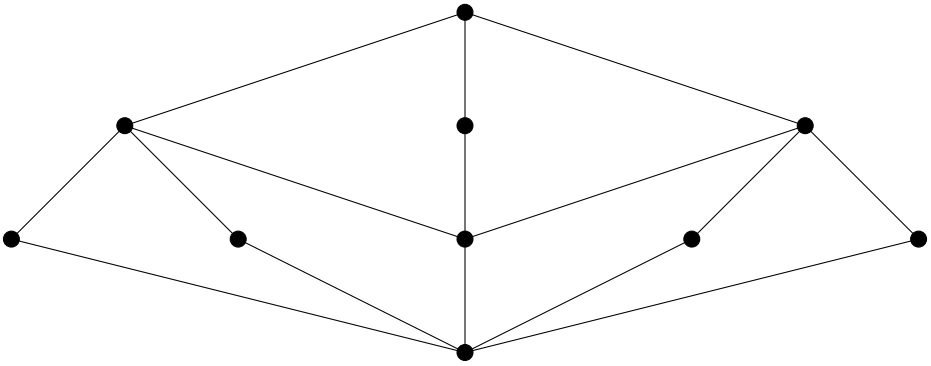
Some Abstract Algebra



A Primer and Interactive Workbook

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Preface

This book consists of two parts: one, a primer designed to provide an adequate introduction to the essentials of abstract algebra and to some related number theory, and two, a workbook designed to enable the reader to interactively engage with colleagues in exploring the fascinating world of abstract algebra.

We have taken a problem solving approach – the primer alone contains over 130 problems. So be prepared for minimal text material to read, combined with worksheets that extend and enhance text topics. These worksheets are designed to encourage discovery of interesting relationships between algebraic structures, geometry, mappings, and proofs.

Very little, if any, background in abstract algebra is needed for a course based on this Primer and the workbook. This material has been used successfully for over a decade with in-service secondary teachers seeking licensure or an MA degree in teaching mathematics.

In this book we embrace the oft-quoted maxim - "You learn mathematics by doing mathematics." Such an effort leads to better understanding and deeper learning.

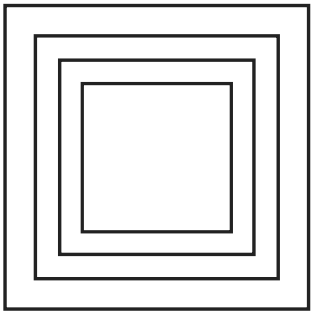
Finally, a valuable by-product: A significant number of teachers who have studied this material have incorporated a variety of the worksheets into their secondary curriculum as they encounter topics like closure, binary operations and their properties, modular arithmetic, and the structure of the integers (yes, GCD and LCM show up), and the rational and real numbers.

Richard Grassl

May 2018

Some Abstract Algebra - A Primer

and some Number Theory



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Contents

Introduction

An abstract algebra consists of a set of objects (integers, real numbers, permutations, polynomials, matrices, ...), various binary operations, along with some properties (closure, inverses, commutativity,...). Examples of abstract algebras include groups, rings, integral domains, and fields. Operations include rotations of regular geometrical figures, ordinary and modular addition and multiplication, addition and multiplication of matrices and of polynomials, composition of permutation cycles, direct products and others.

In one sense the core ideas of algebra are abstracted out and viewed from a much larger lens. For example, the problem of finding analogues of the quadratic formula, around the mid 1500's, led to the study of the symmetric groups which shed light on the nonsolvability of the general quintic.

Applications are plentiful. Among the many fields of study making significant use of algebraic structures we include cryptography, genetics, mineralogy, the study of molecular structures in chemistry, elementary particle theory in physics, Latin squares in statistical experiments, and finally, architecture and art.

Important contributors over the past several centuries include Joseph Lagrange, Niels Abel, Arthur Cayley, Emmy Noether, Gauss, Galois, Sylow among many others.

Closure

We say that a binary operation $\square$ on a set S is CLOSED if whenever a and b are any two elements in S then $a \square b$ is also in S. For example, the operation $+$ is closed on the set $S = Z = \{0, \pm 1, \pm 2, \dots\}$ since $a + b$ is in Z. But subtraction is not closed on the set $N = \{0, 1, 2, 3, \dots\}$ since $1 - 3 = -2$ is not in N.

The following gives the two part procedure for determining if a particular set S is closed under a particular binary operation $\square$:

1. Choose two arbitrary elements for S; label them a, b.

2. Show that $a \square b$ is a number of S.

Example. The set $E = \{0, 2, 4, 6, \dots\}$ is closed under addition. Let $a = 2m$ and $b = 2n$ be arbitary elements of E. Then since $2m + 2n = 2(m + n)$ is an even integer the sum of $2m$ and $2n$ is in E. E is also closed under multiplication since $(2m)(2n) = 4mn = 2(2mn)$ is even.

PROBLEMS.

In each of the following if the set is closed under the operation give reasons (actually a proof); if not, provide a counterexample.

1. Is $A = \{0, 1, 4, 9, 16, \dots\}$ closed
   a. under addition? b. under subtraction? c. under multiplication?

2. Is $B = \{0, \pm 5, \pm 10, \pm 15, \dots\}$ closed
   a. under addition? b. under multiplication?

3. Is $C = \{0, 2, 4, 6\}$, a finite set, closed under addition?

4. Is $D = \{1, 3, 5, 7, \dots\}$ closed

   a. under addition? b. under multiplication?

5. Is $E = \{1, 4, 7, 10, 13, \dots\}$, the positive integers having remainder 1 upon division by 3 closed

   a. under addition? b. under multiplication?

6. Repeat #5 with $F = \{2, 5, 8, 11, 14, \dots\}$.

7. The set $G = \{0, \pm 4, \pm 8, \pm 12, \dots\}$ is closed under subtraction. Give another set H that is closed under subtraction. Show that $G \cap H$ is also closed under subtraction.

8. Is the set of all rational numbers of the form 2^n, where n is in Z, closed under multiplication?

9. Is the set of all positive rational numbers closed under addition? Under multiplication?

10. Let $I = \{2^m \cdot 3^n : m, n \text{ are in } Z\}$. Is I closed under multiplication?
    Hint: Is $3/8$ in I? Is $1/9$ in I?

11. Are the irrationals closed under multiplication?

12. Prove that if S and T are sets of integers closed under subtraction so is the intersection $S \cap T$. Is the union of S and T also closed under subtraction?

13. Why does 0 always have to be a member of any set that is closed under subtraction?

14. Let R denote a 120° rotation of an equilateral triangle. Is the set $\{I, R, R^2\}$ closed under "rotation"? Here, I means do nothing and R^2 means a 240° rotation.

UNITS MULTIPLICATION

Under units multiplication the product of any two positive integers, denoted by $a \square b$, is the units digit of the product under ordinary multiplication. So $5 \square 9 = 5$, $7 \square 8 = 6$.

15. Is the set $\{0, 2, 4, 6, 8\}$ closed under units multiplication?

16. Is the set $\{1, 3, 7, 9\}$ closed under units multiplication?

17. Is $\{1, 4, 6\}$ closed under units multiplication? How about the set $\{2, 4, 8\}$? How about $\{1, 5, 9\}$?

18. What would you have to add to the set $\{1, 3, 5\}$ to make it closed under units multiplication?

Binary Operations

The concepts of being commutative and associative are usually introduced to students as they study the four basic arithmetic operations of addition, subtraction, multiplication and division of integers. These four operations denoted by $+, -, \times, \div$ are examples of binary operations.

Let S be any set. A binary operation on S is a function $f : S \times S \mapsto S$. Sometimes a binary operation is depicted using the infix notation $(m, n) \rightarrow m \,\square\, n$, rather than the prefix notation $f(m, n)$. The following are examples of binary operations on $N = \{0, 1, 2, 3, \dots\}$.

