash (q, y_i, \beta\chi) = (q, y_i, vB\mu\chi).$$

If β has a prefix of terminals (in our example, only has v), we can pop them with matching terminals in a prefix of y_i, ending up in configuration $(q, y_{i+1}, \alpha_{i+1})$, where α_{i+1} (e.g. $B\mu\chi$) is the tail of the sentential γ_{i+1}.

($\subseteq$-direction). We shall show by an induction on the length of $\overset{*}{\vdash}$ that

$$\text{If } (q, x, A) \overset{*}{\vdash} (q, \epsilon, \epsilon), \text{ then } A \overset{*}{\Rightarrow} x.$$

Suppose $w \in N(P_G)$, choose $A = S$, and $x = w$, then $(q, w, S) \overset{*}{\vdash} (q, \epsilon, \epsilon)$. We have $S \overset{*}{\Rightarrow} w$, meaning $w \in L(G)$.

Basis step: Length 1. Then it must be that $A \to \epsilon$ is in G, and we have $(q, \epsilon) \in \delta(q, \epsilon, A)$. In this case, $x = \epsilon$, and we know that $A \overset{*}{\Rightarrow} \epsilon$.

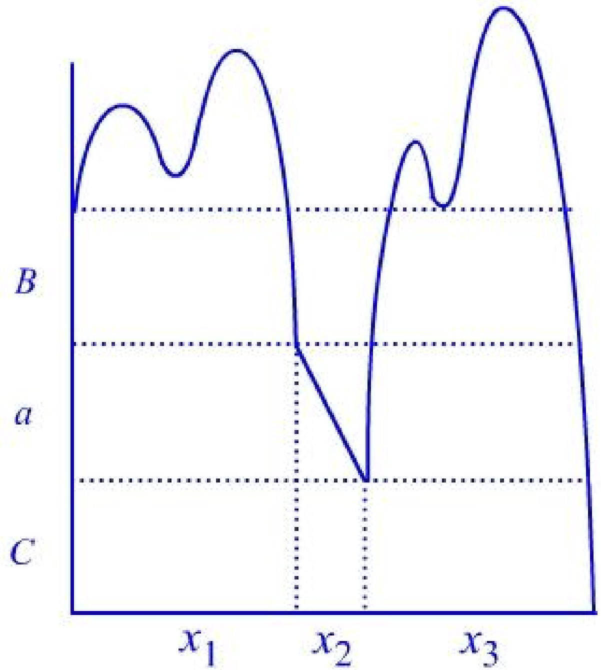
Inductive step: Length is $n > 1$, and the induction hypothesis holds for lengths $< n$.

Since A is a variable, we must have

$$(q, x, A) \vdash (q, x, Y_1 Y_2 \cdots Y_k) \vdash \cdots \vdash (q, \epsilon, \epsilon)$$

where $A \to Y_1 Y_2 \cdots Y_k$ is in G.

We can now write x as $x_1 x_2 \cdots x_k$, according to the figure in next slide, where $Y_1 = B$, $Y_2 = a$ and $Y_3 = C$.


B
a
C
x_1
x_2
x_3

Now we can conclude that

$$(q, x_i x_{i+1} \cdots x_k, Y_i) \overset{*}{\vdash} (q, x_{i+1} \cdots x_k, \epsilon)$$

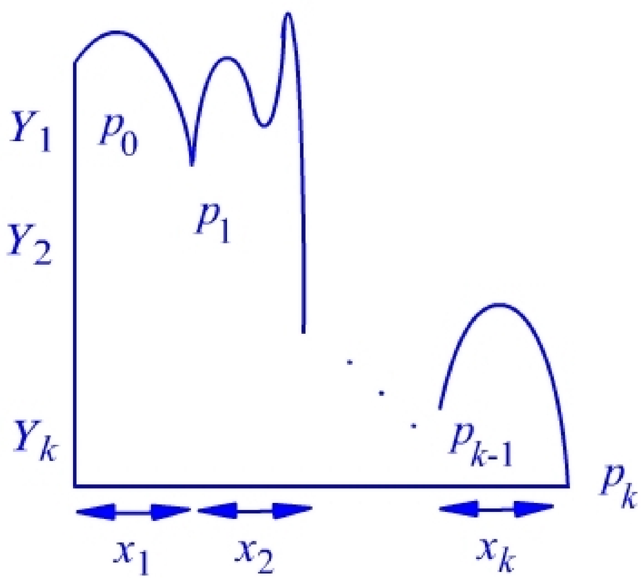
is less than n steps, for all $i = 1, 2, \cdots, k$.

