May 16 Lec—Beta, Mgfs of Uniform, Normal, and Gamma

Beta distributions

Note that the beta integral

$$\int_0^1 x^{lpha-1} (1-x)^{eta-1} \, dx = rac{\Gamma(lpha)\Gamma(eta)}{\Gamma(lpha+eta)}$$

for $\alpha > 0$, $\beta > 0$.

Example

$$\int_0^1 x^2 (1-x)^3 \, dx$$

is a beta integral with $\alpha=3, \beta=4$. The integral is equal to $\frac{\Gamma(3)\Gamma(4)}{\Gamma(7)}=\frac{2!3!}{6!}$

Definition: beta distribution

The function

$$f_X(x) = egin{cases} rac{\Gamma(lpha+eta)}{\Gamma(lpha)\Gamma(eta)} x^{lpha-1} (1-x)^{eta-1} & x \in (0,1) \ 0 & ext{otherwise} \end{cases}$$

is the pdf of the **beta distribution** with parameters α, β . We write $X \sim Beta(\alpha, \beta)$ if X has a beta distribution with parameters α, β .

Remark

$$U(0,1) = Beta(1,1)$$

Expected value and variance of beta distribution

If $X \sim Beta(\alpha, \beta)$, then

1.
$$\mathbb{E}(X) = \frac{\alpha}{\alpha + \beta}$$

2.
$$V(X) = \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$$

Proof

$$\mathbb{E}(X) = \frac{\Gamma(\alpha + \beta)}{\Gamma(\alpha)\Gamma(\beta)} \int_0^1 x \cdot x^{\alpha - 1} (1 - x)^{\beta - 1} dx$$

$$\frac{\Gamma(\alpha + \beta)}{\Gamma(\alpha)\Gamma(\beta)} \int_0^\infty x^{\alpha} (1 - x)^{\beta - 1} dx$$

$$\frac{\Gamma(\alpha + \beta)}{\Gamma(\alpha)\Gamma(\beta)} \cdot \frac{\Gamma(\alpha + 1)\Gamma(\beta)}{\Gamma(\alpha + \beta + 1)}$$

$$= \frac{\alpha}{\alpha + \beta}$$

and

$$egin{aligned} \mathbb{E}(X^2) &= rac{\Gamma(lpha+eta)}{\Gamma(lpha)\Gamma(eta)} \int_0^1 x^2 \cdot x^{lpha-1} (1-x)^{eta-1} \, dx \ &= rac{\Gamma(lpha+eta)}{\Gamma(lpha)\Gamma(eta)} \int_0^1 x^{lpha+1} (1-x)^{eta-1} \, dx \ &= rac{\Gamma(lpha+eta)}{\Gamma(lpha)\Gamma(eta)} \cdot rac{\Gamma(lpha+2)\Gamma(eta)}{\Gamma(lpha+eta+2)} \ &= rac{lpha(lpha+1)}{(lpha+eta+1)(lpha+eta)} \end{aligned}$$

So

$$egin{split} V(X) &= \mathbb{E}(X^2) - (\mathbb{E}(X))^2 \ &= rac{lpha(lpha+1)}{(lpha+eta+1)(lpha+eta)} - \left(rac{lpha}{lpha+eta}
ight)^2 \end{split}$$

Moment generating functions of pdfs

(For those of you who have taken ODEs, you will see that this is just the Laplace transform of the pdf.)

Definition: Moment generating function of a pdf

For a pdf f_X ,

$$m_X(t) = \mathbb{E}(e^{tx})$$

$$=\int_{-\infty}^{\infty}e^{tx}f_X(x)\,dx$$

Properties of the mgf

- 1. $m_X(0) = 1$
- 2. Domain of m_X is an interval
- 3. If the domain of m_X contains an interval centered at t=0, then

$$rac{d^n}{dt^n}(m_X(t))|_{t=0}=\mathbb{E}(X^n)$$

Also recall this property:

Mgf of a linear transformation of a (continuous random) variable

If
$$X = aY + b$$
 then

$$m_X(t)=e^{bt}m_Y(at)$$

Let's calculate some mgfs!

Uniform

X=(b-a)Y+a , where $Y\sim U(0,1)$.

The mgf of Y is

$$m_Y(s) = \mathbb{E}(e^{sy})$$

$$=\int_0^1 e^{sy}\,dy$$

$$=egin{cases} 1 & s=0 \ rac{1}{s}(e^s-1) & s
eq 0 \end{cases}$$

So the mgf of X is

$$egin{aligned} m_X(t) &= e^{at} m_Y((b-a)(t)) \ &= egin{cases} 1 & t = 0 \ rac{e^{at} \cdot e^{(b-a)t} - 1}{(b-a)t} & t
eq 0 \end{cases} \end{aligned}$$

Normal

Let
$$X\sim N(\mu,\sigma^2)$$
 so $X=\sigma Z+\mu$ where $Z\sim N(0,1)$. So $m_X(t)=e^{\mu t}m_Z(\sigma t)$, and

$$egin{align} m_Z(s) &= \int_{-\infty}^\infty e^{sz} f_Z(z) \, dz \ &= \int_{-\infty}^\infty e^{sz} \cdot rac{1}{\sqrt{2\pi}} e^{-z^2/2} \, dx \ &= rac{1}{\sqrt{2\pi}} \int_{-\infty}^\infty e^{-rac{1}{2} \cdot [z^2 - 2sz]} \, dx \ \end{aligned}$$

Completing the square:

$$egin{aligned} &= rac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-rac{1}{2}\cdot[(z-s)^2-s^2]}\,dx \ &= rac{e^{1/2\cdot s^2}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-1/2(z-s)^2}\,dz \end{aligned}$$

Make change of variable z-s=u:

$$=rac{e^{1/2\cdot s^2}}{\sqrt{2\pi}}\underbrace{\int_{-\infty}^{\infty}e^{-1/2u^2}\,du}_{=1}$$
 $=e^{rac{1}{2}s^2}$

Gamma

$$X \sim \Gamma(lpha, eta)$$

$$m_X(t) = \int_0^\infty e^{tx} f_X(x) \, dx$$

$$egin{aligned} &=rac{1}{\Gamma(lpha)eta^lpha}\int_0^\infty e^{tx}x^{lpha-1}e^{rac{t}{eta}}dx \ &=rac{1}{\Gamma(lpha)eta^lpha}\int_0^\infty x^{lpha-1}e^{-x\left(rac{1}{eta}-t
ight)}dx \end{aligned}$$

Let $a=\alpha-1$, $b=\frac{1}{\beta}-t=\frac{1-\beta t}{\beta}$. We need $1-\beta t>0$ (i.e. $t<\frac{1}{\beta}$) for the integral to converge.

$$egin{aligned} &= rac{1}{\Gamma(lpha)eta^lpha}rac{\Gamma(lpha)}{\left(rac{1-eta t}{eta}
ight)^lpha} \ &= rac{1}{(1-eta t)^lpha}, \quad t < rac{1}{eta} \end{aligned}$$