Chapter 0

Introduction and Preliminaries

0.2 Exercises

1. Classify each of the sentences below as an atomic statement, and molecular statement, or not a statement at all. If the statement is molecular, say what kind it is (conjuction, disjunction, conditional, biconditional, negation).

(a) The sum of the first 100 odd positive integers.

(b) Everybody needs somebody sometime.

(c) The Broncos will win the Super Bowl or I'll eat my hat.

(d) We can have donuts for dinner, but only if it rains.

(e) Every natural number greater than 1 is either prime or composite.

(f) This sentence is false.

2. Suppose P and Q are the statements: P: Jack passed math. Q: Jill passed math.

(a) Translate "Jack and Jill both passed math" into symbols.

(b) Translate "If Jack passed math, then Jill did not" into symbols.

(c) Translate "$P \vee Q$" into English.

(d) Translate "$\neg(P \wedge Q) \rightarrow Q$" into English.

(e) Suppose you know that if Jack passed math, then so did Jill. What can you conclude if you know that:

i. Jill passed math?

ii. Jill did not pass math?

3. Geoff Poshingten is out at a fancy pizza joint, and decides to order a calzone. When the waiter asks what he would like in it, he replies, "I want either pepperoni or sausage. Also, if I have sausage, then I must also include quail. Oh, and if I have pepperoni or quail then I must also have ricotta cheese."

(a) Translate Geoff's order into logical symbols.

(b) The waiter knows that Geoff is either a liar or a truth-teller (so either everything he says is false, or everything is true). Which is it?

(c) What, if anything, can the waiter conclude about the ingredients in Geoff's desired calzone?

4. Consider the statement "If Oscar eats Chinese food, then he drinks milk."

(a) Write the converse of the statement.

(b) Write the contrapositive of the statement.

(c) Is it possible for the contrapositive to be false? If it was, what would that tell you?

(d) Suppose the original statement is true, and that Oscar drinks milk. Can you conclude anything (about his eating Chinese food)? Explain.

(e) Suppose the original statement is true, and that Oscar does not drink milk. Can you conclude anything (about his eating Chinese food)? Explain.

5. Which of the following statements are equivalent to the implication, "if you win the lottery, then you will be rich," and which are equivalent to the converse of the implication?

(a) Either you win the lottery or else you are not rich.

(b) Either you don't win the lottery or else you are rich.

(c) You will win the lottery and be rich.

(d) You will be rich if you win the lottery.

(e) You will win the lottery if you are rich.

(f) It is necessary for you to win the lottery to be rich.

(g) It is sufficient to win the lottery to be rich.

(h) You will be rich only if you win the lottery.

(i) Unless you win the lottery, you won't be rich.

(j) If you are rich, you must have won the lottery.

(k) If you are not rich, then you did not win the lottery.

(l) You will win the lottery if and only if you are rich.

6. Consider the implication, "if you clean your room, then you can watch TV." Rephrase the implication in as many ways as possible. Then do the same for the converse.

Hint.
Of course there are many answers. It helps to assume that the statement is true and the converse is *note* true. Think about what that means in the real world and then start saying it in different ways. Some ideas: Use "necessary and sufficient" language, use "only if," consider negations, use "or else" language.

7. Translate into symbols. Use $E(x)$ for "x is even" and $O(x)$ for "x is odd."

(a) No number is both even and odd.

(b) One more than any even number is an odd number.

(c) There is prime number that is even.

(d) Between any two numbers there is a third number.

(e) There is no number between a number and one more than that number.

8. Translate into English:

(a) $\forall x(E(x) \to E(x+2))$.

(b) $\forall x \exists y(\sin(x) = y)$.

(c) $\forall y \exists x(\sin(x) = y)$.

(d) $\forall x \forall y(x^3 = y^3 \to x = y)$.

9. Suppose $P(x)$ is some predicate for which the statement $\forall x P(x)$ is true. Is it also the case that $\exists x P(x)$ is true? In other words, is the statement $\forall x P(x) \to \exists x P(x)$ always true? Is the converse always true? Explain.

10. For each of the statements below, give a domain of discourse for which the statement is true, and a domain for which the statement is false.

(a) $\forall x \exists y(y^2 = x)$.

(b) $\forall x \forall y \exists z(x < z < y)$.

(c) $\exists x \forall y \forall z(y < z \to y \leq x \leq z)$ Hint: domains need not be infinite.

0.3 Exercises

1. Let $A = \{1,2,3,4,5\}$, $B = \{3,4,5,6,7\}$, and $C = \{2,3,5\}$.

(a) Find $A \cap B$.

(b) Find $A \cup B$.

(c) Find $A \setminus B$.

(d) Find $A \cap \overline{(B \cup C)}$.

(e) Find $A \times C$.

(f) Is $C \subseteq A$? Explain.

(g) Is $C \subseteq B$? Explain.

2. Let $A = \{x \in \mathbb{N} : 3 \leq x \leq 13\}$, $B = \{x \in \mathbb{N} : x \text{ is even}\}$, and $C = \{x \in \mathbb{N} : x \text{ is odd}\}$.

(a) Find $A \cap B$.

(b) Find $A \cup B$.

(c) Find $B \cap C$.

(d) Find $B \cup C$.

3. Find an example of sets A and B such that $A \cap B = \{3, 5\}$ and $A \cup B = \{2, 3, 5, 7, 8\}$.

4. Find an example of sets A and B such that $A \subseteq B$ and $A \in B$.

5. Recall $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$ (the integers). Let $\mathbb{Z}^+ = \{1, 2, 3, \ldots\}$ be the positive integers. Let $2\mathbb{Z}$ be the even integers, $3\mathbb{Z}$ be the multiples of 3, and so on.

(a) Is $\mathbb{Z}^+ \subseteq 2\mathbb{Z}$? Explain.

(b) Is $2\mathbb{Z} \subseteq \mathbb{Z}^+$? Explain.

(c) Find $2\mathbb{Z} \cap 3\mathbb{Z}$. Describe the set in words, and using set notation.

(d) Express $\{x \in \mathbb{Z} : \exists y \in \mathbb{Z}(x = 2y \vee x = 3y)\}$ as a union or intersection of two sets already described in this problem.

6. Let A_2 be the set of all multiples of 2 except for 2. Let A_3 be the set of all multiples of 3 except for 3. And so on, so that A_n is the set of all multiple of n except for n, for any $n \geq 2$. Describe (in words) the set $\overline{A_2 \cup A_3 \cup A_4 \cup \cdots}$.

7. Draw a Venn diagram to represent each of the following:

(a) $A \cup \overline{B}$

(b) $\overline{(A \cup B)}$

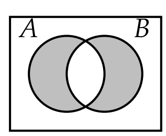
(c) $A \cap (B \cup C)$

(d) $(A \cap B) \cup C$

(e) $\overline{A} \cap B \cap \overline{C}$

(f) $(A \cup B) \setminus C$

8. Describe a set in terms of A and B (using set notation) which has the following Venn diagram:



9. Find the following cardinalities:

(a) $|A|$ when $A = \{4, 5, 6, \ldots, 37\}$

(b) $|A|$ when $A = \{x \in \mathbb{Z} : -2 \leq x \leq 100\}$

(c) $|A \cap B|$ when $A = \{x \in \mathbb{N} : x \leq 20\}$ and $B = \{x \in \mathbb{N} : x \text{ is prime}\}$

10. Let $A = \{a, b, c, d\}$. Find $\mathcal{P}(A)$.

Hint.
We are looking for a set containing 16 sets.

11. Let $A = \{1, 2, \ldots, 10\}$. How many subsets of A contain exactly one element (i.e., how many **singleton** subsets are there). How many **doubleton** (containing exactly two elements) are there?

12. Let $A = \{1, 2, 3, 4, 5, 6\}$. Find all sets $B \in \mathcal{P}(A)$ which have the property $\{2, 3, 5\} \subseteq B$.

13. Find an example of sets A and B such that $|A| = 4$, $|B| = 5$, and $|A \cup B| = 9$.

14. Find an example of sets A and B such that $|A| = 3$, $|B| = 4$, and $|A \cup B| = 5$.

15. Are there sets A and B such that $|A| = |B|$, $|A \cup B| = 10$, and $|A \cap B| = 5$? Explain.

16. In a regular deck of playing cards there are 26 red cards and 12 face cards. Explain, using sets and what you have learned about cardinalities, why there are only 32 cards which are either red or a face card.

0.4 EXERCISES

1. Write out all functions $f : \{1, 2, 3\} \to \{a, b\}$ (using two-line notation). How many are there? How many are injective? How many are surjective? How many are both?

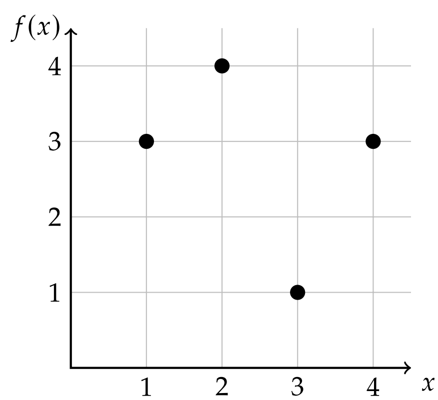
2. Write out all functions $f : \{1, 2\} \to \{a, b, c\}$ (in two-line notation). How many are there? How many are injective? How many are surjective? How many are both?

3. Consider the function $f : \{1, 2, 3, 4, 5\} \to \{1, 2, 3, 4\}$ given by the table below:

x	1	2	3	4	5
$f(x)$	3	2	4	1	2

(a) Is f injective? Explain.
(b) Is f surjective? Explain.
(c) Write the function using two-line notation.

4. Consider the function $f : \{1, 2, 3, 4\} \to \{1, 2, 3, 4\}$ given by the graph below.



(a) Is f injective? Explain.
(b) Is f surjective? Explain.
(c) Write the function using two-line notation.

5. For each function given below, determine whether or not the function is injective and whether or not the function is surjective.

(a) $f : \mathbb{N} \to \mathbb{N}$ given by $f(n) = n + 4$.

(b) $f : \mathbb{Z} \to \mathbb{Z}$ given by $f(n) = n + 4$.

(c) $f : \mathbb{Z} \to \mathbb{Z}$ given by $f(n) = 5n - 8$.

(d) $f : \mathbb{Z} \to \mathbb{Z}$ given by $f(n) = \begin{cases} n/2 & \text{if is even} \\ (n+1)/2 & \text{if is odd.} \end{cases}$

6. Let $A = \{1, 2, 3, \ldots, 10\}$. Consider the function $f : \mathcal{P}(A) \to \mathbb{N}$ given by $f(B) = |B|$. That is, f takes a subset of A as an input and outputs the cardinality of that set.

(a) Is f injective? Prove your answer.

(b) Is f surjective? Prove your answer.

(c) Find $f^{-1}(1)$.

(d) Find $f^{-1}(0)$.

(e) Find $f^{-1}(12)$.

7. Let $A = \{n \in \mathbb{N} : 0 \leq n \leq 999\}$ be the set of all numbers with three or fewer digits. Define the function $f : A \to \mathbb{N}$ by $f(abc) = a + b + c$, where a, b, and c are the digits of the number in A. For example, $f(253) = 2 + 5 + 3 = 10$.

(a) Find $f^{-1}(3)$.

(b) Find $f^{-1}(28)$.

(c) Is f injective. Explain.

(d) Is f surjective. Explain.

8. Let $f : X \to Y$ be some function. Suppose $3 \in Y$. What can you say about $f^{-1}(3)$ if you know,

(a) f is injective? Explain.

(b) f is surjective? Explain.

(c) f is bijective? Explain.

9. Find a set X and a function $f : X \to \mathbb{N}$ so that $f^{-1}(0) \cup f^{-1}(1) = X$.

10. What can you deduce about the sets X and Y if you know ...

(a) there is an injective function $f : X \to Y$? Explain.

(b) there is a surjective function $f : X \to Y$? Explain.

(c) there is a bijectitve function $f : X \to Y$? Explain.

11. Suppose $f : X \to Y$ is a function. Which of the following are possible? Explain.

(a) f is injective but not surjective.

(b) f is surjective but not injective.

(c) $|X| = |Y|$ and f is injective but not surjective.

(d) $|X| = |Y|$ and f is surjective but not injective.

(e) $|X| = |Y|$, X and Y are finite, and f is injective but not surjective.

(f) $|X| = |Y|$, X and Y are finite, and f is surjective but not injective.

12. Let $f : X \to Y$ and $g : Y \to Z$ be functions. We can define the **composition** of f and g to be the function $g \circ f : X \to Z$ which the image of each $x \in X$ is $g(f(x))$. That is, plug x into f, then plug the result into g (just like composition in algebra and calculus).

(a) If f and g are both injective, must $g \circ f$ be injective? Explain.

(b) If f and g are both surjective, must $g \circ f$ be surjective? Explain.

(c) Suppose $g \circ f$ is injective. What, if anything, can you say about f and g? Explain.

(d) Suppose $g \circ f$ is surjective. What, if anything, can you say about f and g? Explain.

Hint.

Work with some examples. What if $f = \begin{pmatrix} 1 & 2 & 3 \\ a & a & b \end{pmatrix}$ and $g = \begin{pmatrix} a & b & c \\ 5 & 6 & 7 \end{pmatrix}$?

13. Consider the function $f : \mathbb{Z} \to \mathbb{Z}$ given by $f(n) = \begin{cases} n+1 & \text{if is even} \\ n-3 & \text{if is odd.} \end{cases}$

(a) Is f injective? Prove your answer.

