May 14 Lec—Linear Trans. of Random Variables, Uniform Cont., Normal, Gamma

Quiz 2 is tonight/tomorrow morning, on Chapter 3.

Continuing the discussion of uniform distributions:

Proposition (linear transformations of random variables)

If $Y = \alpha X + \beta$, $\alpha \neq 0$, the pdf of X and Y are linked through the equation

$$f_Y(y) = \frac{1}{|\alpha|} f_X\left(\frac{y-\beta}{\alpha}\right)$$

Proof

Case 1: $\alpha < 0$. The case $\alpha > 0$ is left as an exercise.
We have

$$\begin{aligned} F_Y(y) &= P(Y \leq y) \\ &= P(\alpha X + \beta \leq y) \\ &= P\left(X \geq \frac{y-\beta}{\alpha}\right) \end{aligned}$$

(notice that we switched the sign of the inequality because $\alpha < 0$)

$$\begin{aligned} &= 1 - P\left(X \leq \frac{y-\beta}{\alpha}\right) \\ &= 1 - F_X\left(\frac{y-\beta}{\alpha}\right) \end{aligned}$$

So the pdf of Y is

$$f_Y(y) = F_Y'(y) = \frac{d}{dy}\left[1 - F_X\left(\frac{y-\beta}{\alpha}\right)\right]$$

Applying the Chain Rule (and making the ghastly assumption that

$\frac{d}{dx}F_X = f_X$ exists at y), we have

$$f_Y(y) = -\frac{1}{\alpha}F_X'\left(\frac{y-\beta}{\alpha}\right)$$

$$= -\frac{1}{\alpha}f_X\left(\frac{y-\beta}{\alpha}\right)$$

By the definition of absolute value,

$$\frac{1}{|\alpha|}f_X\left(\frac{y-\beta}{\alpha}\right)$$

Back to the uniform distribution:

Remark

If $X \sim U(a,b)$ then $Y = \frac{1}{b-a}(X-a) \sim U(0,1)$ (i.e. $X = (b-a)Y + a$)

Proof

Apply the previous proposition with

$$f_X(x) = \begin{cases} \frac{1}{b-a} & a < x < b \\ 0 & \text{otherwise} \end{cases}$$

and $\alpha = |\frac{1}{b-a}| = \frac{1}{b-a}$ (since we assume $b > a$), $\beta = \frac{-a}{b-a}$.
So by the previous proposition,

$$f_Y(y) = \frac{1}{|\alpha|}f_X\left(\frac{y-\beta}{\alpha}\right)$$

We have $\frac{1}{|\alpha|} = \frac{1}{\alpha} = b-a$, and
$\frac{y-\beta}{\alpha} = \frac{1}{\alpha}\left(y + \frac{a}{b-a}\right) = (b-a)\left(y + \frac{a}{b-a}\right) = (b-a)y + a$
So

$$f_Y(y) = (b-a)f_X((b-a)y + a)$$

Proposition

If $X \sim U(a,b)$ then

1. $\mathbb{E}(X) = \frac{b+a}{2}$
2. $V(X) = \frac{(b-a)^2}{12}$

Proof

We have the relationship

$$X = (b-a)Y + a$$

thus

$$\mathbb{E}(X) = (b-a)\mathbb{E}(Y) + a$$

, where $Y \sim U(0,1)$
and

$$V(X) = (b-a)^2 V(Y)$$

We have

$$\mathbb{E}(Y) = \int_0^1 y \cdot 1 \, dy = \frac{1}{2}$$

and

$$\mathbb{E}(Y^2) = \int_0^1 y^2 \cdot 1 \, dy = \frac{1}{3}$$

and

$$V(Y) = \mathbb{E}(Y^2) - (\mathbb{E}(Y))^2$$

$$= \frac{1}{3} - \frac{1}{4} = \frac{1}{12}$$

So

$$\mathbb{E}(X) = (b-a) \cdot \frac{1}{2} + a = \frac{b+a}{2}$$

and

$$V(X) = (b-a)^2 \cdot \frac{1}{12}$$

Example

Assume $X \sim U(0,5)$.