- If Y_i is a variable we have by the induction hypothesis and Theorem 6.2 that $Y_i \overset{*}{\Rightarrow} x_i$.
- If Y_i is a terminal, we have $|x_i| = 1$, and $Y_i = x_i$. Thus $Y_i \overset{*}{\Rightarrow} x_i$ by the reflexivity of $\overset{*}{\Rightarrow}$.

So, $A \Rightarrow Y_1 Y_2 \cdots Y_k \overset{*}{\Rightarrow} x_1 Y_2 \cdots Y_k \overset{*}{\Rightarrow} x_1 x_2 \cdots Y_k \overset{*}{\Rightarrow} x_1 x_2 \cdots x_k = x.$ ◀

From PDA's to CFG's

Let's first look at how a PDA can consume $x = x_1 x_2 \cdots x_k$ and empty the stack.



We shall define a grammar with variables of the form $[p_{i-1}Y_ip_i]$ representing going from p_{i-1} to p_i with net effect of popping Y_i.

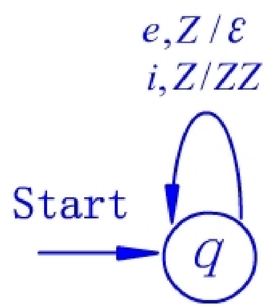
Formally, let $P = (Q, \Sigma, \Gamma, \delta, q_0, Z_0)$ be a PDA. Define $G_P = (V, \Sigma, R, S)$, where

- $V = \{[pXq] \mid \{p, q\} \subseteq Q,\ X \in \Gamma\} \cup \{S\}$.
- $R = \{S \to [q_0Z_0p] \mid p \in Q\} \cup$

  $\{[qXr_k] \to a[rY_1r_1] \cdots [r_{k-1}Y_kr_k] \mid a \in \Sigma \cup \{\epsilon\}, \{r_1, \cdots, r_k\} \subseteq Q, (r, Y_1 \cdots Y_k) \in \delta(q, a, X)\}$.

Note that k can be 0, in which case $(r, \epsilon) \in \delta(q, a, X)$. Then production is $[qXr] \to a$.

Example Let's convert $P = (\{q\}, \{i, e\}, \{Z\}, \delta, q, Z)$, where $\delta(q, i, Z) = \{(q, ZZ)\}$ and $\delta(q, e, Z) = \{(q, \epsilon)\}$ to a grammar.



By the construction above, we get $G_P = (V, \{i, e\}, R, S)$ where $V = \{[qZq], S\}$, and $R = \{S \to [qZq],\ [qZq] \to i[qZq][qZq],\ [qZq] \to e\}$.

Replacing $[qZq]$ by A, we get productions $S \to A$ and $A \to iAA|e$, or $S \to iSS|e$.

Example Let $P = (\{p, q\}, \{0, 1\}, \{X, Z_0\}, \delta, q, Z_0)$, where δ is given by

1. $\delta(q, 1, Z_0) = \{(q, XZ_0)\}$
2. $\delta(q, 1, X) = \{(q, XX)\}$
3. $\delta(q, 0, X) = \{(p, X)\}$
4. $\delta(q, \epsilon, Z_0) = \{(q, \epsilon)\}$
5. $\delta(p, 1, X) = \{(p, \epsilon)\}$
6. $\delta(p, 0, Z_0) = \{(q, Z_0)\}$

Construct the CFG G_P.

