DISCRETE MATHEMATICS
（离散数学）

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Chapter 3 COUNTING

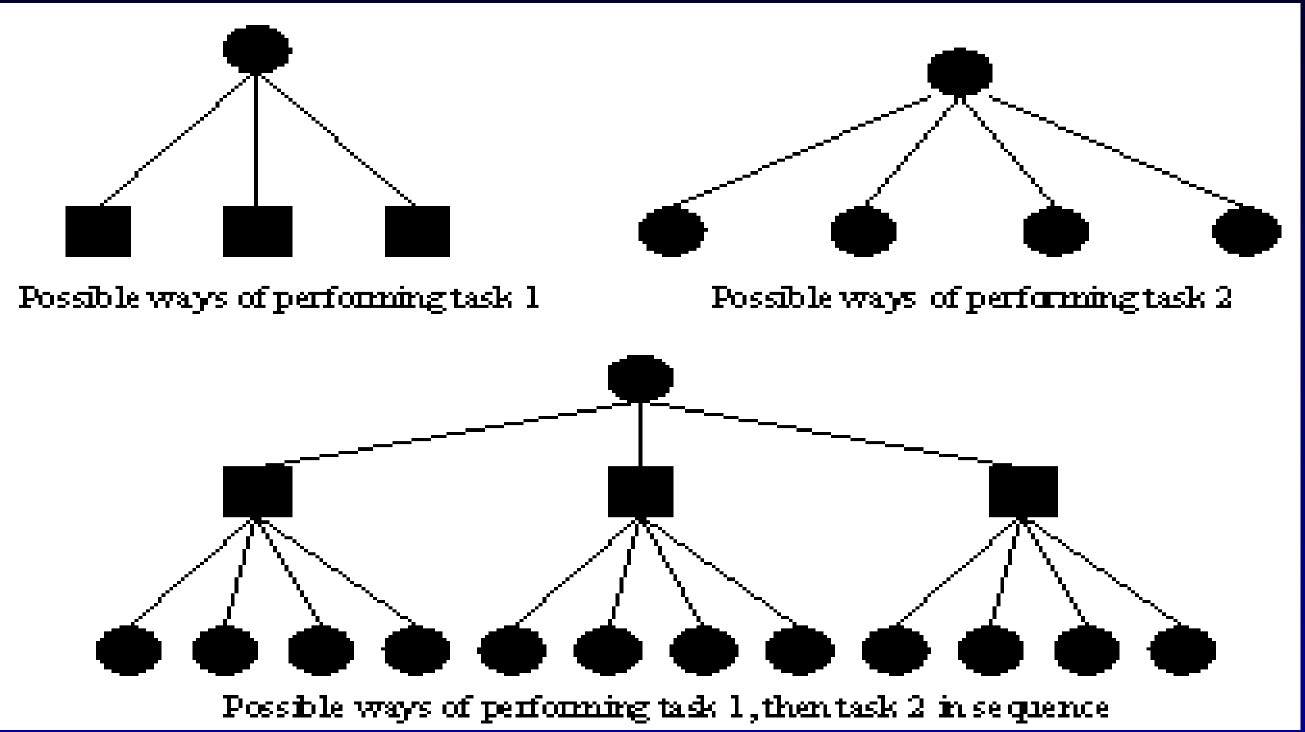
3.1 PERMUTATIONS (排列)

Theorem 1 Suppose that two tasks T_1 and T_2 are to be performed in sequence. If T_1 can be performed in n_1 ways, and for each of these ways T_2 can be performed in n_2 ways, then the sequence T_1T_2 can be performed in n_1n_2 ways.

Proof: Each choice of a method of performing T_1 will result in a different way of performing the task sequence. There are n_1 such methods, and for each of these we may choose n_2 ways of performing T_2. Thus, in all, there will be n_1n_2 ways of performing the sequence T_1T_2.

See Figure 3.1 for the case where n_1 is 3 and n_2 is 4.

Figure 3.1



Theorem 1 is sometimes called the **multiplication principle of counting** (乘法原理).