1. Find $P(-2 \leq x \leq 3)$.
2. Find $\mathbb{E}(X^4)$.

$P(-2 \leq X \leq 3) = P(0 \leq X \leq 3) = \frac{1}{5} \cdot (3-0) = 0.6$
The professor ended up showing $\mathbb{E}(X^5) = \int_0^5 x^5 \cdot \frac{1}{5}\, dx$. You can also compute this in another way: since $X = 5Y$, we have $\mathbb{E}(X^5) = 5^5\mathbb{E}(Y^5) = 5^5\mathbb{E}(Y^5) = 5^5 \cdot \int_0^1 y^5\, dy = \frac{5^5}{5}$.

Normal distribution

Definition: Normal distribution

We say that a continuous random variable X has a **normal distribution** with mean μ and variance σ^2 ($X \sim N(\mu, \sigma^2)$) if the pdf of X is

$$f_X(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$

for all $x \in \mathbb{R}$.

The reason for this weird formula is

$$\int_{-\infty}^{\infty} e^{-t^2/2}\, dt = \sqrt{2\pi}$$

(this can be shown with Calc 3 knowledge). So

$$\Phi_Z(t) := \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}}\, dx = 1$$

which is the cdf of the *standard normal distribution* ($Z \sim N(0,1)$). Accordingly, its pdf is $f(t) = \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}}$.

Because there is no closed form to the integral $\int_{-\infty}^{x} e^{-t^2/2}\,dt$, the values of $\Phi_Z(t)$ can only be computed numberically (usually with a table of pre-computed values). But we do know trivially that $\Phi_Z(0) = \frac{1}{2}$ and $\Phi_Z(x) + \Phi_Z(-x) = 1$, by symmetry.

Proposition

If $X \sim N(\mu, \sigma^2)$ then $Z := \frac{X-\mu}{\sigma} \sim N(0,1)$

Proof

We have

$$f_X(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}, \quad x \in \mathbb{R}$$

and $Z = \underbrace{\frac{1}{\sigma}}_{\alpha} x \underbrace{-\frac{\mu}{\sigma}}_{\beta}$.

By the proposition we had earlier, we have

$$f_Z(z) = \frac{1}{|\alpha|} f_X\left(\frac{z-\beta}{\alpha}\right)$$

$$= \sigma \cdot f_X\left(\left(z + \frac{\mu}{\sigma}\right) \cdot \sigma\right)$$

$$= \sigma \cdot f_X(\sigma z + \mu)$$

$$\sigma \cdot \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{1}{2}\left(\frac{\sigma z + \mu - \mu}{\sigma}\right)^2}$$

$$= \frac{1}{\sqrt{2\pi}} \cdot e^{\frac{1}{2}z^2}$$

which is the pdf of a standard normal distribution. So $Z \sim N(0,1)$.

Proposition

If $X \sim N(\mu, \sigma^2)$ then

1. $\mathbb{E}(X) = \mu$
2. $V(X) = \sigma^2$

Proof

Since $X = \sigma Z + \mu$ where $Z \sim N(0, 1)$, proving the above is equivalent to proving tht $\mathbb{E}(Z) = 0$ and $V(Z) = 1$.
We observe that

$$\mathbb{E}(Z) = \int_{-\infty}^{\infty} \frac{t}{\sqrt{2\pi}} e^{-\frac{1}{2}t^2}\, dx$$

This integral converges, and since the integrand is an odd function we have that the integral is 0.
On the other hand,

$$\mathbb{E}(Z)^2 = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} t^2 e^{-\frac{1}{2}} t^2\, dx$$

The integral converges, and the integrand is an even function. So

$$\mathbb{E}(Z^2) = 2 \cdot \frac{1}{\sqrt{2\pi}} \int_0^{\infty} t^2 e^{-\frac{1}{2}t^2}\, dt$$

Using integration by parts,

$$= \frac{2}{\sqrt{2\pi}} \left[\int_0^{\infty} t \cdot t e^{-t^2/2}\, dt \right]$$

$$= \frac{2}{\sqrt{2\pi}} \cdot \left[\cancel{-te^{-t^2/2}|_0^{\infty}} + \int_0^{\infty} e^{-t^2/2}\, dt \right]$$

Because $e^{-t^2/2}$ is an even function and $\int_{-\infty}^{\infty} e^{-t^2/2}\, dt = \sqrt{2\pi}$, we have

$$\mathbb{E}(Z^2) = \frac{2}{\sqrt{2\pi}} \cdot \frac{1}{2} \cdot \sqrt{2\pi} = 1$$

So $V(Z) = \mathbb{E}(Z^2) - (\mathbb{E}(Z))^2 = 1$.