Question What's the language $N(P)$?

$$\{\epsilon\} \cup \{1^{n_1}01^{n_1}01^{n_2}01^{n_2}0 \cdots 1^{n_k}01^{n_k}0 \mid k, n_1, \cdots n_k \geq 1\}$$

We get $G_P = (V, \{0, 1\}, R, S)$ where

$$V = \{S, [pXp], [pXq], [pZ_0p], [pZ_0q], [qXp], [qXq], [qZ_0p], [qZ_0q]\}$$

and the productions in R are

$$S \rightarrow [qZ_0q] \mid [qZ_0p]$$

$$[qZ_0q] \rightarrow 1[qXq][qZ_0q] \mid 1[qXp][pZ_0q], \quad [qZ_0p] \rightarrow 1[qXq][qZ_0p] \mid 1[qXp][pZ_0p]$$

From rule (1): $\delta(q, 1, Z_0) = \{(q, XZ_0)\}$

$[qXq] \to 1[qXq][qXq] \mid 1[qXp][pXq], \quad [qXp] \to 1[qXq][qXp] \mid 1[qXp][pXp]$

From rule (2): $\delta(q, 1, X) = \{(q, XX)\}$

$[qXq] \to 0[pXq], \; [qXp] \to 0[pXp]$ From rule (3): $\delta(q, 0, X) = \{(p, X)\}$

$[qZ_0q] \to \epsilon$ From rule (4): $\delta(q, \epsilon, Z_0) = \{(q, \epsilon)\}$

$[pXp] \to 1$ From rule (5): $\delta(p, 1, X) = \{(p, \epsilon)\}$

$[pZ_0q] \to 0[qZ_0q], \; [pZ_0p] \to 0[qZ_0p]$ From rule (6): $\delta(p, 0, Z_0) = \{(q, Z_0)\}$

Theorem 6.6 *If CFG G_P constructed from PDA P by the construction above, then $L(G_P) = N(P)$.*

Proof (⊇-direction). We shall show by an induction on the length of the sequence $\overset{*}{\vdash}$ that

$$\text{If } (q, w, X) \overset{*}{\vdash} (p, \epsilon, \epsilon), \text{ then } [qXp] \overset{*}{\Rightarrow} w.$$

Suppose $w \in N(P)$, then $(q_0, w, Z_0) \overset{*}{\vdash} (p, \epsilon, \epsilon)$ for some p. We have $[q_0 Z_0 p] \overset{*}{\Rightarrow} w$, i.e. $S \overset{*}{\Rightarrow} w$, because of the way the rules for start symbol S are constructed. That means $w \in L(G_P)$.

Basis step: Length 1. Then w is an a or ϵ, and $(p, \epsilon) \in \delta(q, w, X)$. By the construction of G_P we have $[qXp] \to w$ and thus $[qXp] \overset{*}{\Rightarrow} w$.

Inductive step: Length is $n > 1$, and the induction hypothesis holds for lengths $< n$. We must have

$$(q, w, X) \vdash (r_0, x, Y_1Y_2 \cdots Y_k) \vdash \cdots \vdash (p, \epsilon, \epsilon)$$

where $w = ax$ (a is either a symbol in Σ or $a = \epsilon$). It follows that $(r_0, Y_1Y_2 \cdots Y_k) \in \delta(q, a, X)$. Then we have a production

$$[qXr_k] \to a[r_0Y_1r_1] \cdots [r_{k-1}Y_kr_k],$$

for all $\{r_1, \cdots, r_k\} \subseteq Q$.

We may now choose r_i to be the state in the sequence $\overset{*}{\vdash}$ when Y_i is popped, finally, $r_k = p$ when Y_k is popped. Let $x = w_1 w_2 \cdots w_k$, where w_i is consumed while Y_i is popped. Then

$$(r_{i-1}, w_i, Y_i) \overset{*}{\vdash} (r_i, \epsilon, \epsilon).$$

By the induction hypothesis we get $[r_{i-1} Y_i r_i] \overset{*}{\Rightarrow} w_i$.

We then get the following derivation sequence:

$$[qXr_k] \Rightarrow a[r_0 Y_1 r_1] \cdots [r_{k-1} Y_k r_k] \overset{*}{\Rightarrow} aw_1 [r_1 Y_2 r_2][r_2 Y_3 r_3] \cdots [r_{k-1} Y_k r_k] \overset{*}{\Rightarrow}$$
$$aw_1 w_2 [r_2 Y_3 r_3] \cdots [r_{k-1} Y_k r_k] \overset{*}{\Rightarrow} \quad \cdots \quad \overset{*}{\Rightarrow} aw_1 w_2 \cdots w_k = w$$

where $r_k = p$.

($\subseteq$-direction). We shall show by an induction on the length of the derivation $\overset{*}{\Rightarrow}$ that

$$\text{If } [qXp] \overset{*}{\Rightarrow} w, \text{ then } (q, w, X) \overset{*}{\vdash} (p, \epsilon, \epsilon).$$

Suppose $w \in L(G_P)$, then $S \overset{*}{\Rightarrow} w$. There is a state p such that $[q_0 Z_0 p] \overset{*}{\Rightarrow} w$, because we have only productions $S \to [q_0 Z_0 p]$ for the start symbol S.

Now we have $(q_0, w, Z_0) \overset{*}{\vdash} (p, \epsilon, \epsilon)$. That means $w \in N(P)$.

Basis step: One step. Then we have a production $[qXp] \rightarrow w$. From the construction of G_P it follows that $(p, \epsilon) \in \delta(q, a, X)$, where $w = a$. But then $(q, w, X) \vdash (p, \epsilon, \epsilon)$.

Inductive step: Length is $n > 1$, and the induction hypothesis holds for lengths $< n$. We must have

$$[qXr_k] \Rightarrow a[r_0Y_1r_1][r_1Y_2r_2] \cdots [r_{k-1}Y_kr_k] \stackrel{*}{\Rightarrow} w$$

where $r_k = p$.

We can break w into $aw_1 \cdots w_k$ such that $[r_{i-1}Y_ir_i] \stackrel{*}{\Rightarrow} w_i$. From the induction hypothesis we get $(r_{i-1}, w_i, Y_i) \stackrel{*}{\vdash} (r_i, \epsilon, \epsilon)$.

From Theorem 6.1 we get

$$(r_{i-1}, w_i w_{i+1} \cdots w_k, Y_i Y_{i+1} \cdots Y_k) \overset{*}{\vdash} (r_i, w_{i+1} \cdots w_k, Y_{i+1} \cdots Y_k).$$

Since this holds for all $i = 1, 2, \cdots, k$, we put all these sequences together and get

$$(q, aw_1 w_2 \cdots w_k, X) \vdash (r_0, w_1 w_2 \cdots w_k, Y_1 Y_2 \cdots Y_k) \overset{*}{\vdash}$$
$$(r_1, w_2 \cdots w_k, Y_2 \cdots Y_k) \overset{*}{\vdash} (r_2, w_3 \cdots w_k, Y_3 \cdots Y_k) \overset{*}{\vdash} \quad \cdots \quad \overset{*}{\vdash} (r_k, \epsilon, \epsilon).$$

Since $r_k = p$, we have shown that $(q, w, X) \overset{*}{\vdash} (p, \epsilon, \epsilon)$. ◀


Break for 15 minutes

Deterministic

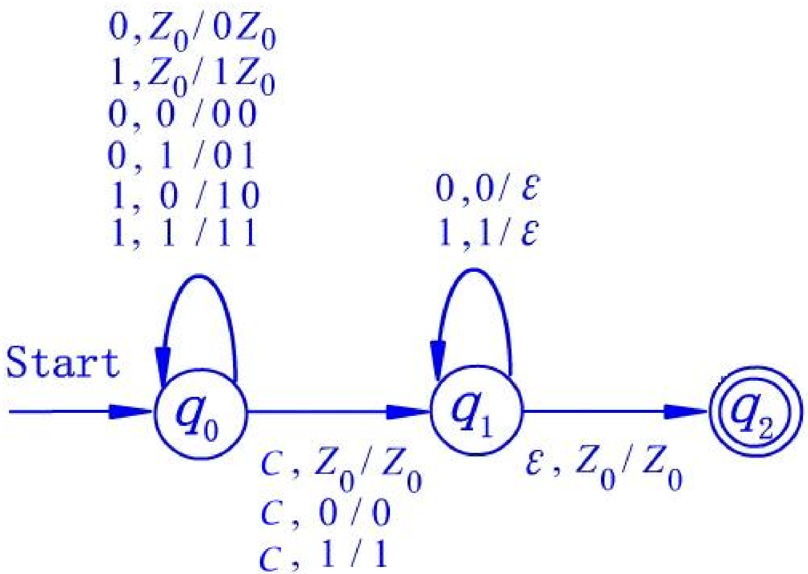
Pushdown Automata

A PDA $P = (Q, \Sigma, \Gamma, \delta, q_0, Z_0, F)$ is deterministic (DPDA) if and only if

1. $\delta(q, a, X)$ is always empty or a singleton for any $q \in Q$, $a \in \Sigma$ or $a = \epsilon$, and $X \in \Gamma$.
2. If $\delta(q, a, X)$ is nonempty, then $\delta(q, \epsilon, X)$ must be empty.

