May 22 Lec—Independence, Expected Value (Bivariate), Mgfs of Sum of Two Random Variables

We did this example before:

Example

$$f(x,y)=\begin{cases}6x & 0<x<y<1\\ 0 & \text{otherwise}\end{cases}$$

and saw that $X\sim Beta(2,2), Y\sim Beta(3,1)$ (marginal distributions).
Let us find $f(y|X=x)=\frac{f(x,y)}{f_X(x)}$

$$f(y|X=x)=\begin{cases}\frac{6x}{f_X(x)} & x<y<1\\ 0 & \text{otherwise}\end{cases}$$

$$=\begin{cases}\frac{6x}{6x(1-x)}=\frac{1}{1-x} & x<y<1\\ 0 & \text{otherwise}\end{cases}$$

Given $X=x$, we see $Y\sim U(x,1)$.
Now calculating $f(x|Y=y)=\frac{f(x,y)}{f_Y(y)}$:

$$f(x|Y=y)=\begin{cases}\frac{6x}{3y^2}=\frac{2x}{y^2} & 0<x<y\\ 0 & \text{otherwise}\end{cases}$$

Given $Y=y$, we have $X\sim Y\cdot Beta(2,1)$ (you can easily verify this by using the formula $f_V(v)=\frac{1}{|\alpha|}f_U\left(\frac{v-\beta}{\alpha}\right)$ for $V=\alpha U+\beta$, taught in May 14)

Example 2

$$f(x,y)=\begin{cases}e^{-x} & 0<y<x\\ 0 & \text{otherwise}\end{cases}$$

We found yesterday that $X\sim G(\alpha=2,\beta=1)$, $Y\sim E(\beta=1)$.
Let $x>0$. Given $X=x$, we have

$$f(y|X=x)=\frac{f(x,y)}{f_X(x)}=\begin{cases}\frac{e^{-x}}{xe^{-x}} & 0<y<x\\ 0 & \text{otherwise}\end{cases}$$

So given $X = x$, we have $Y \sim U(0, x)$.
Now for $f(x|Y = y)$:

$$f(x|Y = y) = \begin{cases} \frac{e^{-x}}{e^{-y}} = e^{-(x+y)} & y < x \\ 0 & \text{otherwise} \end{cases}$$

So given $Y = y$, we have $X + Y \sim Exponential(1)$.
Intuitively, X and Y are dependent. We'll formally define this notation now.

Independence

Definition:

1. Let (X, Y) be a pair of discrete random variables, and p be their jpf. We say X and Y are independent if

$$p(x, y) = p_X(x)p_Y(y)$$

2. If (X, Y) is a pair of continuous random variables and f is their jpdf, then X and Y are independent if

$$f(x, y) = f_X(x)f_Y(y)$$

The motivation of this definition is the fact that, in the discrete case,

$$p(x, y) = P(\{X = x\} \cap \{Y = y\})$$

and if X and Y are independent, then $\{X = x\}$ and $\{Y = y\}$ should be independent events too. So

$$\begin{aligned} p(x, y) &= P(\{X = x\}) \cdot P(\{Y = y\}) \\ &= p_X(x) \cdot p_Y(y) \end{aligned}$$

So the above definition of independence is equivalent to saying that (in the discrete case, and vice versa for Y)

$$p(x|Y = y) = \frac{p(x, y)}{p_Y(y)} = p_X(x)$$

and (in the continuous case, vice versa for Y)