Theorem 2 Suppose that tasks $T_1, T_2, \ldots, T_k$ are to be performed in sequence. If T_1 can be performed in n_1 ways, and for each of these ways T_2 can be performed in n_2 ways, and for each of these $n_1 n_2$ ways of performing $T_1 T_2$ in sequence, T_3 can be performed in n_3 ways, and so on, then the sequence $T_1 T_2 \cdots T_k$ can be performed in exactly $n_1 n_2 \cdots n_k$ ways.

Proof: This result can be proved by using the principle of mathematical induction on k.

Example Let A be a set with n elements. How many subsets does A have?

Solution: We know from Section 1.3 that each subset of A is determined by its characteristic function, and if A has n elements, this function may be described as an array of 0's and 1's having length n. The first element of the array can be filled in two ways (with a 0 or a 1), and this is true for all succeeding elements as well. Thus, by the extended multiplication principle, there are

$$\underbrace{2 \cdot 2 \cdot \cdots \cdot 2}_{\text{n factors}} = 2^n$$

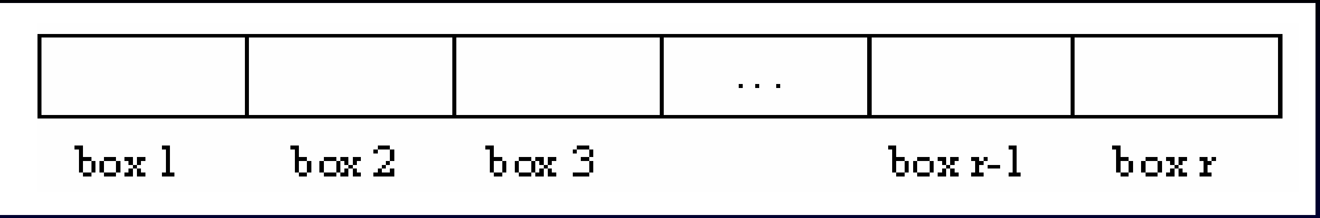
ways of filling the array, and therefore 2^n subsets of A.

Problem 1: How many different sequences, each of length r, can be formed using elements from A if

(a) elements in the sequence may be repeated?

(b) all elements in the sequence must be distinct?

First we note that any sequence of length r can be formed by filling r boxes in order from left to right with elements of A. In case (a) we may use copies of elements of A.



Let T_1 be the task “fill box 1,” let T_2 be the task “fill box 2,” and so on. Then the combined task $T_1 T_2 \cdots T_r$ represents the formation of the sequence.

Case (a). T_1 can be accomplished in n ways, since we may copy any element of A for the first position of the sequence. The same is true for each of the tasks $T_2, T_3, \cdots, T_r$. Then by the extended multiplication principle, the number of sequences that can be formed is

$$\underbrace{n \cdot n \cdot \ \cdots \ \cdot n}_{\text{r factors}} = n^r$$

We have therefore proved the following result.

Theorem 3: Let A be a set with n elements and $1 \leqq r \leqq n$. Then the number of sequences of length r that can be formed from elements of A, allowing repetitions, is n^r.

Now we consider case (b) of Problem 1. Here also T_1 can be performed in n ways, since any element of A can be chosen for the first position. Whichever element is chosen, only (n-1) elements remain, so that T_2 can be performed in (n-1) ways, and so on, until finally T_r can be performed in (n-(r-1)) or (n-r+1) ways. Thus, by the extended principle of multiplication, a sequence of r distinct elements from A can be formed in n-(r-1) or (n-r+1) ways. Thus, by the extended principle of multiplication, a sequence of r distinct elements from A can be formed in n(n-1)(n-2)···(n-r+1) ways.

A sequence of r distinct elements of a is often called a **permutation** of A taken r at a time. The preceding discussion shows that the number of such sequences depends only on n and r, not on A. This number is often written P_n^r and is called the **number of permutations of n objects taken r at a time**（排列数）.

Theorem 4: If 1≤r≤n, then P_n^r, the number of permutations of n objects taken r at a time, is

$$n \cdot (n-1) \cdot (n-2) \cdots (n-r+1).$$

When r=n, we are counting the distinct arrangements of the elements of A, with |A|=n, into sequences of length n. Such a sequence is simply called a **permutation**（置换）of A. The number of permutations of A is thus P_n^n or n · (n-1) · (n-2) · · · 2 ·1, if n≥1. This number is also written n! and is read **n factorial**（阶乘）.

For convenience, we define 0! to be 1. Then for every n≥0 the number of permutations of n objects is n!.

$$\begin{aligned} P_n^r &= n\cdot(n-1)\cdot(n-2)\cdot\ \cdots\ \cdot(n-r+1) \\ &= \frac{n\cdot(n-1)\cdot(n-2)\cdot\ \cdots\ \cdot(n-r+1)\cdot(n-r)\cdot(n-r-1)\cdot\ \cdots\ \cdot 2\cdot 1}{(n-r)\cdot(n-r-1)\cdot\ \cdots\ \cdot 2\cdot 1} \\ &= \frac{n!}{(n-r)!}. \end{aligned}$$

The following theorem describes the general situation for permutations with limited repeats.

Theorem 5: The number of distinguishable permutations that can be formed from a collection of n objects where the first object appears k_1 times, the second object k_2 times, and so on, is $\frac{n!}{k_1!k_2!\ \cdots\ k_t!}$., where $k_1+k_2+\cdots+k_t=n$.

3.2 CONNMBINATIONS（组合）

Problem 2: Let A be any set with n elements and $1 \leqslant r \leqslant n$. How many different subsets of A are there, each with r elements?

We are trying to compute the number of ways to choose B. Call this number C. Then task 1 can be performed in r! ways. Thus the total number of ways of performing both tasks is, by the multiplication principle, C·r!. But it is also P_n^r. Hence,

$$C \cdot r! = P_n^r = \frac{n!}{(n-r)!}.$$

Therefore,

$$C = \frac{n!}{r!(n-r)!}.$$

Theorem 1: Let A be a set with |A|=n, and let $1 \leqslant r \leqslant n$. Then the number of combinations of the elements of A, taken r at a time, that is the number of r-element subsets of A is $\dfrac{n!}{r!(n-r)!}$.

The number of combinations of A, taken r at a time, does not depend on A, but only on n and r. C_n^r is called the **number of combinations of n objects taken r at a time**. $C_n^r = \dfrac{n!}{r!(n-r)!}$.

Theorem 2: Suppose k selections are to be made from n items without regard to order and that repeats are allowed, assuming at least k copies of each of the n items. The number of ways these selections can be made is C_{n+k-1}^{k}.

Example Suppose that a valid computer password consists of seven characters, the first of which is a letter chosen from the set {A, B, C, D, E, F, G} and the remaining six characters are letters chosen from the English alphabet or a digit. How many different passwords are possible?

Solution: A password can be constructed by performing the tasks T_1 and T_2 is sequence.

TASK 1: Choose a starting letter from the set given.

TASK 2: Choose a sequence of letter and digits. Repeats are allowed.

Task T_1 can be performed in C_7^1 or 7 ways. Since there are 26 letters and 10 digits that can be chosen for each of the remaining six characters, and since repeats are allowed, task T_2 can be performed in 36^6 or 2,176,782,336 ways. By the multiplication principle, there are $7 \cdot 2176782336$ or 15,237,476,352 different passwords.

3.3 PIGEONHOLE PRINCIPLE（抽屉原理）

Theorem 1: (The Pigeonhole Principle) If n pigeons are assigned to m pigeonholes, and m<n, then at least one pigeonhole contains two or more pigeons.

Proof: Suppose each pigeonhole contains at most 1 pigeon. Then at most m pigeons have been assigned. But since m<n, not all pigeons have been assigned pigeonholes. This is a contradiction. At least one pigeonhole contains two or more pigeons.

Example Show that if any five numbers from 1 to 8 are chosen, then two of them will add to 9.

Solution: Construct four different sets, each containing two numbers that add up to 9 as follows: $A_1=\{1,8\}$, $A_2=\{2,7\}$, $A_3=\{4,5\}$. Each of the five numbers chosen must belong to one of these sets. Since there are only four sets, the pigeonhole principle tells us that two of the chosen numbers belong to the same set. These numbers add up to 9.

- The Extended Pigeonhole Principle

If n and m are positive integers, then $\lfloor n/m \rfloor$ stands for the largest integer less than or equal to the rational number $\lfloor n/m \rfloor$.

Theorem 2: (The Extended Pigeonhole Principle) If n pigeons are assigned to m pigeonholes, then one of the pigeonholes must contain at least $\lfloor (n-1)/m \rfloor + 1$ pigeons.

Proof (by contradiction): If each pigeonhole contains no more than $\lfloor (n-1)/m \rfloor$ pigeons, then there are at most $\lfloor (n-1)/m \rfloor \le m \cdot (n-1)/m = n-1$ pigeons in all. This contradicts our hypothesis, so one of the pigeonholes must contain at least $\lfloor (n-1)/m \rfloor + 1$ pigeons.

3.5 RECURRENCE RELATIONS

When the problem is to find an explicit formula for a recursively defined sequence, the recursive formula is called a recurrence relation（递推关系）. A recursive formula must be accompanied by information about the beginning of the sequence. This information is called the **initial condition** or **conditions** for the sequence.

Example The recurrence relation $f_n=f_{n-1}+f_{n-2}$, $f_1=f_2=1$, defines the **Fibonacci sequence** 1, 1, 2, 3, 5, 8, 13, 21,… .The initial conditions are $f_1=1$ and $f_2=1$.

Example The recurrence relation $a_n=a_{n-1}+3$ with $a_1=2$ defines the sequence 2,5,8,… . We back track the value of an by substituting the definition of a_{n-1}, a_{n-2}, and so on until a pattern is clear.

$$\begin{aligned} a_n &= a_{n-1}+3 \\ &= (a_{n-2}+3)+3 \\ &= ((a_{n-3}+3)+3)+3 \end{aligned}$$

or

$$\begin{aligned} a_n &= a_{n-1}+3 \\ &= a_{n-2}+2\cdot 3 \\ &= a_{n-3}+3\cdot 3 \end{aligned}$$

Eventually this process will produce

$$\begin{aligned} a_n &= a_{n\cdot(n-1)}+(n-1)\cdot 3 \\ &= a_1+(n-1)\cdot 3 \\ &= 2+(n-1)\cdot 3 \end{aligned}$$

An explicit formula for the sequence is $a_n=2+(n-1)3$.

Example Find an explicit formula for the sequence defined the recurrence relation $b_n=2b_{n-1}+1$ with initial condition $b_1=7$.

Solution: We begin by substituting the definition of the previous term in the defining formula.

$$
\begin{aligned}
b_n &= 2b_n + 1 \\
&= 2(2b_{n-2} + 1) + 1 \\
&= 2[2(2b_{n-3} + 1) + 1] + 1 \\
&= 2^3 b_{n-3} + 4 + 2 + 1 \\
&= 2^3 b_{n-3} + 2^2 + 2^1 + 1.
\end{aligned}
$$

A pattern is emerging with these rewriting of b_n.

(Note: There are no set rules for how to rewrite these expressions and a certain amount of experimentation may be necessary.) The backtracking will end at

$$b_n = 2^{n-1} b_{n-(n-1)} + 2^{n-2} + 2^{n-3} + \cdots + 2^2 + 2^1 + 1$$

$$= 2^{n-1} b_{n-2} + 2^{n-1} - 1 \qquad \text{Using Exercise 3, Section 2.4}$$

$$= 7 \cdot 2^{n-1} + 2^{n-1} - 1 \qquad \text{Using } b_1 = 7$$

$$= 8 \cdot 2^{n-1} - 1 \quad \text{or} \quad 2^{n+2} - 1.$$

Two useful summing rules were proved in Section 2.4.

$$S1. \quad 1 + a + a^2 + a^3 + \cdots + a^{n-1} = \frac{a^n - 1}{a - 1}.$$

$$S2. \quad 1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}.$$

A recurrence relation is a **linear homogeneous relation**（线性齐次关系）of degree k if it is of the form

$$a_n = r_1 a_{n-1} + r_2 a_{n-2} + \cdots + r_k a_{n-k} \quad \text{with the } r_i\text{'s constants.}$$

Example

(a) The relation $c_n=(-2)c_{n-1}$ is a linear homogeneous recurrence relation of degree 1.