(b) Is f surjective? Prove your answer.

14. At the end of the semester a teacher assigns letter grades to each of her students. Is this a function? If so, what sets make up the domain and codomain, and is the function injective, surjective, bijective, or neither?

15. In the game of *Hearts*, four players are each dealt 13 cards from a deck of 52. Is this a function? If so, what sets make up the domain and codomain, and is the function injective, surjective, bijective, or neither?

16. Suppose 7 players are playing 5-card stud. Each player initially receives 5 cards from a deck of 52. Is this a function? If so, what sets make up the domain and codomain, and is the function injective, surjective, bijective, or neither?

Chapter 1

Counting

1.1 Exercises

1. Your wardrobe consists of 5 shirts, 3 pairs of pants, and 17 bow ties. How many different outfits can you make?

2. For your college interview, you must wear a tie. You own 3 regular (boring) ties and 5 (cool) bow ties.

(a) How many choices do you have for your neck-wear?

(b) You realize that the interview is for clown college, so you should probably wear both a regular tie and a bow tie. How many choices do you have now?

(c) For the rest of your outfit, you have 5 shirts, 4 skirts, 3 pants, and 7 dresses. You want to select either a shirt to wear with a skirt or pants, or just a dress. How many outfits do you have to choose from?

3. Your Blu-ray collection consists of 9 comedies and 7 horror movies. Give an example of a question for which the answer is:

(a) 16.

(b) 63.

4. We usually write numbers in decimal form (or base 10), meaning numbers are composed using 10 different "digits" $\{0, 1, \ldots, 9\}$. Sometimes though it is useful to write numbers **hexadecimal** or base 16. Now there are 16 distinct digits that can be used to form numbers: $\{0, 1, \ldots, 9, \mathrm{A}, \mathrm{B}, \mathrm{C}, \mathrm{D}, \mathrm{E}, \mathrm{F}\}$. So for example, a 3 digit hexadecimal number might be 2B8.

(a) How many 2-digit hexadecimals are there in which the first digit is E or F? Explain your answer in terms of the additive principle (using either events or sets).

(b) Explain why your answer to the previous part is correct in terms of the multiplicative principle (using either events or sets). Why do both the additive and multiplicative principles give you the same answer?

(c) How many 3-digit hexadecimals start with a letter (A-F) and end with a numeral (0-9)? Explain.

(d) How many 3-digit hexadecimals start with a letter (A-F) or end with a numeral (0-9) (or both)? Explain.

5. Suppose you have sets A and B with $|A| = 10$ and $|B| = 15$.

(a) What is the largest possible value for $|A \cap B|$?

(b) What is the smallest possible value for $|A \cap B|$?

(c) What are the possible values for $|A \cup B|$?

6. If $|A| = 8$ and $|B| = 5$, what is $|A \cup B| + |A \cap B|$?

7. A group of college students were asked about their TV watching habits. Of those surveyed, 28 students watch *The Walking Dead*, 19 watch *The Blacklist*, and 24 watch *Game of Thrones*. Additionally, 16 watch *The Walking Dead* and *The Blacklist*, 14 watch *The Walking Dead* and *Game of Thrones*, and 10 watch *The Blacklist* and *Game of Thrones*. There are 8 students who watch all three shows. How many students surveyed watched at least one of the shows?

8. In a recent survey, 30 students reported whether they liked their potatoes Mashed, French-fried, or Twice-baked. 15 liked them mashed, 20 liked French fries, and 9 liked twice baked potatoes. Additionally, 12 students liked both mashed and fried potatoes, 5 liked French fries and twice baked potatoes, 6 liked mashed and baked, and 3 liked all three styles. How many students *hate* potatoes? Explain why your answer is correct.

9. For how many $n \in \{1, 2, \ldots, 500\}$ is n a multiple of one or more of 5, 6, or 7?

Hint.
To find out how many numbers are divisible by 6 and 7, for example, take 500/42 and round down.

10. Let A, B, and C be sets.

(a) Find $|(A \cup C) \setminus B|$ provided $|A| = 50$, $|B| = 45$, $|C| = 40$, $|A \cap B| = 20$, $|A \cap C| = 15$, $|B \cap C| = 23$, and $|A \cap B \cap C| = 12$.

(b) Describe a set in terms of A, B, and C with cardinality 26.

11. Consider all 5 letter "words" made from the letters a through h. (Recall, words are just strings of letters, not necessarily actual English words.)

(a) How many of these words are there total?

(b) How many of these words contain no repeated letters?

(c) How many of these words start with the sub-word "aha"?

(d) How many of these words either start with "aha" or end with "bah" or both?

(e) How many of the words containing no repeats also do not contain the sub-word "bad"?

12. For how many three digit numbers (100 to 999) is the *sum of the digits* even? (For example, 343 has an even sum of digits: $3 + 4 + 3 = 10$ which is even.) Find the answer and explain why it is correct in at least two *different* ways.

13. The number 735000 factors as $2^3 \cdot 3 \cdot 5^4 \cdot 7^2$. How many divisors does it have? Explain your answer using the multiplicative principle.

1.2 Exercises

1. Let $S = \{1, 2, 3, 4, 5, 6\}$

(a) How many subsets are there total?

(b) How many subsets have $\{2, 3, 5\}$ as a subset?

(c) How many subsets contain at least one odd number?

(d) How many subsets contain exactly one even number?

2. Let $S = \{1, 2, 3, 4, 5, 6\}$

(a) How many subsets are there of cardinality 4?

(b) How many subsets of cardinality 4 have $\{2, 3, 5\}$ as a subset?

(c) How many subsets of cardinality 4 contain at least one odd number?

(d) How many subsets of cardinality 4 contain exactly one even number?

3. Let $A = \{1, 2, 3, \ldots, 9\}$.

(a) How many subsets of A are there? That is, find $|\mathcal{P}(A)|$. Explain.

(b) How many subsets of A contain exactly 5 elements? Explain.

(c) How many subsets of A contain only even numbers? Explain.

(d) How many subsets of A contain an even number of elements? Explain.

4. How many 9-bit strings (that is, bit strings of length 9) are there which:

(a) Start with the sub-string 101? Explain.

(b) Have weight 5 (i.e., contain exactly five 1's) and start with the sub-string 101? Explain.

(c) Either start with 101 or end with 11 (or both)? Explain.

(d) Have weight 5 and either start with 101 or end with 11 (or both)? Explain.

5. You break your piggy-bank to discover lots of pennies and nickels. You start arranging these in rows of 6 coins.

(a) You find yourself making rows containing an equal number of pennies and nickels. For fun, you decide to lay out every possible such row. How many coins will you need?

(b) How many coins would you need to make all possible rows of 6 coins (not necessarily with equal number of pennies and nickels)?

6. How many 10-bit strings contain 6 or more 1's?

7. How many subsets of $\{0, 1, \ldots, 9\}$ have cardinality 6 or more?

Hint.
Break the question into five cases.

8. What is the coefficient of x^{12} in $(x + 2)^{15}$?

9. What is the coefficient of x^9 in the expansion of $(x + 1)^{14} + x^3(x + 2)^{15}$?

10. How many shortest lattice paths start at (3,3) and

(a) end at (10,10)?

(b) end at (10,10) and pass through (5,7)?

(c) end at (10,10) and avoid (5,7)?

11. Gridtown USA, besides having excellent donut shoppes, is known for its precisely laid out grid of streets and avenues. Streets run east-west, and avenues north-south, for the entire stretch of the town, never curving and never interrupted by parks or schools or the like.
Suppose you live on the corner of 1st and 1st and work on the corner of 12th and 12th. Thus you must travel 22 blocks to get to work as quickly as possible.

(a) How many different routes can you take to work, assuming you want to get there as quickly as possible?

(b) Now suppose you want to stop and get a donut on the way to work, from your favorite donut shoppe on the corner of 8th st and 10th ave. How many routes to work, via the donut shoppe, can you take (again, ensuring the shortest possible route)?

(c) Disaster Strikes Gridtown: there is a pothole on 4th avenue between 5th and 6th street. How many routes to work can you take avoiding that unsightly (and dangerous) stretch of road?

(d) How many routes are there both avoiding the pothole and visiting the donut shoppe?

12. Suppose you are ordering a large pizza from *D.P. Dough*. You want 3 distinct toppings, chosen from their list of 11 vegetarian toppings.

(a) How many choices do you have for your pizza?

(b) How many choices do you have for your pizza if you refuse to have pineapple as one of your toppings?

(c) How many choices do you have for your pizza if you *insist* on having pineapple as one of your toppings?

(d) How do the three questions above relate to each other?

13. Explain why the coefficient of x^5y^3 the same as the coefficient of x^3y^5 in the expansion of $(x + y)^8$?

1.3 Exercises

1. A pizza parlor offers 10 toppings.

(a) How many 3-topping pizzas could they put on their menu? Assume double toppings are not allowed.

(b) How many total pizzas are possible, with between zero and ten toppings (but not double toppings) allowed?

(c) The pizza parlor will list the 10 toppings in two equal-sized columns on their menu. How many ways can they arrange the toppings in the left column?

2. A combination lock consists of a dial with 40 numbers on it. To open the lock, you turn the dial to the right until you reach a first number, then to the left until you get to second number, then to the right again to the third number. The numbers must be distinct. How many different combinations are possible?

3. Using the digits 2 through 8, find the number of different 5-digit numbers such that:

(a) Digits can be used more than once.

(b) Digits cannot be repeated, but can come in any order.

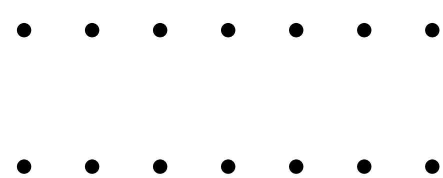
(c) Digits cannot be repeated and must be written in increasing order.

(d) Which of the above counting questions is a combination and which is a permutation? Explain why this makes sense.

4. How many triangles are there with vertices from the points shown below? Note, we are not allowing degenerate triangles - ones with all three vertices on the same line, but we do allow non-right triangles. Explain why your answer is correct.

Hint.
You need at exactly two points on either the x- or y-axis, but don't over-count the right triangles.

5. How many quadrilaterals can you draw using the dots below as vertices (corners)?



6. How many of the quadrilaterals possible in the previous problem are:

(a) Squares?

(b) Rectangles?

(c) Parallelograms?

(d) Trapezoids?[1]

(e) Trapezoids that are not parallelograms?

7. An *anagram* of a word is just a rearrangement of its letters. How many different anagrams of "uncopyrightable" are there? (This happens to be the longest common English word without any repeated letters.)

8. How many anagrams are there of the word "assesses" that start with the letter "a"?

9. How many anagrams are there of "anagram"?

10. On a business retreat, your company of 20 businessmen and businesswomen go golfing.

(a) You need to divide up into foursomes (groups of 4 people): a first foursome, a second foursome, and so on. How many ways can you do this?

[1]Here, as in calculus, a trapezoid is defined as a quadrilateral with *at least* one pair of parallel sides. In particular, parallelograms are trapezoids.

(b) After all your hard work, you realize that in fact, you want each foursome to include one of the five Board members. How many ways can you do this?

11. How many different seating arrangements are possible for King Arthur and his 9 knights around their round table?

12. Consider sets A and B with $|A| = 10$ and $|B| = 17$.

(a) How many functions $f : A \to B$ are there?

(b) How many functions $f : A \to B$ are injective?

13. Consider functions $f : \{1, 2, 3, 4\} \to \{1, 2, 3, 4, 5, 6\}$.

(a) How many functions are there total?

(b) How many functions are injective?

(c) How many of the injective functions are *increasing*? To be increasing means that if $a < b$ then $f(a) < f(b)$, or in other words, the outputs get larger as the inputs get larger.

14. We have seen that the formula for $P(n, k)$ is $\dfrac{n!}{(n-k)!}$. Your task here is to explain *why* this is the right formula.

(a) Suppose you have 12 chips, each a different color. How many different stacks of 5 chips can you make? Explain your answer and why it is the same as using the formula for $P(12, 5)$.

(b) Using the scenario of the 12 chips again, what does 12! count? What does 7! count? Explain.

(c) Explain why it makes sense to divide 12! by 7! when computing $P(12, 5)$ (in terms of the chips).

(d) Does your explanation work for numbers other than 12 and 5? Explain the formula $P(n, k) = \frac{n!}{(n-k)!}$ using the variables n and k.

1.4 Exercises

1. Prove the identity $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$ using a question about subsets.

2. Give a combinatorial proof of the identity $2 + 2 + 2 = 3 \cdot 2$.

3. Give a combinatorial proof for the identity $1 + 2 + 3 + \cdots + n = \binom{n+1}{2}$.

4. Suppose you own x fezzes and y bow ties. Of course, x and y are both greater than 1.

(a) How many combinations of fez and bow tie can you make? You can wear only one fez and one bow tie at a time. Explain.

(b) Explain why the answer is *also* $\binom{x+y}{2} - \binom{x}{2} - \binom{y}{2}$. (If this is what you claimed the answer was in part (a), try it again.)

(c) Use your answers to parts (a) and (b) to give a combinatorial proof of the identity

$$\binom{x+y}{2} - \binom{x}{2} - \binom{y}{2} = xy$$

5. A woman is getting married. She has 15 best friends but can only select 6 of them to be her bridesmaids, one of which needs to be her maid of honor. How many ways can she do this?

(a) What if she first selects the 6 bridesmaids, and then selects one of them to be the maid of honor?

(b) What if she first selects her maid of honor, and then 5 other bridemaids?