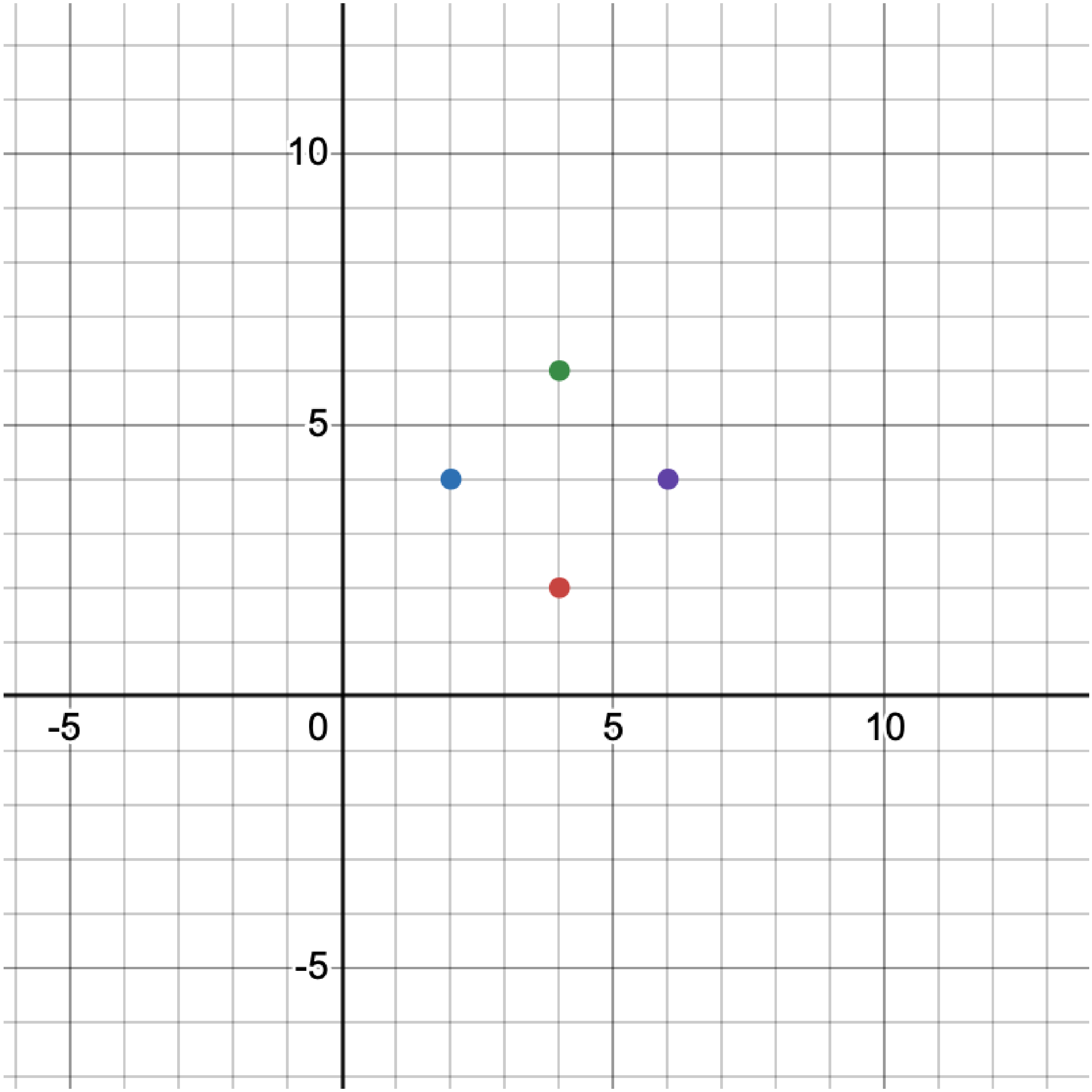
$$f(x|Y=y)=\frac{f(x,y)}{f_Y(y)}=f_X(x)$$

Example

y-**value** x-**value**	-1	0	1	$p_Y(y)$
0	$\frac{1}{4}$	$\frac{1}{6}$	$\frac{1}{12}$	$\frac{1}{2}$
1	$\frac{1}{4}$	0	$\frac{1}{4}$	$\frac{1}{2}$
$p_X(x)$	$\frac{1}{2}$	$\frac{1}{6}$	$\frac{1}{3}$	

Y and X cannot possibly be independent because $p(0,1)=0\neq p_X(0)\cdot p_Y(1)$.

Also consider these two supports, the first for a discrete distribution and the second for a continuous one:



10
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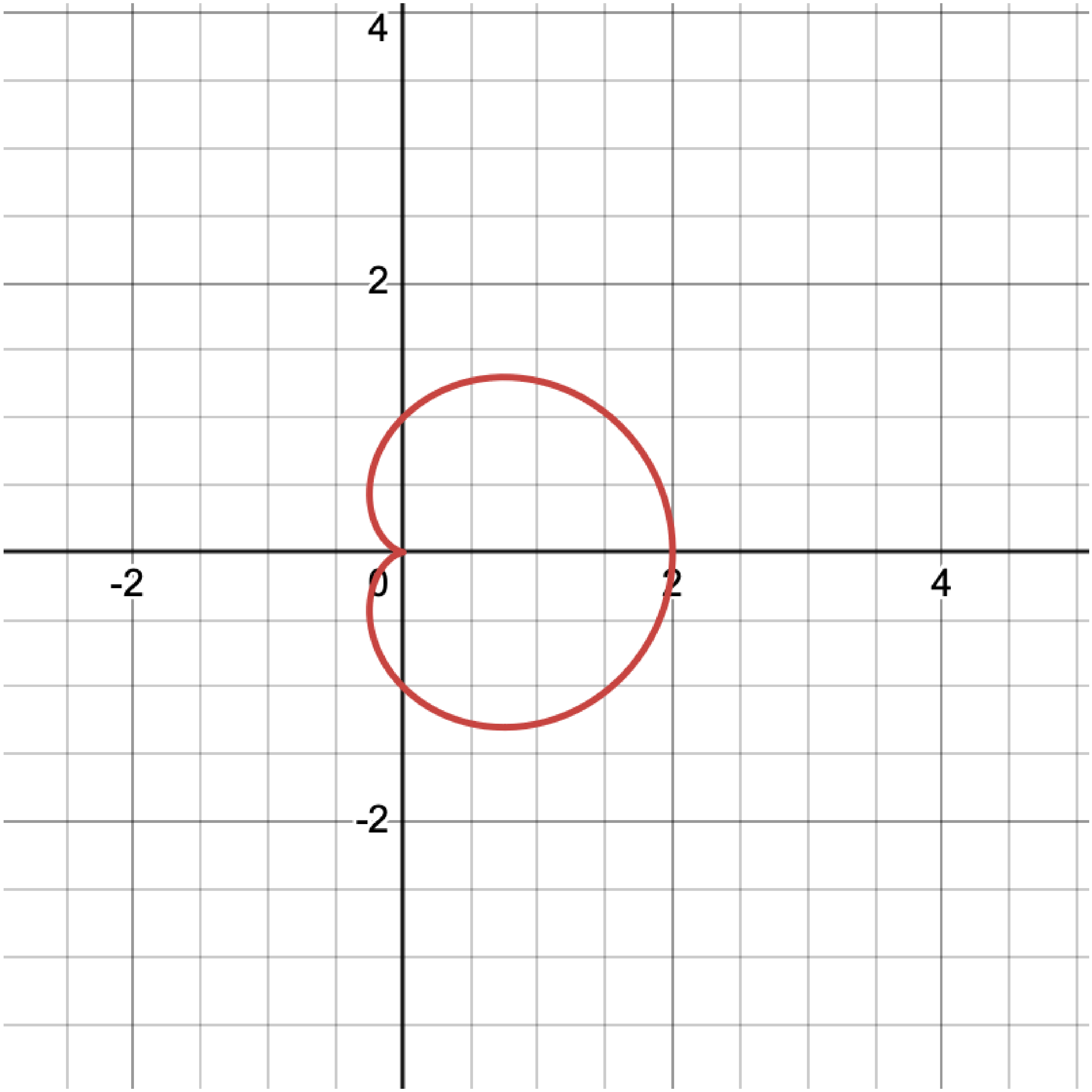


In the first case, if $X = 4$ then $p(Y = 4|X = 4) = 0$, but if $X = 2$ then $p(Y = 4|X = 2) \neq 0$.
In the second case, a similar thing happens if we choose $y = 0$ and $X = -0.2$.

Proposition

If two continuous random variables X, Y are independent, then their joint support is rectangular type.
If the joint support is rectangular type, a sufficient condition is $f(x, y) = h(x)g(y)$.

In total, two continuous random variables X, Y are independent if and only if their joint support is of rectangular type and $f(x, y) = h(x)g(y)$ for some functions $h(x), g(y)$.

Examples

1. $$f(x, y) = \begin{cases} 6x & 0 < x < y < 1 \\ 0 & \text{otherwise} \end{cases}$$

   We drew the support of this before—it's not rectangular, so X and Y aren't independent.

2. $$f(x, y) = \begin{cases} x + y & x, y \in (0, 1) \\ 0 & \text{otherwise} \end{cases}$$

   Rectangular support, but for the sake of your dear life you can't write the kernel $x + y$ as a product of a function of x and a function of y.

3. $$f(x, y) = \begin{cases} 2x^2 y e^{-(x+2y)} & x, y > 0 \\ 0 & \text{otherwise} \end{cases}$$

   Rectangular type support, but that fact alone is not enough.
   We want to write it as functions of X and Y in the form of $h(x) = C_1 \cdot x^2 e^{-x}$ and $g(y) = C_2 \cdot y e^{-2y}$.
   If X and Y's marginal distributions have these kinds of kernel, we see that $X \sim G(\alpha = 3, \beta = 1)$ so $C_1 = \frac{1}{2}$, and $Y \sim G\left(\alpha = 2, \beta = \frac{1}{2}\right)$ so $C_2 = 4$. Hence we should split the 2 in $f(x, y)$ as $4 \cdot \frac{1}{2}$.