Exercises

For $Z \sim N(0,1)$, what is $\mathbb{E}(Z^{2n+1})$? (0)

Find $\mathbb{E}(Z^{2n})$ by iteration, i.e. find a recursive formula dependent on n. (will be solved in class tomorrow)

Example

Refer to 4.71.
For $X \sim N(\mu = 0.13, \sigma^2 = (0.005)^2)$, compute $P(0.12 < X < 0.14)$.

$$P(0.12 < X < 0.14) = P(0.12 < 0.005Z + 0.13 < 0.14)$$
$$= P\left(-\frac{0.01}{0.005} < Z < \frac{0.01}{0.005}\right)$$
$$= P(-2 < Z < 2)$$
$$= 1 - 2\Phi_Z(-2)$$

which (after consulting the table) we found to be 0.944.

Let the above be the probability that a wire manufactured by a certain factory passes the specifications. Suppose you buy 4 of those wires. What is the probability that all 4 pass specifications?
Let X be the number of wires that pass specifications among the 4.
We see that $X \sim B(4, 0.944)$. So $p_X(4) = C_4^4(0.944)^4(1-0.944)^0 = 0.944^4$.

A quick review of the gamma function

Definition: Gamma function

$$\Gamma(\alpha) = \int_0^\infty t^{\alpha-1}e^{-t}\,dt$$

We have

$$\Gamma(\alpha) = \underbrace{\int_0^1 t^{\alpha-1}e^{-t}\,dt}_{\text{comparable to } \int_0^1 t^{\alpha-1}\,dt \text{ which converges for } \alpha \cancel{-1+1} > 0} + \underbrace{\int_1^\infty t^{\alpha-1}e^{-t}\,dt}_{\text{always converges}}$$

So $\Gamma(\alpha)$ is defined for $\alpha > 0$.

Properties of the gamma function

For all $\alpha > 0$, we have $\Gamma(\alpha+1) = \alpha\Gamma(\alpha)$.

Proof

$$\Gamma(\alpha+1) = \int_0^\infty t^\alpha e^{-t}\,dt$$

Applying integration by parts,

$$= [-t^\alpha e^{-t}]_0^\infty + \int_0^\infty \alpha t^{\alpha-1}e^{-t}\,dt$$

$$= \alpha\Gamma(\alpha)$$

We compute that $\Gamma(1) = \int_0^\infty e^{-t}\,dt = -e^{-t}|0^\infty = 1$. By induction (here the scribe skips a thousand steps which are left as exercise to the reader), for $n \in \mathbb{N}$, we have $\Gamma(n) = (n-1)!$.

Given that $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$, we have $\Gamma\left(\frac{3}{2}\right) = \frac{1}{2}\cdot\Gamma\left(\frac{1}{2}\right) = \frac{1}{2}\sqrt{\pi}$, $\Gamma\left(\frac{5}{2}\right) = \frac{3}{2}\cdot\frac{1}{2}\sqrt{\pi}$, ... We observe that

$$\Gamma\left(\frac{2n+1}{2}\right) = \underbrace{\frac{2n-1}{2}\cdot\frac{2n-3}{2}\cdot\dots\cdot\frac{1}{2}}_{n\text{ terms}}\cdot\sqrt{\pi}$$

$$= \frac{1}{2^n}(2n-1)(2n-3)\dots(3)(1)\sqrt{\pi}$$

$$= \frac{\sqrt{\pi}}{2^n}\left[\frac{(2n)!}{\underbrace{(2n)(2n-2)\dots(4)(2)}_{n\text{ terms}}}\right]$$

$$= \frac{\sqrt{\pi}}{2^n}\left[\frac{(2n)!}{2^n(n)(n-1)\dots(2)(1)}\right]$$

$$= \frac{\sqrt{\pi}}{2^n}\left[\frac{(2n)!}{2^n \cdot n!}\right]$$

$$= \frac{(2n)!}{2^{2n} \cdot n!}\sqrt{\pi}$$

Key formula

$$\int_0^\infty t^a e^{-bt}\,dt = \frac{1}{b^{a+1}}\Gamma(a+1)$$

for $b > 0$, $a+1 > 0$.