(c) Explain why $6\binom{15}{6} = 15\binom{14}{5}$.

6. Consider the identity:

$$k\binom{n}{k} = n\binom{n-1}{k-1}$$

(a) Is this true? Try it for a few values of n and k.

(b) Use the formula for $\binom{n}{k}$ to give an algebraic proof of the identity.

(c) Give a combinatorial proof of the identity.

Hint. How many ways can you select a team of k people from a group of n people *and* select one of them to be the team captain?

7. Give a combinatorial proof of the identity $\binom{n}{2}\binom{n-2}{k-2} = \binom{n}{k}\binom{k}{2}$.

Hint.
What if you wanted a pair of co-bridesmades?

8. Consider the bit strings in $\mathbf{B}_2^6$ (bit strings of length 6 and weight 2).

(a) How many of those bit strings start with 01?

(b) How many of those bit strings start with 001?

(c) Are there any other strings we have not counted yet? Which ones, and how many are there?

(d) How many bit strings are there total in $\mathbf{B}_2^6$?

(e) What binomial identity have you just given a combinatorial proof for?

9. Let's count **ternary** digit strings, that is, strings in which each digit can be 0, 1, or 2.

(a) How many ternary digit strings contain exactly n digits?

(b) How many ternary digit strings contain exactly n digits and n 2's.

(c) How many ternary digit strings contain exactly n digits and $n-1$ 2's. (Hint: where can you put the non-2 digit, and then what could it be?)

(d) How many ternary digit strings contain exactly n digits and $n-2$ 2's. (Hint: see previous hint)

(e) How many ternary digit strings contain exactly n digits and $n-k$ 2's.

(f) How many ternary digit strings contain exactly n digits and no 2's. (Hint: what kind of a string is this?)

(g) Use the above parts to give a combinatorial proof for the identity

$$\binom{n}{0} + 2\binom{n}{1} + 2^2\binom{n}{2} + 2^3\binom{n}{3} + \cdots + 2^n\binom{n}{n} = 3^n.$$

10. How many ways are there to rearrange the letters in the word "rearrange"? Answer this question in at least two different ways to establish a binomial identity.

11. Give a combinatorial proof for the identity $P(n,k) = \binom{n}{k}k!$

12. Establish the identity below using a combinatorial proof.

$$\binom{2}{2}\binom{n}{2} + \binom{3}{2}\binom{n-1}{2} + \binom{4}{2}\binom{n-2}{2} + \cdots + \binom{n}{2}\binom{2}{2} = \binom{n+3}{5}.$$

1.5 Exercises

1. A **multiset** is a collection of objects, just like a set, but can contain an object more than once (the order of the elements still doesn't matter). For example, $\{1,1,2,5,5,7\}$ is a multiset of size 6.

(a) How many *sets* of size 5 can be made using the 10 numeric digits 0 through 9?

(b) How many *multi*sets of size 5 can be made using the 10 numeric digits 0 through 9?

2. Each of the counting problems below can be solved with stars and bars. For each, say what outcome the diagram

$$* * * | * || * * |$$

represents, if there are the correct number of stars and bars for the problem. Otherwise, say why the diagram does not represent any outcome, and what a correct diagram would look like.

(a) How many ways are there to select a handful of 6 jellybeans from a jar that contains 5 different flavors?

(b) How many ways can you distribute 5 identical lollipops to 6 kids?

(c) How many 6-letter words can you make using the 5 vowels?

(d) How many solutions are there to the equation $x_1 + x_2 + x_3 + x_4 = 6$.

3. After gym class you are tasked with putting the 14 identical dodgeballs away into 5 bins.

(a) How many ways can you do this if there are no restrictions?

(b) How many ways can you do this if each bin must contain at least one dodgeball?

4. How many integer solutions are there to the equation $x + y + z = 8$ for which

(a) x, y, and z are all positive?

(b) x, y, and z are all non-negative?

(c) x, y, and z are all greater than -3.

5. Using the digits 2 through 8, find the number of different 5-digit numbers such that:

(a) Digits cannot be repeated and must be written in increasing order. For example, 23678 is okay, but 32678 is not.

(b) Digits *can* be repeated and must be written in *non-decreasing* order. For example, 24448 is okay, but 24484 is not.

6. When playing Yahtzee, you roll five regular 6-sided dice. How many different outcomes are possible from a single roll? The order of the dice does not matter.

7. Your friend tells you she has 7 coins in her hand (just pennies, nickels, dimes and quarters). If you guess how many of each kind of coin she has, she will give them to you. If you guess randomly, what is the probability that you will be correct?

8. How many integer solutions to $x_1 + x_2 + x_3 + x_4 = 25$ are there for which $x_1 \geq 1$, $x_2 \geq 2$, $x_3 \geq 3$ and $x_4 \geq 4$?

9. Solve the three counting problems below. Then say why it makes sense that they all have the same answer. That is, say how you can interpret them as each other.

(a) How many ways are there to distribute 8 cookies to 3 kids?

(b) How many solutions in non-negative integers are there to $x + y + z = 8$?

(c) How many different packs of 8 crayons can you make using crayons that come in red, blue and yellow?

10. Consider functions $f : \{1, 2, 3, 4, 5\} \to \{0, 1, 2, \ldots, 9\}$.

(a) How many of these functions are strictly increasing? Explain. (A function is strictly increasing provided if $a < b$, then $f(a) < f(b)$.)

(b) How many of the functions are non-decreasing? Explain. (A function is non-decreasing provided if $a < b$, then $f(a) \leq f(b)$.)

11. *Conic*, your favorite math themed fast food drive-in offers 20 flavors which can be added to your soda. You have enough money to buy a large soda with 4 added flavors. How many different soda concoctions can you order if:

(a) You refuse to use any of the flavors more than once?

(b) You refuse repeats but care about the order the flavors are added?

(c) You allow yourself multiple shots of the same flavor?

(d) You allow yourself multiple shots, and care about the order the flavors are added?

1.6 Exercises

1. The dollar menu at your favorite tax-free fast food restaurant has 7 items. You have \$10 to spend. How many different meals can you buy if you spend all your money and:

(a) Purchase at least one of each item.

(b) Possibly skip some items.

(c) Don't get more than 2 of any particular item.

2. After a late night of math studying, you and your friends decide to go to your favorite tax-free fast food Mexican restaurant, *Burrito Chime*. You decide to order off of the dollar menu, which has 7 items. Your group has \$16 to spend (and will spend all of it).

(a) How many different orders are possible? Explain. (The *order* in which the order is placed does not matter - just which and how many of each item that is ordered.)

(b) How many different orders are possible if you want to get at least one of each item? Explain.

(c) How many different orders are possible if you don't get more than 4 of any one item? Explain.

3. After another gym class you are tasked with putting the 14 identical dodgeballs away into 5 bins. This time, no bin can hold more than 6 balls. How many ways can you clean up?

4. Consider the equation $x_1 + x_2 + x_3 + x_4 = 15$. How many solutions are there with $2 \leq x_i \leq 5$ for all $i \in \{1, 2, 3, 4\}$?

5. Suppose you planned on giving 7 gold stars to some of the 13 star students in your class. Each student can receive at most one star. How many ways can you do this? Use PIE, and also an easier method, and compare your results.

6. Based on the previous question, give a combinatorial proof for the identity:

$$\binom{n}{k} = \binom{n+k-1}{k} - \sum_{j=1}^{n} (-1)^{j+1} \binom{n}{j} \binom{n+k-(2j+1)}{k}.$$

7. Illustrate how the counting of derangements works by writing all permutations of $\{1, 2, 3, 4\}$ and the crossing out those which are not derangements. Keep track of the permutations you cross out more than once, using PIE.

8. How many permutations of $\{1, 2, 3, 4, 5\}$ leave exactly 1 element fixed?

9. Ten ladies of a certain age drop off their red hats at the hat check of a museum. As they are leaving, the hat check attendant gives the hats back randomly. In how many ways can exactly six of the ladies receive their own hat (and the other four not)? Explain.

10. The Grinch sneaks into a room with 6 Christmas presents to 6 different people. He proceeds to switch the name-labels on the presents. How many ways could he do this if:

(a) No present is allowed to end up with its original label? Explain what each term in your answer represents.

(b) Exactly 2 presents keep their original labels? Explain.

(c) Exactly 5 presents keep their original labels? Explain.

11. Consider functions $f : \{1,2,3,4\} \to \{a,b,c,d,e,f\}$. How many functions have the property that $f(1) \neq a$ or $f(2) \neq b$, or both?

12. Consider sets A and B with $|A| = 10$ and $|B| = 5$. How many functions $f : A \to B$ are surjective?

13. Let $A = \{1,2,3,4,5\}$. How many injective functions $f : A \to A$ have the property that for each $x \in A$, $f(x) \neq x$?

14. Let d_n be the number of derangements of n objects. For example, using the techniques of this section, we find

$$d_3 = 3! - \left(\binom{3}{1}2! - \binom{3}{2}1! + \binom{3}{3}0!\right)$$

We can use the formula for $\binom{n}{k}$ to write this all in terms of factorials. After simplifying, for d_3 we would get

$$d_3 = 3!\left(1 - \frac{1}{1} + \frac{1}{2} - \frac{1}{6}\right)$$

Generalize this to find a nicer formula for d_n. Bonus: For large n, approximately what fraction of all permutations are derangements? Use your knowledge of Taylor series from calculus.

1.7 Chapter Review

1. You have 9 presents to give to your 4 kids. How many ways can this be done if:

(a) The presents are identical, and each kid gets at least one present?

(b) The presents are identical, and some kids might get no presents?

(c) The presents are unique, and some kids might get no presents?

(d) The presents are unique and each kid gets at least one present?

2. For each of the following counting problems, say whether the answer is $\binom{10}{4}$, $P(10,4)$, or neither. If you answer is "neither," say what the answer should be instead.

(a) How many shortest lattice paths are there from $(0,0)$ to $(10,4)$?

(b) If you have 10 bow ties, and you want to select 4 of them for next week, how many choices do you have?

(c) Suppose you have 10 bow ties and you will wear one on each of the next 4 days. How many choices do you have?

(d) If you want to wear 4 of your 10 bow ties next week (Monday through Sunday), how many ways can this be accomplished?

(e) Out of a group of 10 classmates, how many ways can you rank your top 4 friends?

(f) If 10 students come to their professor's office but only 4 can fit at a time, how different combinations of 4 students can see the prof first?

(g) How many 4 letter words can be made from the first 10 letters of the alphabet?

(h) How many ways can you make the word "cake" from the first 10 letters of the alphabet?

(i) How many ways are there to distribute 10 apples among 4 children?

(j) If you have 10 kids (and live in a shoe) and 4 types of cereal, how many ways can your kids eat breakfast?

(k) How many ways can you arrange exactly 4 ones in a string of 10 binary digits?

(l) You want to select 4 single digit numbers as your lotto picks. How many choices do you have?

(m) 10 kids want ice-cream. You have 4 varieties. How many ways are there to give the kids as much ice-cream as they want?

(n) How many 1-1 functions are there from $\{1,2,\ldots,10\}$ to $\{a,b,c,d\}$?

(o) How many surjective functions are there from $\{1,2,\ldots,10\}$ to $\{a,b,c,d\}$?

(p) Each of your 10 bow ties match 4 pairs of suspenders. How many outfits can you make?

(q) After the party, the 10 kids each choose one of 4 party-favors. How many outcomes?

(r) How many 6-elements subsets are there of the set $\{1, 2, \ldots, 10\}$

(s) How many ways can you split up 11 kids into 5 teams?

(t) How many solutions are there to $x_1 + x_2 + \cdots + x_5 = 6$ where each x_i is non-negative?

(u) Your band goes on tour. There are 10 cities within driving distance, but only enough time to play 4 of them. How many choices do you have for the cities on your tour?

(v) In how many different ways can you play the 4 cities you choose?

(w) Out of the 10 breakfast cereals available, you want to have 4 bowls. How many ways can you do this?

(x) There are 10 types of cookies available. You want to make a 4 cookie stack. How many different stacks can you make?

(y) From you home at (0,0) you want to go to either the donut shop at (5,4) or the one at (3,6). How many paths could you take?

(z) How many 10-digit numbers do not contain a sub-string of 4 repeated digits?

3. Recall, you own 3 regular ties and 5 bow ties. You realize that it would be okay to wear more than two ties to your clown college interview.

(a) You must select some of your ties to wear. Everything is okay, from no ties up to all ties. How many choices do you have?

(b) If you want to wear at least one regular tie and one bow tie, but are willing to wear up to all your ties, how many choices do you have for which ties to wear?

(c) How many choices do you have if you wear exactly 2 of the 3 regular ties and 3 of the 5 bow ties?

(d) Once you have selected 2 regular and 3 bow ties, in how many orders could you put the ties on, assuming you must have one of the three bow ties on top?

4. Give a counting question where the answer is $8 \cdot 3 \cdot 3 \cdot 5$. Give another question where the answer is $8 + 3 + 3 + 5$.

5. Consider five digit numbers $\alpha = a_1a_2a_3a_4a_5$, with each digit from the set $\{1, 2, 3, 4\}$.

(a) How many such numbers are there?

(b) How many such numbers are there for which the *sum* of the digits is even?

(c) How many such numbers contain more even digits than odd digits?

6. In a recent small survey of airline passengers, 25 said they had flown American in the last year, 30 had flown Jet Blue, and 20 had flown Continental. Of those, 10 reported they had flown on American and Jet Blue, 12 had flown on Jet Blue and Continental, and 7 had flown on American and Continental. 5 passengers had flown on all three airlines.
How many passengers were surveyed? (Assume the results above make up the entire survey.)