OPERATION	INFIX NOTATION
$f(m,n) = m + n$	$m \,\square\, n = m + n$
$f(m,n) = mn$	$m \,\square\, n = mn$
$f(m,n) = \text{GCD}(m,n)$	$m \,\square\, n = \text{GCD}(m,n)$
$f(m,n) = 5^m \cdot n$	$m \,\square\, n = 5^m \cdot n$

Additional binary operations that will be of importance include

$$f(x, y) = x \div y \text{ on } S = \text{Reals}$$

$$f(m, n) = m - n \text{ on } S = Z = \{0, \pm 1, \pm 2, \dots\}$$

$$f(m, n) = m + n - mn \text{ on } S = Z$$

$$f(m, n) = m + n - 1 \text{ on } S = Z$$

$$f(A, B) = A \cup B \text{ on } P(S) \text{ for } S = \{\text{a,b,c}\}$$

$$f(A, B) = A \cap B \text{ on } P(S) \text{ for } S = \{\text{a,b,c}\}$$

Here, $\cup$ denotes union and $\cap$ denotes intersection. Also, $P(S)$, the power set for S, is the set of all subsets of S. As an example, if S={a,b},then $P(S)$ = { $\emptyset$, {a}, {b}, {a,b} }.

Properties of Binary Operations

Binary operations on a set X may or may not satisfy the following properties:

Commutativity: $x \square y = y \square x$ for all x, y in X.

Associativity: $x \square (y \square z) = (x \square y) \square z$ for all x, y, z in X.

Identity: An element $e \in X$ such that $x \square e = e \square x = x$ for x in X is called an identity for the binary operation $\square$.

Inverses: If e is an identity under $\square$, an inverse of an element a in X is an element b in X such that $a \square b = e = b \square a$.

Example 1. The operation + on Z is associative since $a + (b + c) = (a + b) + c$ for all a, b, c in Z. Since $a + b = b + a$, + is commutative. The element 0 serves as an identity e since $0 + a = a + 0 = a$. Each element $a \in Z$ has an inverse, namely $-a$.

One important characterization or consequence of the notation $f : A \times A \mapsto A$ is that the result $f(m, n)$, or $m \square n$, must be an element in A; i.e., A must be closed under the binary operation $\square$. So divisibility, denoted $\div$, is not a binary operation on $N = \{0, 1, 2, \dots\}$ since $m \div n = \frac{m}{n}$ is not necessarily an integer. But $\div$ is a binary operation on the set R^+ of positive real numbers. Notice that since $2 \div 3 \neq 3 \div 2$, $\div$ is not commutative.

Consider the binary operation $m \square n = 3^m \cdot n$ on $N = \{0, 1, 2, \dots\}$; show that $\square$ is not associative. The following single example accomplishes this:

$$1 \square (0 \square 1) = 1 \square 1 = 3 \text{ but } (1 \square 0) \square 1 = 0 \square 1 = 1$$

Does $\square$ have an identity? it might be natural to try 0 or 1. Since $0 \square n = n$ but $n \square 0 = 0$, 0 is not an identity. Since $1 \square n = 3n$, 1 is not an identity. No other element in n works either; without an identity, there are no inverses.

Example 2. Union, $\cup$, is a binary operation on $P(S)$ where S={a,b,c}. Since $A \cup (B \cup C) = (a \cup B) \cup C$, $\cup$ is associative. Since $\emptyset \cup A = A = A \cup \emptyset$ for all $A \in P(S)$, the empty set $\emptyset$ serves as an identity.

When the set A is finite, a binary operation $\square$ can be given by a matrix table where the element $x \square y$ is found at the intersection of row x and column y.

Example 3. Let $A = \{1, 3, 7, 9\}$ and let $x \square y$ be the digit in the units position upon ordinary multiplication of x and y. This is sometimes written as $x \square y \equiv xy \pmod{10}$. The matrix table is

$\square$	1	3	7	9
1	1	3	7	9
3	3	9	1	7
7	7	1	9	3
9	9	7	3	1

This binary operation $\square : A \times A \mapsto A$ is associative (you need to check $4 \cdot 4 \cdot 4 = 64$

cases), is commutative (from the symmetry of the table), has identity 1, and each element has an inverse as shown in the following table:

x	1	3	7	9
inverse of x	1	7	3	9

Example 4. The binary operation $\circ$ on A={e, a, b} given by the table below is associative, commutative and has e as an identity. In the exercises you are asked to find inverses.

$\circ$	e	a	b
e	e	a	b
a	a	b	e
b	b	e	a

$\square$	1	2	5	10
1	1	1	1	1
2	1	2	1	2
5	1	1	5	5
10	1	2	5	10

Example 5. Let $A = \{1, 2, 5, 10\}$ and $m \square n = \text{GCD}(m, n)$, the greatest common divisor of m and n. The matrix table for $\square$ is given. With some effort, you can show that $\square$ is associative.The table's symmetry verifies that the operation $\square$ is commutative. In the exercises, you are asked to determine an identity and to see if there are inverses.

"An" Identity versus "The" Identity

Throughout this discussion of properties we have been saying "an" identity. Here is some good news! We can replace "an" with "the" whenever an identity exists.

Theorem: If the binary operation $\square$ on X has an identity, then it is unique.
Proof: Proceed using a proof by contradiction. Suppose there are two different identities; call them e and f. Since $x \square e = x = e \square x$ for all x in X, it must hold for $x = f$, i.e., $f \square e = f = e \square f$. Since $x \square f = x = f \square x$ for all x in X, it must hold for $x = e$, i.e., $e \square f = e = f \square e$. But then $e \square f = f$ and $e \square f = e$, or $e = f$ contradicting the fact that they were different.

Similarly inverses are unique. Let e be the identity for an associative operation $\circ$ on X, and let g and h be two inverses for some a in X. Then $g = g \circ e = g \circ (a \circ h) = (g \circ a) \circ h = e \circ h = h$. You should give reasons for each step.

Meet and Join

There are two binary operations on B={0,1} that are basic in computer design and operation. The meet $\bigwedge$ and join $\bigvee$operations are given by the tables below.

$\bigwedge$	0	1
0	0	0
1	0	1

$\bigvee$	0	1
0	0	1
1	1	1

The meet operation is similar to intersection $\cap$ and join is similar to union $\cup$, and behave like the logical connectives "and" and "or" respectively. Each of $\bigwedge$ and $\bigvee$ are commutative and associative. Are there identities, inverses?

BITWISE ADDITION MODULO 2

Another binary operation having applications in coding theory is based on the table below.

$\oplus$	0	1
0	0	1
1	1	0

Here, $a \oplus b$ is 0 if $a + b$ is even and $a \oplus b = 1$ if $a + b$ is odd. Equivalently, $a \oplus b = 0$ if $a = b$, and $a \oplus b = 1$ if $a \neq b$. The binary operation $\oplus$ on $B = \{0, 1\}$ is called "bitwise addition modulo 2". On $B^2 = \{00, 01, 10, 11\}$, the table for $\oplus$ is given.

$\oplus$	00	01	10	11
00	00	01	10	11
01	01	00	11	10
10	10	11	00	01
11	11	10	01	00

The operation is performed bitwise; $10 \oplus 01 = 11$ since $1 \oplus 0 = 1$ and $0 \oplus 1 = 1$. You are asked to investigate properties of $\oplus$ in the exercises.

PROBLEMS.

1. (a) Is subtraction a commutative binary operation on Z? Explain.

   (b) Is subtraction associative on Z?

   (c) is multiplication commutative on Z?

   (d) Does multiplication have an identity on Z? Are there inverses?

2. Give an example of subsets A and B of Z so that

   (a) subtraction is not a binary operation on A.

(b) multiplication is not a binary operation on B.

3. Let S={a,b,c}, and $P(S)$ be the power set of S.

   (a) Is $\cup$ on $P(S)$ commutative Explain.

   (b) Does {a} have an inverse?

4. Let S={a,b,c,d}.

   (a) Is $\cap$ on $P(S)$ commutative, associative? Explain.

   (b) Does $\cap$ have an identity?

   (c) What is the inverse of A={b,d} under $\cap$?

5. Let $\circ$ be the binary operation on $N = \{0, 1, 2, 3, \dots\}$ with $m \circ n = (5^m)(2n + 1)$.

   (a) Compute $2 \circ 3$ and $3 \circ 2$. Is $\circ$ commutative?

   (b) Is $\circ$ associative? Does $\circ$ have an identity?

6. Let $\circ$ be the binary operation $m \circ n = m + n - mn$ on Z. Is $\circ$ commutative, associative? Does $\circ$ have an identity?

7. Make a table of inverses for the operation in example 4.

8. Let $A = \{1, -1, i, -i\}$ and let $\square$ denote ordinary complex multiplication.
   (a) Make the matrix table for $\square$. (b) Is $\square$ associative, commutative?
   (c) Does $\square$ have an identity?
   (d) Give a table of inverses for the elements of A.