Proof

Make the change of variables $s = bt$. We have

$$\int_0^\infty t^a e^{-bt}\,dt = \frac{1}{b}\int_0^\infty \left(\frac{s}{b}\right)^a e^{-s}\,ds$$

$$= \frac{1}{b^{a+1}}\int_0^\infty s^a e^{-s}\,ds$$

$$= \frac{1}{b^{a+1}}\Gamma(a+1)$$

Remark

For $\alpha > 0, \beta > 0$, we have

$$\int_0^\infty x^{\alpha-1}e^{-x/\beta}\,dx = \frac{\Gamma(\alpha)}{\left(\frac{1}{\beta}\right)^\alpha} = \beta^a\Gamma(\alpha)$$

hence $\frac{1}{\beta^\alpha\Gamma(\alpha)}\int_0^\infty x^{\alpha-1}e^{-x/\beta}\,dx = 1$.

Definition: Gamma distribution

The function

$$f_X(x) = \begin{cases} \frac{1}{\beta^\alpha \Gamma(\alpha)} x^{\alpha-1} e^{-x/\beta} & x \in (0, \infty) \\ 0 & \text{otherwise} \end{cases}$$

is a pdf. X is said to have a Gamma distribution with parameter α, β, written as $X \sim \text{Gamma}(\alpha, \beta)$.

Exercise

If $X \sim G(\alpha, \beta)$, then $Y := \frac{X}{\beta} \sim G(\alpha, 1)$.

Proposition

If $X \sim G(\alpha, \beta)$, then

1. $\mathbb{E}(X) = \beta\alpha$
2. $V(X) = \beta^2\alpha$

Proof

$$\mathbb{E}(X) = \int_0^\infty x f_X(x)\, dx$$

$$\int_0^\infty \frac{1}{\Gamma(\alpha)\beta^\alpha} x^\alpha e^{-x/\beta}\, dx$$

$$= \frac{1}{\Gamma(\alpha)\beta^\alpha} \int_0^\infty x^\alpha e^{-x/\beta}\, dx$$

$$= \frac{1}{\Gamma(\alpha)\beta^\alpha} \cdot (\beta)^{\alpha+1} \cdot \Gamma(\alpha+1)$$

$$\frac{1}{\Gamma(\alpha)\beta^\alpha} \cdot \beta^{\alpha+1} \cdot \alpha\Gamma(\alpha)$$

$$= \beta\alpha$$

and

$$\mathbb{E}(X^2) = \frac{1}{\beta^\alpha \Gamma(\alpha)} \int_0^\infty x^{\alpha+1} e^{-x/\beta}\, dx$$

$$= \frac{1}{\beta^\alpha \Gamma(\alpha)} \cdot \beta^{\alpha+2}\Gamma(\alpha+2)$$

$$= \frac{1}{\beta^\alpha \Gamma(\alpha)} \cdot \beta^{\alpha+2}(\alpha+1)(\alpha)\Gamma(\alpha)$$

$$= \beta^2(\alpha+1)(\alpha)$$

So $V(X) = \mathbb{E}(X^2) - (\mathbb{E}(X))^2 = \beta^2\alpha$

Exercise

For $X \sim G(\alpha, \beta)$ we have

$$\mathbb{E}(X^n) = \beta^n(\alpha)(\alpha+1)(\alpha+2)\ldots(\alpha+n-1)$$

For tomorrow:
Many distributions come from the gamma distribution, such as the exponential distribution $(\alpha = 1)$ and chi-square distribution $(\beta = 2, \alpha = \frac{p}{2})$.