# May 28 Lec—Conditional Variance

Recall we proved that

$$Cov(Y - \mathbb{E}(Y|X), H(X)) = 0$$

so $Y - \mathbb{E}(Y|X) \perp \mathbb{E}(Y|X)$.
If we transform the Pythagorean theorem of vector spaces to here ( $\|u\|^2 = \|w\|^2 + \|u-w\|^2$ for $u \perp w$), and keeping in mind that $\|X\|^2 = Cov(X) = V(X)$, we have

$$V(Y) = V(\mathbb{E}(Y|X)) + V(Y - \mathbb{E}(Y|X))$$

**Proposition**

$$V(Y) = V(\mathbb{E}(Y|X)) + V(Y - \mathbb{E}(Y|X))$$

**Proof**

$$V(Y) = Cov(Y, Y)$$

Adding and subtracting $\mathbb{E}(Y|X)$:

$$= Cov[(Y - \mathbb{E}(Y|X)) + \mathbb{E}(Y|X), (Y - \mathbb{E}(Y|X)) + \mathbb{E}(Y|X)]$$
$$= V(Y - \mathbb{E}(Y|X)) + V(\mathbb{E}(Y|X)) + \underbrace{2Cov(\mathbb{E}(Y|X), Y - \mathbb{E}(Y|X))}_{=0}$$

**Remark**

Since by the Tower Property we have $\mathbb{E}(Y - \mathbb{E}(Y|X)) = 0$, we have $V(Y - \mathbb{E}(Y|X)) = \mathbb{E}((Y - \mathbb{E}(Y|X))^2)$.
Again by the Tower Property,

$$V(Y - \mathbb{E}(X)) = \mathbb{E}(\mathbb{E}([Y - \mathbb{E}(Y|X)]^2|X))$$

**Definition**

The random variable

$$\mathbb{E}[(Y - \mathbb{E}(Y|X))^2|X]$$

is called the **conditional variance** of $Y$ given $X$ and is denoted $V(Y|X)$.

Applying the Proposition, the Remark, and the Definition, we conclude

$$V(Y) = V(\mathbb{E}(Y|X)) + \mathbb{E}(V(Y|X))$$

**Example**

Continuing the restaurant example we had yesterday. Find $\mathbb{E}(X_1)$ and $V(X_1)$.

Given $X$ fixed, we knew that $X_1 \sim B(X, p)$.
So $\mathbb{E}(X_1|X) = Xp$ and $V(X_1|X) = Xp(1-p)$.
$\mathbb{E}(X_1) = \mathbb{E}(\mathbb{E}(X_1|X)) = \mathbb{E}(Xp) = p\mathbb{E}(X) = \lambda p$ (we would have gotten the same answer if we used the fact that $X \sim P(\lambda p)$)
and

$$V(X_1) = \mathbb{E}(V(X_1|X)) + V(\mathbb{E}(X_1|X))$$

$$\mathbb{E}(p(1-p)X) + V(pX)$$

$$p(1-p)\mathbb{E}(X) + p^2 V(X)$$

$$= \lambda p(1-p) + p^2\lambda = \lambda p$$

as we expected from the fact that $X \sim P(\lambda p)$.

**Another example**

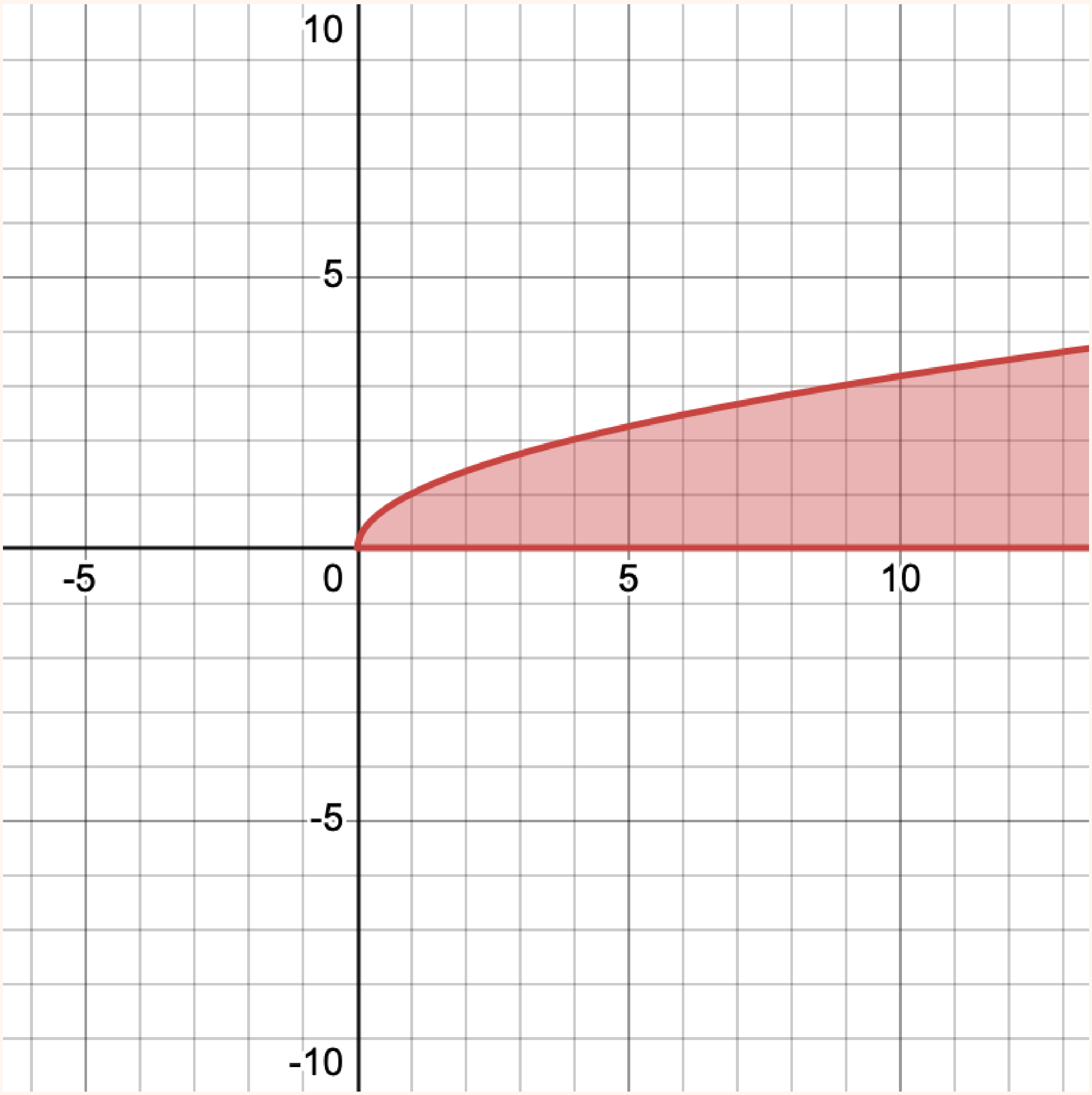
Suppose $X \sim \Gamma(\alpha = 2, \beta = 2)$
Given $X = x$, suppose $Y \sim U(0, \sqrt{x})$.
Find $\mathbb{E}(Y)$ and $V(Y)$.