Expected value

You have seen that, for discrete cases,

$$\mathbb{E}(h(x)) = \sum_x h(x) p_X(x)$$

and for continuous cases,

$$\mathbb{E}(h(x)) = \int_{-\infty}^{\infty} h(x) f_X(x)\, dx$$

For joint random variables, this is a generalization:

Definition: Expected value (joint random variables)

1. Let (X, Y) be a pair of discrete random variables and $p(x, y)$ be their jpf. If $h : \mathbb{R}^2 \to \mathbb{R}$, then

$$\mathbb{E}(h(x, y)) = \sum_{x,y} h(x, y) p(x, y)$$

2. Let (X, Y) be a pair of continuous random variable and $f(x, y)$ be their jpdf. Again, if $h : \mathbb{R}^2 \to \mathbb{R}$, then

$$\mathbb{E}(h(x, y)) = \iint_{\mathbb{R}^2} h(x, y) f(x, y)\; dA$$

The univariate case is a special case of the bivariate case: Let $h(x, y) = g(x)$. Then $\mathbb{E}(h(x, y)) = \mathbb{E}(g(x))$. So

$$\begin{aligned}\iint_{\mathbb{R}^2} h(x, y) f(x, y)\; dA &= \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} g(x) f(x, y)\, dy \right) dx \\ &= \int_{-\infty}^{\infty} g(x) \left(\int_{-\infty}^{\infty} f(x, y)\, dy \right) dx \\ &= \int_{-\infty}^{\infty} g(x) f_X(x)\, dx = \mathbb{E}(g(x))\end{aligned}$$

Proposition

1. If $h(x, y) = C$ a constant, then $\mathbb{E}(h(x, y)) = C$.
2. $\mathbb{E}(h_1(x, y) + h_2(x, y)) = \mathbb{E}(h_1(x, y)) + \mathbb{E}(h_2(x, y))$. In particular,

$$\mathbb{E}(X + Y) = \mathbb{E}(X) + \mathbb{E}(Y)$$

3. If $\alpha \in \mathbb{R}$ is a constant, then $\mathbb{E}(\alpha \cdot h(x, y)) = \alpha \cdot \mathbb{E}(h(x, y))$.

Example

y-value\ x-value	-1	0	1	$p_Y(y)$
0	$\frac{1}{4}$	$\frac{1}{6}$	$\frac{1}{12}$	$\frac{1}{2}$
1	$\frac{1}{4}$	0	$\frac{1}{4}$	$\frac{1}{2}$
$p_X(x)$	$\frac{1}{2}$	$\frac{1}{6}$	$\frac{1}{3}$	

We calculate $\mathbb{E}(XY)$:

$$\mathbb{E}(XY) = \sum_{x,y} xyp(x,y)$$

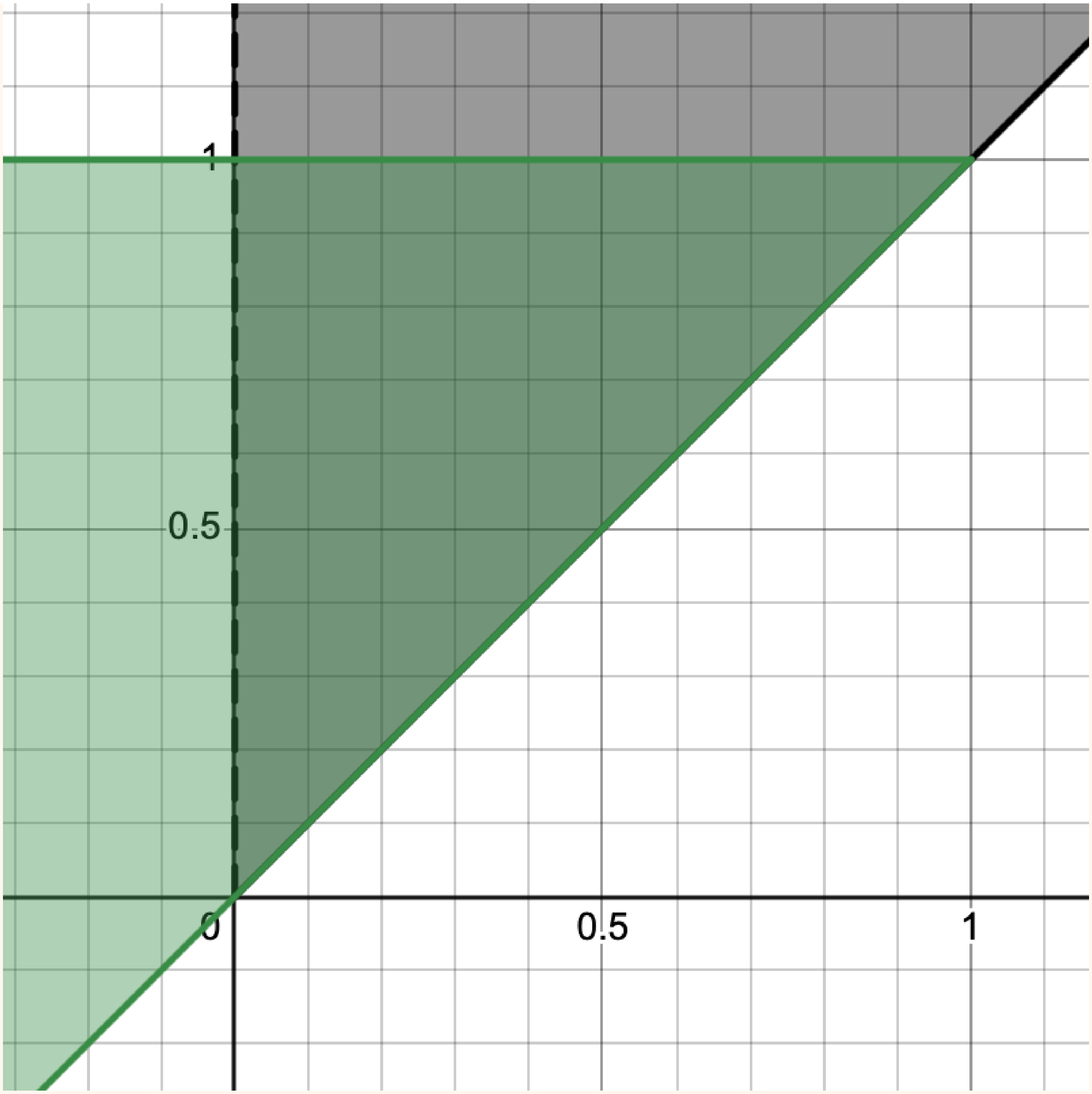
A term is 0 if any one of its factors is 0. So the sum consists of only two nonzero terms:

$$\mathbb{E}(XY) = (-1)(1)\left(\frac{1}{4}\right) + (1)(1)\left(\frac{1}{4}\right) = 0$$

A continuous example

$$f(x,y)\begin{cases} 6x & 0 < x < y < 1 \\ 0 & \text{otherwise} \end{cases}$$

The support we have drawn very many times, it is the upper left triangle of a unit square:



We compute $\mathbb{E}(X^2Y + Y^2)$:

$$\mathbb{E}(X^2Y + Y^2) = \mathbb{E}(X^2Y) + \mathbb{E}(Y^2)$$

Calculating each of the right-hand-side terms:

$$\mathbb{E}(X^2Y) = \iint_{\mathbb{R}^2} x^2 y f(x, y)\, dA$$

$$= \int_0^1 \int_0^y x^2 y \cdot 6x\, dx\, dy$$

$$= \int_0^1 \frac{3}{2} y^5\, dy = \frac{1}{4}$$

and

$$\mathbb{E}(Y^2) = \iint_{\mathbb{R}^2} y^2 f(x,y)\; dA$$

$$= \int_0^1 \int_0^y y^2 6x \, dx \, dy$$

$$= \int_0^1 3y^4 \, dy = \frac{3}{5}$$

so

$$\mathbb{E}(X^2 Y + Y^2) = \frac{1}{4} + \frac{3}{5} = \frac{17}{20}$$

The next thing is SUPER IMPORTANT!!!

Proposition

Let X and Y be two independent random variables. Let h, g be two functions $\mathbb{R} \to \mathbb{R}$. Then

$$\mathbb{E}(h(x)g(y)) = \mathbb{E}(h(x))\mathbb{E}(g(y))$$

Scribe's aside

We shall see in a few lectures what this formula would look like if X and Y are not independent.

