May 16 Lec—Beta, Mgfs of Uniform, Normal, and Gamma

Beta distributions

Note that the **beta integral**

$$\int_0^1 x^{\alpha-1}(1-x)^{\beta-1}\,dx = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$$

for $\alpha > 0$, $\beta > 0$.

Example

$$\int_0^1 x^2(1-x)^3\,dx$$

is a beta integral with $\alpha = 3, \beta = 4$. The integral is equal to $\frac{\Gamma(3)\Gamma(4)}{\Gamma(7)} = \frac{2!3!}{6!}$

Definition: beta distribution

The function

$$f_X(x) = \begin{cases} \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1} & x \in (0,1) \\ 0 \quad \text{otherwise} \end{cases}$$

is the pdf of the **beta distribution** with parameters α, β. We write $X \sim Beta(\alpha, \beta)$ if X has a beta distribution with parameters α, β.

Remark

$U(0,1) = Beta(1,1)$

Expected value and variance of beta distribution

If $X \sim Beta(\alpha, \beta)$, then

1. $\mathbb{E}(X) = \frac{\alpha}{\alpha+\beta}$
2. $V(X) = \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$

Proof

$$\mathbb{E}(X) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \int_0^1 x \cdot x^{\alpha-1}(1-x)^{\beta-1}\,dx$$

$$\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \int_0^\infty x^{\alpha}(1-x)^{\beta-1}\,dx$$

$$\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \cdot \frac{\Gamma(\alpha+1)\Gamma(\beta)}{\Gamma(\alpha+\beta+1)}$$

$$= \frac{\alpha}{\alpha+\beta}$$

and

$$\mathbb{E}(X^2) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \int_0^1 x^2 \cdot x^{\alpha-1}(1-x)^{\beta-1}\,dx$$

$$= \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \int_0^1 x^{\alpha+1}(1-x)^{\beta-1}\,dx$$

$$\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \cdot \frac{\Gamma(\alpha+2)\Gamma(\beta)}{\Gamma(\alpha+\beta+2)}$$

$$= \frac{\alpha(\alpha+1)}{(\alpha+\beta+1)(\alpha+\beta)}$$

So

$$V(X) = \mathbb{E}(X^2) - (\mathbb{E}(X))^2$$

$$= \frac{\alpha(\alpha+1)}{(\alpha+\beta+1)(\alpha+\beta)} - \left(\frac{\alpha}{\alpha+\beta}\right)^2$$

Moment generating functions of pdfs

(For those of you who have taken ODEs, you will see that this is just the Laplace transform of the pdf.)

Definition: Moment generating function of a pdf

For a pdf f_X,

$$m_X(t) = \mathbb{E}(e^{tx})$$

$$= \int_{-\infty}^{\infty} e^{tx} f_X(x)\, dx$$

Properties of the mgf

1. $m_X(0) = 1$
2. Domain of m_X is an interval
3. If the domain of m_X contains an interval centered at $t = 0$, then

$$\frac{d^n}{dt^n}(m_X(t))|_{t=0} = \mathbb{E}(X^n)$$

Also recall this property:

Mgf of a linear transformation of a (continuous random) variable

If $X = aY + b$ then

$$m_X(t) = e^{bt} m_Y(at)$$

Let's calculate some mgfs!

Uniform

$X = (b-a)Y + a$, where $Y \sim U(0,1)$.
The mgf of Y is

$$m_Y(s) = \mathbb{E}(e^{sy})$$

$$= \int_0^1 e^{sy}\, dy$$

$$= \begin{cases} 1 & s = 0 \\ \frac{1}{s}(e^s - 1) & s \neq 0 \end{cases}$$

So the mgf of X is

$$m_X(t) = e^{at} m_Y((b-a)(t))$$

$$= \begin{cases} 1 & t = 0 \\ \frac{e^{at} \cdot e^{(b-a)t} - 1}{(b-a)t} & t \neq 0 \end{cases}$$

Normal

Let $X \sim N(\mu, \sigma^2)$
so $X = \sigma Z + \mu$ where $Z \sim N(0,1)$. So $m_X(t) = e^{\mu t} m_Z(\sigma t)$, and

$$m_Z(s) = \int_{-\infty}^{\infty} e^{sz} f_Z(z) \, dz$$

$$= \int_{-\infty}^{\infty} e^{sz} \cdot \frac{1}{\sqrt{2\pi}} e^{-z^2/2} \, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2} \cdot [z^2 - 2sz]} \, dx$$

Completing the square:

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2} \cdot [(z-s)^2 - s^2]} \, dx$$

$$= \frac{e^{1/2 \cdot s^2}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-1/2(z-s)^2} \, dz$$

Make change of variable $z - s = u$:

$$= \frac{e^{1/2 \cdot s^2}}{\sqrt{2\pi}} \underbrace{\int_{-\infty}^{\infty} e^{-1/2u^2} \, du}_{=1}$$

$$= e^{\frac{1}{2}s^2}$$

Gamma

$X \sim \Gamma(\alpha, \beta)$

$$m_X(t) = \int_0^{\infty} e^{tx} f_X(x) \, dx$$

$$= \frac{1}{\Gamma(\alpha)\beta^\alpha} \int_0^\infty e^{tx} x^{\alpha-1} e^{\frac{t}{\beta}}\, dx$$

$$= \frac{1}{\Gamma(\alpha)\beta^\alpha} \int_0^\infty x^{\alpha-1} e^{-x\left(\frac{1}{\beta}-t\right)}\, dx$$

Let $a = \alpha - 1$, $b = \frac{1}{\beta} - t = \frac{1-\beta t}{\beta}$.
We need $1 - \beta t > 0$ (i.e. $t < \frac{1}{\beta}$) for the integral to converge.

$$= \frac{1}{\Gamma(\alpha)\beta^\alpha} \frac{\Gamma(\alpha)}{\left(\frac{1-\beta t}{\beta}\right)^\alpha}$$

$$= \frac{1}{(1-\beta t)^\alpha}, \quad t < \frac{1}{\beta}$$