9. What is the identity for the binary operation in example 5? Are there inverses?

10. Let $A = \{1, 2, 5, 10\}$ and define a binary operation on A by $m \square n = \text{LCM}(m, n)$. Make the matrix table for $\square$ and decide whether $\square$ is commutative, has an identity, inverses.

11. Let $A = Z = \{0, \pm 1, \pm 2, \pm 3, \dots\}$ and define a binary operation on Z by $m \square n = m + n - 1$.
    (a) Is $\square$ commutative? (b) Does $\square$ have an identity?
    (c) Are there inverses?

12. Define the operation $\square$ on Z^+ as follows: $m \square n = \text{GCD}(m, n) + \text{LCM}(m, n)$. Is $\square$ associative?

13. (a) Is $m \square n = 3^m \cdot n$ commutative on $N = \{0, 1, 2, 3, \dots\}$?
    (b) How many ordered pairs (m, n) can you find so that $m \square n = 18$?

14. Which properties are satisfied by $\oplus$, bitwise addition modulo 2?

15. Show that $m \square n = n\, 2^m$ is not associative on $N = \{0, 1, 2, \dots\}$. Is $\square$ commutative?

16. Let $m \circ n = m + n - 3$ on Z. What is the identity? What is the inverse of an element p?

Groups

A group is an algebraic structure that consists of two items: a set of elements G, and a binary operation $\circ$. This structure satisfies FOUR axioms:

CLOSURE: For any elements a and b in G, $a \circ b$ is also in G.

IDENTITY: There is a unique element e in G such that for any $a \in G$ we have $a \circ e = a = e \circ a$. INVERSES: For every element a in G there is an element a^{-1} in G so that $a \circ a^{-1} = e = a^{-1} \circ a$

ASSOCIATIVITY: For any three elements a, b, c in G we have $(a \circ b) \circ c = a \circ (b \circ c)$.

A general group can be denoted as $(G, \circ)$ indicating the importance of having both a carrier set G and a binary operation $\circ$. When the operation is clear from a particular context we may write just G for that group.

Example 1. $(Z, +)$ is a group under the usual addition operation. Choose a, b in Z. Since $a + b$ is an integer CLOSURE holds. The identity e is 0 since $0 + a = a = a + 0$ for any $a \in G$. The inverse of a is $-a$ (we could write $a^{-1} = -a$) since $a + (-a) = 0 = (-a) + a$. ASSOCIATIVITY holds since $(a + b) + c = a + (b + c)$.

Example 2. In the exercises you will show that the following are groups: $(Q, +)$, $(R, +)$, $(Q^+, \times)$, $(R^+, \times)$ where Q=Rationals, R=Reals, Q^+=positive Rationals, R^+=positive Reals.

Example 3. In the exercises you will show that the following are not groups: $(Z, -)$, $(Z, \div)$.

ABELIAN GROUPS

Some groups have an additional fifth property called commutativity. A binary operation $\circ$ on a set G is commutative if $a \circ b = b \circ a$ for all a, b in G. We also say that the group $(G, \circ)$ is an abelian group, named after Niels Abel, a major contributor in the development of group theory. He also proved the insolvability of the fifth-degree polynomial equation, one of his greatest achievements.

The examples involving Z, Q, and R above are all abelian groups.

GROUP TABLES

When the set G is finite (most of the above examples were infinite) the four group properties and be readily detected from an operation table. We saw earlier that the set $G = \{1, 3, 7, 9\}$ was closed under units multiplication denoted $\otimes$. The operation table follows.

$\otimes$	1	3	7	9
1	1	3	7	9
3	3	9	1	7
7	7	1	9	3
9	9	7	3	1

The symbol $\otimes$ means take a from the left most column and "multiply" by b from the very top row. The 16 interior elements are just 1, 3, 7 and 9 showing closure. The element 1 acts like the identity – look at the top interior row and the left most interior column. Inverses are easy to find; just look for the 1's in the table. Since $1 \otimes 1 = 1$, $3 \otimes 7 = 1$ and $9 \otimes 9 = 1$ we have the following table

of inverses.

x	1	3	7	9
x^{-1}	1	7	3	9

Associativity is inherited from multiplication in Z; we now know that $(G, \otimes)$ is a group. in fact, the symmetry of the table shows that G is an abelian group.

Example 4. Here is an example of a group that involves functions and algebra. The operation is composition of functions. Let $f(x) = x$, $g(x) = \frac{1}{1-x}$ and $h(x) = \frac{x-1}{x}$. $G = \{f, g, h\}$ is a group with identity function f. The "product" gh is $g(h(x)) = g(\frac{x-1}{x}) = \frac{1}{1-(\frac{x-1}{x})} = \frac{x}{x-(x-1)} = x$. Conclusion: $gh = f$. You should form the other products and make the group table.

Consequences of the 4 Group Axioms

Theorem 1. If $ab = ac$ in a group G then $b = c$. (This is called left cancellation).
Proof: Multiply each side of $ab = ac$ on the left by a^{-1}. You get $(a^{-1}a)b = (a^{-1}a)c$ or $b = c$. A similar result holds for right cancellation. But be careful "mixed" cancellation may not work, i.e. $ab = ca$ does not necessarily imply $b = c$.

Theorem 2. In the multiplication table for a group $G = \{g_1, g_2, \dots\}$ each element of G appears exactly once in each row.
Proof: The entries on row r are $rg_1, rg_2, rg_3, \dots rg_n$. If two of these are the same, say $rg_i = rg_j$ then $g_i = g_j$ upon left multiplication by r^{-1}. But this is a contradiction. Why? It can also be shown that elements in any column are distinct.

Theorem 3. For any $a \in G$, $(a^{-1})^{-1} = a$.
Proof: $a\,a^{-1} = e$. This is like saying that the inverse of 2 in $(Q, \times)$ is $2^{-1} = \frac{1}{2}$ since

$2 \cdot \frac{1}{2} = 1$, the identity.

Theorem 4. $(abc)^{-1} = c^{-1}b^{-1}a^{-1}$. (The sock-shoe theorem).

Proof: $(abc)(c^{-1}b^{-1}a^{-1} = ab(c\,c^{-1})b^{-1}a^{-1} = a(b\,b^{-1})a^{-1} = a\,a^{-1} = e$

Theorem 5. If $a = a^{-1}$ for all $a \in G$, then G is abelian.

Proof: $ab = a^{-1}b^{-1} = (ba)^{-1} = ba$. This result is equivalent to saying that if the operation table has e, the identity, down the main diagonal then G is abelian. The reason; $a = a^{-1}$ implies $a^2 = e$.

Theorem 6. Let a and b be in a group G. Show that $(ab)^{-1} = a^{-1}b^{-1}$ if and only if $ab = ba$.

Theorem 7. Prove that if $(ab)^2 = a^2b^2$ in a group G if and only if G is abelian.

A GROUP GENERATOR

Sometimes there is an element in a group G whose powers (sums) generate the entire group. In $G = \{1, 3, 7, 9\}$ under units multiplication, 3 is such an element since $3^0 = 1, 3^1 = 3, 3^2 = 9, 3^3 = 7$. We can write $[3] = \{1, 3, 7, 9\}$ to show that 3 is a generator. It is also true that $[7] = [3]$, but $[9] \neq \{1, 3, 7, 9\}$. A group that has a generator is called cyclic. Hence {1,3,7,9} is a cyclic group. $\{1, -1, i, -i\}$ is also a cyclic group under complex multiplication, generated by either i or $-i$. $(Z, +)$ is an additive cyclic group generated by 1 or -1. For an additive group, powers are replaced by sums: $2 = 1 + 1, 3 = 1 + 1 + 1, 4 = 1 + 1 + 1 + 1, \ldots$ and so on. In summary, for a group whose operation is multiplication a^m means $a \cdot a \cdot a \cdot \cdots \cdot a$; if the operation is addition, $m \cdot a$ means $a + a + a + \cdots + a$.

Subgroups

Let $(G, \circ)$ be a group and let H be a subset of the set G. $(H, \circ)$ is a subgroup of $(G, \circ)$ if $(H, \circ)$ is closed under $\circ$, has the e of G as the identity and contains inverses. Associativity is inherited from $(G, \circ)$. Examples are easy to find. $(Z, +)$ is a subgroup of $(Q, +)$. {1,9} is a subgroup of {1,3,7,9} under mod 10 multiplication. The set $\{1, -1\}$ is a subgroup of $\{1, -1, i, -i\}$ which itself is a subgroup of all complex numbers under multiplication. The set of all integral multiples of 3, $H = \{0, \pm 3, \pm 6, \dots\}$ is a subgroup of $(Z, +)$. Can you give a subset S of $(Z, +)$ such that S is closed under addition but is not a subgroup?