First, try to brute-force it.
We know $f_X(x) = \frac{1}{\beta^a \Gamma(\alpha)} x^{\alpha-1} e^{-x/\beta} = \frac{1}{4} x e^{-\frac{x}{2}}$ with domain $(0, \infty)$
and $f(y|X = x) = \frac{1}{\sqrt{x}}$, domain $(0, \sqrt{x})$

So

$$f(x,y) = f_X(x)\cdot f(y|X=x) = \begin{cases} \frac{1}{4}xe^{-\frac{x}{2}}\cdot\frac{1}{\sqrt{x}} & 0<y<\sqrt{x} \\ 0 & \text{otherwise} \end{cases}$$

$$\begin{cases} \frac{1}{4}\sqrt{x}e^{-\frac{x}{2}} & 0<y<\sqrt{x} \\ 0 & \text{otherwise} \end{cases}$$




So

$$f_Y(y) = \int_{-\infty}^{\infty} f(x,y)\,dx$$

$$= \begin{cases} \int_{y^2}^{\infty} \frac{1}{4}\sqrt{x}e^{-x/2}\,dx & y>0 \\ 0 & \text{otherwise} \end{cases}$$

(Good luck trying to find a closed form for that)
and

$$\mathbb{E}(Y) = \int_0^\infty y f_Y(y)\, dy$$

$$\int_0^\infty y \left( \int_{y^2}^\infty \frac{1}{4} \sqrt{x} e^{-x/2}\, dx \right) dx$$

Swap the order of integration (which is what we did in the Tower Property anyway):

$$= \int_0^\infty \int_0^{\sqrt{x}} y \cdot \frac{1}{4} \sqrt{x} e^{-x/2}\, dy\, dx$$

$$\int_0^\infty \frac{1}{4} \sqrt{x} e^{-x/2} \cdot \frac{x}{2}\, dx$$

$$= \mathbb{E}\left( \frac{\sqrt{x}}{2} \right)$$

(you can compute it, which we'll do below)

On the other hand, we could have just used the Tower Property and the formulas derived from it:
$\mathbb{E}(Y|X) = \mathbb{E}(Y|X = x) = \frac{\sqrt{x}}{2}$
and $V(Y|X) = V(Y|X = x) = \frac{(\sqrt{x})^2}{12} = \frac{x}{12}$ by the properties of the uniform distribution.
So,

$$\mathbb{E}(Y) = \mathbb{E}(\mathbb{E}(Y|X)) = \frac{1}{2}\mathbb{E}(\sqrt{x})$$

which is a lot lot easier to do:

$$\frac{1}{2}\mathbb{E}(\sqrt{x}) = \frac{1}{2} \int_0^\infty \frac{1}{4} x e^{-\frac{x}{2}} \sqrt{x}\, dx$$

$$= \frac{1}{8} \int_0^\infty x^{3/2} e^{-x/2}\, dx$$

$$\frac{1}{8} \cdot 2^{5/2} \cdot \Gamma\left(\frac{5}{2}\right)$$

$$= \frac{3\sqrt{2\pi}}{8}$$

and

$$V(Y) = \mathbb{E}(V(Y|X)) + V(\mathbb{E}(Y|X))$$

$$= \mathbb{E}\left(\frac{x}{3}\right) + V\left(\frac{\sqrt{x}}{2}\right)$$

$$\frac{1}{3} \cdot \mathbb{E}(X) + \mathbb{E}\left(\left(\frac{\sqrt{x}}{2}\right)^2\right) - \left(\mathbb{E}\left(\frac{\sqrt{x}}{2}\right)\right)^2$$

$$\frac{1}{3}\mathbb{E}(X) + \mathbb{E}\left(\frac{X}{4}\right) - \left(\frac{1}{2}\mathbb{E}(\sqrt{x})\right)$$

$$= \frac{7}{12}\mathbb{E}(X) - \frac{1}{2}\mathbb{E}(\sqrt{x})$$

$$= \frac{7}{12} \cdot 4 + \frac{3\sqrt{2\pi}}{8} = \frac{56 + 9\sqrt{2\pi}}{24}$$