7. Recall, by 8-bit strings, we mean strings of binary digits, of length 8.

(a) How many 8-bit strings are there total?

(b) How many 8-bit strings have weight 5?

(c) How many subsets of the set $\{a, b, c, d, e, f, g, h\}$ contain exactly 5 elements?

(d) Explain why your answers to parts (b) and (c) are the same. Why are these questions equivalent?

8. What is the coefficient of x^{10} in the expansion of $(x + 1)^{13} + x^2(x + 1)^{17}$?

9. How many 8-letter words contain exactly 5 vowels (a,e,i,o,u)? What if repeated letters were not allowed?

10. For each of the following, find the number of shortest lattice paths from $(0, 0)$ to $(8, 8)$ which:

(a) pass through the point $(2,3)$.

(b) avoid (do not pass through) the point $(7,5)$.

(c) either pass through $(2,3)$ or $(5,7)$ (or both).

11. You live in Grid-Town on the corner of 2nd and 3rd, and work in a building on the corner of 10th and 13th. How many routes are there which take you from home to work and then back home, but by a different route?

12. How many 10-bit strings start with 111 or end with 101 or both?

13. How many 10-bit strings of weight 6 start with 111 or end with 101 or both?

14. How many 6 letter words made from the letters a, b, c, d, e, f without repeats do not contain the subword "bad" in (a) consecutive letters? or (b) not-necessarily consecutive letters (but in order)?

15. Explain using lattice paths why $\sum_{k=0}^{n} \binom{n}{k} = 2^n$.

16. Suppose you have 20 one-dollar bills to give out as prizes to your top 5 discrete math students. How many ways can you do this if:

(a) Each of the 5 students gets at least 1 dollar?

(b) Some students might get nothing?

(c) Each student gets at least 1 dollar but no more than 7 dollars?

Hint.
Stars and bars.

17. How many functions $f : \{1,2,3,4,5\} \to \{a, b, c, d, e\}$ are there satisfying:

(a) $f(1) = a$ or $f(2) = b$ (or both)?

(b) $f(1) \neq a$ or $f(2) \neq b$ (or both)?

(c) $f(1) \neq a$ *and* $f(2) \neq b$, and f is injective?

(d) f is surjective, but $f(1) \neq a$, $f(2) \neq b$, $f(3) \neq c$, $f(4) \neq d$ and $f(5) \neq e$?

18. How many functions map $\{1,2,3,4,5,6\}$ *onto* $\{a, b, c, d\}$ (i.e., how many *surjections* are there)?

19. To thank your math professor for doing such an amazing job all semester, you decide to bake Oscar cookies. You know how to make 10 different types of cookies.

(a) If you want to give your professor 4 different types of cookies, how many different combinations of cookie type can you select? Explain your answer.

(b) To keep things interesting, you decide to make a different number of each type of cookie. If again you want to select 4 cookie types, how many ways can you select the cookie types and decide for which there will be the most, second most, etc. Explain your answer.

(c) You change your mind again. This time you decide you will make a total of 12 cookies. Each cookie could be any one of the 10 types of cookies you know how to bake (and it's okay if you leave some types out). How many choices do you have? Explain.

(d) You realize that the previous plan did not account for presentation. This time, you once again want to make 12 cookies, each one could be any one of the 10 types of cookies. However, now you plan to shape the cookies into the numerals 1, 2, . . . , 12 (and probably arrange them to make a giant clock, but you haven't decided on that yet). How many choices do you have for which types of cookies to bake into which numerals? Explain.

(e) The only flaw with the last plan is that your professor might not get to sample all 10 different varieties of cookies. How many choices do you have for which types of cookies to make into which numerals, given that each type of cookie should be present at least once? Explain.

20. For which of the parts of the previous problem (Exercise 1.7.19) does it make sense to interpret the counting question as counting some number of functions? Say what the domain and codomain should be, and whether you are counting all functions, injections, surjections, or something else.

Chapter 2

Sequences

2.1 Exercises

1. Find the closed formula for each of the following sequences by relating them to a well know sequence. Assume the first term given is a_1.

(a) $2, 5, 10, 17, 26, \ldots$

(b) $0, 2, 5, 9, 14, 20, \ldots$

(c) $8, 12, 17, 23, 30, \ldots$

(d) $1, 5, 23, 119, 719, \ldots$

2. For each sequence given below, find a closed formula for a_n, the nth term of the sequence (assume the first terms are a_0) by relating it to another sequence for which you already know the formula. In each case, briefly say how you got your answers.

(a) $4, 5, 7, 11, 19, 35, \ldots$

(b) $0, 3, 8, 15, 24, 35, \ldots$

(c) $6, 12, 20, 30, 42, \ldots$

(d) $0, 2, 7, 15, 26, 40, 57, \ldots$ (Cryptic Hint: these might be called "house numbers")

3. The Fibonacci sequence is $0, 1, 1, 2, 3, 5, 8, 13, \ldots$ (where $F_0 = 0$).

(a) Give the recursive definition for the sequence.

(b) Write out the first few terms of the sequence of partial sums: $0, 0+1, 0+1+1, \ldots$

(c) Give a closed formula for the sequence of partial sums in terms of F_n (for example, you might say $F_0 + F_1 + \cdots + F_n = 3F_{n-1}^2 + n$, although that is definitely not correct).

4. Consider the three sequences below. For each, find a recursive definition. How are these sequences related?

(a) $2, 4, 6, 10, 16, 26, 42, \ldots.$

(b) $5, 6, 11, 17, 28, 45, 73, \ldots.$

(c) $0, 0, 0, 0, 0, 0, 0, \ldots.$

5. Show that $a_n = 3 \cdot 2^n + 7 \cdot 5^n$ is a solution to the recurrence relation $a_n = 7a_{n-1} + 10a_{n-2}$. What would the initial conditions need to be for this to be the closed formula for the sequence?

6. Write out the first few terms of the sequence given by $a_1 = 3$; $a_n = 2a_{n-1} + 4$. Then find a recursive definition for the sequence $10, 24, 52, 108, \ldots.$

7. Write out the first few terms of the sequence given by $a_n = n^2 - 3n + 1$. Then find a closed formula for the sequence (starting with a_1) $0, 2, 6, 12, 20, \ldots.$

8. Find a closed formula for the sequence with recursive definition $a_n = 2a_{n-1} - a_{n-2}$ with $a_1 = 1$ and $a_2 = 2$.

9. Find a recursive definition for the sequence with closed formula $a_n = 3 + 2n$. Bonus points if you can give a recursive definition in which makes use of two previous terms and no constants.

2.2 Exercises

1. Consider the sequence $5, 9, 13, 17, 21, \ldots$ with $a_1 = 5$

(a) Give a recursive definition for the sequence.

(b) Give a closed formula for the nth term of the sequence.

(c) Is 2013 a term in the sequence? Explain.

(d) How many terms does the sequence $5, 9, 13, 17, 21, \ldots, 533$ have?

(e) Find the sum: $5 + 9 + 13 + 17 + 21 + \cdots + 533$. Show your work.

(f) Use what you found above to find b_n, the n^{th} term of $1, 6, 15, 28, 45, \ldots$, where $b_0 = 1$

2. Consider the sequence $(a_n)_{n \geq 0}$ which starts $8, 14, 20, 26, \ldots$.

(a) What is the next term in the sequence?

(b) Find a formula for the nth term of this sequence.

(c) Find the sum of the first 100 terms of the sequence: $\sum_{k=0}^{99} a_k$.

3. Consider the sum $4 + 11 + 18 + 25 + \cdots + 249$.

(a) How many terms (summands) are in the sum?

(b) Compute the sum. Remember to show all your work.

4. Consider the sequence $1, 7, 13, 19, \ldots, 6n + 7$.

(a) How many terms are there in the sequence?

(b) What is the second-to-last term?

(c) Find the sum of all the terms in the sequence.

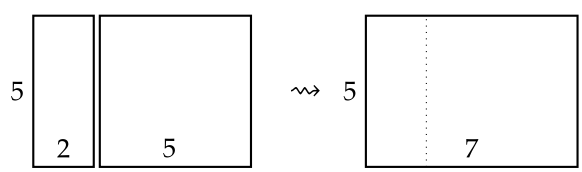
5. Find $5 + 7 + 9 + 11 + \cdots + 521$.

6. Find $5 + 15 + 45 + \cdots + 5 \cdot 3^{20}$.

7. Find $1 - \frac{2}{3} + \frac{4}{9} - \cdots + \frac{2^{30}}{3^{30}}$.

8. Find x and y such that $27, x, y, 1$ is part of an arithmetic sequence. Then find x and y so that the sequence is part of a geometric sequence. (Warning: x and y might not be integers.)

9. Starting with any rectangle, we can create a new, larger rectangle by attaching a square to the longer side. For example, if we start with a 2×5 rectangle, we would glue on a 5×5 square, forming a 5×7 rectangle:



(a) Create a sequence of rectangles using this rule starting with a 1×2 rectangle. Then write out the sequence of *perimeters* for the rectangles (the first term of the sequence would be 6, since the perimeter of a 1×2 rectangle is 6 - the next term would be 10).

(b) Repeat the above part this time starting with a 1×3 rectangle.

(c) Find recursive formulas for each of the sequences of perimeters you found in parts (a) and (b). Don't forget to give the initial conditions as well.

(d) Are the sequences arithmetic? Geometric? If not, are they *close* to being either of these (i.e., are the differences or ratios *almost* constant)? Explain.

10. Consider the sequence $2, 7, 15, 26, 40, 57, \ldots$ (with $a_0 = 2$). By looking at the differences between terms, express the sequence as a sequence of partial sums. Then find a closed formula for the sequence by computing the nth partial sum.

11. If you have enough toothpicks, you can make a large triangular grid. Below, are the triangular grids of size 1 and of size 2. The size 1 grid requires 3 toothpicks, the size 2 grid requires 9 toothpicks.

(a) Let t_n be the number of toothpicks required to make a size n triangular grid. Write out the first 5 terms of the sequence $t_1, t_2, \ldots$.

(b) Find a recursive definition for the sequence. Explain why you are correct.

(c) Is the sequence arithmetic or geometric? If not, is it the sequence of partial sums of an arithmetic or geometric sequence? Explain why your answer is correct.

(d) Use your results from part (c) to find a closed formula for the sequence. Show your work.

12. Use summation ($\sum$) or product ($\prod$) notation to rewrite the following.

(a) $2 + 4 + 6 + 8 + \cdots + 2n$.

(b) $1 + 5 + 9 + 13 + \cdots + 425$.

(c) $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{50}$.

(d) $2 \cdot 4 \cdot 6 \cdot \cdots \cdot 2n$.

(e) $(\frac{1}{2})(\frac{2}{3})(\frac{3}{4}) \cdots (\frac{100}{101})$.

13. Expand the following sums and products. That is, write them out the long way.

(a) $\displaystyle\sum_{k=1}^{100} (3 + 4k)$.

(b) $\displaystyle\sum_{k=0}^{n} 2^k$.

(c) $\displaystyle\sum_{k=2}^{50} \frac{1}{(k^2 - 1)}$.

(d) $\displaystyle\prod_{k=2}^{100} \frac{k^2}{(k^2 - 1)}$.

(e) $\displaystyle\prod_{k=0}^{n} (2 + 3k)$.

2.3 Exercises

1. Use polynomial fitting to find the formula for the nth term of the sequences $(a_n)_{n \geq 0}$ below.

(a) 2, 5, 11, 21, 36,. . .

(b) 0, 2, 6, 12, 20,. . .

(c) 1, 2, 4, 8, 15, 26 . . .

(d) 3, 6, 12, 22, 37, ... After finding a formula here, compare to part (a).

2. Make up a sequences that have

(a) 3, 3, 3, 3, . . . as its second differences.

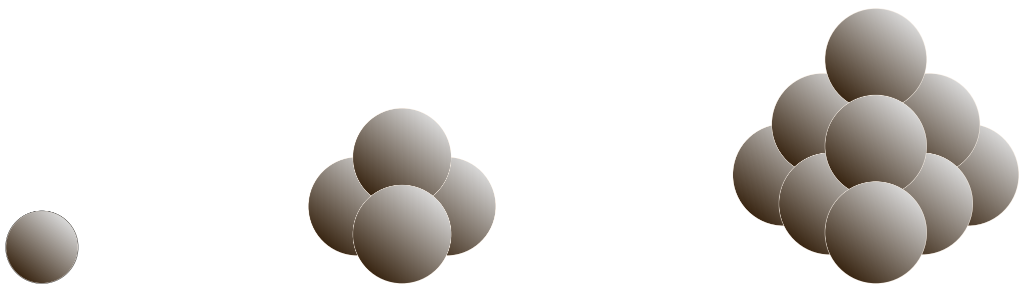
(b) 1, 2, 3, 4, 5, . . . as its third differences.

(c) 1, 2, 4, 8, 16, . . . as its 100th differences.

3. Consider the sequence $1, 3, 7, 13, 21, \ldots$. Explain how you know the closed formula for the sequence will be quadratic. Then "guess" the correct formula by comparing this sequence to the squares $1, 4, 9, 16, \ldots$ (do not use polynomial fitting).

4. Use a similar technique as in the previous exercise to find a closed formula for the sequence $2, 11, 34, 77, 146, 247, \ldots$.

5. In their down time, ghost pirates enjoy stacking cannonballs in triangular based pyramids (aka, tetrahedrons), like those pictured here:



Note, in the picture on the right, there are some cannonballs (actually just one) you cannot see. The next picture would have 4 cannonballs you cannot see. The stacks are *not* hollow.
The pirates wonder how many cannonballs would be required to build a pyramid 15 layers high (thus breaking the world cannonball stacking record). Can you help?

(a) Let $P(n)$ denote the number of cannonballs needed to create a pyramid n layers high. So $P(1) = 1$, $P(2) = 4$, and so on. Calculate $P(3)$, $P(4)$ and $P(5)$.

(b) Use polynomial fitting to find a closed formula for $P(n)$. Show your work.

(c) Answer the pirate's question: how many cannonballs do they need to make a pyramid 15 layers high?

6. Suppose $a_n = n^2 + 3n + 4$. Find a closed formula for the sequence of differences by computing $a_n - a_{n-1}$.

7. Repeat the above assuming this time $a_n = an^2 + bn + c$. That is, prove that every quadratic sequence has arithmetic differences.

8. Can you use polynomial fitting to find the formula for the nth term of the sequence 4, 7, 11, 18, 29, 47, . . . ? Explain why or why not.

9. Will the nth sequence of differences of $2, 6, 18, 54, 162, \ldots$ ever be constant? Explain.

10. Consider the sequences $2, 5, 12, 29, 70, 169, 408, \ldots$ (with $a_0 = 2$).

(a) Describe the rate of growth of this sequence.

(b) Find a recursive definition for the sequence.

(c) Find a closed formula for the sequence.

(d) If you look at the sequence of differences between terms, and then the sequence of second differences, the sequence of third differences, and so on, will you ever get a constant sequence? Explain how you know.

2.4 Exercises

1. Find the next two terms in $(a_n)_{n \geq 0}$ beginning $3, 5, 11, 21, 43, 85 \ldots$.. Then give a recursive definition for the sequence. Finally, use the characteristic root technique to find a closed formula for the sequence.

2. Solve the recurrence relation $a_n = a_{n-1} + 2^n$ with $a_0 = 5$.

3. Show that 4^n is a solution to the recurrence relation $a_n = 3a_{n-1} + 4a_{n-2}$.

4. Find the solution to the recurrence relation $a_n = 3a_{n-1} + 4a_{n-2}$ with initial terms $a_0 = 2$ and $a_1 = 3$.

5. Find the solution to the recurrence relation $a_n = 3a_{n-1} + 4a_{n-2}$ with initial terms $a_0 = 5$ and $a_1 = 8$.

6. Solve the recurrence relation $a_n = 2a_{n-1} - a_{n-2}$.

(a) What is the solution if the initial terms are $a_0 = 1$ and $a_1 = 2$?

(b) What do the initial terms need to be in order for $a_9 = 30$?

(c) For which x are there initial terms which make $a_9 = x$?

7. Solve the recurrence relation $a_n = 3a_{n-1} + 10a_{n-2}$ with initial terms $a_0 = 4$ and $a_1 = 1$.

8. Suppose that r^n and q^n are both solutions to a recurrence relation of the form $a_n = \alpha a_{n-1} + \beta a_{n-1}$. Prove that $c \cdot r^n + d \cdot q^n$ is also a solution to the recurrence relation, for any constants c, d.

9. Think back to the magical candy machine at your neighborhood grocery store. Suppose that the first time a quarter is put into the machine 1 Skittle comes out. The second time, 4 Skittles, the third time 16 Skittles, the fourth time 64 Skittles, etc.

(a) Find both a recursive and closed formula for how many Skittles the nth customer gets.

(b) Check your solution for the closed formula by solving the recurrence relation using the Characteristic Root technique.

10. You have access to 1×1 tiles which come in 2 different colors and 1×2 tiles which come in 3 different colors. We want to figure out how many different $1 \times n$ path designs we can make out of these tiles.

(a) Find a recursive definition for the sequence a_n of paths of length n.

(b) Solve the recurrence relation using the Characteristic Root technique.

11. Let a_n be the number of $1 \times n$ tile designs you can make using 1×1 squares available in 4 colors and 1×2 dominoes available in 5 colors.

(a) First, find a recurrence relation to describe the problem. Explain why the recurrence relation is correct (in the context of the problem).

(b) Write out the first 6 terms of the sequence $a_1, a_2, \ldots$.

(c) Solve the recurrence relation. That is, find a closed formula for a_n.

12. Consider the recurrence relation $a_n = 4a_{n-1} - 4a_{n-2}$.

(a) Find the general solution to the recurrence relation (beware the repeated root).

(b) Find the solution when $a_0 = 1$ and $a_1 = 2$.

(c) Find the solution when $a_0 = 1$ and $a_1 = 8$.

2.5 Exercises

1. Use induction to prove for all $n \in \mathbb{N}$ that $\sum_{k=0}^{n} 2^k = 2^{n+1} - 1$.

2. Prove that $7^n - 1$ is a multiple of 6 for all $n \in \mathbb{N}$.

3. Prove that $1 + 3 + 5 + \cdots + (2n - 1) = n^2$ for all $n \geq 1$.

4. Prove that $F_0 + F_2 + F_4 + \cdots + F_{2n} = F_{2n+1} - 1$ where F_n is the nth Fibonacci number.

5. Prove that $2^n < n!$ for all $n \geq 4$. (Recall, $n! = 1 \cdot 2 \cdot 3 \cdot \cdots \cdot n$.)

6. Prove, by mathematical induction, that $F_0 + F_1 + F_2 + \cdots + F_n = F_{n+2} - 1$, where F_n is the nth Fibonacci number ($F_0 = 0$, $F_1 = 1$ and $F_n = F_{n-1} + F_{n-2}$).

7. Zombie Euler and Zombie Cauchy, two famous zombie mathematicians, have just signed up for Twitter accounts. After one day, Zombie Cauchy has more followers than Zombie Euler. Each day after that, the number of new followers of Zombie Cauchy is exactly the same as the number of new followers of Zombie Euler (and neither lose any followers). Explain how a proof by mathematical induction can show that on every day after the first day, Zombie Cauchy will have more followers than Zombie Euler. That is, explain what the base case and inductive case are, and why they together prove that Zombie Cauchy will have more followers on the 4th day.

8. Find the largest number of points which a football team cannot get exactly using just 3-point field goals and 7-point touchdowns (ignore the possibilities of safeties, missed extra points, and two point conversions). Prove your answer is correct by mathematical induction.

9. Prove that the sum of n squares can be found as follows

$$1^2 + 2^2 + 3^2 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

10. What is wrong with the following "proof" of the "fact" that $n + 3 = n + 7$ for all values of n (besides of course that the thing it is claiming to prove is false)?

Proof. Let $P(n)$ be the statement that $n + 3 = n + 7$. We will prove that $P(n)$ is true for all $n \in \mathbb{N}$. Assume, for induction that $P(k)$ is true. That is, $k + 3 = k + 7$. We must show that $P(k + 1)$ is true. Now since $k + 3 = k + 7$, add 1 to both sides. This gives $k + 3 + 1 = k + 7 + 1$. Regrouping $(k + 1) + 3 = (k + 1) + 7$. But this is simply $P(k + 1)$. Thus by the principle of mathematical induction $P(n)$ is true for all $n \in \mathbb{N}$. QED

11. The proof in the previous problem does not work. But if we modify the "fact," we can get a working proof. Prove that $n+3 < n+7$ for all values of $n \in \mathbb{N}$. You can do this proof with algebra (without induction), but the goal of this exercise is to write out a valid induction proof.

12. Find the flaw in the following "proof" of the "fact" that $n < 100$ for every $n \in \mathbb{N}$.

Proof. Let $P(n)$ be the statement $n < 100$. We will prove $P(n)$ is true for all $n \in \mathbb{N}$. First we establish the base case: when $n = 0$, $P(n)$ is true, because $0 < 100$. Now for the inductive step, assume $P(k)$ is true. That is, $k < 100$. Now if $k < 100$, then k is some number, like 80. Of course $80 + 1 = 81$ which is still less than 100. So $k + 1 < 100$ as well. But this is what $P(k + 1)$ claims, so we have shown that $P(k) \rightarrow P(k + 1)$. Thus by the principle of mathematical induction, $P(n)$ is true for all $n \in \mathbb{N}$. QED

13. While the above proof does not work (it better not since the statement it is trying to prove is false!) we can prove something similar. Prove that there is a strictly increasing sequence $a_1, a_2, a_3, \ldots$ of numbers (not necessarily integers) such that $a_n < 100$ for all $n \in \mathbb{N}$. (By **strictly increasing** we mean $a_n < a_{n+1}$ for all n. So each term must be larger than the last.)

14. What is wrong with the following "proof" of the "fact" that for all $n \in \mathbb{N}$, the number $n^2 + n$ is odd?

Proof. Let $P(n)$ be the statement "$n^2 + n$ is odd." We will prove that $P(n)$ is true for all $n \in \mathbb{N}$. Suppose for induction that $P(k)$ is true, that is, that $k^2 + k$ is odd. Now consider the statement $P(k + 1)$. Now $(k + 1)^2 + (k + 1) = k^2 + 2k + 1 + k + 1 = k^2 + k + 2k + 2$. By the inductive hypothesis, $k^2 + k$ is odd, and of course $2k + 2$ is even. An odd plus an even is always odd, so therefore $(k + 1)^2 + (k + 1)$ is odd. Therefore by the principle of mathematical induction, $P(n)$ is true for all $n \in \mathbb{N}$. QED

15. Now give a valid proof (by induction, even though you might be able to do so without using induction) of the statement, "for all $n \in \mathbb{N}$, the number $n^2 + n$ is even."

16. Prove that there is a sequence of positive real numbers $a_0, a_1, a_2, \ldots$ such that the partial sum $a_0 + a_1 + a_2 + \cdots + a_n$ is strictly less than 2 for all $n \in \mathbb{N}$. Hint: think about how you could define what a_{k+1} is to make the induction argument work.

17. Prove that every positive integer is either a power of 2, or can be written as the sum of distinct powers of 2.

19. Use induction to prove that if n people all shake hands with each other, that the total number of handshakes is $\frac{n(n-1)}{2}$.

20. Suppose that a particular real number x has the property that $x + \frac{1}{x}$ is an integer. Prove that $x^n + \frac{1}{x^n}$ is an integer for all natural numbers n.

21. Use induction to prove that $\displaystyle\sum_{k=0}^{n} \binom{n}{k} = 2^n$. That is, the sum of the nth row of Pascal's Triangle is 2^n.

22. Use induction to prove $\binom{4}{0} + \binom{5}{1} + \binom{6}{2} + \cdots + \binom{4+n}{n} = \binom{5+n}{n}$. (This is an example of the hockey stick theorem.)

23. Use the product rule for logarithms ($\log(ab) = \log(a) + \log(b)$) to prove, by induction on n, that $\log(a^n) = n\log(a)$, for all natural numbers $n \geq 2$.

24. Let $f_1, f_2, \ldots, f_n$ be differentiable functions. Prove, using induction, that

$$(f_1 + f_2 + \cdots + f_n)' = f_1' + f_2' + \cdots + f_n'$$

You may assume $(f + g)' = f' + g'$ for any differentiable functions f and g.

Hint.
You are allowed to assume the base case. For the inductive case, group all but the last function together as one sum of functions, then apply the usual sum of derivatives rule, and then the inductive hypothesis.

25. Suppose $f_1, f_2, \ldots, f_n$ are differentiable functions. Use mathematical induction to prove the generalized product rule:

$$(f_1 f_2 f_3 \cdots f_n)' = f_1' f_2 f_3 \cdots f_n + f_1 f_2' f_3 \cdots f_n + f_1 f_2 f_3' \cdots f_n + \cdots + f_1 f_2 f_3 \cdots f_n'$$

You may assume the product rule for two functions is true.

Hint.
For the inductive step, we know by the product rule for two functions that

$$(f_1 f_2 f_3 \cdots f_k f_{k+1})' = (f_1 f_2 f_3 \cdots f_k)' f_{k+1} + (f_1 f_2 f_3 \cdots f_k) f_{k+1}'$$

Then use the inductive hypothesis on the first summand, and distribute.

2.6 Chapter Review

1. Find $3 + 7 + 11 + \cdots + 427$.

2. Consider the sequence $2, 6, 10, 14, \ldots, 4n + 6$.

(a) How many terms are there in the sequence?

(b) What is the second-to-last term?

(c) Find the sum of all the terms in the sequence.

3. Consider the sequence given by $a_n = 2 \cdot 5^{n-1}$.

(a) Find the first 4 terms of the sequence. What sort of sequence is this?

(b) Find the *sum* of the first 25 terms. That is, compute $\sum_{k=1}^{25} a_k$.

4. Consider the sequence $5, 11, 19, 29, 41, 55, \ldots$. Assume $a_1 = 5$.

(a) Find a closed formula for a_n, the nth term of the sequence, by writing each term as a sum of a sequence. Hint: first find a_0, but ignore it when collapsing the sum.

(b) Find a closed formula again, this time using either polynomial fitting or the characteristic root technique (whichever is appropriate). Show your work.

(c) Find a closed formula once again, this time by recognizing the sequence as a modification to some well known sequence(s). Explain.

5. Use polynomial fitting to find a closed formula for the sequence $(a_n)_{n\geq 1}$:

$$4, 11, 20, 31, 44, \ldots.$$

6. Suppose the closed formula for a particular sequence is a degree 3 polynomial. What can you say about the closed formula for:

(a) The sequence of partial sums.

(b) The sequence of second differences.

7. Consider the sequence given recursively by $a_1 = 4$, $a_2 = 6$ and $a_n = a_{n-1} + a_{n-2}$.

(a) Write out the first 6 terms of the sequence.

(b) Could the closed formula for a_n be a polynomial? Explain.

8. The sequence $-1, 0, 2, 5, 9, 14\ldots$ has closed formula $a_n = \frac{(n+1)(n-2)}{2}$. Use this fact to find a closed formula for the sequence $4, 10, 18, 28, 40, \ldots$.

9. The in song *The Twelve Days of Christmas*, my true love gave to me first 1 gift, then 2 gifts and 1 gift, then 3 gifts, 2 gifts and 1 gift, and so on. How many gifts did my true love give me all together during the twelve days?

10. Consider the recurrence relation $a_n = 3a_{n-1} + 10a_{n-2}$ with first two terms $a_0 = 1$ and $a_1 = 2$.

(a) Write out the first 5 terms of the sequence defined by this recurrence relation.

(b) Solve the recurrence relation. That is, find a closed formula for a_n.

11. Consider the recurrence relation $a_n = 2a_{n-1} + 8a_{n-2}$, with initial terms $a_0 = 1$ and $a_1 = 3$.

(a) Find the next two terms of the sequence (a_2 and a_3).

(b) Solve the recurrence relation. That is, find a closed formula for the nth term of the sequence.

12. Your magic chocolate bunnies reproduce like rabbits: every large bunny produces 2 new mini bunnies each day, and each day every mini bunny born the previous day grows into a large bunny. Assume you start with 2 mini bunnies and no bunny ever dies (or gets eaten).

(a) Write out the first few terms of the sequence.

(b) Give a recursive definition of the sequence and explain why it is correct.

(c) Find a closed formula for the nth term of the sequence.

13. Prove the following statements my mathematical induction:

(a) $n! < n^n$ for $n \geq 2$

(b) $\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \cdots + \frac{1}{n\cdot(n+1)} = \frac{n}{n+1}$ for all $n \in \mathbb{Z}^+$.

(c) $4^n - 1$ is a multiple of 3 for all $n \in \mathbb{N}$.

(d) The *greatest* amount of postage you *cannot* make exactly using 4 and 9 cent stamps is 23 cents.

(e) Every even number squared is divisible by 4.

14. Prove $1^3 + 2^3 + 3^3 + \cdots + n^3 = \left(\frac{n(n+1)}{2}\right)^2$ holds for all $n \geq 1$, by mathematical induction.

15. Suppose $a_0 = 1$, $a_1 = 1$ and $a_n = 3a_{n-1} - 2a_{n-1}$. Prove, using strong induction, that $a_n = 1$ for all n.

16. Prove, using strong induction, that every positive integer can be written as the sum of distinct powers of 2. For example, $13 = 1 + 4 + 8 = 2^0 + 2^2 + 2^3$.

17. Prove using induction that every set containing n elements has 2^n different subsets for any $n \geq 1$.

Chapter 3

Symbolic Logic and Proofs

3.1 Exercises

1. Consider the statement about a party, "If it's your birthday or there will be cake, then there will be cake."

(a) Translate the above statement into symbols. Clearly state which statement is P and which is Q.

(b) Make a truth table for the statement.

(c) Assuming the statement is true, what (if anything) can you conclude if there will be cake?

(d) Assuming the statement is true, what (if anything) can you conclude if there will not be cake?

(e) Suppose you found out that the statement was a lie. What can you conclude?

2. Make a truth table for the statement $(P \vee Q) \rightarrow (P \wedge Q)$.

3. Make a truth table for the statement $\neg P \wedge (Q \rightarrow P)$. What can you conclude about P and Q if you know the statement is true?

4. Make a truth table for the statement $\neg P \rightarrow (Q \wedge R)$.

Hint.
Like above, only now you will need 8 rows instead of just 4.

5. Determine whether the following two statements are logically equivalent: $\neg(P \rightarrow Q)$ and $P \wedge \neg Q$. Explain how you know you are correct.

6. Are the statements $P \rightarrow (Q \vee R)$ and $(P \rightarrow Q) \vee (P \rightarrow R)$ logically equivalent?

7. Simplify the following statements (so that negation only appears right before variables).

(a) $\neg(P \rightarrow \neg Q)$.

(b) $(\neg P \vee \neg Q) \rightarrow \neg(\neg Q \wedge R)$.

(c) $\neg((P \rightarrow \neg Q) \vee \neg(R \wedge \neg R))$.

(d) It is false that if Sam is not a man then Chris is a woman, and that Chris is not a woman.

8. Use De Morgan's Laws, and any other logical equivalence facts you know to simplify the following statements. Show all your steps. Your final statements should have negations only appear directly next to the sentence variables or predicates (P, Q, $E(x)$, etc.), and no double negations. It would be a good idea to use only conjunctions, disjunctions, and negations.

(a) $\neg((\neg P \wedge Q) \vee \neg(R \vee \neg S))$.

(b) $\neg((\neg P \rightarrow \neg Q) \wedge (\neg Q \rightarrow R))$ (careful with the implications).

9. Tommy Flanagan was telling you what he ate yesterday afternoon. He tells you, "I had either popcorn or raisins. Also, if I had cucumber sandwiches, then I had soda. But I didn't drink soda or tea." Of course you know that Tommy is the worlds worst liar, and everything he says is false. What did Tommy eat?
Justify your answer by writing all of Tommy's statements using sentence variables (P, Q, R, S, T), taking their negations, and using these to deduce what Tommy actually ate.

10. Determine if the following deduction rule is valid:

$$\begin{array}{rc} & P \vee Q \\ & \neg P \\ \hline \therefore & Q \end{array}$$

11. Determine if the following is a valid deduction rule:

$$\begin{array}{rc} & P \to (Q \vee R) \\ & \neg(P \to Q) \\ \hline \therefore & R \end{array}$$

12. Determine if the following is a valid deduction rule:

$$\begin{array}{rc} & (P \wedge Q) \to R \\ & \neg P \vee \neg Q \\ \hline \therefore & \neg R \end{array}$$

13. Can you chain implications together? That is, if $P \to Q$ and $Q \to R$, does that means the $P \to R$? Can you chain more implications together? Let's find out:

(a) Prove that the following is a valid deduction rule:

$$\begin{array}{rc} & P \to Q \\ & Q \to R \\ \hline \therefore & P \to R \end{array}$$

(b) Prove that the following is a valid deduction rule for any $n \geq 2$:

$$\begin{array}{rc} & P_1 \to P_2 \\ & P_2 \to P_3 \\ & \vdots \\ & P_{n-1} \to P_n \\ \hline \therefore & P_1 \to P_n. \end{array}$$

I suggest you don't go through the trouble of writing out a 2^n row truth table. Instead, you should use part (a) and mathematical induction.

14. We can also simplify statements in predicate logic using our rules for passing negations over quantifiers, and then applying propositional logical equivalence to the "inside" propositional part. Simplify the statements below (so negation appears only directly next to predicates).

(a) $\neg\exists x \forall y(\neg O(x) \vee E(y))$.

(b) $\neg\forall x \neg\forall y \neg(x < y \wedge \exists z(x < z \vee y < z))$.

(c) There is a number n for which no other number is either less n than or equal to n.

(d) It is false that for every number n there are two other numbers which n is between.

15. Suppose P and Q are (possibly molecular) propositional statements. Prove that P and Q are logically equivalent if any only if $P \leftrightarrow Q$ is a tautology.

Hint.
What do these concepts mean in terms of truth tables?

16. Suppose $P_1, P_2, \ldots, P_n$ and Q are (possibly molecular) propositional statements. Suppose further that

$$\begin{array}{rc} & P_1 \\ & P_2 \\ & \vdots \\ & P_n \\ \hline \therefore & Q \end{array}$$

is a valid deduction rule. Prove that the statement

$$(P_1 \wedge P_2 \wedge \cdots \wedge P_n) \rightarrow Q$$

is a tautology.

3.2 Exercises

1. Consider the statement "for all integers a and b, if $a + b$ is even, then a and b are even"

(a) Write the contrapositive of the statement.

(b) Write the converse of the statement.

(c) Write the negation of the statement.

(d) Is the original statement true or false? Prove your answer.

(e) Is the contrapositive of the original statement true or false? Prove your answer.

(f) Is the converse of the original statement true or false? Prove your answer.

(g) Is the negation of the original statement true or false? Prove your answer.

2. Consider the statement: for all integers n, if n is even then $8n$ is even.

(a) Prove the statement. What sort of proof are you using?

(b) Is the converse true? Prove or disprove.

3. Your "friend" has shown you a "proof" he wrote to show that $1 = 3$. Here is the proof:

Proof. I claim that $1 = 3$. Of course we can do anything to one side of an equation as long as we also do it to the other side. So subtract 2 from both sides. This gives $-1 = 1$. Now square both sides, to get $1 = 1$. And we all agree this is true. QED

What is going on here? Is your friends argument valid? Is the argument a proof of the claim $1 = 3$? Carefully explain using what we know about logic. Hint: What implication follows from the given proof?

4. Suppose you have a collection of 5-cent stamps and 8-cent stamps. We saw earlier that it is possible to make any amount of postage greater than 27 cents using combinations of both these types of stamps. But, let's ask some other questions:

(a) What amounts of postage can you make if you only use an even number of both types of stamps? Prove your answer.

(b) Suppose you made an even amount of postage. Prove that you used an even number of at least one of the types of stamps.

(c) Suppose you made exactly 72 cents of postage. Prove that you used at least 6 of one type of stamp.

5. Suppose that you would like to prove the following implication:

For all numbers n, if n is prime then n is solitary.

Write out the beginning and end of the argument if you were to prove the statement,

(a) Directly

(b) By contrapositive

(c) By contradiction

You do not need to provide details for the proofs (since you do not know what solitary means). However, make sure that you provide the first few and last few lines of the proofs so that we can see that logical structure you would follow.

6. Prove that $\sqrt{3}$ is irrational.

7. Consider the statement: for all integers a and b, if a is even and b is a multiple of 3, then ab is a multiple of 6.

(a) Prove the statement. What sort of proof are you using?

(b) State the converse. Is it true? Prove or disprove.

8. Prove the statement: For all integers n, if $5n$ is odd, then n is odd. Clearly state the style of proof you are using.

9. Prove the statement: For all integers a, b, and c, if $a^2 + b^2 = c^2$, then a or b is even.

10. Prove: $x = y$ if and only if $xy = \dfrac{(x+y)^2}{4}$. Note, you will need to prove two "directions" here: the "if" and the "only if" part.

11. The game TENZI comes with 40 six-sided dice (each numbered 1 to 6). Suppose you roll all 40 dice.

(a) Prove that there will be at least seven dice that land on the same number.

(b) How many dice would you have to roll before you were guaranteed that some four of them would all match or all be different? Prove your answer.

12. Prove that $\log(7)$ is irrational.

13. Prove that there are no integer solutions to the equation $x^2 = 4y + 3$.

14. Prove that every prime number greater than 3 is either one more or one less than a multiple of 6.

Hint. Prove the contrapositive by cases.

15. For each of the statements below, say what method of proof you should use to prove them. Then say how the proof starts and how it ends. Bonus points for filling in the middle.

(a) There are no integers x and y such that x is a prime greater than 5 and $x = 6y + 3$.

(b) For all integers n, if n is a multiple of 3, then n can be written as the sum of consecutive integers.

(c) For all integers a and b, if $a^2 + b^2$ is odd, then a or b is odd.

16. A standard deck of 52 cards consists of 4 suites (hearts, diamonds, spades and clubs) each containing 13 different values (Ace, 2, 3, . . . , 10, J, Q, K). If you draw some number of cards at random you might or might not have a pair (two cards with the same value) or three cards all of the same suit. However, if you draw enough cards, you will be guaranteed to have these. For each of the following, find the smallest number of cards you would need to draw to be guaranteed having the specified cards. Prove your answers.

(a) Three of a kind (for example, three 7's).

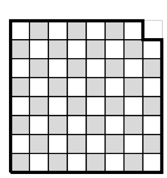
(b) A flush of five cards (for example, five hearts).

(c) Three cards that are either all the same suit or all different suits.

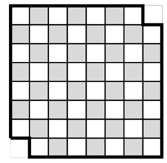
17. Suppose you are at a party with 19 of your closest friends (so including you, there are 20 people there). Explain why there must be least two people at the party who are friends with the same number of people at the party. Assume friendship is always reciprocated.

18. Your friend has given you his list of 115 best Doctor Who episodes (in order of greatness). It turns out that you have seen 60 of them. Prove that there are at least two episodes you have seen that are exactly four episodes apart.

19. Suppose you have an $n \times n$ chessboard but your dog has eaten one of the corner squares. Can you still cover the remaining squares with dominoes? What needs to be true about n? Give necessary and sufficient conditions (that is, say exactly which values of n work and which do not work). Prove your answers.



20. What if your $n \times n$ chessboard is missing two opposite corners? Prove that no matter what n is, you will not be able to cover the remaining squares with dominoes.



3.3 Chapter Review

1. Complete a truth table for the statement $\neg P \to (Q \wedge R)$.

2. Suppose you know that the statement "if Peter is not tall, then Quincy is fat and Robert is skinny" is false. What, if anything, can you conclude about Peter and Robert if you know that Quincy is indeed fat? Explain (you may reference problem 3.3.1).

