Strategic Timing and Resource Allocation for Optimal Isolation and Travel Restrictions in Infectious Disease Control: In Progress

by

© **George Adu-Boahen**

A thesis to be submitted to the School of Graduate Studies in the partial fulfilment of the requirements for the degree of Master of Science

Supervisor: Dr. Amy Hurford

Department of Mathematics and Statistics
Memorial University

August 2024

St. John's, Newfoundland and Labrador, Canada

Abstract

Acknowledgements

Dedication

Table of Contents

List of Figures

List of Tables

List of Abbreviations

IVP Initial-Value Problem 5

ODEs Ordinary Differential Equations v, 4, 5

PMP Pontryagin's Maximum Principle v, 8–10, 15, 17, 22

SIR Susceptibles Infected Recovered 11

Chapter 1

Introduction

Despite significant progress in prevention and control, containing and mitigating infectious diseases remains a challenging task in the modern day due to their complicated spreading patterns and increasing speed of spread [6]. Many efforts to curb a disease attempt to cut the transmission path or control it at its source. For example, the COVID-19 pandemic led many countries to implement social distancing and travel restrictions to prevent the spread of the disease. Effectual measures are imperative to control its transmission and alleviate its impact on public health and society. While these measures have proven effective as a control strategy in some countries, they come with substantial economic and social drawbacks and are frequently met with opposition during implementation. Additionally, it is challenging to ascertain the ideal duration and extent of implementing controls, as these considerations hinge on various factors, such as preparedness for epidemics and pandemics, the transmission rate, healthcare system capacity, population density and the public's adherence to preventive measures. In this context, we need help making choices concerning devising interventions that yield optimal results while working within resource constraints.

Decision makers responsible for managing epidemic policy have faced difficult choices in balancing the competing claims of saving lives and the high economic cost of control strategies. Some balance has to be struck between saving lives and the socio-economic cost of implementing control strategies. During the COVID-19 pandemic, many countries introduced extensive economic and social control measures to limit the spread of the disease. While these measures are costly, they successfully stopped the epidemic's growth in most regions where they were applied relatively early and with sufficient stringency [22].

Modelling techniques help us understand the observed epidemiological patterns and predict the consequences of introducing intervention measures to contain disease spread. Mathematical epidemiology made its debut in 1760 when Daniel Bernoulli developed a model to explain how smallpox spreads [9]. However, because well-known mathematicians were not interested in solving application-based problems, the field's advancement was modest. The area did not receive much attention again until 1906, when Hamer developed a discrete model for the propagation of measles. Ronald Ross, who developed a model for malaria transmission through mosquito bites in 1922, carried on this trend [5]. When biologist McKendrick and physician Kermack added mortality and birth rates to the conventional epidemic model in 1927 [21], the field made much more progress, establishing epidemic modelling as an important use in mathematical literature. The field gained even more prominence with the emergence of AIDS and HIV in 1985, becoming a crucial tool for public health.

Experts in the biological sciences have authored most of the work on mathematical epidemic modelling, which frequently presents models that are challenging to solve using the tools available for analytical mathematics today. On the other hand, little study has been done explicitly on epidemic control. "Control" is often considered to imply running a model with parameter sets intended to lower the reproduction number below one, despite the fact that it is mentioned as an objective in many papers [1], [8], [31].

The modelling approach pioneered by Dirk Brockmann and Dirk Helbing [6] was expanded by researchers in 2021 [17] to theoretically elucidate how prevention and control strategies, including mobility restrictions and non-pharmaceutical interventions, influence the trajectory of an epidemic's spread. Findings indicated that while these approaches are generally effective in curbing the progression of the epidemic, their efficacy is markedly varied across different regions. In areas with a low risk of spread, the measures were found to be predominantly mitigating against imported cases rather than local transmission. Conversely, regions with a higher risk required defences against the introduction of new cases from outside and the escalation of existing cases within the region. In 2021, it was shown that the contribution of travel restrictions to reducing the spread of an epidemic is minimal, given that imported cases have a limited influence on local transmission [26].

However, as a disease becomes more prevalent, one issue of practical concern is the limited resources available to prevent people from being infected by a particular virus or to treat those infected. Significant progress has been made by proposing mathematical models, which offer valuable information for decision-making in global health [14], [35], [37], [34], [38],[28], [33], [7], [25], [3], [27], [32].

One common aim when modelling resource constraints is to describe how changes in intervention measures will affect the characteristics of the infection dynamics and consequently affect disease control. Many control programs, such as isolation and vaccination, have been modelled [14], [2], [23].

Hansen and Day (2011) provided optimal control policies for an isolation-only model, a vaccination-only model and a combined isolation–vaccination model, with analytic solutions for the controls that minimize the infectious burden under the assumption that there are limited control resources [14].

This research builds upon the foundation laid by Hansen and Day [14], focusing on how changes in control methods affect the disease dynamics. Specifically, we consider an isolation-only model, a travel restrictions-only model, and a combined isolation-travel restriction model. We also seek to characterize how these novel phenomena in our targeted model affect disease control. We start our model at an early stage of the epidemic. We explore a pragmatic and efficient strategy rooted in optimal control theory to minimize cost and curb disease transmission. We answer the question of "when" and "how" control measures can be implemented within resource constraints.

The foundation of our modelling involves using a system of ODEs, which provides a robust framework to describe the dynamic processes inherent in the spread of the disease. The mathematical model makes it possible to describe the rates at which people shift over time between several disease states, such as susceptible, infectious, and removed (SIR). The essence of using ODEs in this context lies in their ability to capture the continuous change in population compartments, which correspond to the aforementioned disease states and other critical variables. This is achieved through rate equations that integrate various factors, including transmission rates, recovery rates, and death rates. By adjusting these rates, we can mimic real-world scenarios across diverse populations and conditions. The resulting systems of ODEs yield rich dynamics that can be studied analytically and numerically to predict disease trajectories and evaluate the potential impact of health policies.

Understanding how solutions to ODEs depend on initial conditions and parameters, such as transmission coefficients or contact patterns, is central to using ODEs in infectious disease modelling. The system's ability to reliably forecast future states stems from the existence and uniqueness of solutions, which hinges upon the initial conditions and parameters. With the notion of admissibility of control functions, we set a framework that ensures the system's responses remain within feasible bounds, aligning with realistic scenarios. The practical application of this model rests on these controls being seamlessly adjustable within the compact set's constraints over the chosen time frame. Verifying the solutions' existence and uniqueness, which we will delve into in the upcoming section, is a critical step that reinforces the model's integrity and application to real-world systems.

1.1 Basic Fundamental Properties of ODEs

We state some essential characteristics of the solutions to ordinary differential equations, including existence, uniqueness, continuous dependence on initial conditions, and continuous dependence on parameters. Consider the nonlinear dynamical control system

$$\dot{x}(t) = f(t, x(t), u(t), .); \quad x(t_0) = x_0, \tag{1.1}$$

where $x(t) = (x_1(t), \cdots, x_n(t)) \in \mathbb{R}^n, u(t) = (u_1(t), \cdots, u_m(t)) \in \mathbb{R}^m$. For the mathematical model to predict the system's future state from its current state, the Initial-Value Problem (IVP) (1.1) must have a unique solution. A trajectory of the system (1.1) corresponding to a control $u(t)$ is a continuous curve $x(t)$ solving (1.1) for almost all t. We also refer to $x(t) \in \mathbb{R}^n$ for some t as the state. An admissible control $u(t)$ will be a piecewise-continuous vector-valued function on time interval $\mathbb{I} = [0, T]$, which will have values at time $t \in \mathbb{I}$ in a nonempty compact set. We establish the question of existence and uniqueness in the next section.

1.1.1 Existence and Uniqueness of Solutions to ODEs

The first goal of this section is to establish the local existence and uniqueness of solutions.

Thus, we are interested in solutions to the differential equations (1.1) that appear to take a more general form. Let $\mathbb{I} \subset \mathbb{R}^n$ an interval of time, $U \subset \mathbb{R}^n$ and $\Theta \subset \mathbb{R}^m$ be open sets and let $f : \mathbb{I} \times U \times \Theta \to \mathbb{R}^n$ be a continuous function. We focus on solutions to the initial value problem

$$\dot{x} = f(t, x, \theta), \quad x(t_0) = x_0, \tag{1.2}$$

that is, the existence of a solution $x : \mathbb{I} \to U$ such that $t_0 \in \mathbb{I}$, $\theta \in \Theta$ and $x(t_0) = x_0$.

It is known from the theory of ordinary differential equations that under certain regularity assumptions, a (nonlinear) differential equation (1.2) has a unique solution passing through x_0 at $t = t_0$. The regularity conditions are

1. $f(t, x, \theta)$ is a continuous function.

2. $f(t, x, \theta)$ satisfies a global Lipschitz condition.

Definition 1.1.1. *Consider metric spaces (X, d_X) (Y, d_Y). A function $f : X \to Y$ is Lipschitz if there exists a real constant $K \geq 0$ such that, for all $x_1, x_2 \in X$*

$$d_Y(f(x_1), f(x_2) \leq K d_X(x_1, x_2)$$

The smallest K satisfying this inequality is denoted by Lip(f) $:= K$ and is called the Lipschitz constant of f.

The corresponding existence and uniqueness theorem is as follows. The proofs can be found in [15],[19], [30].