(b) The relation $a_n=a_{n-1}+3$ is not a linear homogeneous recurrence relation.

(c) The recurrence relation $f_n=f_{n-1}+f_{n-2}$ is a linear homogeneous relation of degree 2.

(d) The recurrence relation $g_n=g^2_{n-1}+g_{n-2}$ is not a linear homogeneous relation.

Theorem 1

(a) If the characteristic equation $x^2-r_1x-r_2=0$ of the recurrence relation $a_n=r_1a_{n-1}+r_2a_{n-2}$ has two distinct roots, s_1 and s_2, then $a_n=us_1^n+vs_2^n$, where u and v depend on the initial conditions, is the explicit formula for the sequence.

(b) If the characteristic equation $x^2-r_1x-r_2=0$ has a single root s, the explicit formula is $a_n=us^n+vns^n$, where u and v depend on the initial conditions.

Proof

(a) Suppose that s_1 and s_2 are roots of $x^2-r_1x-r_2=0$, so $s_1^2-r_1s_1-r_2=0$, $s_2^2-r_1s_2-r_2=0$, and $a_n=us_1^n+vs_2^n$, for $n\geqslant 1$. We show that this definition of a_n defines the same sequence as $a_n=r_1a_{n-1}+r_2a_{n-2}$. First we note that u and v are chosen so that $a_1=us_1+vs_2$ and $a_2=us_1^2+vs_2^2$ and so the initial conditions are satisfied. Then

$$
\begin{aligned}
a_n &= us_1^n + vs_2^n && \text{Split out } s_1^2 \text{ and } s_2^2. \\
&= us_1^{n-2}s_1^2 + vs_2^{n-2}s_2^2 && \text{Substitute for } s_1^2 \text{ and } s_2^2. \\
&= us_1^{n-2}(r_1s_1 + r_2) + vs_2^{n-2}(r_1s_2 + r_2) \\
&= r_1us_1^{n-1} + r_2us_1^{n-2} + r_1vs_2^{n-1} + r_2vs_2^{n-2} \\
&= r_1(us_1^{n-1} + vs_2^{n-1}) + r_2(us_1^{n-2} + vs_2^{n-2}) \\
&= r_1a_{n-1} + r_2a_{n-2} && \text{Use definitions of } a_{n-1} \text{ and } a_{n-2}.
\end{aligned}
$$

(b) This part may be proved in a similar way.

The **Fibonacci sequence** is defined by a linear homogeneous recurrence relation of degree 2, so by Theorem 1, the roots of the associated equation are needed to describe the explicit formula for the sequence. From $f_n=f_{n-1}+f_{n-2}$ and $f_1=f_2=1$, we have $x^2-x-1=0$. Using the quadratic formula to obtain the roots, we find

$$s_1 = \frac{1+\sqrt{5}}{2} \quad \text{and} \quad s_2 = \frac{1-\sqrt{5}}{2}.$$

It remains to determine the u and v of Theorem 1. We solve

$$1 = u\left(\frac{1+\sqrt{5}}{2}\right) + v\left(\frac{1-\sqrt{5}}{2}\right) \quad \text{and} \quad 1 = u\left(\frac{1+\sqrt{5}}{2}\right)^2 + v\left(\frac{1-\sqrt{5}}{2}\right)^2.$$

For the given initial conditions, u is $\frac{1}{\sqrt{5}}$ and v is $-\frac{1}{\sqrt{5}}$.

The explicit formula for the Fibonacci sequence is

$$f_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n.$$

The end !