May 15 Lec—Exponential, Chi-Square, Midterm Review

A fumbled-up quiz question:
Choose one ball from 3 reds and 6 blues. If you get a red, flip an unfair coin with $P(H|R) = p$, $P(T|R) = 1 - p$. If you get a blue, flip another unfair coin with $P(T|B) = q$, $P(H|B) = 1 - q$. If H is $+2$ and T is -1, find the expected value.
$P(H) = P(R)P(H|R) + P(B)P(H|B) = \frac{1}{3} \cdot p + \frac{2}{3}(1 - q)$
$P(T) = P(R)P(T|R) + P(B)P(T|B) = \frac{1}{3}(1 - p) + \frac{2}{3}q$
$\mathbb{E}(\text{score}) = 2\left(\frac{1}{3}p + \frac{2}{3}(1-q)\right) - \left(\frac{1}{3}(1-p) + \frac{2}{3}q\right) = p - 2q + 1$

Exponential distributions

Recall that

$$\int_0^\infty x^a e^{-bx}\, dx = \frac{\Gamma(a+1)}{b^{a+1}}$$

for $a > -1, b > 0$.

Definition: Exponential distribution

Exponential distributions are gamma distributions with $\alpha = 1$. We write $X \sim \text{Exponential}(\beta)$, and the pdf of X is

$$f_X(x) = \begin{cases} \frac{1}{\beta} e^{-x/\beta} & x \in (0, \infty) \\ 0 & \text{otherwise} \end{cases}$$

The cdf of X is

$$F_X(t) = \int_{-\infty}^x f_X(t)\, dt = \begin{cases} 1 - e^{-x/\beta} & x \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

Remark

Just writing

$$f(x) = \frac{1}{\beta} e^{-x/\beta}$$

without specifying the domain does *not* make $f(x)$ a pdf.

Exponential distributions are used to model lifetimes.

Definition: Survival function

$$S(x) = P(X > x) = 1 - F_X(x)$$
$$= \begin{cases} 1 & x \leq 0 \\ e^{-x/\beta} & x \geq 0 \end{cases}$$

Because $S(a+b) = S(a)S(b)$ for $a, b > 0$ due to the properties of the exponential distribution, we have

$$P(X > a+b) = P(X > a)P(X > b)$$
$$\frac{P(X > a+b)}{P(X > a)} = P(X > b)$$

Since $P(X > a+b) = P([X > a+b] \cap [X > a])$,

$$\frac{P([X > a+b] \cap [X > a])}{P(X > a)} = P(X > b)$$

The right hand side is $P(X > a+b | X > a)$, so we have

$$P(X > a+b | X > a) = P(X > b)$$

which is the famous **memoryless property**.

Remark

Exercise for those that love analysis: It can be prove that any continuous function that satisfies $F(a+b) = F(a)F(b)$ is exponential. So this is the only distribution with this property.

Properties of the exponential distribution

1. $\mathbb{E}(X) = \beta$
2. $V(X) = \beta^2$

Proof

Obviously elementary from the fact that X also has a gamma pdf.

Example

Let $X \sim E(\beta)$ with $\mathbb{E}(X) = \beta$. Consider

$$Y = \begin{cases} 10 & 0 < x < 2 \\ 15 & 2 < x < 4 \\ 50 & x > 4 \end{cases}$$

Find $\mathbb{E}(Y)$.
We see that $\beta = 4$ $p_Y(10) = F_X(2) = 1 - e^{-1/2}$,
$p_Y(15) = F_X(4) - F_X(2) = e^{-1/2} - e^{-1}$
$p_Y(50) = 1 - F_X(4) = e^{-1}$.
So $\mathbb{E}(Y) =$

Chi-square distribution

Let $Z \sim N(0, 1)$. We find the distribution of $X = Z^2$.
We first find the cdf of X.

$$F_X(x) = P(X = Z^2 \leq x) = 0,\ x < 0$$

If $x > 0$,

$$F_X(x) = P(Z^2 \leq x)$$

$$= P(-\sqrt{x} \leq Z \leq \sqrt{x})$$

$$= \Phi_Z(\sqrt{x}) - \Phi_Z(-\sqrt{x})$$

By symmetry,

$$= 2\Phi_Z(\sqrt{x}) - 1$$

So we have

$$F_X(x) = \begin{cases} 0 & x < 0 \\ 2\Phi_Z(\sqrt{x}) - 1 & x \geq 0 \end{cases}$$

The pdf of X is

$$f_X(x) = \begin{cases} 0 & x < 0 \\ \frac{1}{\sqrt{x}} \Phi'_Z(\sqrt{x}) & x > 0 \end{cases}$$

Recall that $\Phi_Z(t) = f_Z(t) = \frac{1}{\sqrt{2\pi}} e^{-t^2/2}$

So

$$f_X(x) = \begin{cases} 0 & x < 0 \\ \frac{1}{\sqrt{2\pi}} x^{-1/2} e^{-x/2} & x > 0 \end{cases}$$

We therefore observe that $X \sim G\left(\alpha = \frac{1}{2}, \beta = 2\right)$

(Aside: From here we learn that $\frac{1}{\Gamma\left(\frac{1}{2}\right) \cdot \left(\frac{1}{2}\right)^{1/2}} = \frac{1}{\sqrt{2\pi}}$ so $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$.)

Definition: Chi-square distribution

X is said to have a chi-square distribution with p degrees of freedom ($X \sim \chi^2(p)$, p a positive integer) if $X \sim G\left(\alpha = \frac{p}{2}, \beta = 2\right)$, i.e. the pdf of X is

$$f_X(x) = \begin{cases} \frac{1}{\Gamma\left(\frac{p}{2}\right) 2^{p/2}} x^{p/2-1} e^{-x/2} & x > 0 \\ 0 & \text{otherwise} \end{cases}$$

Expected value and variance

If $X \sim \chi^2(p)$, then

1. $\mathbb{E}(X) = 2 \cdot \frac{p}{2} = p$
2. $V(X) = 2^2 \cdot \frac{p}{2} = 2p$

Example

Find $\mathbb{E}(Z^{2n})$.
We have $\mathbb{E}(Z^{2n}) = \mathbb{E}((Z^2)^n) = \mathbb{E}(X^n)$ where $X = Z^2 \sim G\left(\frac{1}{2}, 2\right)$
and

$$\mathbb{E}(X^n) = \int_0^\infty x^n f_X(x)\,dx$$

$$= \int_0^\infty x^n \cdot \frac{1}{\sqrt{2\pi}} x^{-1/2} e^{-x/2}\,dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_0^\infty x^{n-1/2} e^{-x/2}\,dx$$

$$= \frac{1}{\sqrt{2\pi}} \cdot \frac{\Gamma\left(n+\frac{1}{2}\right)}{\left(\frac{1}{2}\right)^{n+1/2}}$$

$$= \frac{2^n}{\sqrt{\pi}} \Gamma\left(n+\frac{1}{2}\right)$$

Midterm Review

1. Consider the jumbled up REDDDIITT. What is the probability that in any shuffling of the letters the two T's are together?
   There are $\frac{9!}{3!2!2!}$ unique placements up to the same letters.
   There are 8 possible positions of the two T's. For the other 7 letters, there are $\frac{7!}{3!2!}$ ways of numbering them up to the same letters. So the probability of this event happening is

   $$\frac{8 \cdot \frac{7!}{3!2!}}{\frac{9!}{3!2!2!}}$$

   What if we specify that the two T's are together and only two of the three D's are together? (this would be very complicated)

Side note: The most probable/most likely outcome is the outcome with the highest probability, not the expected value.

2. Consider X with mgf $\frac{e^{-2t}}{8}(1+e^{2t})^3$. What is the distribution of X?

$$\frac{e^{-2t}}{8}(1+e^{2t})^3 = e^{-2t}\left(\frac{1}{2}+\frac{1}{2}e^{2t}\right)^3$$

$X = -2+2Y$ where $Y \sim B\left(3, \frac{1}{2}\right)$

3. Let $X \sim B\left(4, \frac{1}{5}\right)$, $X =$ number of heads in 4 tosses. If head, triple your current fortune (10). If tail, half your current fortune. Let Y be your fortune after the 4 tosses. Find the expected fortune.

$$Y = 10 \cdot 3^X \cdot \left(\frac{1}{2}\right)^{4-X} = \frac{10}{16} \cdot 6^X$$

$$\mathbb{E}(Y) = \frac{10}{16}\mathbb{E}(6^X) = \frac{10}{16}\mathbb{E}(e^{\ln(6)X}) = \frac{10}{16}m_X(\ln(6))$$

$$= \frac{10}{16}\left(\frac{4}{5}+\frac{1}{5}e^{\ln(6)}\right)^4 = 10$$

4. Consider the probability function $p_Y(n) = (\frac{1}{2})^{n+1}$. It isn't a geometric distribution. But consider $X = Y+1$. We see that $X \in \{1, 2, 3, \ldots\}$ and $P(X=k) = P(Y=k-1)$.