3. Are the statements $P \to (Q \vee R)$ and $(P \to Q) \vee (P \to R)$ logically equivalent? Explain your answer.

4. Is the following a valid deduction rule? Explain.

$$\begin{array}{rl} & P \to Q \\ & P \to R \\ \hline \therefore & P \to (Q \wedge R). \end{array}$$

5. Write the negation, converse and contrapositive for each of the statements below.

(a) If the power goes off, then the food will spoil.

(b) If the door is closed, then the light is off.

(c) $\forall x(x < 1 \to x^2 < 1)$.

(d) For all natural numbers n, if n is prime, then n is solitary.

(e) For all functions f, if f is differentiable, then f is continuous.

(f) For all integers a and b, if $a \cdot b$ is even, then a and b are even.

(g) For every integer x and every integer y there is an integer n such that if $x > 0$ then $nx > y$.

(h) For all real numbers x and y, if $xy = 0$ then $x = 0$ or $y = 0$.

(i) For every student in Math 228, if they do not understand implications, then they will fail the exam.

6. Consider the statement: for all integers n, if n is even and $n \leq 7$ then n is negative or $n \in \{0, 2, 4, 6\}$.

(a) Is the statement true? Explain why.

(b) Write the negation of the statement. Is it true? Explain.

(c) State the contrapositive of the statement. Is it true? Explain.

(d) State the converse of the statement. Is it true? Explain.

7. Consider the statement: $\forall x(\forall y(x + y = y) \to \forall z(x \cdot z = 0))$.

(a) Explain what the statement says in words. Is this statement true? Be sure to state what you are taking the universe of discourse to be.

(b) Write the converse of the statement, both in words and in symbols. Is the converse true?

(c) Write the contrapositive of the statement, both in words and in symbols. Is the contrapositive true?

(d) Write the negation of the statement, both in words and in symbols. Is the negation true?

8. Write each of the following statements in the form, "if ..., then ...." Careful, some of the statements might be false (which is alright for the purposes of this question).

(a) To loose weight, you must exercise.

(b) To loose weight, all you need to do is exercise.

(c) Every American is patriotic.

(d) You are patriotic only if you are American.

(e) The set of rational numbers is a subset of the real numbers.

(f) A number is prime if it is not even.

(g) Either the Broncos will win the Super Bowl, or they won't play in the Super Bowl.

9. Simplify the following.

(a) $\neg(\neg(P \wedge \neg Q) \rightarrow \neg(\neg R \vee \neg(P \rightarrow R)))$.

(b) $\neg\exists x \neg\forall y \neg\exists z(z = x + y \rightarrow \exists w(x - y = w))$.

10. Consider the statement: for all integers n, if n is odd, then $7n$ is odd.

(a) Prove the statement. What sort of proof are you using?

(b) Prove the converse. What sort of proof are you using?

11. Suppose you break your piggy bank and scoop up a handful of 22 coins (pennies, nickels, dimes and quarters).

(a) Prove that you must have at least 6 coins of a single denomination.

(b) Suppose you have an odd number of pennies. Prove that you must have an odd number of at least one of the other types of coins.

(c) How many coins would you need to scoop up to be sure that you either had 4 coins that were all the same or 4 coins that were all different? Prove your answer.

12. You come across four trolls playing bridge. They declare:

Troll 1: All trolls here see at least one knave.
Troll 2: I see at least one troll that sees only knaves.
Troll 3: Some trolls are scared of goats.
Troll 4: All trolls are scared of goats.

Are there any trolls that are not scared of goats? Recall, of course, that all trolls are either knights (who always tell the truth) or knaves (who always lie).

Chapter 4

Graph Theory

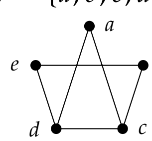
4.1 Exercises

1. If 10 people each shake hands with each other, how many handshakes took place? What does this question have to do with graph theory?

2. Among a group of 5 people, is it possible for everyone to be friends with exactly 2 of the people in the group? What about 3 of the people in the group?

3. Is it possible for two *different* (non-isomorphic) graphs to have the same number of vertices and the same number of edges? What if the degrees of the vertices in the two graphs are the same (so both graphs have vertices with degrees 1, 2, 2, 3, and 4, for example)? Draw two such graphs or explain why not.

4. Are the two graphs below equal? Are they isomorphic? If they are isomorphic, give the isomorphism. If not, explain.
Graph 1: $V = \{a, b, c, d, e\}$, $E = \{\{a, b\}, \{a, c\}, \{a, e\}, \{b, d\}, \{b, e\}, \{c, e\}\}$.

Graph 2: 

5. Consider the following two graphs:

G_1 $\quad V_1 = \{a, b, c, d, e, f, g\}$
$E_1 = \{\{a, b\}, \{a, d\}, \{b, c\}, \{b, d\}, \{b, e\}, \{b, f\}, \{c, g\}, \{d, e\}, \{e, f\}, \{f, g\}\}$.

G_2 $\quad V_2 = \{v_1, v_2, v_3, v_4, v_5, v_6, v_7\}$,
$E_2 = \{\{v_1, v_4\}, \{v_1, v_5\}, \{v_1, v_7\}, \{v_2, v_3\}, \{v_2, v_6\}, \{v_3, v_5\}, \{v_3, v_7\}, \{v_4, v_5\}, \{v_5, v_6\}, \{v_5, v_7\}\}$

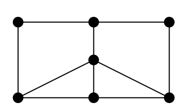
(a) Let $f : G_1 \to G_2$ be a function that takes the vertices of Graph 1 to vertices of Graph 2. The function is given by the following table:

x	a	b	c	d	e	f	g
$f(x)$	v_4	v_5	v_1	v_6	v_2	v_3	v_7

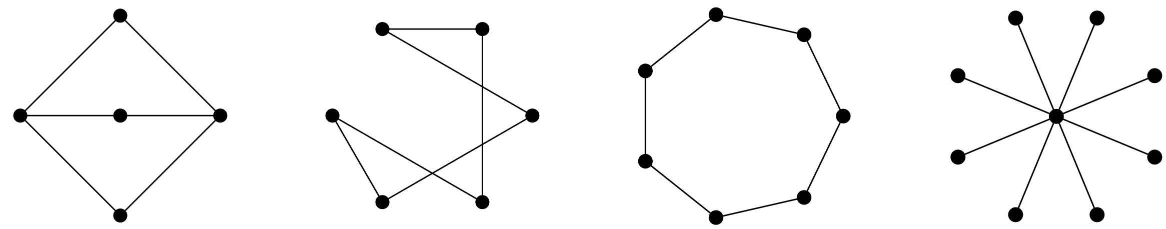
Does f define an isomorphism between Graph 1 and Graph 2? Explain.

(b) Define a new function g (with $g \neq f$) that defines an isomorphism between Graph 1 and Graph 2.

(c) Is the graph pictured below isomorphic to Graph 1 and Graph 2? Explain.



6. Which of the graphs below are bipartite? Justify your answers.



7. For which $n \geq 3$ is the graph C_n bipartite?

8. For each of the following, try to give two different unlabeled graphs with the given properties, or explain why doing so is impossible.

(a) Two different trees with the same number of vertices and the same number of edges. A tree is a connected graph with no cycles.

(b) Two different graphs with 8 vertices all of degree 2.

(c) Two different graphs with 5 vertices all of degree 4.

(d) Two different graphs with 5 vertices all of degree 3.

4.2 Exercises

1. Is it possible for a planar graph to have 6 vertices, 10 edges and 5 faces? Explain.

2. The graph G has 6 vertices with degrees $2, 2, 3, 4, 4, 5$. How many edges does G have? Could G be planar? If so, how many faces would it have. If not, explain.

3. I'm thinking of a polyhedron containing 12 faces. Seven are triangles and four are squares. The polyhedron has 11 vertices other than those around the mystery face. How many sides does the last face have?

4. Consider some classic polyhedrons.

(a) An *octahedron* is a regular polyhedron made up of 8 equilateral triangles (it sort of looks like two pyramids with their bases glued together). Draw a planar graph representation of an octahedron. How many vertices, edges and faces does an octahedron (and your graph) have?

(b) The traditional design of a soccer ball is in fact a (spherical projection of a) truncated icosahedron. This consists of 12 regular pentagons and 20 regular hexagons. No two pentagons are adjacent (so the edges of each pentagon are shared only by hexagons). How many vertices, edges, and faces does a truncated icosahedron have? Explain how you arrived at your answers. Bonus: draw the planar graph representation of the truncated icosahedron.

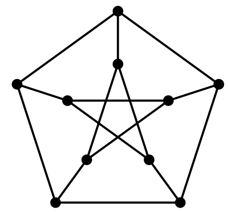
(c) Your "friend" claims that he has constructed a convex polyhedron out of 2 triangles, 2 squares, 6 pentagons and 5 octagons. Prove that your friend is lying. Hint: each vertex of a convex polyhedron must border at least three faces.

5. Prove Euler's formula using induction on the number of edges in the graph.

6. Prove Euler's formula using induction on the number of *vertices* in the graph.

7. Euler's formula ($v - e + f = 2$) holds for all *connected* planar graphs. What if a graph is not connected? Suppose a planar graph has two components. What is the value of $v - e + f$ now? What if it has k components?

8. Prove that the *Petersen graph* (below) is not planar. Hint: what is the length of the shortest cycle?



Hint.
What is the length of the shortest cycle? (This quantity is usually called the **girth** of the graph.)

9. Prove that any planar graph with v vertices and e edges satisfies $e \leq 3v - 6$.

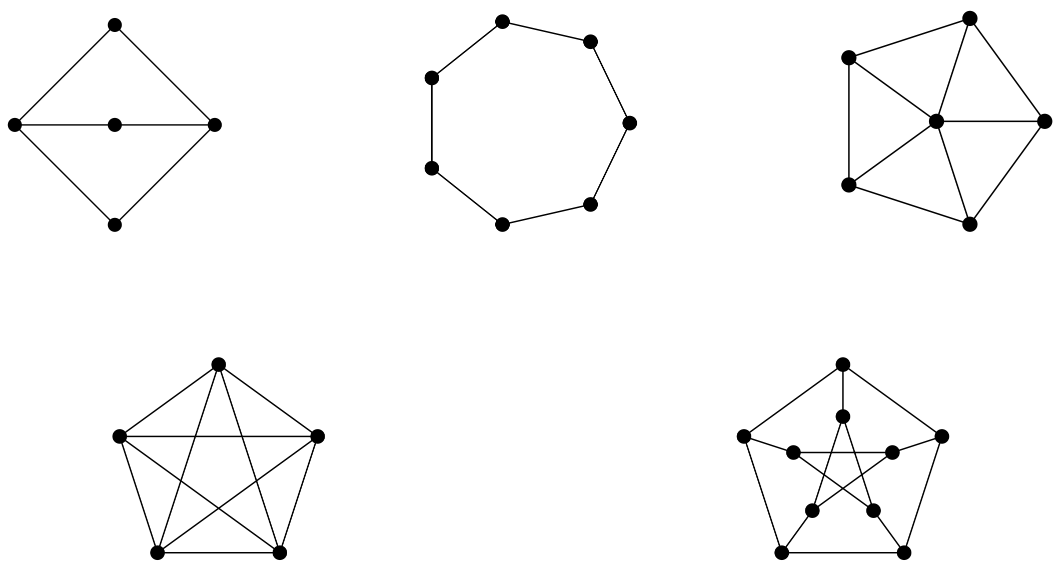
10. Prove that any planar graph must have a vertex of degree 5 or less.

4.3 Exercises

1. What is the smallest number of colors you need to properly color the vertices of $K_{4,5}$? That is, find the chromatic number of the graph.

2. Draw a graph with chromatic number 6 (i.e., which requires 6 colors to properly color the vertices). Could your graph be planar? Explain.

3. Find the chromatic number of each of the following graphs.



4. A group of 10 friends decides to head up to a cabin in the woods (where nothing could possibly go wrong). Unfortunately, a number of these friends have dated each other in the past, and things are still a little awkward. To get the cabin, they need to divide up into some number of cars, and no two people who dated should be in the same car.

(a) What is the smallest number of cars you need if all the relationships were strictly heterosexual? Represent an example of such a situation with a graph. What kind of graph do you get?

(b) Because a number of these friends dated there are also conflicts between friends of the same gender, listed below. Now what is the smallest number of conflict-free cars they could take to the cabin?

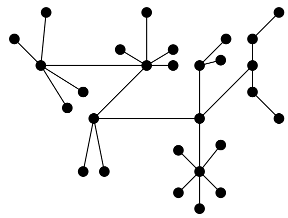
Friend	A	B	C	D	E	F	G	H	I	J
Conflicts with	BEJ	ADG	HJ	BF	AI	DJ	B	CI	EHJ	ACFI

(c) What do these questions have to do with coloring?

5. What is the smallest number of colors that can be used to color the vertices of a cube so that no two adjacent vertices are colored identically?

6. Prove the chromatic number of any tree is two. Recall, a tree is a connected graph with no cycles.

(a) Describe a procedure to color the tree below.



(b) The chromatic number of C_n is two when n is even. What goes wrong when n is odd?

(c) Prove that your procedure from part (a) always works for any tree.

(d) Now, prove using induction that every tree has chromatic number 2.

7. Prove the 6-color theorem: every planar graph has chromatic number 6 or less. Do not assume the 4-color theorem (whose proof is MUCH harder), but you may assume the fact that every planar graph contains a vertex of degree at most 5.

8. Not all graphs are perfect. Give an example of a graph with chromatic number 4 that does not contain a copy of K_4. That is, there should be no 4 vertices all pairwise adjacent.