Theorem 1.1.2. *Let $\mathbb{I} \subset \mathbb{R}$, $U \subset \mathbb{R}^n$ and $\Theta \subset \mathbb{R}^m$ be open sets, and assume $f : \mathbb{I} \times U \times \Theta \to \mathbb{R}^n$ is a Lipschitz function. If $(t_0, x_0, \theta_0) \in \mathbb{I} \times U \times \Theta$, then there exists an open* *neighbourhood of the form* $\mathbb{I}_0 \times U_0 \times \Theta_0$ *of* (t_0, x_0, θ_0) *and a Lipschitz continuous function* $\varphi : \mathbb{I}_0 \times U_0 \times \Theta_0 \to \mathbb{R}^n$ *such that for every* $(t_0, x_0, \theta_0) \in \mathbb{I}_0 \times U_0 \times \Theta_0$

$$\varphi(., t_0, x_0, \theta_0) : \mathbb{I}_0 \to \mathbb{R}^n$$

is a solution to the initial value problem

$$\dot{x} = f(t, x, \theta_0), \quad x(t_0) = x_0. \tag{1.3}$$

Furthermore, if $\psi(., t_0, x_0, \theta_0)$ *is another solution to the initial value problem* (1.3)*, then* $\psi(t) = \varphi(t)$ *on the intersection of their domains of definition.*

By solving the relevant differential equation, we can use Gronwall's inequality to constrain a function known to meet a particular differential inequality. It offers a comparison theorem, which can be utilized to demonstrate the uniqueness of a solution to the initial value problem (1.3).

Theorem 1.1.3. ***(The Gronwall's Inequality)*** *Let* $\alpha, \beta : (a, b) \to [0, \infty)$ *be continuous functions. Assume*

$$\alpha(t) \leq C + \left| \int_{t_0}^{t} \alpha(s)\beta(s)ds \right|, \quad t_0, t \in (a, b)$$

for some constant $C \geq 0$*. Then,*

$$\alpha(t) \leq C \exp\left(\left| \int_{t_0}^{t} \beta(s)ds \right| \right)$$

Applying the Gronwall's inequality to our initial value problem (1.3), let $\alpha(t) := \|\varphi(., x_0) - \psi(., y_0)\|$, $C = \|x_0 - y_0\|$ and $\beta(t) = K$. We obtain the proposition below.

Proposition 1.1.4. *Let* $U \subset \mathbb{R}^n$ *be an open set and assume* $f : U \to \mathbb{R}^n$ *is a Lipschitz continuous function with* $Lip(f) = K$*. If* $\varphi(., x_0) : \mathbb{I}_{x_0} \to \mathbb{R}^n$ *and* $\psi(., y_0) : \mathbb{I}_{y_0} \to \mathbb{R}^n$ *are* *solutions to the initial value problem* (1.3) *with* $x(t_0) = x_0$ *and* $x(t_0) = y_0$*, respectively, then*

$$\|\varphi(t, x_0, \theta_0) - \psi(t, y_0, \theta_0)\| \leq \|x_0 - y_0\| e^{K|t-t_0|} \tag{1.4}$$

for all $t \in \mathbb{I}_{x_0} \cap \mathbb{I}_{y_0}$.

Remarks 1.1.5. *Proposition 1.1.4 guarantees the existence and uniqueness of solutions. To show that two solutions to the same initial value problem* (1.3) *agree on the intersection of their domains of definition, we let* $\varphi : \mathbb{I}_0 \to \mathbb{R}^n$ *and* $\psi : \mathbb{I}_1 \to \mathbb{R}^n$ *denote two solutions to the initial value problem* (1.3)*. Given that* $x(t_0) = x_0 = y_0$*, from equation* (1.4) *for all* $t \in \mathbb{I}_0 \cap \mathbb{I}_1$,

$$\|\varphi(t) - \psi(t)\| = 0,$$

which establishes the uniqueness of the solution to the initial value problem (1.3).

1.2 Pontryagin Maximum Principle (PMP)

The Pontryagin Maximum Principle is a fundamental mathematical principle in the field of optimal control theory. It provides necessary conditions that an optimal control and corresponding state trajectory must satisfy for a wide class of optimal control problems. [24].

The PMP is applied to problems where the objective is to maximize a particular performance criterion or cost, typically expressed as the integral of a given performance index over a specified time interval, and subject to given constraints. The principle states that, under certain regularity conditions, an optimal control strategy and the corresponding state trajectory must satisfy a set of differential equations known as the canonical equations [16].

The canonical equations involve the system dynamics, the costate variables (Lagrange multipliers), and the partial derivatives of the Hamiltonian, which is a function combining the system dynamics and the cost function. The optimal control is determined by maximizing the Hamiltonian over the set of feasible controls.

The Pontryagin Maximum Principle is widely used to analyze and solve optimization problems, where the goal is to find the best control strategy for a dynamic system.

The basic optimal control problem for ordinary differential equations consists of finding a piecewise control $u(t)$ and the associated state variable $x(t)$ to maximize the given objective functional below,

$$\max J = \int_{t_0}^{T} L(t, x(t), u(t))\, dt \tag{1.5}$$

$$\text{subject to } \ \dot{x} = f(t, x(t), u(t)), \quad x(t_0) = x_0. \tag{1.6}$$

where equation (1.6) models the system dynamics, and the term $L(t, x(t), u(t))$ is referred to as the integral cost. The function $L(t, x(t), u(t))$ is assumed to be non-negative and continuous in all arguments for $t \in [t_0, T]$. In solving the optimal control problem above (1.5)-(1.6), the first step is to form the Hamiltonian. The Hamiltonian is defined as

$$H(t, x(t), u(t), \lambda(t)) = \lambda_0 L(t, x(t), u(t) + \lambda f(t, x(t), u(t)) \tag{1.7}$$

Theorem 1.2.1. *(PMP) If $u^*(t)$ and $x^*(t)$ are the optimal solution of the control problem, then there exist piecewise differentiable adjoint variables $\lambda(t)$ such that*

$$H(t, x^*(t), u(t), \lambda(t)) \leq H(t, x^*(t), u^*(t), \lambda(t)) \tag{1.8}$$

for all controls u at each time t, where H is the Hamiltonian and

$$\dot{\lambda}(t) = \frac{\partial H(t, x^*(t), u^*(t)}{\partial x} \tag{1.9}$$

$$\lambda(T) = 0 \tag{1.10}$$

are the costate and transversality conditions, respectively.

We focus on the application of the PMP theorem, excluding detailed proof. We refer [4], [24] for the proof.

Definition 1.2.2. *A triple (x^*, u^*, λ) is called extremal if (x^*, u^*) is admissible and the equations $\dot{x} = H_\lambda$ and $\dot{\lambda} = -H_x$ hold along (x^*, u^*).*

Theorem 1.2.3. *Suppose that $f(t, x, u)$ is a continuously differentiable function in its three arguments and concave in u. Suppose u^* is an optimal control with associated state x^*, and λ a piecewise differentiable function with $\lambda(t) \geq 0 \; \forall \, t$. Suppose for all $t_0 \leq t \leq T$*

$$0 = H_u(t, x^*(t), u^*(t), \lambda(t)). \tag{1.11}$$

Then for all controls u and each $t_0 \leq t \leq T$, we have

$$H(t, x^*(t), u(t), \lambda(t)) \leq H(t, x^*(t), u^*(t), \lambda(t)) \tag{1.12}$$

The same essential conditions are derived through similar reasoning when the problem involves minimizing rather than maximizing. In a minimization problem, we minimize the Hamiltonian pointwise and the inequality in PMP is reversed [36]. Indeed, for a minimization problem with f being convex in u, we can derive

$$H(t, x^*(t), u(t), \lambda(t)) \geq H(t, x^*(t), u^*(t), \lambda(t)) \tag{1.13}$$

by the same argument as in Theorem 1.2.3

This concludes our elementary derivation of the Pontryagin maximum principle.

Chapter 2

Mathematical Model

2.1 Problem description and assumptions

We incorporate the widely used epidemiological compartmental model, Susceptibles Infected Recovered (SIR) model. In this model, births and deaths are neglected, and the recovered population is assumed to no longer infect others and cannot be reinfected. Here, the SIR model captures the importation (infected non-resident travellers). Our model and analysis follow from Hansen and Day (2011) [14]. We address epidemics with no vaccination, where the possible controls are isolation and travel restrictions. We model these non-pharmaceutical interventions via finite time controls $(u_1(t), u_2(t)) \in [0, u_{1\max}] \times [0, u_{2\max}]$, where 0 corresponds to no control and $u_{1\max}, u_{2\max}$ correspond to when we have maximum daily rate of community isolation and traveller isolation respectively. We denote S, I_1, I_2 the number of susceptibles, community prevalence, traveller's prevalence (imported cases), respectively, and their evolution is governed by the following system of nonlinear ordinary differential equations

$$\frac{dS}{dt} = -\beta S(I_1 + cI_2) \tag{2.1}$$

$$\frac{dI_1}{dt} = \beta S(I_1 + cI_2) - (\mu + u_1(t))I_1 \tag{2.2}$$

$$\frac{dI_2}{dt} = \theta - (\gamma + u_2(t))I_2 \tag{2.3}$$

with $S(t_0) > 0, I_1(t_0) \geq 0, I_2(t_0) \geq 0,\ \beta, \mu, \theta, \gamma, c \geq 0$, where β is the transmission rate, μ is the per capita loss rate of infected community members through both mortality and recovery, θ is the baseline number of infected non-resident travellers per unit time, γ is the removal rate of non-resident travellers and c is the relative transmissibility of travellers. We assume that non-resident travellers who are infected have contracted the infection before entering the community, and therefore are promptly either recovered or removed from the community.

Moreover, $u_2(t)$ is defined as a post-arrival traveller isolation (daily rate) representing the post-arrival travel isolation measure. This is implemented after travellers arrive to reduce the spread of infection from incoming cases. It Includes actions like quarantine, isolation, and testing of travellers. $u_2(t)$ is integrated into the model as a control variable affecting the rate of change of I_2 (infected non-resident travellers).

The set of admissible controls is given by

$$U_{ad} = \{u = (u_1, u_2) \text{ such that } (u_1, u_2) \text{ measurable}; (u_1(t), u_2(t)) \in [0, u_{1\max}] \times [0, u_{2\max}]\} \tag{2.4}$$

being a compact convex subset of $\mathbb{R}^m$ and the controls are bounded and Lebesgue measurable. Thus, all possible set u must be contained in the set of admissible controls U_{ad}.

Given the set of admissible controls U_{ad}, we have U_{ad} closed by the definition of a closed set. Further, let $u_1, u_2 \in U_{ad}$, then it follows from the definition of a convex set that

$$bu_1 + (1-b)u_2 \in [0, u_{\max}]^2$$

for all $b \in [0, 1]$. Consequently $bu_1 + (1-b)u_2 \in U_{ad}$, implying the convexity of U_{ad}.

2.2 Objectives and contributions

The modelling and assumptions in the present work are motivated by epidemics, which are mainly managed through broad NPIs with limited resources and without vaccines. To understand the effects of NPIs on an entire population, we stick to model (2.1)-(2.3) with isolation of infected community members and travellers.

Our primary objective is to understand the timing and best control strategies for implementing isolation and travel restrictions. That is;

1. Identify when different public health strategies are optimal, as defined by optimal control theory.

2. Extend existing results to consider imported infections and travel measures.

To this end, we also investigate solutions in specific scenarios (such as isolation-only (no importation), isolation-only (with importations), traveller isolation only, and mixed strategies).

2.3 Defining an Outbreak End-Point

One aspect of this work is that the problem is posed in terms of the infinite time limit but formulated in a way that only requires a solution over a finite time interval. We assume that the interventions last a finite time T, the maximum time that the population will adhere. The selection of the terminal time T is one possible issue with optimal control [14]. Hansen and Day defined an outbreak as over at $t = T$ if prevalence is less than a small value, $I_{\min}$. This approach prevents a second wave of infection arising from a fractional individual [14]. Defining an outbreak end-point in this way is necessary to consider elimination strategies as a possible recommended strategy [20].

2.4 Resource constraints (Optimal Control Problem)

Let $U_{1[u_1,u_2]}(T)$, $U_{2[u_1,u_2]}(T)$ denote the total number of community residents and travellers that need to be isolated at time T respectively and $U_{1\max}, U_{2\max}$ the total resources available for community isolation and traveller isolation respectively.

In keeping with [14], we assume that resources are limited, such that,

$$U_{1[u_1,u_2]}(T) = \int_0^T u_1(t) I_{1[u_1,u_2]} \; dt \leq U_{1\max} \tag{2.5}$$

and

$$U_{2[u_1,u_2]}(T) = \int_0^T u_2(t) I_{2[u_1,u_2]} \; dt \leq U_{2\max} \tag{2.6}$$

The aim of public health measures is assumed to minimize the number of new infections,

$$J = \int_0^T \beta S_{[u_1,u_2]}(I_{1[u_1,u_2]} + c I_{2[u_1,u_2]}) \; dt \tag{2.7}$$

subject to the resource constraints (2.5)-(2.6).

Without constraints, thus when we have no resource limitations on the controls, we get:

$$U_{1[u_{1\max},u_2^*]}(T) = \int_0^T u_1(t) I_{1[u_1,u_2]}\ dt \tag{2.8}$$

and

$$U_{2[u_1^*,u_{2\max}]}(T) = \int_0^T u_2(t) I_{2[u_1,u_2]}\ dt \tag{2.9}$$

To apply the PMP, we define the Hamiltonian as

$$H(t) = \lambda_0 \beta S(I_1 + I_2) + \lambda_{I_1}\frac{dI_1}{dt} + \lambda_{I_2}\frac{dI_2}{dt} + \lambda_{U_1}\frac{dU_1}{dt} + \lambda_{U_2}\frac{dU_2}{dt} \tag{2.10}$$

2.4.1 Existence of Optimal Controls

The PMP only provides necessary conditions for optimality, and the fulfilment of necessary conditions alone does not guarantee optimality. For an optimal control to exist, we want to have compactness of feasible solution sets. We provide a result stating the existence of at least one optimal solution to the optimal control problem (2.5)-(2.7) under some appropriate compactness and convexity assumptions. Precisely, we follow the standard Filippov's approach [11]. Filippov's existence theorem is a result of the theory of differential inclusions, which are generalizations of ordinary differential equations that allow for multiple possible trajectories at a single point in the state space. Filippov's existence theorem addresses the existence of solutions for differential inclusions [10], [11].

Theorem 2.4.1. ***(Filippov's existence theorem)*** *Consider an optimal control problem defined by a differential inclusion* $\dot{x} \in F(t, x, u)$*, where* $F : [t_0, T] \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ *is a* *set-valued mapping representing the dynamics, t is time, x is the state variable and u is the control input. Assume that the set-valued map F is upper semicontinuous in x and continuous in u for each fixed t. If the optimal control problem has nonempty, compact, and convex solution sets for all t, then an optimal control exists for almost every initial point in $\mathbb{R}^n$.*

To establish the existence of the optimal control, we rely on findings presented in [12] and [18]. Initially, we address the boundedness of the state variables in the system (2.1)-(2.2). By summing up all the equations in the model (2.1)-(2.2), we obtain $N(t) \leq N(t_0)$ where $N(t) = S(t) + I_1(t) + I_2(t)$. Considering the characteristics of the infectious disease model, for $\mu \geq 0$, it is evident that $0 \leq S(t), I_1(t), I_2(t) \leq N(t_0)$. In other words, the state variables of the system are bounded. The assurance of the existence of an optimal control solution is ensured by satisfying the following conditions.

(a) The set of control variables and corresponding state variables is not empty.

(b) The admissible control set U_{ad} is compact and bounded.

c) The vector function $f(t, x, u)$ is continuous.

By examining the control set U_{ad} (2.4), it becomes apparent that the system state equation's solution remains continuous and bounded for every permissible control function. Therefore, condition (a) and (b) are satisfied. The function $f(t, x, u)$, as defined in (2.1)-(2.2), adheres to the Lipschitz condition (1.1.1) concerning the state variables, ensuring the existence of the model's solution. The system's (2.1)-(2.2) solution is evidently continuous, thereby fulfilling condition (c).

2.5 Bang-Bang Optimal Controls

Now we concentrate on scenarios characterized by linearity in the controls. Linear optimization problem is defined as that class of optimal control problems in which the control function appears only linearly [13]. In these cases, optimal solutions often incorporate discontinuities in the control variables. Notice that equations (2.1) - (2.3) and the integrand in (2.7) are both linear functions of the controls $u_1(t), u_2(t)$. Thus, the Hamiltonian (2.10) is also a linear function of the controls; hence, the optimality condition contains no information on the controls. The PMP, when applied to bounded control problems that are linear in the control variable, explicitly defines the bang-bang control. However, the bang-bang control is undefined when the switching function is identically zero. The consequence of this problem is that we are not able to find a characterization of the optimal controls. Hansen and Day [14] show that the optimal control for our problem is bang-bang. Bang-bang control involves switching between two extreme values, typically represented as $u_{\max}$ and 0.

$$u_1^*(t) = \begin{cases} u_{1\max}, & \text{maximum rate of community isolation} \\ 0, & \text{no community isolation} \end{cases} \tag{2.11}$$

$$u_2^*(t) = \begin{cases} u_{2\max}, & \text{maximum rate of traveller isolation} \\ 0, & \text{no traveller restrictions} \end{cases} \tag{2.12}$$

2.6 Definition of Public Health Strategies

To control disease outbreaks and protect the population from health risks, public health initiatives are crucial. The use of elimination, suppression, and circuit breakers stand out as three significant strategies. Safeguarding public health and reducing the impact of infectious diseases requires an understanding of and effective implementation of these strategies.

Public Health Strategy	**Description**	**Our Definition**
Elimination	Strict public health measures reduce infection prevalence to zero locally, but not in all regions, such that there remains a risk of disease importation (Baker, Wilson, and Blakely 2020; Metcalf et al. 2021).	(a) The outbreak is eliminated by public health measures, i.e., $U_{1[u_{1\max}]}(T) \leq U_{1\max}$ and $U_{2[u_{2\max}]}(T) \leq U_{2\max}$. (b) $\frac{dI_1}{dt} < 0$ shortly after $u_1^*(t)$ and/or $u_2^*(t)$ are implemented.
Suppression	Infection is kept at low levels (Baker, Wilson, and Blakely 2020).	(a) The outbreak is eliminated by public health measures, i.e., $U_{1[u_{1\max}]}(T) \leq U_{1\max}$ and/or $U_{2[u_{2\max}]}(T) \leq U_{2\max}$; (b) $\frac{dI_1}{dt} \geq 0$ shortly after $u_1(t)$ and/or $u_2(t)$ are implemented.
Circuit Breaker	Public health measures are intermittent with breaks in between.	An optimal control involves at least two switches between public health measures of different intensities.

Table 2.1: Definitions of public health strategies

2.7 Problem Classification

Next, we classify our general problem into the following four parts:

Problem	**Description**	**Special values of parameters**
1	Community member isolation only, no importations	$\theta = 0, I_2(0) = 0, U_{2\max} = 0$
2	Community member isolation, with importations	$U_{2\max} = 0.$
3	Travel measures only	$U_{1\max} = 0.$
4	Both community member isolation and travel measures	None

Table 2.2: The four problems that we analyze

2.8 Problem 1: Community Isolation Only (no case importation)

The derivation and proof of the theorems of Problem 1 is a direct result of the work of Hansen and Day 2011 [14].

Considering community isolation as the only control in the model, our model now becomes;

$$\frac{dS}{dt} = -\beta S I_1 \tag{2.13}$$

$$\frac{dI_1}{dt} = \beta S I_1 - (\mu + u_1(t))I \tag{2.14}$$

Our objective is;

$$\min J = \min \int_0^T \beta S_{[u_1]} I_{1[u_1]} \, dt \tag{2.15}$$

subject to equations (2.13)-(2.14), $T = \inf\{t | I_{[u_1]}(t) = 1\}, u_1(t) \in [0, u_{1\max}]$ for all $t \in [0, T]$ and subject to the resource constraint;

$$U_{1[u_1]}(T) = \int_0^T u_1(t) I_{1[u_1]} \, dt \leq U_{1\max} \tag{2.16}$$

From Eq. (2.13), we have

$$dS = \beta S I_1 \, dt \tag{2.17}$$

integrating both sides, we get

$$\int_0^T dS = -\int_0^T \beta S I_1 \, dt \tag{2.18}$$

$$S(T) - S(0) = -\int_0^T \beta S I_1 \, dt \tag{2.19}$$

$$S(0) - S(T) = \int_0^T \beta S I_1 \, dt \tag{2.20}$$

on the other hand, rearranging Eq. (2.13), we get

$$\frac{1}{S} \, dS = -\beta I_1 \, dt \tag{2.21}$$

Taking integral on both sides, we have

$$\int_0^T \frac{1}{S} \, dS = -\beta \int_0^T I_1 \, dt \tag{2.22}$$

$$-\frac{1}{\beta} \ln \left(\frac{S(T)}{S(0)} \right) = \int_0^T I_1 \, dt \tag{2.23}$$

We observe from equations (2.20) and (2.23) that the terms on the right-hand side are both minimized by maximizing $S(T)$ since S_0 is a fixed quantity. Next, we state and proof the main result of the isolation policy.

Theorem 2.8.1. ***(Optimal Community Isolation Strategy)*** *If* $U_{1[u_{1max}]}(T) \leq U_{1max}$, *then the optimal community isolation strategy for Problem 1 is* $u_1^*(t) = u_{1max}$. *If* $U_{1[u_{1max}]}(T) > U_{1max}$, *then the optimal control* $u_1^*(t)$ *is any bang-bang control* $u_1(t)$ *such that* $U_{1[u_1^*]}(T) = U_{1max}$.

Rephrasing Theorem (2.8.1) in terms of public health terminology, the optimal isolation strategy is:

1. **Elimination:** if $U_{1[u_{1\max}]}(T) \leq U_{1\max}$ and community infections decline shortly after the implementation of public health measures.

2. **Suppression:** if $U_{1[u_{1\max}]}(T) \leq U_{1\max}$ and community infections increase shortly after the implementation of public health measures.

3. **Suppression or circuit-breaker:** if $U_{1[u_{1\max}]}(T) > U_{1\max}$.

The optimal isolation policy, as outlined in theorem (2.8.1), is to implement maximal isolation efforts throughout the epidemic, provided sufficient resources are available. Without adequate resources, the optimal policy defaults to any strategy that utilizes all available resources [14].

***Proof* (Theorem** (2.8.1): Following equations (2.13), (2.14) and (2.16), the isolation model with limited resources is described by the system of ordinary differential equations:

$$\frac{dS}{dt} = -\beta S I_1 \tag{2.24}$$
$$\frac{dI}{dt} = \beta S I_1 - (\mu + u_1(t)) I_1 \tag{2.25}$$
$$\frac{dU_1}{dt} = u_1(t) I_1 \tag{2.26}$$

Next, we formulate Problem 1 Sec.2.8 as a maximization problem and apply the PMP. The objective now becomes

$$\max J = \max \left(-\int_0^T \beta S_{[u_1]} I_{[u_1]}\, dt \right) \tag{2.27}$$

We derive the necessary conditions for optimality and the associated adjoint variables. The Hamiltonian is

$$H(t) = -\lambda_0 \beta S I_1 - \lambda_S \beta S I_1 + \lambda_{I_1} \beta S I_1 - \lambda_{I_1}(\mu + u_1 I_1 + \lambda_{U_1} u_1 I_1 \tag{2.28}$$
$$= -\dot{\lambda}_{I_1} I_1 = -\dot{\lambda}_S S - \lambda_{I_1}\mu + (\lambda_{U_1} - \lambda_{I_1}) u_1 I_1 = 0 \tag{2.29}$$

There are associated adjoint variables, $\lambda_S, \lambda_{I_1}, \lambda_{U_1}$, which correspond to the states S, I_1, and U_1 respectively such that;

$$\dot{\lambda}_S = -\frac{\partial H}{\partial S} = -(\lambda_{I_1} - \lambda_0 - \lambda_S)\beta I_1 \tag{2.30}$$

$$\dot{\lambda}_{I_1} = -\frac{\partial H}{\partial I_1} = -(\lambda_{I_1} - \lambda_0 - \lambda_S)\beta S - (\lambda_{U_1} - \lambda_{I_1})u_1 + \lambda_{I_1}\mu \tag{2.31}$$

$$\dot{\lambda}_{U_1} = -\frac{\partial H}{\partial U_1} = 0 \tag{2.32}$$

and the optimality condition is obtained as follows:

$$\frac{\partial H}{\partial u_1} = \psi_1(t) = (\lambda_{U_1} - \lambda_{I_1})I_1 \text{ at } u_1^*(t) \tag{2.33}$$

with the boundary conditions $(\lambda_0, \lambda_S(T), \lambda_{I_1}(T), \lambda_{U_1} = (\lambda_0, 0, \lambda_{I_1}(T), q)$ known as the transversality conditions, where $q \leq 0$, and $\psi_1(t)$ is called the switching function. Equations (2.28)-(2.33) form the necessary conditions that an optimal control must satisfy.

Remarks 2.8.2. *Pontryagin defines the Hamiltonian with two co-state variables λ_0 and λ_x (x represents the state variables). Hence, λ_x represents the adjoint variable with respect to the state variables. Subsequently, λ_0 turns out to be constant in time; its value is determined in the Pontryagin theory as follows:*

$\lambda_0 = -1$ *if $u(t)$ is feasible and the objective functional (2.27) is to be minimized.*

$\lambda_0 = +1$ *if $u(t)$ is feasible and the objective functional (2.27) is to be maximized.*

$\lambda_0 = 0$ *if $u(t)$ is unfeasible.*

For our simplified problem, we have shown that all admissible controls are feasible, and we are to look for a maximum of the objective function, thus $\lambda_0 = +1$.

We now summarize the control characterization as follows:

$$u_1^*(t) = \begin{cases} u_{1\max}, & \text{if} \quad \lambda_{U_1} > \lambda_{I_1} \\ 0, & \text{if} \quad \lambda_{U_1} < \lambda_{I_1} \end{cases} \tag{2.34}$$

which follows from equation (2.11).

From equation (2.29), we observe that $\dot{\lambda}_{I_1} = 0$ and therefore the optimal control is either $u_1^*(t) \equiv 0, u_1^*(t) \equiv u_{1\max}$ or $u_1^*(t)$ is singular.

We observe that without the constraint equation Problem 1 (2.8) and equation (2.16) become an unconstrained optimal control problem, and its solution is $u_1^*(t) \equiv u_{1\max}$.

Claim 2.8.3. *The Optimal control for Problem 1 with $U_{1max} = \infty$ is $u_1^*(t) \equiv u_{1max}$.*

Proof: substituting equation (2.30) into (2.31) with $\lambda_{U_1} = 0$ and making $\dot{\lambda}_S$ the subject gives,

$$\dot{\lambda}_S = \dot{\lambda}_{I_1} \frac{I_1}{S} - \lambda_{I_1}(\mu + u_1)\frac{I_1}{S} \tag{2.35}$$

We have already established that the optimal control is purely bang-bang (no singular components) [14].

The optimal control will be determined once the sign of λ_{I_1} is determined. To determine the sign of λ_{I_1}, we use the transversality condition $\lambda_S(T) = 0$. Since λ_{I_1} is a constant, equation (2.31) gives

$$\lambda_{I_1} = \frac{(\lambda_0 + \lambda_S)\beta S}{\beta S - u_1 - \mu} = \frac{(\lambda_0)\beta S(T)}{\beta S(T) - u_1(T) - \mu} \tag{2.36}$$

This implies that $\text{sign}(\lambda_{I_1}) = \text{sign}\Big(S(T) - \frac{u_1(T)+\mu}{\beta}\Big)$. From equation (2.14), λ_{I_1} is negative if and only if $\frac{dI_1}{dt} < 0$. Since T is the smallest time that $I_1(t) = 1$ and $I(0) > 1$, it must be that $\frac{dI_1}{dt}$ is negative. Therefore, $u_1^*(t) \equiv u_{1\max}$.

Our second observation is that the total number of isolated individuals can be calculated as;

From equation (2.24), we can write $-\dot{S} = \beta S I_1$ and $I_1 = -\frac{\dot{S}}{\beta S}$. Substituting these two expressions into equation (2.25) gives;

$$\dot{I} = -\dot{S} + \frac{\mu}{\beta}\frac{\dot{S}}{S} - u_1 I_1. \tag{2.37}$$

Rearranging equation (2.37) and integrating from 0 to T, we obtain:

$$\int_0^T u_1 I_1 \, dt = S(0) - S(T) + I_1(0) - I_1(T) + \frac{\mu}{\beta}\ln\left(\frac{S(T)}{S(0)}\right) \tag{2.38}$$

Equation (2.38) shows that the constraint value $U_{1[u_1]}(T) = \int_0^T u_1 I_{1[u_1]} \, dt$ depends only on $S_{[u_1]}(T)$.

Again, the cost function can be rewritten as

$$\int_0^T \beta I_{1[u_1]} S_{[u_1]} \, dt = S(0) - S_{[u_1]}(T) \tag{2.39}$$

and therefore minimizing the cost function is equivalent to maximizing $S_{[u_1]}(T)$.

Now to determine the optimal control when $U_{1[u_{1\max}]}(T) > U_{1\max}$, we rewrite equation (2.38) as

$$\frac{\mu}{\beta}\ln(S_{[u_1]}(T)) - S_{[u_1]}(T) = I_{1[u_1]}(T) - I_1(0) - S(0) + \frac{\mu}{\beta}\ln(S(0)) + U_{1[u_1]}(T). \tag{2.40}$$

There are two possible scenarios:

1. If $U_{1[u_{1\max}]}(T) > U_{1\max}$ and $S_{[u_{1\max}]}(T) < \frac{\mu}{\beta}$, then as long as $U_{1[u_1]}(T) \leq U_{1\max} < U_{1[u_{1\max}]}(T)$, the function $S_{[u_1]}(T)$ shows an upward trend concerning $U_{[u_1]}(T)$. This implies that any control strategy $u_1^*(t)$ utilizing the entire available resource set will be optimal.

2. If $U_{1[u_{1\max}]}(T) > U_{1\max}$ and $S_{[u_{1\max}]}(T) > \frac{\mu}{\beta}$, considering the convex downward function $f(S) = \frac{\mu}{\beta}\ln(S) - S$ with a maximum at $S = \frac{\mu}{\beta}$, for any $u_1(t)$ with $U_{1[u_1]}(T) < U_{1\max}$, it implies that $S_{[u_1]}(T) < \frac{\mu}{\beta}$. Consequently, $S_{[u_1]}(T)$ increases with $U_{1[u_1]}(T)$ for any $U_{1[u_1]}(T) < U_{1\max}$. Thus, $u_1^*(t)$ denotes any control strategy utilizing all available resources.

This concludes the proof of Theorem 2.8.1.

2.9 Problem 2: Community Isolation Only (with case importation)

In this model, we assume that there are case importations yet the only control measure is community isolation.

$$\frac{dS}{dt} = -\beta S(I_1 + cI_2) \tag{2.41}$$

$$\frac{dI_1}{dt} = \beta S(I_1 + cI_2) - (\mu + u_1(t))I_1 \tag{2.42}$$

$$\frac{dI_2}{dt} = \theta - \gamma I_2 \tag{2.43}$$

$$\frac{dU_1}{dt} = u_1(t)I_1 \tag{2.44}$$

The optimal control is similar to Problem 1 [14] and the resulting theorem as Theorem 2.8.1. The problem of minimizing the total number of new cases becomes a problem of finding the control $u_1(t)$ that minimizes:

$$J = \int_0^T \beta S(I_1 + I_2)\ dt \tag{2.45}$$

subject to the constraints $I_1(T) = 1$ and $U_1(T) \leq U_{1\max}$.

The Hamiltonian is given by

$$H(t) = (\lambda_0 - \lambda_S + \lambda_{I_1})\beta S(I_1 + I_2) - \lambda_{I_1}(\mu + u_1(t))I_1 + \lambda_{I_2}(\theta - \gamma I_2) + \lambda_{U_1}u_1(t)I_1 \tag{2.46}$$

where the adjoint variables are defined by

$$\dot{\lambda}_S = -\frac{\partial H}{\partial S} = -(\lambda_0 - \lambda_S + \lambda_{I_1})\beta(I_1 + I_2) \tag{2.47}$$

$$\dot{\lambda}_{I_1} = -\frac{\partial H}{\partial I_1} = -(\lambda_0 - \lambda_S + \lambda_{I_1})\beta S - (\lambda_{U_1} - \lambda_{I_1})u_1 + \lambda_{I_1}\mu \tag{2.48}$$

$$\dot{\lambda}_{I_2} = -\frac{\partial H}{\partial I_2} = -(\lambda_0 - \lambda_S + \lambda_{I_1})\beta S + \lambda_{I_2}\gamma \tag{2.49}$$

$$\dot{\lambda}_{U_1} = -\frac{\partial H}{\partial U_1} = 0 \tag{2.50}$$

The transversality conditions are $(\lambda_0, \lambda_S(T), \lambda_{I_1}(T), \lambda_{I_2}(T), \lambda_{U_1}) = (\lambda_0, 0, \lambda_{I_1}(T), 0, q)$ where $q \leq 0,\ \lambda_{I_1}(T) \geq 0$, which implies $\lambda_{U_1} \leq 0$.

The optimal control therefore satisfies

$$u_1^*(t) = \begin{cases} u_{1\max}, & \text{if} \quad \lambda_{U_1} - \lambda_{I_1} < 0 \\ ? & \text{if} \quad \lambda_{U_1} - \lambda_{I_1} = 0 \\ 0, & \text{if} \quad \lambda_{U_1} - \lambda_{I_1} > 0 \end{cases} \tag{2.51}$$

Since the optimal control is bang-bang, it implies that the optimal control is either $u_1^*(t) \equiv u_{1\max}$ or $u_1^*(t) \equiv 0$. To further constrain the form of the optimal control, let consider the case when $\lambda_{U_1} = 0$. Since $\lambda_{I_1}(T) \geq 0$ and $\dot{\lambda}_{I_1} < 0$, it must be that $\lambda_{I_1}(t) > 0$ for all $t < T$. Therefore, if $\lambda_{U_1} = 0$, the optimal control is to isolate with maximum effort for the entire epidemic.

Now, let consider the case when $\lambda_{U_1} < 0$. If $\lambda_{U_1} < 0$, then all of the control resources are used therefore the optimal control is to isolate with the maximum effort until all of the available resources are used.

When we integrate the two scenarios, we observe that the optimal control strategy is to isolate with maximum effort until all available resources are depleted or the epidemic ends (refer to Theorem 2.8.1).

2.10 Problem 3: Post-Arrival Traveller isolation Only

In Problem 3, we consider traveller isolation as the only control in the model. Our model now becomes

$$\frac{dS_1}{dt} = -\beta S(I_1 + cI_2) \tag{2.52}$$
$$\frac{dI_1}{dt} = \beta S(I_1 + cI_2) - \mu I_1 \tag{2.53}$$
$$\frac{dI_2}{dt} = \theta - (u_2(t) + \gamma)I_2 \tag{2.54}$$
$$\frac{dU_2}{dt} = u_2(t)I_2 \tag{2.55}$$

Our objective function now becomes;

$$\min J = \min \int_0^T \beta S_{[u_2]}(I_{1[u_2]} + cI_{2[u_2]})\ dt \tag{2.56}$$

subject to equations (2.52)-(2.55), $T = \inf\{t | I_{1[u_2]}(t) = 1\}$, $u_2(t) \in [0, u_{2\max}]$ for all $t \in [0, T]$ and subject to the resource constraint;

$$U_{2[u_2]}(T) = \int_0^T u_2 I_2\ dt \leq U_{2\max} \tag{2.57}$$

Theorem 2.10.1. ***(Optimal Traveller Isolation Strategy)*** *If $U_{2[u_{2max}]}(T) \leq U_{2max}$, then the optimal traveller isolation policy for Problem 3 is $u_2^*(t) = u_{2max}$. If $U_{2[u_{2max}]}(T) > U_{2max}$, then the optimal control $u_2^*(t)$ is any bang-bang control $u_2(t)$ such that $U_{2[u_2^*]}(T) = U_{2max}$.*

The proof of the optimal traveller isolation strategy mirrors and adheres to the methodologies of Theorem 2.8.1 and Theorem 2.10.1, with the PMP relations for problem 3 outlined below.

The Hamiltonian is given by

$$H(t) = (\lambda_0 - \lambda_S + \lambda_{I_1})\beta S(I_1 + I_2) - \lambda_{I_1}\mu I_1 + \lambda_{I_2}\theta - \lambda_{I_2}(u_2(t) + \gamma)I_2 + \lambda_{U_2}u_2(t)I_2 \tag{2.58}$$

where the adjoint variables are defined by

$$\dot{\lambda}_S = -\frac{\partial H}{\partial S} = -(\lambda_0 - \lambda_S + \lambda_{I_1})\beta(I_1 + I_2) \tag{2.59}$$

$$\dot{\lambda}_{I_1} = -\frac{\partial H}{\partial I_1} = -(\lambda_0 - \lambda_S + \lambda_{I_1})\beta S + \lambda_{I_1}\mu \tag{2.60}$$

$$\dot{\lambda}_{I_2} = -\frac{\partial H}{\partial I_2} = -(\lambda_0 - \lambda_S + \lambda_{I_1})\beta S - (\lambda_{U_2} - \lambda_{I_2})u_2(t) + \lambda_{I_2}\gamma \tag{2.61}$$

$$\dot{\lambda}_{U_2} = -\frac{\partial H}{\partial U_2} = 0 \tag{2.62}$$

The transversality conditions are $(\lambda_0, \lambda_S(T), \lambda_{I_1}(T), \lambda_{I_2}(T), \lambda_{U_2}) = (\lambda_0, 0, \lambda_{I_1}(T), \lambda_{I_2}(T), p)$ where $p \leq 0$.

The optimal control therefore satisfies

$$u_2^*(t) = \begin{cases} u_{2\max}, & \text{if} \quad \lambda_{U_2} - \lambda_{I_2} < 0 \\ 0, & \text{if} \quad \lambda_{U_2} - \lambda_{I_2} > 0 \end{cases} \tag{2.63}$$

The conclusion drawn from Theorem 2.10.1 is that the optimal traveller isolation strategy is to isolate with maximum effort throughout the epidemic, provided sufficient resources are available. Without adequate resources, the optimal strategy is any strategy that fully utilizes all available resources. An intriguing insight from the theorem is that, in specific situations, stringent restrictions might be inefficient, leading to unnecessarily high costs [29]. This is exemplified by scenarios where the impact of traveller isolation is minimal due to the limited contribution of imported cases to local transmission [26]. Hence, it is significant that policymakers consider the local incidence of the disease, the growth of local epidemics, and the volume of travel before implementing such strategies.

2.11 Problem 4: Combined Strategies

The model for the combined strategies with limited resources is described by the system of ordinary differential equations:

$$\frac{dS}{dt} = -\beta S(I_1 + cI_2) \tag{2.64}$$
$$\frac{dI_1}{dt} = \beta S(I_1 + cI_2) - (\mu + u_1(t))I_1 \tag{2.65}$$
$$\frac{dI_2}{dt} = \theta - (\gamma + u_2)I_2 \tag{2.66}$$
$$\frac{dU_1}{dt} = u_1(t)I_1 \tag{2.67}$$
$$\frac{dU_2}{dt} = u_2(t)I_2 \tag{2.68}$$

Theorem 2.11.1. ***(Optimal Mixed Strategies)***
If $U_{1[u_{1max},u_{2max}]}(T) \leq U_{1max}$ *and* $U_{2[u_{1max},u_{2max}]}(T) \leq U_{2max}$, *then the optimal control* $u_1^*(t) = u_{1max}$ *and* $u_2^*(t) = u_{2max}$.
If $U_{1[u_{1max},u_{2max}]}(T) \leq U_{1max}$ *and* $U_{2[u_{1max},u_{2max}]}(T) > U_{2max}$, *then the optimal control*

$u_1^*(t) = u_{1max}$ *and* $u_2^*(t)$ *is any bang-bang control such that* $U_{2[u_1^*,u_2^*]}(T) = U_{2max}$.

...

Claim 2.11.2. *If there exists a* $t_k \geq 0$ *such that* $u_1(t) = 0$ *for all* $t \in (t_k, T]$, *then* $u_2(t) = u_2^*(t)$ *for all* $t \in (t_k, T]$.

Proof: Once $t > t_k$, the mixed strategies model becomes a traveller isolation-only model, and therefore, $u_2^*(t)$ is the optimal control for the traveller isolation-only model where $\tilde{U}_{2\max} = U_{2\max} - \int_{t_0}^{t_k} u_2 \; I_2 \; dt$.

Claim 2.11.3. *If there exists a* $t_k \geq 0$ *such that* $u_2(t) = 0$ *for all* $t \in (t_k, T]$, *then* $u_1(t) = u_1^*(t)$ *for all* $t \in (t_k, T]$.

Proof: Once $t > t_k$, the mixed strategies model becomes community isolation-only model, and therefore, $u_1^*(t)$ is the optimal control for the community isolation-only model where $\tilde{U}_{1\max} = U_{1\max} - \int_{t_0}^{t_k} u_1 \; I_1 \; dt$.

Chapter 3

Numerical Results and Discussion

In this chapter, we delve into our comprehensive findings on the optimal strategies for community and traveler isolation in the context of resource limitations. Our investigation is structured around the four distinct problem sets: community isolation only, community isolation with case importation, traveler isolation only, and mixed strategies incorporating both community and traveler isolation. These sets of problems act as the basis for our study, enabling us to assess and contrast the effectiveness of different public health strategies. We provide a thorough presentation of the findings for every problem set, explaining the best controls and how they relate to public health initiatives. The underlying assumption of our study is that these strategies are shaped in large part by resource limitations. A significant insight from our findings is that, in most scenarios, the performance of the optimal strategy is largely independent of the timing of its implementation. This phenomenon is evident across the different problem sets, highlighting the equivalence of various circuit breakers. By presenting these insights, we contribute to the broader discourse on optimizing public health strategies during pandemics and other health crises.

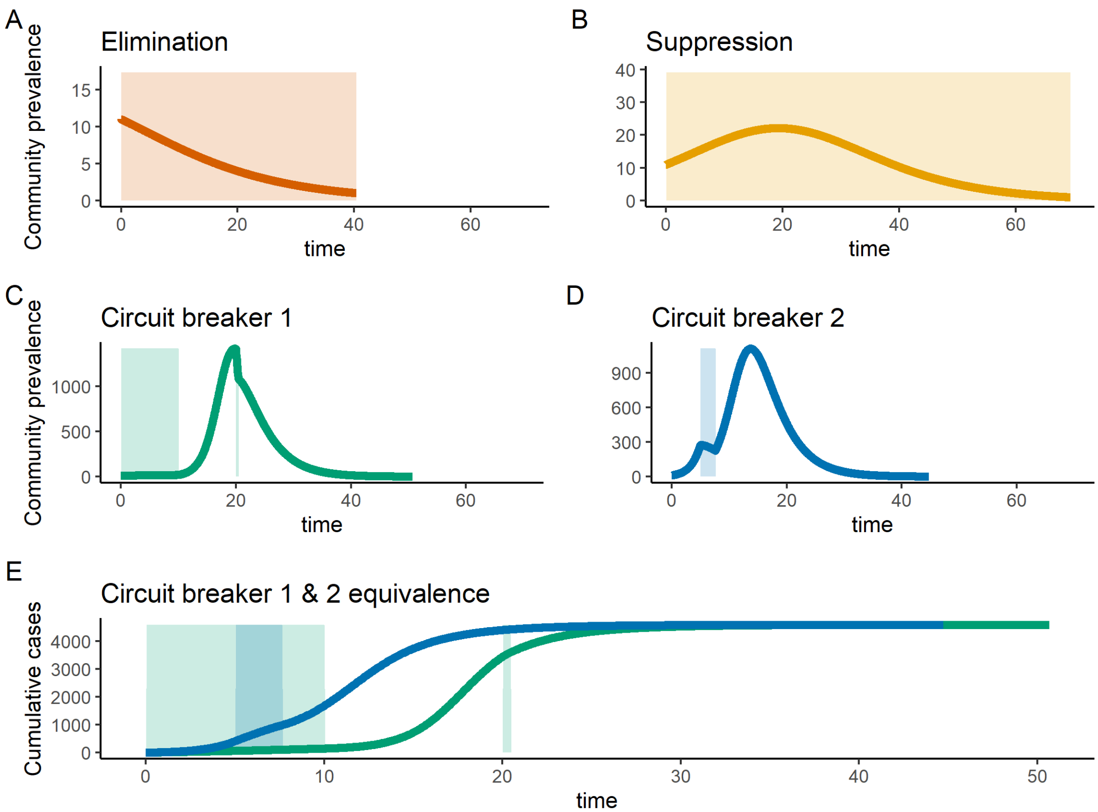

Figure 3.1: Optimal control for the community isolation-only model described in terms of public health strategies. In Figure 3.1 (A and B), there are sufficient resources to implement the control for the entire outbreak. In C - D, resource limitations mean isolation measures cannot remain in place for the entire outbreak. Here, as shown in [14], the cumulative number of cases in the outbreak does not depend on when isolation requirements are implemented, and any strategy that uses all the resources is equivalent (E). Parameters are $u_{1\max} = 0.7\ (A)$ or $0.6\ (B - E); U_{1\max} = 500\ (A, B)$ or $400\ (C - E)$ and for all panels $\beta = 0.002, \mu = 0.334, S(0) = 5000$ and $I_1(0) = 10$, and all other initial conditions are zero. The optimal control is to isolate community members at the maximum rate (shaded region) or not at all (unshaded regions), and the outbreak is over when community infection prevalence (solid lines in A-D) is sufficiently low ($I_1(T) = 1$).

In problem 2, which considers the importation of cases without traveler isolation, we observed that when the rate of case importation is relatively high, elimination of the outbreak becomes unfeasible due to the uncontrolled influx of infected travelers (Figure 3.4(B)). Consequently, the outbreak persists for a longer duration (Figure 3.5(B)), resulting in a higher number of new cases (Figure 3.5(D)). On the other hand, when importation rates are relatively low, elimination can be achieved with a higher community isolation rate, u_{1max} (Figure 3.4(A)). However, given our constraints on resources, it is likely that we will deplete available resources before the epidemic end, leading to another peak in infections. This depletion results in a subsequent increase in infections (Figure 3.5(C)) after the exhaustion of the available resources.

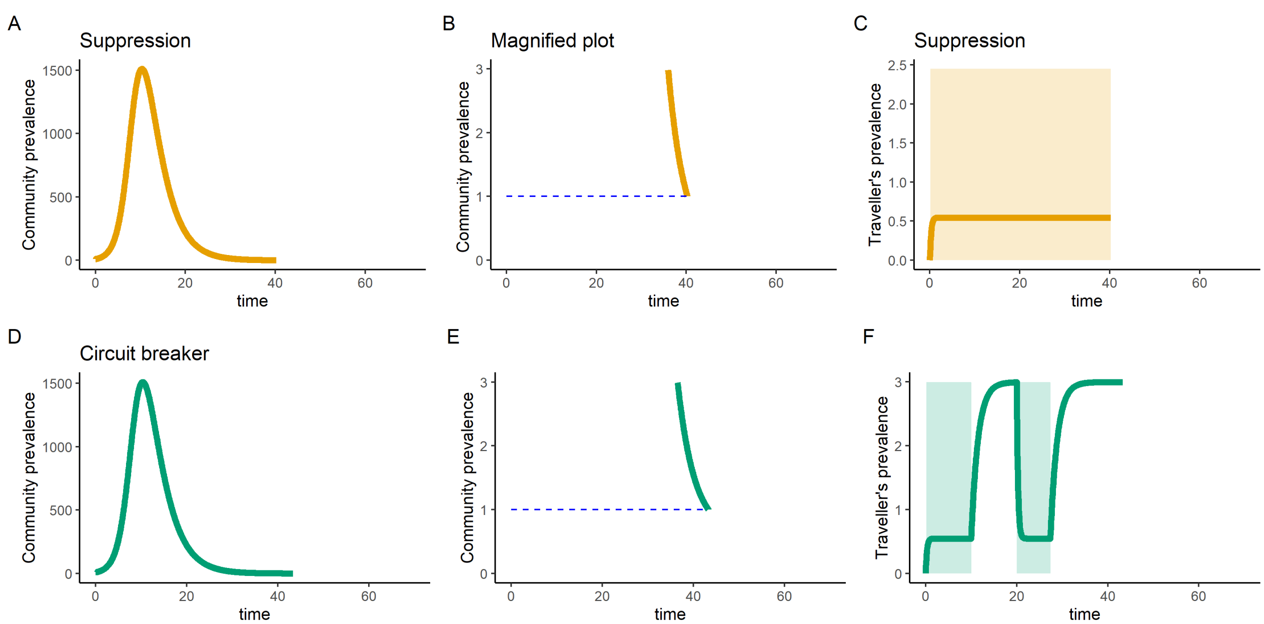

Figure 3.2: Optimal control for the Traveller isolation-only model described in terms of public health strategies

In the post traveller isolation-only model, despite maintaining controls for nearly the entire duration, we observe no significant differences in the state of infection (community prevalence) between scenarios with no control and those with control measures in place (see Figure 3.2). In cases where the outbreak intensifies, the number of imported cases remains relatively low, thereby rendering traveller isolation alone largely ineffective in significantly impacting the outbreak [26].

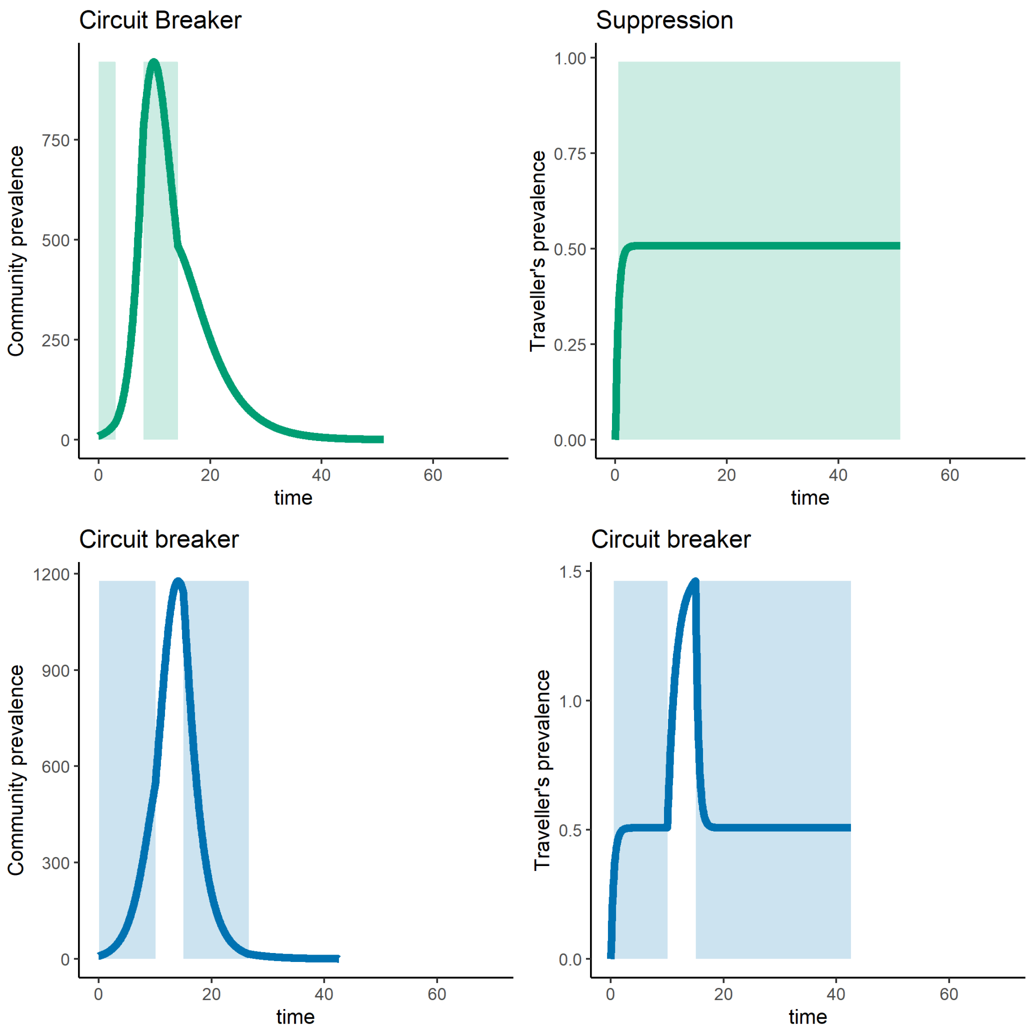

Figure 3.3: Optimal control for the mixed strategies model described in terms of public health strategies. Parameter values used are $u_{1\max} = 1.15$ for elimination (A), $u_{1\max} = 0.2$ for suppression and circuit breaker (B-C). $u_{2\max} = 1.3$ for elimination, suppression and circuit breaker (A-C). $C_{1\max} = 1500$, $C_{2\max} = 50$ for elimination and suppression (A-B), $C_{1\max} = 1000, C_{2\max} = 40$ for circuit breaker (C).

A prominent finding of our research is the effectiveness of the mixed strategies model, which demonstrates that implementing both community and traveler isolation simultaneously from the outset can significantly reduce the number of new infections. Our results highlight that community isolation measures are considerably more effective than post traveler isolation. The model further indicates that the optimal mixed strategy involves applying maximal isolation efforts until either community or traveler isolation resources are depleted.

The mixed strategies model exhibits exceptional adaptability, especially in scenarios where community isolation resources are exhausted first. In such cases, maintaining optimal traveler isolation proves to be the most effective approach. This adaptability emphasizes the advantage of the mixed strategies, which is not simply a combination of optimal community and traveler isolation measures, but a dynamic and responsive strategy tailored to varying resource constraints.

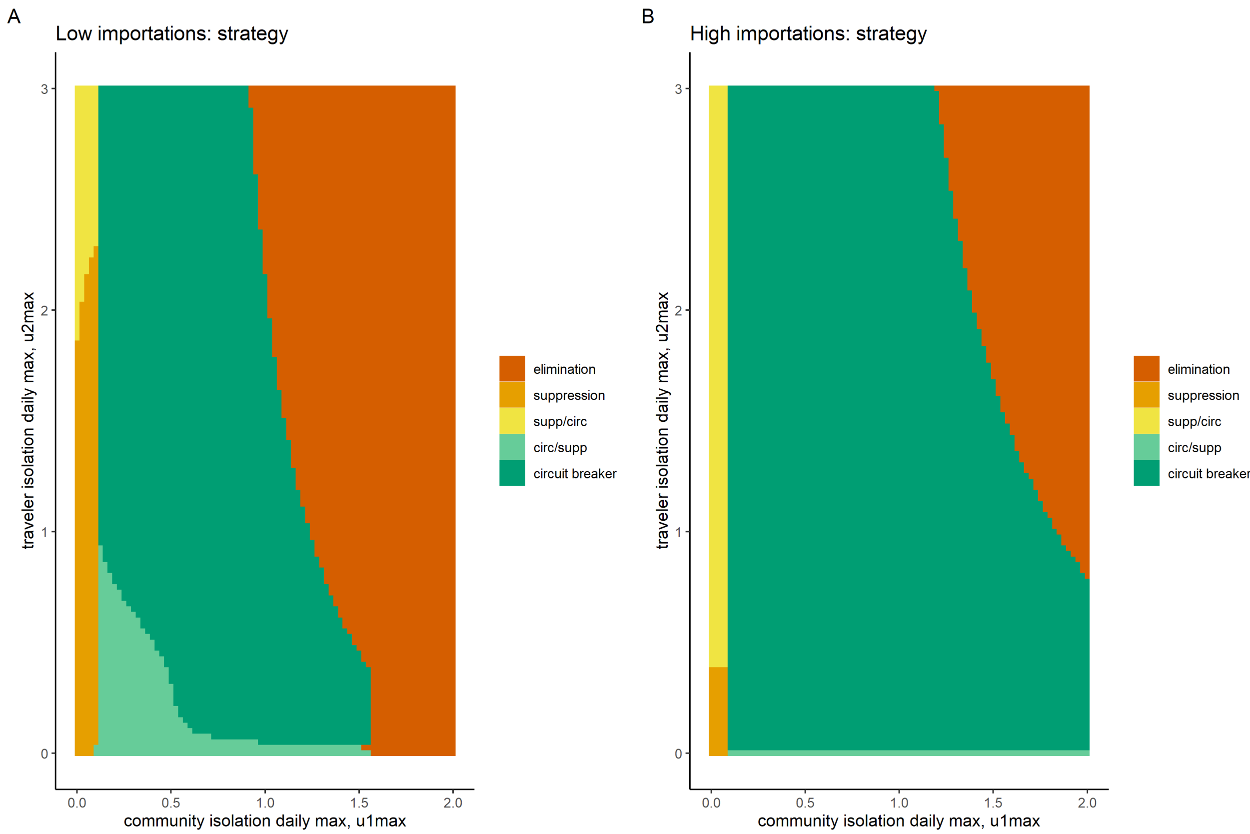

Figure 3.4: Heat map depicting the various public health strategies with respect to community and traveler isolation.

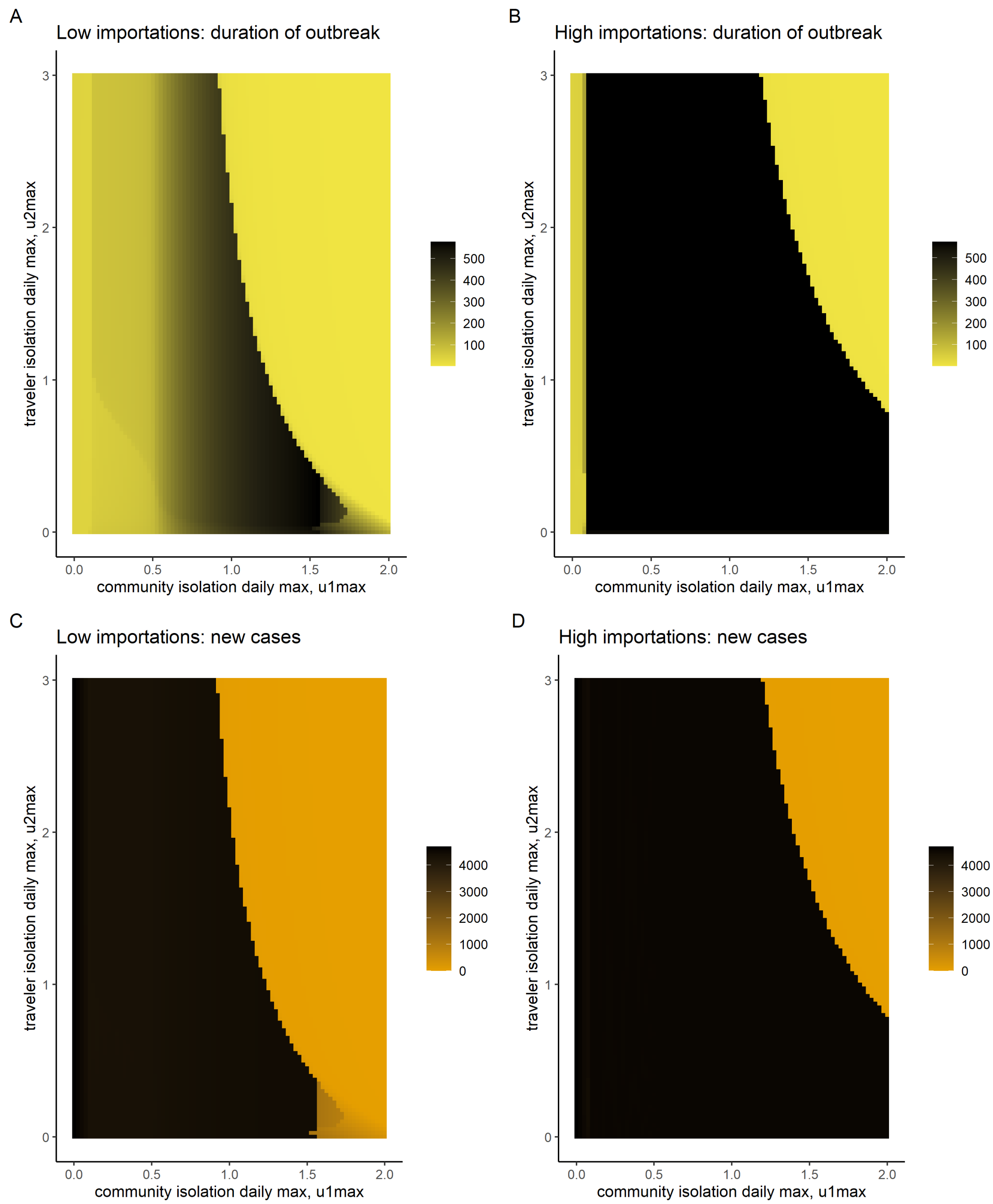

Figure 3.5: Heat map illustrating the effects of various public health strategies on the duration and incidence of new cases.

Elimination can be achieved with sufficient isolation resources and maximal efforts, resulting in a shorter outbreak duration and fewer new cases (see Figure 3.4 and 3.5).

For suppression and suppression/circuit strategies, public health measures have a minimal effect and do not delay the occurrence of infections. This causes the epidemic to grow rapidly and end swiftly.

For circuit breaker and circuit breaker/suppression strategies, the total number of cases is similar to those seen with suppression strategies. However, the number of cases is significantly higher compared to scenarios where the maximum daily isolation rates are adequate to achieve elimination.

Hansen and Day [14] found that if resources are insufficient to maintain isolation throughout the entire outbreak, any strategy that maximizes the utilization of available resources is optimal.

Our results indicate that, given sufficient resources, initiating maximum isolation efforts immediately is the best course of action. Additionally, higher values of u_{1max} and u_{2max} are necessary to achieve elimination when the importation rate, θ, is elevated. If resources are insufficient for the entire duration of the outbreak, any strategy that maximizes the use of available resources remains optimal (Theorem 2.8.1 and 2.11.1). Despite the importation of cases, the optimal strategy aligns with the findings of Hansen and Day (2011) [14].

Even small increases in $u_{1\max}$ and $u_{2\max}$ can make elimination possible, significantly reducing the duration of the outbreak and the number of cases. Additionally, while the timing of interventions does not drastically alter the outcome, delaying actions can incur substantial costs.

Chapter 4

Conclusion

We expanded on the work of Hansen and Day [14] by considering case importation from infected travelers and implementing post traveller isolation as a control measure. Our model indicates that the most effective approach for managing an outbreak depends on the availability of isolation resources. When resources are abundant, we recommend putting in the highest effort to isolate infected individuals throughout the outbreak. Conversely, when resources are scarce, it's critical to employ a strategy that optimizes all available resources for isolation. This adaptable strategy guarantees the best possible outcome under resource-constrained conditions. For optimal results, we recommend initiating isolation efforts at the highest level of effort as soon as an epidemic begins and maintaining this stringent level until all resources are fully expended.

4.1 Study Limitations

4.2 Future Work

References

[1] Modelling Disease Ecology with Mathematics (Differential Equations & Dynamical Systems) (Aims Series on Differential Equations & Dynamical Systems) ... Equations & Dynamical Systems, Volume 2, 2) - Robert Smith?: 9781601330048 - AbeBooks.

[2] Andris Abakuks. Optimal Immunisation Policies for Epidemics. *Advances in Applied Probability*, 6(3):494–511, 1974. Publisher: Applied Probability Trust.

[3] Ahmed Abdelrazec, Jacques Bélair, Chunhua Shan, and Huaiping Zhu. Modeling the spread and control of dengue with limited public health resources. *Mathematical Biosciences*, 271:136–145, January 2016.

[4] Andrei A. Agrachev and Yuri L. Sachkov. *Control Theory from the Geometric Viewpoint*, volume 87 of *Encyclopaedia of Mathematical Sciences*. Springer Berlin Heidelberg, Berlin, Heidelberg, 2004.

[5] Fred Brauer. Mathematical epidemiology: Past, present, and future. *Infectious Disease Modelling*, 2(2):113–127, May 2017.

[6] Dirk Brockmann and Dirk Helbing. The Hidden Geometry of Complex, Network-Driven Contagion Phenomena. *Science*, 342(6164):1337–1342, December 2013. Publisher: American Association for the Advancement of Science.

[7] L. Böttcher, O. Woolley-Meza, N. a. M. Araújo, H. J. Herrmann, and D. Helbing. Disease-induced resource constraints can trigger explosive epidemics. *Scientific Reports*, 5(1):16571, November 2015. Number: 1 Publisher: Nature Publishing Group.

[8] Vincenzo Capasso. *Mathematical Structures of Epidemic Systems*, volume 97 of *Lecture Notes in Biomathematics*. Springer, Berlin, Heidelberg, 1993.

[9] P Driessche and Jianhong Wu. Lecture Notes in Mathematical Epidemiology. January 2008.

[10] A. F. Filippov. On Certain Questions in the Theory of Optimal Control. *Journal of the Society for Industrial and Applied Mathematics Series A Control*, 1(1):76–84, January 1962. Publisher: Society for Industrial and Applied Mathematics.

[11] A. F. Filippov. *Differential Equations with Discontinuous Righthand Sides: Control Systems*. Springer Science & Business Media, September 1988. Google-Books-ID: KBDyZSwpQpQC.

[12] Wendell Fleming and Raymond Rishel. