Práctica en Julia y Optimización

Taller 3

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Métodos Cuantitativos

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1 Optimización

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Problem 1

Piense en el modelo de optimización:

$$Max x_1^a x_2^b x_3^c$$

$$s.a$$

$$p_1x_1 + p_2x_2 + p_3x_3 \leq I$$

$$x_3 \geq 2x_2$$

$$x_1, x_2, x_3 \geq 0$$

1. Resuelva el problema utilizando los conocimientos de las condiciones de Karush Kuhn Tucker. Suponga a,b,c >1, los precios p1, p2, p3 >0. Y el párametro I > 1.
2. Resuelva en Julia, usando a=b=c=2, p1=100, p2=120 y p3=200, con I=100000.

Solution. 1. Planteamos el lagrangiano:

$$L = x_1^a x_2^b x_3^c - \lambda_1 [p_1x_1 + p_2x_2 + p_3x_3 - I] - \lambda_2 [2x_2 - x_3]$$

C.P.O

$\frac{\partial L}{\partial x_1} = ax_1^{a-1}x_2^b x_3^c - \lambda_1 p_1 = 0$

$\frac{\partial L}{\partial x_2} = bx_1^a x_2^{b-1} x_3^c - \lambda_1 p_2 + 2\lambda_2 = 0$

$\frac{\partial L}{\partial x_3} = cx_1^a x_2^b x_3^{c-1} - \lambda_1 p_3 + \lambda_2 = 0$

$\frac{\partial L}{\partial \lambda_1} = I - p_1x_1 - p_2x_2 - p_3x_3 = 0$

$\frac{\partial L}{\partial \lambda_2} = 2x_2 - x_3 = 0$

$\lambda_1 \geq 0$

$\lambda_2 \geq 0$

$p_1x_1 + p_2x_2 + p_3x_3 \leq I$

$2x_2 - x_3 \geq 0$

$x_1 \geq 0, x_2 \geq 0, x_3 \geq 0$

Caso I: $\lambda_1 = \lambda_2 = 0$

$ax_1^{a-1}x_2^b x_3^c = 0 \quad x_1 = x_2 = x_3 = 0$

$2x_2 - x_3 = 2(0) - 0 = 0 < 0$

No se cumple.

Caso II: $\lambda_1 > 0, \lambda_2 = 0$

$ax_1^{a-1}x_2^b x_3^c - \lambda_1 p_1 = 0$ (1)

$bx_1^a x_2^{b-1} x_3^c - \lambda_1 p_2 = 0$ (2)

$cx_1^a x_2^b x_3^{c-1} - \lambda_1 p_3 = 0$ (3)

$p_1x_1 + p_2x_2 + p_3x_3 = I \rightarrow x_1 = \frac{I - p_2x_2 - p_3x_3}{p_1}$ (4)

Resolviendo (2) y (1):
$\frac{ax_1^{a-1}x_2^bx_3^c}{p_1} = \frac{bx_1^ax_2^{b-1}x_3^c}{p_2} \to x_1 = x_2 * \frac{p_2}{p_2} * \frac{a}{b}$ (5)
Resolviendo (3) y (1):
$\frac{ax_1^{a-1}x_2^bx_3^c}{p_1} = \frac{cx_1^ax_2^bx_3^{c-1}}{p_3} \to x_1 = x_3 * \frac{p_3}{p_1} * \frac{a}{c}$ (6)
Resolviendo (5) y (6):
$x_2 * \frac{p_2}{p_1} * \frac{a}{c} = x_3 * \frac{p_3}{p_1} * \frac{a}{c} \to x_2 = x_3 * \frac{p_3}{p_2} * \frac{b}{c}$ (7)
Resolviendo (5) y (4):
$x_2 * \frac{p_2}{p_1} * \frac{a}{b} = \frac{I-p_2x_2-p_3x_3}{p_1} \to p_2x_2 + p_2x_2 * \frac{a}{b} = I - p_3x_3$
$p_2x_2 + p_2x_2\left(1+\frac{a}{b}\right) = I - p_3x_2 \to x_2 = \frac{I-p_3x_3}{p_2(1+\frac{a}{b})}$ (8)
Resolviendo (7) y (8):
$x_3 * \frac{p_3}{p_2} * \frac{b}{c} = \frac{I-p_3x_3}{p_2(1+\frac{a}{b})} \to p_3x_3 + p_3x_3\left(1+\frac{a}{b}\right) * \frac{b}{c} = I$
$p_3x_3\left(1+\frac{a+b}{c}\right) = I \to x_3 = \frac{I}{p_3\left(\frac{a+b+c}{c}\right)}$ (9)
Resolviendo (9) y (7):
$x_2 = \frac{I}{p_3\left(\frac{a+b+c}{c}\right)} * \frac{p_3}{p_2} * \frac{b}{c} \to x_2 = \frac{I}{p_2(a+b+c)}$
Resolviendo (9) y (6):
$x_1 = \frac{I}{p_3\left(\frac{a+b+c}{c}\right)} * \frac{p_2}{p_1} * \frac{a}{c} \to x_1 = \frac{I}{p_1(a+b+c)}$
Caso III: $\lambda_1 = 0, \lambda_2 > 0$
$ax_1^{a-1}x_2^bx_3^c - \lambda_1p_1 = 0$
$bx_1^ax_2^{b-1}x_3^c - 2\lambda_2 = 0$
$cx_1^ax_2^bx_3^{c-1} + \lambda_2 = 0 \to \lambda_2 = 0$
No se cumple.
Caso IV: $\lambda_1 > 0, \lambda_2 > 0$
$ax_1^{a-1}x_2^bx_3^c - \lambda_1p_1 = 0$ (1)
$bx_1^ax_2^{b-1}x_3^c - \lambda_1p_2 - 2\lambda_2 = 0$ (2)
$cx_1^ax_2^bx_3^{c-1} - \lambda_1p_3 + \lambda_2 = 0$ (3)
$2x_2 - x_3 = 0$ (4)
$p_1x_1 + p_2x_2 + p_3x_3 - I = 0$ (5)
Resolviendo (1) y (2):
$\frac{ax_1^{a-1}x_2^bx_3^c}{p_1} = (bx_1^ax_2^{b-1}x_3^c - 2\lambda_2) * \frac{1}{p_2}$
$ax_1^{a-1}x_2^bx_3^c * \frac{p_2}{p_1} - bx_1^ax_2^{b-1}x_3^c = -2\lambda_2$
$\left(-\frac{1}{2}\right)x_1^ax_2^bx_3^c\left(\frac{ap_2}{x_1p_1} - \frac{b}{x_2}\right) = \lambda_2$ (6)
Resolviendo (1) y (3):
$\frac{ax_1^{a-1}x_2^bx_3^c}{p_1} = \frac{cx_1^ax_2^bx_3^{c-1}+\lambda_2}{p_3}$
$\frac{p_3}{p_1} * ax_1^{a-1}x_2^bx_3^c - cx_1^ax_2^bx_3^{c-1} = \lambda_2$
$x_1^ax_2^bx_3^c\left(\frac{p_3a}{p_1x_1} - \frac{c}{x_3}\right) = \lambda_2$ (7)
Resolviendo (6) y (7):
$x_1^ax_2^bx_3^c\left(\frac{p_3a}{p_1x_1} - \frac{c}{x_3}\right) = \left(-\frac{1}{2}\right)x_1^ax_2^bx_3^c\left(\frac{ap_2}{x_1p_1} - \frac{b}{x_2}\right)$
$\frac{p_3}{p_1} * \frac{a}{x_1} - \frac{c}{x_3} = -\frac{1}{2}\left(\frac{a}{x_1} * \frac{p_2}{p_1} - \frac{b}{x_2}\right)$ (8)
Resolviendo (4) y (5):
$p_1 + x_1 + p_2x_2 + p_3(2x_2) - I = 0 \to x_1 = \frac{I-p_2x_2-p_3*2x_2}{p_1}$ (9)

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Resolviendo (4) y (8):

$\frac{p_3}{p_1} * \frac{a}{x_1} - \frac{c}{2x_1} = \left(-\frac{1}{2}\right) * \frac{a}{x_1} * \frac{p_2}{p_1} - \frac{b}{x_2} \rightarrow x_2 = \frac{ax_2p_2+2ax_2p_3}{p_1(c+b)}$ (10)

Resolviendo (9) y (10:

$\frac{I-p_2x_2-p_3 2x_2}{p_1} = \frac{ax_2p_2+2ax_2p_3}{p_1(c+b)}$

$I(c+b) - p_2x_2(c+b) - 2x_2p_3(c+b) - ax_2p_2 - 2ax_2p_3 = 0$

$I(c+b) = x_2(cp_2 + bp_2 + 2cp_3 + 2bp_3 + ap_2 + 2ap_3)$

$x_2 = \frac{I(c+b)}{p_2(a+b+c)2p_3(a+b+c)} \rightarrow \frac{I(c+b)}{(p_2+2p_3)(a+b+c)}$

$x_1 = I - p_2\left[\frac{I(c+b)}{(p_2+2p_3)(a+b+c)}\right] - 2p_3\left[\frac{I(c+b)}{(p_2+2p_3)(a+b+c)}\right]$

$x_3 = 2\left[\frac{I(c+b)}{(p_2+2p_3)(a+b+c)}\right]$

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2. La especificación del modelo es:

```
using JuMP
using Ipopt

function solve_optimization()
    a = 2
    b = 2
    c = 2
    p1 = 100
    p2 = 120
    p3 = 200
    I = 100000

    model = Model(Ipopt.Optimizer)
    @variables(model, begin
        x1 >= 0
        x2 >= 0
        x3 >= 0
    end)

    @NLobjective(
        model,
        Max,
        x1^a * x2^b * x3^c
    )

    @constraint(
        model,
        p1 * x1 + p2 * x2 + p3 * x3 <= I
    )

    @constraint(
        model,
        x3 >= 2 * x2
    )

    optimize!(model)

    println("Termination status: ", termination_status(model))
    println("Objective value: ", objective_value(model))
    println("x1: ", value(x1))
    println("x2: ", value(x2))
    println("x3: ", value(x3))
end

solve_optimization()
```