Example Let's define $L_{wcwr} = \{wcw^R \mid w \in \{0, 1\}^*\}$. This language can be recognized by a DPDA.

The DPDA for L_{wcwr} is shown as follows.



DPDA's and Regular Languages as Well as CFL's

We'll show that

$$\text{Regular Languages} \subset \text{L(DPDA)} \subset \text{Context-free Languages}$$

Theorem 6.7 *If L is regular language, then $L = L(P)$ for some DPDA P.*

Proof Since L is regular there is a DFA A s.t. $L = L(A)$. Let $A = (Q, \Sigma, \delta_A, q_0, F)$. We define the DPDA $P = (Q, \Sigma, \{Z_0\}, \delta_P, q_0, Z_0, F)$, where the transition function $\delta_P(q, a, Z_0) = \{(\delta_A(q, a), Z_0)\}$, for all $q \in Q$, and $a \in \Sigma$.

We shall show by an induction on the length of w that

$$(q, w, Z_0) \overset{*}{\vdash} (p, \epsilon, Z_0) \quad \text{iff} \quad \hat{\delta}_A(q, w) = p$$

Choose $q = q_0$, since both A and P accept by entering one of the states of F, we conclude their languages are same.

We only give a proof for “only if” part.

Basis step: Let $w = \epsilon$. From $(q, \epsilon, Z_0) \vdash (p, \epsilon, Z_0)$, we know $(p, Z_0) \in \delta_P(q, \epsilon, Z_0)$. Since $\delta_P(q, \epsilon, Z_0)$ is a singleton, we must have $p = \delta_A(q, \epsilon) = q$.

Inductive step: Let $w = ax$. We have

$$(q, w, Z_0) = (q, ax, Z_0) \vdash (\delta_A(q, a), x, Z_0) \overset{*}{\vdash} (p, \epsilon, Z_0).$$

By the induction hypothesis, we get $\hat{\delta}_A(\delta_A(q, a), x) = p$.

Notice that we mentioned the formula for the $\hat{\delta}$:

$$\hat{\delta}(q, ax) = \hat{\delta}(\delta(q, a), x)$$

for any state q, string x, and input symbol a.

We conclude that $\hat{\delta}_A(q, w) = p$. ◀

- We have shown that the DPDA languages include all the regular languages.
- We have already seen that a DPDA can accept language like L_{wcwr} that are not regular.
- Are there CFL's that can not be accepted by any DPDA?

  Yes, for example L_{wwr} ! But a formal proof is complex.

The two modes of acceptance – final state and empty stack – are not same for DPDA's. What about DPDA's that accept by empty stack?

A language L has the prefix property if there are no two distinct string in L, such that one is a prefix of the other.

Example L_{wcwr} has the prefix property. But $\{0\}^*$ does not have the prefix property.

Theorem 6.8 *L is $N(P)$ for some DPDA P if and only if L has the prefix property and L is $L(P')$ for some DPDA P'.*

That is, the languages accepted by empty stack are exactly those of the languages accepted by final state that have the prefix property. We'll show this in three parts.