Modular Groups

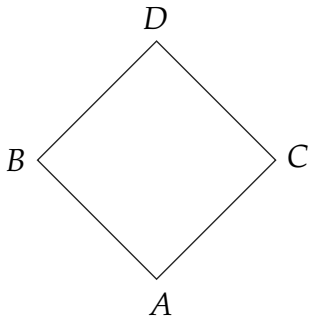
Clock arithmetic provides a fruitful source of nice finite groups. Recalling that 13 o'clock is really just 1 o'clock upon subtraction of 12, we can make a clock with just the four numbers 0,1,2,3 the remainders when any integer n is divided by 4. We can write $7 \equiv 3 (\text{mod } 4)$ for example. This is read as 7 is congruent to 3 modulo 4. In general $a \equiv b (\text{mod } m)$ means that a and b have the same remainder when divided by m; or that $a - b$ is divisible by m. Definition. Let $Z_n = \{0, 1, 2, 3, \dots, n - 1\}$. The sum $a + b$ of any two elements in Z_n is just the remainder when $a + b$ is divided by n. With this definition of sum we can see

that $(Z_n, +)$ is a group. Lets form the operation table for $Z_4 = \{0, 1, 2, 3\}$.

+	0	1	2	3
0	0	1	2	3
1	1	2	3	0
2	2	3	0	1
3	3	0	1	2

The table shows closure, and that 0 is the identity. Locate the 0's in the table to see that the inverse of 0 is 0, the inverse of 1 is 3 (since 1+3=0), the inverse of 2 is 2, and the inverse of 3 is 1. This new sum rule is an associative binary operation on Z_4 since ordinary addition is associative on Z, the usual integers. In similar analysis, $(Z_n, +)$ is an additive group.

Example 5. $Z_6 = \{0, 1, 2, 3, 4, 5\}$ is a group with a number of subgroups. $A = \{0\}$, $B = \{0, 3\}$, $C = \{0, 2, 4\}$, and $D = \{0, 1, 2, 3, 4, 5\}$ are all subgroups of Z_6. The following picture, called a lattice of subgroups, shows the relationships between the subgroups.



The upward sloping lines indicate subgroup inclusion: $A \subseteq B, A \subseteq C, B \subseteq D, C \subseteq D$ and $A \subseteq B \subseteq D, A \subseteq C \subseteq D$, two chains of length three. Example 6.

The modular groups Z_n are cyclic. For Z_4, 1 is a generator since

$$1 \cdot 1 = 1 = 1$$

$$2 \cdot 1 = 1 + 1 = 2$$

$$3 \cdot 1 = 1 + 1 + 1 = 3$$

$$4 \cdot 1 = 1 + 1 + 1 + 1 = 4$$

In additive notation $3 \cdot 1$ means add three 1's together. In Z_6, only 1 and 5 are generators. For example, 3 is not a generator since you can only make 0 and 3 using multiples $m \cdot 3$ of 3. Try it! Likewise, the element 2 will only generate 0, 2, 4. The previous example prompts the question: which elements in Z_n are generators? The following chart might provide a clue as you pursue the question in the exercises.

n	Z_n	generators
2	$\{0, 1\}$	1
3	$\{0, 1, 2\}$	1, 2
4	$\{0, 1, 2, 3\}$	1, 3
5	$\{0, 1, 2, 3, 4\}$	1, 2, 3, 4
6	$\{0, 1, 2, 3, 4, 5\}$	1, 5

PROBLEMS.

1. Show that each of the following are groups.

   (a) $(Q, +)$ (b) $(R, +)$ (c) $(Q^+, \times)$ (d) $(R^+, \times)$

2. Show that the following are not groups

   (a) $(Z, -)$ (b) $(Z, \div)$ (c) $(Z, \times)$

3. Make the multiplication table for $A = \{4, 8, 12, 16\}$ under multiplication mod 20. Does A have an identity?

4. is $B = \{2, 4, 6, 8\}$ under units multiplication a group? Is B cyclic? Is B abelian?

5. Verify that 7 is a generator of the group {1,3,7,9} under units multiplication, but that 9 is not a generator.

6. Let $S = \{e, a, b, c\}$. Make the 4×4 group operation table assuming that e is the identity and that $a^2 = b^2 = c^2 = e$. Is S cyclic? Abelian?

7. Show that $G = \{1, -1, i, -i\}$ is a group under ordinary complex multiplication. Is G cyclic?

8. Show that $G = \{00, 01, 10, 11\}$ is a group using bitwise addition mod 2. Is G cyclic?

9. Give an example of a group that illustrates Theorem 5.

10. Prove that in a group $(abcd)^{-1} = d^{-1}c^{-1}b^{-1}a^{-1}$. Why is this called the sock-shoe theorem?

11. Make the group table for $G = \{000, 001, 010, 011, 100, 101, 110, 111\}$ using bitwise addition mod 2. Is G abelian? Is G cyclic?

12. Is {1,3} a subgroup of {1,3,7,9} under mod 10 multiplication?

13. Verify that $H = \{0, \pm 7, \pm 14, \dots\}$ is a subgroup of Z under addition.

14. Is the set of all complex numbers α with $|\alpha| \leq 1$ a subgroup of all nonzero complex numbers under multiplication? Here, if $\alpha = a+bi$, $|\alpha| = \sqrt{a^2 + b^2}$, its distance from the origin.

15. Make the operation table for Z_6 and find all subgroups.

16. Make the operation table for Z_8 and find all subgroups.

17. Without making the addition table for Z_{12} can you give all the closed subsets of Z_{12}? Are these in fact subgroups? Draw the lattice of subgroups of Z_{12}.

18. Show that in $(Z_n, +)$, the additive groups of integers modulo n, that the inverse of any $a \neq 0$ is $n - a$.

19. Explain why $(Z_4, \bullet)$ is not a group where multiplication is modulo 4.

20. Verify associativity for the following sum in Z_7:
    (3+5)+6 and 3+(5+6)

21. Find all the generators for the cyclic group Z_5 and verify that each in fact generates all of Z_5.

22. Determine those elements in Z_n that are generators.

23. Solve the quadratic equation $x(x + 1) = 0$
    (a) in Z_4 (b) in Z_5 (c) in Z_6

24. Make the group multiplication table for $G = \{e, a, a^2, a^3, a^4\}$ where e is the identity and $a^5 = e$. Hint: $a^3 \cdot a^4 = a^7 = a^5 \cdot a^2 = a^2$.

25. Let G be a group. Prove that for any $a \in G, H = \{x \in G : x = a^n \text{ for } n \in Z\}$ is a subgroup of G (generated by a).

More on Cyclic Groups

If a group G is made up of entirely of powers of a particular element, call it a, then G is called a cyclic group denoted by $G = [a]$. The element a is called a generator and the least positive integer s such that $a^s = e$, the identity in G, is called the order of a. G can be a finite or infinite group.

Example 1. The additive group Z is cyclic having two generators: either 1 or -1 will generate all of Z. $5Z = [5] = \{0, \pm 5, \pm 10, \dots\}$ is a cyclic subgroup of Z. The element 5 has infinite order.

Example 2. Let $G = [a]$ be a cyclic group of order 12. Then $[a] = \{e, a, a^2, a^3, \dots, a^{10}, a^{11}\}$. The element a has order 12; it is a generator. The element a^7 is also a generator. The order of a^3 is 4 since 4 is the least power m such that $(a^3)^m = e$.

Example 3. Let $G = [a]$ be a cyclic group of order 30. The order of a^9 is 10 and $[a^9] = \{e, a^9, a^{18}, a^{27}, a^6, a^{15}, a^{24}, a^3, a^{12}, a^{21}\}$.

Example 4. The additive group $Z_n = \{0, 1, 2, \dots, n-1\}$ is cyclic. Z_{12} has 1,5,7 and 11 as generators. We can write $Z_{12} = [1] = [5] = [7] = [11]$. Since GCD(3,12)=3, the element 3 generates a subgroup of $\frac{12}{3} = 4$ elements; [3]={0,3,6,9}.