Proof

$$\mathbb{E}(h(x)g(y)) = \iint_{\mathbb{R}^2} h(x)g(y)f(x,y)\; dA$$

$$= \iint_{\mathbb{R}^2} h(x)g(y)f_X(x)f_Y(y)\; dA$$

$$= \int_{-\infty}^{\infty} g(y)f_Y(y) \left(\int_{-\infty}^{\infty} h(x)f_X(x)\, dx \right) dy$$

$$= \int_{-\infty}^{\infty} g(y)f_Y(y)\mathbb{E}(h(x))\, dy$$

$$= \mathbb{E}(h(x)) \int_{-\infty}^{\infty} g(y) f_Y(y)\, dy$$

$$= \mathbb{E}(h(x)) \mathbb{E}(g(y))$$

Proposition

If X, Y are independent, then $V(X+Y) = V(X) + V(Y)$

Proof

$$V(X+Y) = \mathbb{E}((X+Y)^2) - (\mathbb{E}(X) + \mathbb{E}(Y))^2$$

$$= \mathbb{E}(X^2 + 2XY + Y^2) - ([\mathbb{E}(X)]^2 + 2\mathbb{E}(X)\mathbb{E}(Y) + [\mathbb{E}(Y)]^2)$$

$$= (\mathbb{E}(X^2) - (\mathbb{E}(X))^2) + (\mathbb{E}(Y^2) - (\mathbb{E}(Y))^2) + 2(\mathbb{E}(XY) - \mathbb{E}(X)\mathbb{E}(Y))$$

$$= V(X) + V(Y)$$

Proposition

If X, Y are independent, and $W = X + Y$, the mgf of W is

$$m_W(t) = m_X(t) \cdot m_Y(t)$$

Proof

(Laplace transform and convolutions!)

$$\mathbb{E}(e^{tw}) = \mathbb{E}(e^{t(x+y)})$$

$$= \mathbb{E}(e^{tx} e^{ty})$$

$$= \mathbb{E}(e^{tx}) \mathbb{E}(e^{ty})$$

Examples

1. Sum of two binomials. Let $X_1 \sim B(N_1, p_1)$, $X_2 \sim B(N_2, p_2)$. Suppose X_1, X_2 are independent, and let $X = X_1 + X_2$. Then

$$m_X(t) = m_{X_1}(t) m_{X_2}(t)$$
$$= [p_1 e^t + (1 - p_1)]^{N_1} \cdot [p_2 e^t + (1 - p_2)]^{N_2}$$

If $p_1 = p_2 =: p$, then

$$m_X(t) = [pe^t + (1 - p)]^{N_1 + N_2}$$

so $X \sim B(N_1 + N_2, p)$. Otherwise, you really can't get much information out of this.

2. Let $X_1 \sim P(\lambda_1)$ be independent from $X_2 \sim P(\lambda_2)$. Let $X = X_1 + X_2$. Then

$$m_X(t) = m_{X_1}(t) \cdot m_{X_2}(t)$$
$$= e^{\lambda_1 (e^t - 1)} e^{\lambda_2 (e^t - 1)}$$
$$= e^{(\lambda_1 + \lambda_2)(e^t - 1)}$$

So $X \sim P(\lambda_1 + \lambda_2)$.

3. Let $X_1 \sim N(\mu_1, \sigma_1^2)$ be independent from $X_2 \sim N(\mu_2, \sigma_2^2)$. Let $X = X_1 + X_2$. Then

$$m_X(t) = m_{X_1}(t) \cdot m_{X_2}(t)$$
$$= e^{\{\mu_1 t + \frac{1}{2}\sigma_1^2 t^2\}} e^{\{\mu_2 t + \frac{1}{2}\sigma_2^2 t^2\}}$$
$$= e^{(\mu_1 + \mu_2)t + \frac{1}{2}(\sigma_1^2 + \sigma_2^2)t^2}$$

So $X \sim N(\mu_1 + \mu_2, \sqrt{\sigma_1^2 + \sigma_2^2})$.

4. Let $X_1 \sim \Gamma(\alpha_1, \beta)$ be independent from $X_2 \sim G(\alpha_2, \beta)$ (note that here we require that X_1, X_2 have the same β). Let $X = X_1 + X_2$. Then

$$m_X(t) = m_{X_1}(t) \cdot m_{X_2}(t)$$
$$= \frac{1}{(1 - \beta t)^{\alpha_1}} \cdot \frac{1}{(1 - \beta t)^{\alpha_2}}$$

$$= \frac{1}{(1-\beta t)^{\alpha_1+\alpha_2}}$$

so $X \sim \Gamma(\alpha_1 + \alpha_2, \beta)$.