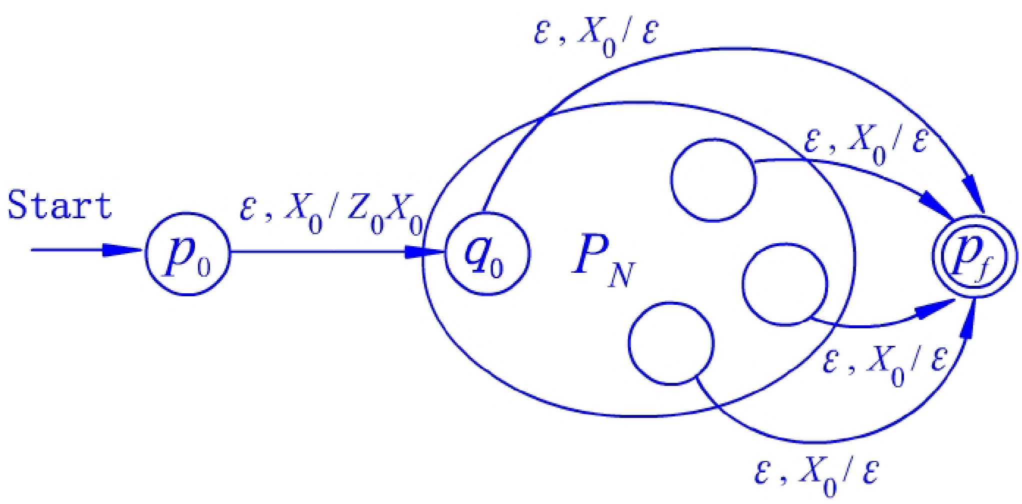
- If $L = N(P)$ for some DPDA P, then L has the prefix property.

Proof   Suppose L does not have the prefix property, i.e. P accepts both w and wx by empty stack, where $x \neq \epsilon$. Then $(q_0, w, Z_0) \overset{*}{\vdash} (q, \epsilon, \epsilon)$ for some state q, where q_0 and Z_0 are the start state and symbol of P. It dose so by a unique sequence of moves because P is deterministic. Thus $(q_0, wx, Z_0) \overset{*}{\vdash} (q, x, \epsilon)$.

However, it is not possible that $(q, x, \epsilon) \overset{*}{\vdash} (p, \epsilon, \epsilon)$ for some state p, because we know x is not ϵ, and a PDA cannot have a move with an empty stack. This observation contradicts the assumption that wx is in $N(P)$.

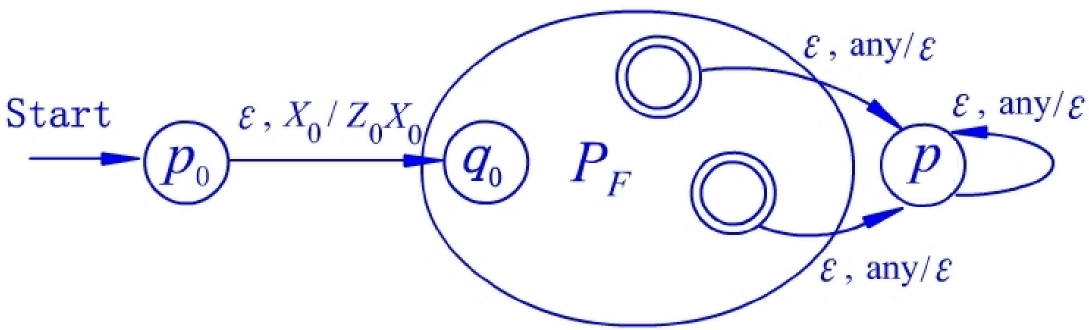
- If $L = N(P)$ for some DPDA P, then there exists a DPDA P' such that $L = L(P')$.

Proof We can convert P to P' just as P_N to P_F.



- If L has the prefix property and is $L(P')$ for some DPDA P', then there exists a DPDA P such that $L = N(P)$.

Proof Converting P' to P just as P_F to P_N, we find that P is not deterministic unless $L(P')$ has the prefix property.



DPDA's and Ambiguous Grammars

We can refine the power of the DPDA's by noting the languages they accept all have unambiguous grammars. Unfortunately, the DPDA languages are not exactly equal to the subset of the CFL's that are not inherently ambiguous.

For instance, L_{wwr} has an unambiguous grammar

$$S \rightarrow 0S0 \mid 1S1 \mid \epsilon$$

even though it is not a DPDA language.

For the converse we have

Theorem 6.9 *If $L = N(P)$ for some DPDA P then L has an unambiguous CFG.*

Proof By inspecting the proof of Theorem 6.6 we see that if the construction is applied to a DPDA the result is a CFG with unique leftmost derivations. ◀

In fact, this theorem can be strengthen as follows, we omit the proof.

Theorem 6.9 *If $L = L(P)$ for some DPDA P then L has an unambiguous CFG.*

Thank you!