PROBLEMS.

1. V_n is the subset of Z_n having multiplicative inverses. Is V_8 cyclic?
2. In Z_9, is the multiplicative group {1, 2, 4, 5, 7, 8} cyclic?
3. Let $[a] = \{e, a, a^2, \dots, a^{23}\}$ be a cyclic group of order 24. List the elements of a subgroup of order 3. What is the order of a^5 in $[a]$?

4. In example 2, which elements of $[a]$ are generators? Why?

5. Is $G = \{1, 3, 7, 9\}$ under units multiplication cyclic?

6. Prove that every cyclic group is abelian.

7. What is the order of the cyclic subgroup of Z_{30} generated by 25?

8. What is the order of the cyclic subgroup $[i]$ of the nonzero complex numbers under multiplication?

9. Find the number of generators of a cyclic group with order:
   (a) 7 (b) 9 (c) 15 (d) 60

10. Let $G = [a]$ be a cyclic group of order 18.
    (a) List all the elements of order 3 in G.
    (b) List all the elements of order 4 in G.

11. Let V be the multiplicative group of the nonzero complex numbers, and let $\omega = (-1 + i\sqrt{3})/2$.
    (a) Show that $\omega = \cos 120^\circ + i \sin 120^\circ$.
    (b) Show that $\omega^3 = 1$ What is the order of ω?
    (c) What is the order of $\cos(5\pi/11) + i \sin(5\pi/11)$ in V?
    (d) What is the order of $(1 + i)/\sqrt{2}$ in V?
    (e) What is the order of $1 + i$ in V?

Lagrange's Theorem

Theorem. Let G be a finite group. The order of any subgroup H divides the order of G.

This will be seen by partitioning the elements of G into non overlapping sets called cosets, as illustrated in the following.

Let $H = \{1, 14\}$ be a subgroup of the group of invertibles $V_{15} = \{1, 2, 4, 7, 8, 11, 13, 14\}$.

DEFINITION. Choose an element a in G. A (left) coset of H is the set aH consisting of all products ah with $h \in H$.

In this example,$1H = \{1, 14\}$, $2H = \{2, 13\}$, $4H = \{4, 11\}$, $7H = \{7, 8\}$

The order of $H = 2$ = order $G/\#$ of cosets. The number of cosets is called the index of H in G.

In general, $H = \{h_1, h_2, \ldots, h_n\}$, $aH = \{ah_1, ah_2, \ldots, ah_n\}$. Cosets have two important properties:

1. The elements of aH are distinct. If $ah_i = ah_j$, left cancellation will produce a contradiction.

2. Distinct left cosets are disjoint – not too hard to prove. Just assume some element is in both aH and bH and see what happens.

Now let order $G = s$ and order $H = r$. Start making all (left) cosets until you exhaust all elements of G.

$$aH = \{ah_1, ah_2, ah_3, \ldots, ah_r\}$$

$$bH = \{bh_1, bh_2, bh_3, \ldots, bh_r\}$$

$$\vdots$$

There can only be a finite number of these cosets, say t. Then $rt = s$. This completes the proof of LaGrange's Theorem.

Here is an example of an additive group where $o(G)$=12, $o(H)$=4 and the index $t = 3$. Let $G = Z_{12} = \{0, 1, 2, \ldots, 11\}$ and $H = \{0, 3, 6, 9\}$. The three cosets are

$$0 + H = \{0, 3, 6, 9\}$$

$$1 + H = \{1, 4, 7, 10\}$$

$$2 + H = \{2, 5, 8, 11\}$$

Verify properties 1 and 2 above, and that every element Z_{12} is in precisely one coset.

As a corollary to Lagrange's Theorem, we have that the order of every element in a finite group must divide $o(G)$.

PROBLEMS.

1. Find all the (left) cosets of $H = \{0, 6\}$ in Z_{12}.
2. Find all the cosets of $H = \{0, 4, 8\}$ in Z_{12}
3. Find all cosets of $H = \{1, 8\}$ in V_9. Does V_9 have any other subgroups.
4. Find all cosets of $H = \{1, 4\}$ in V_{15}. Repeat with $H = \{1, 4, 11, 14\}$.
5. Prove that every group having prime order must be cyclic.
6. Show that a cyclic group of order 22 has one element of order 2 and ten elements of order 11.
7. Find all the cosets of $H = \{1, r, r^2, r^3\}$ in the octic group of the symmetries of a square.

8. Find all cosets of $H = 2Z$ in Z. We call these the evens and odds. One can actually make a group out of these cosets.

9. Let $G = [a]$ be a cyclic group of order 91. Find a subgroup having index 13.

Here is an important consequence of Lagrange's Theorem.

FERMAT'S Theorem – If a is an integer and p is a prime then $a^p \equiv a (\text{mod } p)$.

Proof: $V_p = \{1, 2, \ldots, p-1\}$ has order $o(V_p)=p-1$. By Lagrange's Theorem, every element a satisfies $a^{p-1} \equiv 1 (\text{mod } p)$.

This result extends to

EULER'S Theorem – If $(a, m) = 1$, then $a^{\phi(m)} \equiv 1 (\text{mod } m)$.

Proof. The order of V_m is $\phi(m)$.

Group Isomorphisms

The set of rational numbers is denoted by $Q = \{\frac{a}{b} : b \neq 0, a, b \in Z\}$. The special subset of Q where $b = 1$ is essentially like $Z = \{0, \pm 1, \pm 2, \dots\}$. For all practical purposes, these two sets are the same. This similarity is addressed in the mathematical language that follows:

Definition. An ISOMORPHISM between two groups $(G, \circ)$ and $(G', \square)$ is a mapping $\theta : G \mapsto G'$ such that

(a) θ is one-to-one

(b) θ is onto

(c) $\theta(a \circ b) = \theta(a) \square \theta(b)$ (θ preserves the operation).

For property (c) the operation $a \circ b$ takes place in G, while $\theta(a) \square \theta(b)$ occurs in the image group G'. This same concept is seen in calculus when we write $\lim(f(x) \cdot g(x)) = \lim f(x) \cdot \lim g(x)$ or $(f + g)' = f' + g'$ for derivatives. It is much more notorious when seen as the students' dream: $(x + y)^2 = x^2 + y^2$ or $\log xy = (\log x)(\log y)$.

Here are several examples.

Example 1. Let $Q^* = \{\frac{a}{1} : a \in Z\}$. Then Q^* is isomorphic to Z under the isomorphism $\theta(\frac{a}{1}) = a$. θ is $1-1$, onto and $\theta(\frac{a}{1} + \frac{b}{1}) = \frac{a+b}{1} = a + b = \theta(\frac{a}{1}) + \theta(\frac{b}{1})$.

Example 2. $Q^{**} = \{\frac{a}{2} : a \in Z\}$ is isomorphic to Z.

Example 3. The group $A = \{1, -1, i, -i\}$ under complex multiplication is isomorphic to the cyclic group $B = \{e, a, a^2, a^3\}$ under the following mapping:

x	1	-1	i	$-i$
$\theta(x)$	e	a^2	a	a^3

θ is evidently 1-1 and onto. Here, $\theta(ab) = \theta(a)\theta(b)$ can be checked for each pair a, b. For example, $\theta[i(-i)] = \theta[1] = e$ and $\theta(i) \cdot \theta(-i) = a \cdot a^3 = e$.

Example 4. Let $G = (R, +)$, the real numbers under addition, and $G' = (R^+, \times)$, the positive reals under multiplication. The mapping $\theta(x) = 2^x$ is an isomorphism from G onto G'. Properties of logarithms show 1-1 and onto. $\theta(a + b) = \theta(a) \cdot \theta(b)$ follows from $\theta(a + b) = 2^{a+b} = 2^a \cdot 2^b = \theta(a)\theta(b)$.

Example 5. Again, let $G = (R, +)$. The mapping $\theta : G \mapsto G$ (itself) given by $\theta(x) = x^2$ is not an isomorphism.

PROBLEMS.

1. Find an isomorphism from $(Z, +)$ to $(2Z, +)$.

2. Regarding Example 3, is the following also an isomorphism? $\theta(1) = e$, $\theta(-1) = a$, $\theta(i) = a^2$, $\theta(-i) = a^3$

3. Regarding Example 4, prove that $\theta(x) = 2^x$ is 1-1 and onto.

4. In Example 5, prove that $\theta(x) = x^2$ is not an isomorphism.

5. Can you give an isomorphism from $G' = (R^+, \times)$ onto $G = (R, +)$?

6. If group A is isomorphic to the group B, and if A is abelian, prove the B is abelian.

7. If $\theta : G \mapsto G'$ is an isomorphism and e is the identity of G and e' is the identity in G', then $\theta(e) = e'$ and $\theta(x^{-1}) = [\theta(x)]^{-1}$.

8. Let θ be a group isomorphism from G to G', show that if $\theta(a) = a'$ then $\theta(a^n) = (a')^n$.

9. Show that if $\theta(a) = a'$ under a group isomorphism then a and a' have the same order.

10. Prove that V_{10} is isomorphic to V_5, ie, that the set of invertibles in Z_{10} is isomorphic to the invertibles in Z_5.

11. Let $\theta : (R^+, \times) \mapsto (R^+, \times)$ be defined by $\theta(x) = \sqrt{x}$. Is θ an isomorphism from $(R^+, \times)$ to itself?

12. Show that V_8 is not isomorphic to V_{10}. Hint: Make a table of orders.

13. Show that V_8 is isomorphic to V_{12}.

14. Let $G = \{0, \pm 3, \pm 6, \pm 9, \dots\}$ and $H = \{0, \pm 7, \pm 14, \pm 21, \dots\}$. Are G and H isomorphic under addition? If yes, does that isomorphism preserve multiplication?

From certain of these exercises you can see that essential group properties are preserved under group isomorphisms. We say that these properties are invariant under isomorphisms.

HOW TO PROVE THAT TWO GROUPS ARE ISOMORPHIC:

1. First produce a mapping θ.

2. Check that it is 1-1 and onto.

3. Verify that $\theta(ab) = \theta(a)\theta(b)$.

HOW TO PROVE THAT TWO GROUPS G AND G' ARE NOT ISOMORPHIC:

1. Show that G and G' do not have the same order.

2. Show that one is abelian, the other not.

3. Look at the order of elements of each and note discrepancy.

4. Show that one is cyclic, the other not.

5. In general, look at invariants and see if something is not consistent.

Be careful on what you determine to be structural, ie, like the properites just seen in 1 - 4. For example, you cannot say that Z and $5Z$ under addition are not isomorphic because 13 is in Z but not in $5Z$. That is not a structural property.

Direct Products

Known groups can be building blocks for forming new groups. If G_1 and G_2 are groups then the cartesian product $G_1 \times G_2$ is a group under the operation $(a, b) \circ (c, d) = (ac, bd)$. The identity is (e_1, e_2) and the inverse of (a, b) is (a^{-1}, b^{-1}) since $(a, b) \circ (a^{-1}, b^{-1}) = (e_1, e_2)$. Closure and associativity are easy to see.

Sometimes the operation is viewed as additive. For example, $Z_2 \times Z_3$ is an additive group. It has the six elements: (0, 0), (0, 1), (0, 2), (1, 0), (1, 1), (1, 2). the element (1, 1) is a generator: 2(1, 1) = (1, 1)+(1, 1)=(0, 2), and so on.

PROBLEMS.

1. Show that (1, 1) is a generator of $Z_2 \times Z_3$.

2. Explain why $Z_3 \times Z_3$ is not cyclic.

3. Is $Z_2 \times Z_4$ cyclic? Is $Z_3 \times Z_4$ cyclic?

4. What is the order of (2, 6) in $Z_4 \times Z_{12}$?

5. What is the order of (3, 10, 9) in $Z_4 \times Z_{12} \times Z_{15}$?

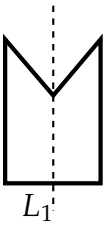
6. What are the orders of elements in $Z_3 \times Z_3 \times Z_3$?

7. Can $Z_2 \times Z_8$ be isomorphic to $Z_4 \times Z_4$?

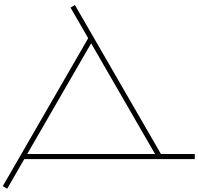
8. When is the group $Z_m \times Z_n$ isomorphic to Z_{mn}?

Groups of Symmetries

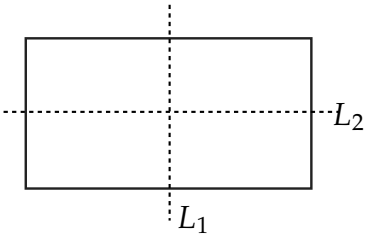
A symmetry of a figure is a rigid motion that leaves the figure unchanged. Lets take the following three figures and give their symmetries.



1. Do nothing
2. Flip about the axis L_1



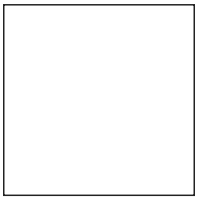
1. Do nothing
2. Rotate 120°
3. Rotate 240°



1. Do nothing
2. Flip about L_1
3. Flip about L_2
4. 180° rotation

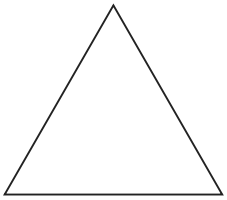
It is convenient to agree that rotation means clockwise.

Task 1. Describe the symmetries of a square.



1. Do nothing
2.
3.
4.
5.
6.
7.
8.

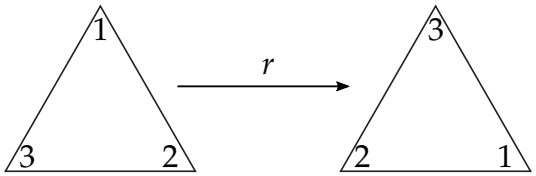
Task 2. Describe the symmetries of an equilateral triangle.



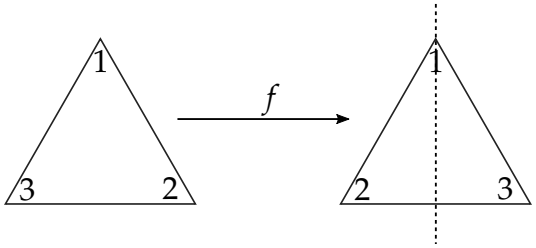
1. Do nothing
2.
3.
4.
5.
6.

Our goal here is to study more deeply such symmetries; introduction of good notation is necessary to stay organized. We will use the symmetries of an

equilateral triangle as our basic example. First, label the vertices and draw the action caused by a rotation and a flip:

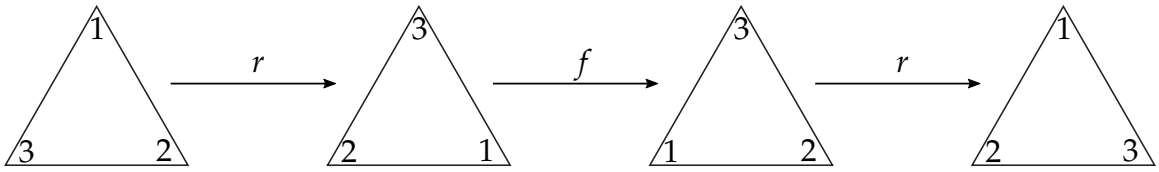


A clockwise rotation through 120°



A flip about an altitude holding vertex 1 fixed

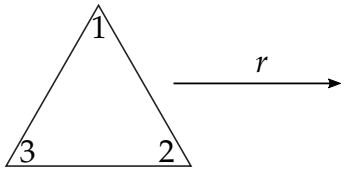
Combinations of rotations and flips give rise to other symmetries:



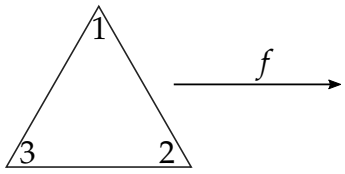
If we view the first or left most triangle as being in original position the movement of the vertices 1 goes to 3, 2 goes to 1, 3 goes to 2 could be expressed in the cycle form (123). Similarly the flip f is (23), where the vertices 2 and 3 are switched and 1 stays fixed.

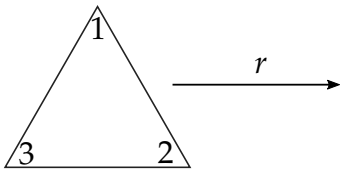
Task 3. Complete the sequence of pictures that will show:

r, followed by r, then f:



f, followed by r, then f:



r, followed by f, then r: 

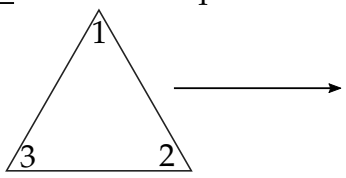
Task 4. For convenience we could let $frrf$ stand for f, then r, then r, then f.

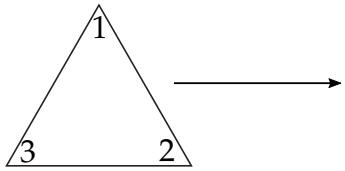
Say in your own words what each of these mean.

$frfrf$ means

r^2frrf means

Task 5. Draw the sequence of triangles to prove that $fr = r^2f$

fr : 

r^2f : 

Task 6. In addition to the flip $f = (23)$, there are two more flips. Express each in terms of just r and f.

Task 7. Show that your six symmetries of an equilateral triangle can be expressed as $\{1, r, r^2, f, rf, r^2f\}$ where 1 means “do nothing".

Task 8. SOME ALGEBRA

You showed in task 5 that $fr = r^2f$. This relationship can be used to show that the “product” of r^2f and rf, in that order, is just r.

$$(r^2f)(rf) = r^2(fr)f = r^2(r^2f)f = r^4f^2 = r \cdot 1 = r$$

Explain why $r^3 = 1$:

Explain why $f^2 = 1$

Similarly, express each of the following products as one of our six symmetries:

$$(rf)(r^2f) =$$

$$f(rf) =$$

$$r^2(r^2f) =$$

$$(r^2f)(r^2) =$$

Task 9. A MULTIPLICATION TABLE

Complete the operation table where the operation is "followed by". Each of the 36 entries should be one of our six symmetries. Multiply row entry by column entry in that order.

Δ	1	r	r^2	f	rf	r^2f
1						
r						
r^2						
f						
rf						
r^2f						

Task 10. We saw that in cycle form $r = (132)$ and $f = (23)$. Express each of the remaining four symmetries in cycle form.

$1 = (1)$ (This is a convention which shows that no vertex moved.)

$r^2 =$

$rf =$

$r^2 f =$

Task 11. You could label four of the square symmetries as $1, r, r^2, r^3$. Interpret these as rotations. What would $1, r, r^2, r^3, r^4$ mean in a regular pentagon?

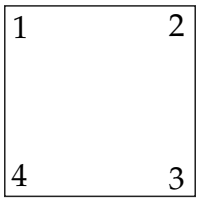
In algebra you have $r^m r^n = r^{m+n}$ and $r^0 = 1$. What do these rules mean in the setting of symmetries of regular n-gons?

What do you think r^{-1} means? What is f^{-1}, the inverse of f?

Using your 6 by 6 multiplication table, determine the inverse of each symmetry:

symmetry	1	r	r^2	f	rf	$r^2 f$
inverse						

Task 12. Use r for a 90° rotation and f to denote a flip about the diagonal fixing 1 and 3 and give the eight symmetries of the square in terms of r and f



A Brief Look at Rings

This type of structure should ring a bell. As in the integers Z, in a ring we can add, subtract, multiply and even distribute. So think of Z as our model of a ring.

Definition. A ring R is a set with two binary operations such that

(a) R is an abelian group under addition.

(b) R is closed and associative under multiplication.

(c) Multiplication is distributive over addition, ie, $a(b + c) = ab + bc$ and $(b + c)a = ba + ca$.

A commutative ring is a ring where $ab = ba$.

Example 1. Z, Q, R, and C are commutative rings.

Example 2. Z_m, the ring of integers modulo m, is commutative.

Example 3. The set $Z[x]$ of all polynomials in x with coefficients in Z is a commutative ring.

Definition. If a ring R has a multiplicative identity 1, then an element a in R is an invertible if there is an a^{-1} such that $a\,a^{-1} = 1$; 1 is called the unity.

Example 4. $2Z$ is a commutative ring with no unity.

A subring (like a subgroup) of a ring R is a subset of R that is a ring. The set $\{0, \pm 5, \pm 10, \dots\}$ is a subring of Z.

PROBLEMS

1. Show that $Z[\sqrt{2}] = \{a + b\sqrt{2} : a, b \in Z\}$is a ring.

2. In a ring R, show that $a^2 - b^2 = (a+b)(a-b)$ if and only if R is commutative.

3. Show that $Z[i] = \{a + bi : a, b \in Z\}$ is a ring. $Z[i]$ is called the ring of Gaussian Integers.

4. Find all invertibles in $Z[i]$.

5. Find the unity in $S = \{0, 2, 4, 6, 8\}$ under addition mod 10.

6. What are the invertibles in $Z \times Z$?

7. Let $R = \{0, 1, c\}$ be a ring with unity.

   (a) Show that $1 + 1 = c$ and that $1 + 1 + 1 = 0$.

   (b) Show that $c^2 = 1$.

   (c) Make the $\times$ and $+$ tables for R.

8. Let $R = \{0, 1, c, d\}$ be a ring (with unity 1) with c, d invertibles. Make the multiplication table for R.

Integral Domains

When you were asked to solve $x^2-7x+12 = 0$, you set each factor in $(x-3)(x-4)$ to zero and solved $x - 3 = 0$ or $x - 4 = 0$. Now try that in Z_{12}; you get 3, 4 and 7 as solutions. This quadratic has three roots.

Definition. if a and b are nonzero elements of a ring and $ab = 0$, we call a and b 0-divisors.

Example 1. in Z_{12}, the 0-divisors are 2, 3, 4, 6, 8, 9, 10 since $2 \cdot 6 = 3 \cdot 4 = 8 \cdot 9 = 6 \cdot 10 = 0$ in Z_{12}. These seven elements are precisely those numbers not relatively prime to 12.

Definition. An integral domain is a commutative ring D with a unity, that has no 0-divisors.

These are	These are not
Z, Q, R	Z_6, Z_{12}
Z_p, p a prime	Z_m, m composite
$Z[\sqrt{2}]$	$Z \times Z$
$Z[x]$	$2Z$
$Z[i]$	$Z_5[i]$

PROBLEMS

1. List all 0-divisors in Z_{20}. What are the invertibles?
2. Find all solutions to $x^2 - 4x + 3 = 0$
   (a) in Z_{12}
   (b) in Z_{11}
3. Find a 0-divisor in $Z_5[i] = \{a + bi : a, b \in Z_5\}$.

4. Show that $Z \times Z$, with multiplication and addition defined coordinatewise, is not an integral domain.

5. Why is $2Z$ not an integral domain?

6. Let $S = \{a, b, c\}$ and $P(S)$ be the power set of S, ie, the set of all subsets of S including ϕ and S. Define the product AB to be $A \cap B$ and the sum $A + B$ to be $(A \cup B) - (A \cap B)$, the elements in $A \cup B$ but not those in $A \cap B$.

   (a) Show that $P(S)$ is a commutative ring.

   (b) What is the unity?

   (c) What acts like a "0"?

   (d) Is $P(S)$ an integral domain?

7. In Z_6 show that $ab = ac$ does <u>not</u> imply $b = c$.

Fields – The Finale

A field is a commutative ring with unity where every nonzero element is an invertible. In other words, a field is an integral domain whose nonzero elements form a multiplicative group. More formally, a field F

(a) is an abelian group under addition

(b) is an abelian group under multiplication (don't count 0)

(c) has the property that multiplication is distributive over addition.

Example 1. Q, R, C, Z_p, $Q[\sqrt{2}]$ are fields.

Example 2. $Z_3[i]$, $Z_7[i]$ and $Z_{11}[i]$ are fields; but $Z_2[i]$, $Z_5[i]$ and $Z_{13}[i]$ are not. [Note that when $p \equiv 3 (\text{mod } 4)$, $Z_p[i]$ is a field].

Theorem. Every finite integral domain D is a field.

Proof. Let $a \in D$; we show that a has a multiplicative inverse. Let $D = \{0, 1, a_1, a_2, \ldots, a_n\}$. The elements $a \cdot 1, a\,a_1, a\,a_2, \ldots, a\,a_n$ are surely distinct since D is an integral doman, and none of these products is 0 since D has no 0-divisors. So one of these must be 1; let's say $a\,a_i = 1$. But then a_i is the inverse of a.

PROBLEMS.

1. Verify that $Q[\sqrt{2}]$ is a field. Show that $2 + 3\sqrt{2}$ has an inverse.

2. Why is $Z_2[i]$ not a field?

3. Why is $Z_{13}[i]$ not a field?

4. Is $Z[\sqrt{2}]$ a field?

5. Let $F = \{0, 2, 4, 6, 8\}$ under addition and multiplication modulo 10. Prove that F is a field.

Appendix: Cosets, Normal Subgroups, and Quotient Groups

LaGrange's Theorem states that the order of any subgroup H divides the order of a (finite) group G. This is (was) proved by partitioning the elements of G into nonoverlaping sets called cosets. For $a \in G$ a (left) coset of H is the set $\{aH\}$ consisting of all products ah with $h \in H$. The number of cosets is called the index of H in G. A right coset is the set $\{Ha : a \in G\}$. If $aH = Ha$ for every $a \in G$, as sets, we say that H is a normal subgroup. . For an additive group, the notation $a + H = H + a$ is used.

Example 1. Let $S_3 = \{1, r, r^2, f, rf, r^2f\}$ be the rigid transformations of an equilateral triangle. $H = \{1, f\}$ is not a normal subgroup: $rH = \{r, rf\}$, but $Hr = \{r, fr\} = \{r, r^2f\}$. As sets, $rH \neq Hr$.

Example 2. $H = \{1, r, r^2\}$ is a normal subgroup of $G = S_3$. $aH = Ha$ if $a \in H$. So now try the other three: $fH = \{f, r^2f, rf\} = Hf$, $rfH = \{rf, f, r^2f\} = Hrf$, and also $r^2fH = Hr^2f$. So $aH = Ha$, for all $a \in G$.

The good news is that if G is abelian, all subgroups are normal. Here is another helpful result: If the index of H in G is 2, H is normal. That's why the H in Example 2 is normal.

It turns out that if H is normal you can multiply cosets! You get the following nice process: $aHbH = abH$. This is not hard to show: $abH \subseteq aHbH$ is easy to show. For $aHbH \subseteq abH$, use normality.

Theorem 1. *Let N be a normal subgroup of G. The product $aNbN$ of cosets is the coset abN.*

Theorem 2. *The (left) cosets of a normal subgroup H in G form a group.*

Proof.

1. Closure: $aHbH = abH$
2. Identity: If e is the identity in G, eH is the identity in the group since $eHaH = eaH = aH = H = aeH = aHeH$.
3. Inverses: The inverse of aH is $a^{-1}H$ since $aHa^{-1}H = a^{-1}aH = H = a^{-1}HaH$.
4. Associativity: $aH(bHcH) = aHbcH = abcH$, and $(aHbH)cH = abHcH = abcH$.

□

Definition 1. If N is a normal subgroup of G, the set of all left cosets aN form a group called the QUOTIENT GROUP, or Factor Group, denoted G/N.

Example 3. Let $G = Z_6$ and $H = \{0, 3\}$. Since Z_6 is abelian, H is a normal subgroup. The three cosets are H, $1 + H = \{1, 4\}$ and $2 + H = \{2, 5\}$ and $G/H = \{H, 1 + H, 2 + H\}$. We next make the addition table for Z_6, with the elements rearranged as $0, 3, 1, 4, 2, 5$; next to it make the operation table for G/H:

$\oplus$	0	3	1	4	2	5
0	0	3	1	4	2	5
3	3	0	4	1	5	2
1	1	4	2	5	3	0
4	4	1	5	2	0	3
2	2	5	3	0	4	1
5	5	2	0	3	1	4

	H	$1+H$	$2+H$
H	H	$1+H$	$2+H$
$1+H$	$1+H$	$2+H$	H
$2+H$	$2+H$	H	$1+H$

These two tables look almost the same!

There are several features of this example to notice:

1. The three cosets partition the elements of Z_6 into three disjoint sets.
2. $H = \{0, 3\}$ is the identity coset.
3. Some funny addition is going on: $(1 + H) + (2 + H) = \{1, 4\} + \{2, 5\} = \{1 + 2, 1 + 5, 4 + 2, 4 + 5\} = \{3, 6, 6, 9\} = \{0, 3\} = H$.
4. The inverse of $1 + H$ is $2 + H$.
5. The index of H in G is 3.

Example 4. Let $G = V_{15} = \{1, 2, 4, 7, 8, 11, 13, 14\}$ and $H = \{1, 11\}$. H is normal in G (why?). The four cosets of H are H, $2H = \{2, 7\}$, $4H = \{4, 14\}$, and $8H = \{8, 13\}$. The quotient group table is:

	H	$2H$	$4H$	$8H$
H	H	$2H$	$4H$	$8H$
$2H$	$2H$	$4H$	$8H$	H
$4H$	$4H$	$8H$	H	$2H$
$8H$	$8H$	H	$2H$	$4H$

Which group of order 4 is this? Z_4 or the Klein 4-group?

Example 5. Let $G = Z$, and $H = 3Z$. Then the cosets in G/H are $0 + H$, $1 + H$ and $2 + H$. If $H = 2Z$ the cosets are simply the set of even integers, and the set of odd integers.

PROBLEMS

1. List the left cosets of the subgroup $3\mathbb{Z}$ in $\mathbb{Z}$. Now list the right cosets. These are the same since the additive group $\mathbb{Z}$ is an abelian group.

2. List the left cosets of $H = \{1, rf\}$ in S_3. Now list the right cosets. What do you notice?

3. Let $H = \{..., -10, -5, 0, 5, 10, ...\}$. Find all the left cosets of H in $\mathbb{Z}$. Are $-3 + H$ and $-8 + H$ in the same coset? Where would you find 21? How about -17?

4. Let $H = \{1, 8\}$ be a subgroup of $V_9 = \{1, 2, 4, 5, 7, 8\}$, the multiplicative group of the invertibles mod 9. Make all left cosets of H in V_9, and make the group table for V_9/H.

5. $H = \{1, r, r^2\}$ is a normal subgroup of $S_3 = \{1, r, r^2, f, rf, r^2f\}$. Make the group table for the quotient group S_3/H.

6. What is the size of $3 + H$ where $H = \{0, 6, 12\}$ is a subgroup of $\mathbb{Z}_{18}$? What is the order of $3 + H$ in $\mathbb{Z}_{18}/H$?

7. If G is an abelian group, explain why every subgroup of G is normal.

8. Determine the elements of the quotient group for each of the following:

   (a) $G = \mathbb{Z}_{12}$ $H = \{0, 4, 8\}$

   (b) $G = \mathbb{Z}$ $H = 2\mathbb{Z}$

   (c) $G = S_4$ $H = \{1, r, r^2, r^3\}$

   (d) $G = V_{15}$ $H = \{1, 4, 11, 14\}$

(e) $G = V_{15}$ $H = \{1, 4\}$

9. Let $G = [a]$ be a cyclic group of order 21 generated by a, and let H be a subgroup having index 3. List the elements of H and the elements of G/H. Make the operation table for the quotient group G/H.

10. Let G be a cyclic group of order 91 and H be a subgroup having index 7. List the cosets of the quotient group G/H.

11. If the index of a subgroup H in G is 2, prove that H is normal.

12. The six roots of $x^6-1 = 0$ form a multiplicative group $G = \{1, r, s, -1, -r, -s\}$ where 1, r, s are roots of $x^3 - 1 = 0$. Form the left cosets of $H = \{1, r, s\}$ and make the operation table for G/H.

13. Let $G = \{000, 001, 010, 011, 100, 101, 110, 111\}$ under bitwise addition mod 2, and $H = \{000, 011\}$. List the left cosets of H.

14. The elements of the Quaternion group G are $\{1, -1, i, -i, j, -j, k, -k\}$. Find a normal subgroup H and make the operation table for G/H.

15. Prove that if G is an abelian group, so is the quotient group G/H for any normal subgroup H.

16. Let $G = Z_4 \times Z_4$ and let N be the cyclic subgroup generated be the element $(3, 2)$. Show that G/N is isomorphic to Z_4.

17. Q/Z is an additive abelian group, with infinite order. What is the order of the coset $2/7 + Z$?

18. $H = \{(x, 5x) : x \in Reals\}$ is a subgroup of the additive group $R \times R$. Give a geometrical description of H and of the coset $(2, 7) + H$. What do

the cosets of H look like? What do the cosets of the circle group in the complex numbers look like?

19. Let $N = \{(x, y) : y = -x\}$ be a subgroup of the additive group $R \times R$. Describe the cosets of N.