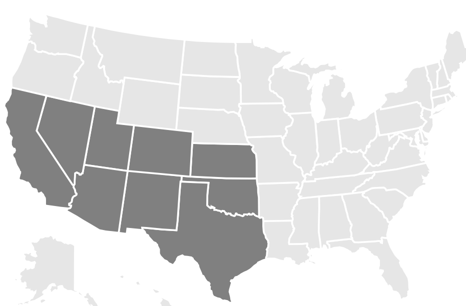
9. Prove by induction on vertices that any graph G which contains at least one vertex of degree less than $\Delta(G)$ (the maximal degree of all vertices in G) has chromatic number at most $\Delta(G)$.

10. You have a set of magnetic alphabet letters (one of each of the 26 letters in the alphabet) that you need to put into boxes. For obvious reasons, you don't want to put two consecutive letters in the same box. What is the fewest number of boxes you need (assuming the boxes are able to hold as many letters as they need to)?

11. Prove that if you color every edge of K_6 either red or blue, you are guaranteed a monochromatic triangle (that is, an all red or an all blue triangle).

4.4 Exercises

1. You and your friends want to tour the southwest by car. You will visit the nine states below, with the following rather odd rule: you must cross each border between neighboring states exactly once (so, for example, you must cross the Colorado-Utah border exactly once). Can you do it? If so, does it matter where you start your road trip? What fact about graph theory solves this problem?



2. Which of the following graphs contain an Euler path? Which contain an Euler circuit?

(a) K_4

(b) K_5.

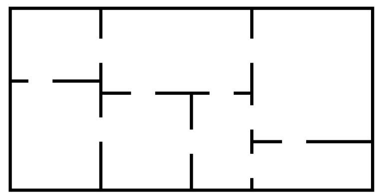
(c) $K_{5,7}$

(d) $K_{2,7}$

(e) C_7

(f) P_7

3. Edward A. Mouse has just finished his brand new house. The floor plan is shown below:



(a) Edward wants to give a tour of his new pad to a lady-mouse-friend. Is it possible for them to walk through every doorway exactly once? If so, in which rooms must they begin and end the tour? Explain.

(b) Is it possible to tour the house visiting each room exactly once (not necessarily using every doorway)? Explain.

(c) After a few mouse-years, Edward decides to remodel. He would like to add some new doors between the rooms he has. Of course, he cannot add any doors to the exterior of the house. Is it possible for each room to have an odd number of doors? Explain.

4. For which n does the graph K_n contain an Euler circuit? Explain.

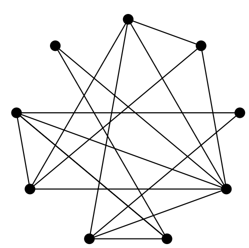
5. For which m and n does the graph $K_{m,n}$ contain an Euler path? An Euler circuit? Explain.

6. For which n does K_n contain a Hamilton path? A Hamilton cycle? Explain.

7. For which m and n does the graph $K_{m,n}$ contain a Hamilton path? A Hamilton cycle? Explain.

8. A bridge builder has come to Königsberg and would like to add bridges so that it *is* possible to travel over every bridge exactly once. How many bridges must be built?

9. Below is a graph representing friendships between a group of students (each vertex is a student and each edge is a friendship). Is it possible for the students to sit around a round table in such a way that every student sits between two friends? What does this question have to do with paths?

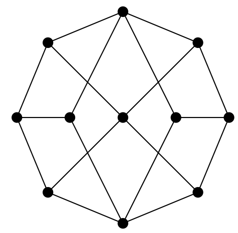


10.

(a) Suppose a graph has a Hamilton path. What is the maximum number of vertices of degree one the graph can have? Explain why your answer is correct.

(b) Find a graph which does not have a Hamilton path even though no vertex has degree one. Explain why your example works.

11. Consider the following graph:



(a) Find a Hamilton path. Can your path be extended to a Hamilton cycle?

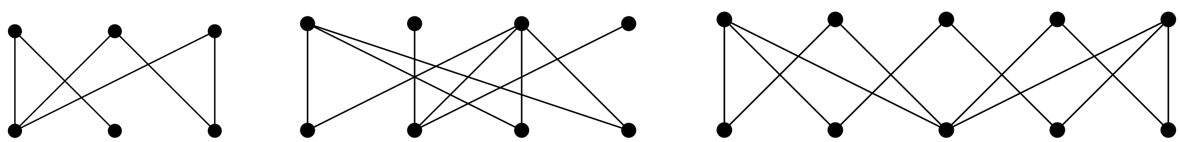
(b) Is the graph bipartite? If so, how many vertices are in each "part"?

(c) Use your answer to part (b) to prove that the graph has no Hamilton cycle.

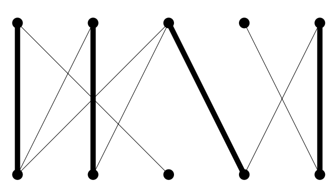
(d) Suppose you have a bipartite graph G in which one part has at least two more vertices than the other. Prove that G does not have a Hamilton path.

4.5 Exercises

1. Find a matching of the bipartite graphs below or explain why no matching exists.

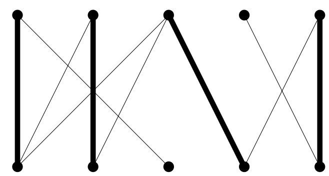


2. A bipartite graph that doesn't have a matching might still have a **partial matching**. By this we mean a set of *edges* for which no vertex belongs to more than one edge (but possibly belongs to none). Every bipartite graph (with at least one edge) has a partial matching, so we can look for the largest partial matching in a graph.
Your "friend" claims that she has found the largest partial matching for the graph below (her matching is in bold). She explains that no other edge can be added, because all the edges not used in her partial matching are connected to matched vertices. Is she correct?

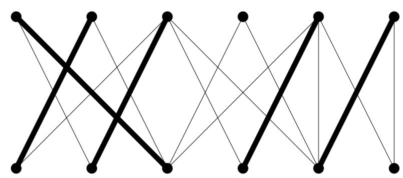


3. One way you might check to see whether a partial matching is maximal is to construct an **alternating path**. This is a sequence of adjacent edges, which alternate between edges in the matching and edges not in the matching (no edge can be used more than once). If an alternating path starts and stops with an edge *not* in the matching, then it is called an **augmenting path**.

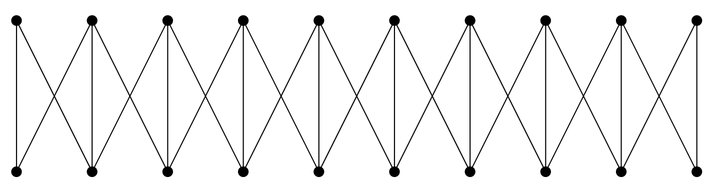
(a) Find the largest possible alternating path for the partial matching of your friend's graph. Is it an augmenting path? How would this help you find a larger matching?



(b) Find the largest possible alternating path for the partial matching below. Are there any augmenting paths? Is the partial matching the largest one that exists in the graph?



4. The two richest families in Westeros have decided to enter into an alliance by marriage. The first family has 10 sons, the second has 10 girls. The ages of the kids in the two families match up. To avoid impropriety, the families insist that each child must marry someone either their own age, or someone one position younger or older. In fact, the graph representing agreeable marriages looks like this:



The question: how many different acceptable marriage arrangements which marry off all 20 children are possible?

(a) How many marriage arrangements are possible if we insist that there are exactly 6 boys marry girls not their own age?

(b) Could you generalize the previous answer to arrive at the total number of marriage arrangements?

(c) How do you know you are correct? Try counting in a different way. Look at smaller family sizes and get a sequence.

(d) Can you give a recurrence relation that fits the problem?

5. We say that a set of vertices $A \subseteq V$ is a **vertex cover** if every edge of the graph is incident to a vertex in the cover (so a vertex cover covers the *edges*). Since V itself is a vertex cover, every graph has a vertex cover. The interesting question is about finding a **minimal** vertex cover, one that uses the fewest possible number of vertices.

(a) Suppose you had a matching of a graph. How can you use that to get a minimal vertex cover? Will your method always work?

(b) Suppose you had a minimal vertex cover for a graph. How can you use that to get a partial matching? Will your method always work?

(c) What is the relationship between the size of the minimal vertex cover and the size of the maximal partial matching in a graph?

6. For many applications of matchings, it makes sense to use bipartite graphs. You might wonder, however, whether there is a way to find matchings in graphs in general.

(a) For which n does the complete graph K_n have a matching?

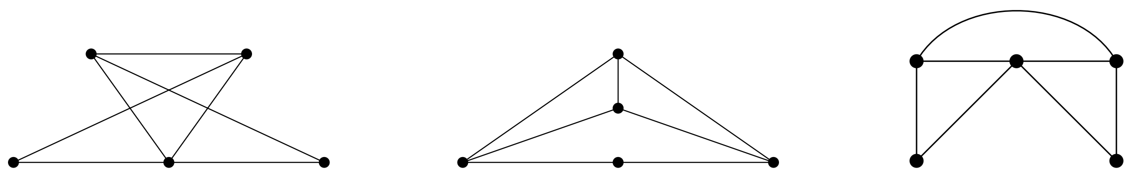
(b) Prove that if a graph has a matching, then $|V|$ is even.

(c) Is the converse true? That is, do all graphs with $|V|$ even have a matching?

(d) What if we also require the matching condition? Prove or disprove: If a graph with an even number of vertices satisfies $|N(S)| \geq |S|$ for all $S \subseteq V$, then the graph has a matching.

4.6 Chapter Review

1. Which (if any) of the graphs below are the same? Which are different? Explain.



2. Which of the graphs in the previous question contain Euler paths or circuits? Which of the graphs are planar?

3. Draw a graph which has an Euler circuit but is not planar.

4. Draw a graph which does not have an Euler path and is also not planar.

5. If a graph has 10 vertices and 10 edges and contains an Euler circuit, must it be planar? How many faces would it have?

6. Suppose G is a graph with n vertices, each having degree 5.

(a) For which values of n does this make sense?

(b) For which values of n does the graph have an Euler path?

(c) What is the smallest value of n for which the graph might be planar? (tricky)

7. At a school dance, 6 girls and 4 boys take turns dancing (as couples) with each other.

(a) How many couples danced if every girl dances with every boy?

(b) How many couples danced if everyone danced with everyone else (regardless of gender)?

(c) Explain what graphs can be used to represent these situations.

8. Among a group of n people, is it possible for everyone to be friends with an odd number of people in the group? If so, what can you say about n?

9. Your friend has challenged you to create a convex polyhedron containing 9 triangles and 6 pentagons.

(a) Is it possible to build such a polyhedron using *only* these shapes? Explain.

(b) You decide to also include one heptagon (seven-sided polygon). How many vertices does your new convex polyhedron contain?

(c) Assuming you are successful in building your new 16-faced polyhedron, could every vertex be the joining of the same number of faces? Could each vertex join either 3 or 4 faces? If so, how many of each type of vertex would there be?

10. Is there a convex polyhedron which requires 5 colors to properly color the vertices of the polyhedron? Explain.

11. How many edges does the graph $K_{n,n}$ have? For which values of n does the graph contain an Euler circuit? For which values of n is the graph planar?

12. The graph G has 6 vertices with degrees $1, 2, 2, 3, 3, 5$. How many edges does G have? If G was planar how many faces would it have? Does G have an Euler path?

13. What is the smallest number of colors you need to properly color the vertices of K_7. Can you say whether K_7 is planar based on your answer?

14. What is the smallest number of colors you need to properly color the vertices of $K_{3,4}$? Can you say whether $K_{3,4}$ is planar based on your answer?

15. A dodecahedron is a regular convex polyhedron made up of 12 regular pentagons.

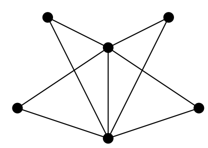
(a) Suppose you color each pentagon with one of three colors. Prove that there must be two adjacent pentagons colored identically.

(b) What if you use four colors?

(c) What if instead of a dodecahedron you colored the faces of a cube?

16. If a planar graph G with 7 vertices divides the plane into 8 regions, how many edges must G have?

17. Consider the graph below:



(a) Does the graph have an Euler path or circuit? Explain.

(b) Is the graph planar? Explain.

(c) Is the graph bipartite? Complete? Complete bipartite?

(d) What is the chromatic number of the graph.

18. For each part below, say whether the statement is true or false. Explain why the true statements are true, and give counterexamples for the false statements.

(a) Every bipartite graph is planar.

(b) Every bipartite graph has chromatic number 2.

(c) Every bipartite graph has an Euler path.

(d) Every vertex of a bipartite graph has even degree.

(e) A graph is bipartite if and only if the sum of the degrees of all the vertices is even.

19. Consider the statement "If a graph is planar, then it has an Euler path."

(a) Write the converse of the statement.

(b) Write the contrapositive of the statement.

(c) Write the negation of the statement.

(d) Is it possible for the contrapositive to be false? If it was, what would that tell you?

(e) Is the original statement true or false? Prove your answer.

(f) Is the converse of the statement true or false? Prove your answer.

20. Remember that a **tree** is a connected graph with no cycles.

(a) Conjecture a relationship between a tree graph's vertices and edges. (For instance, can you have a tree with 5 vertices and 7 edges?)

(b) Explain why every tree with at least 3 vertices has a leaf (i.e., a vertex of degree 1).

(c) Prove your conjecture from part (a) by induction on the number of vertices. Hint: For the inductive step, you will assume that your conjecture is true for all trees with k vertices, and show it is also true for an arbitrary tree with $k + 1$ vertices. Consider what happens when you cut off a leaf and then let it regrow.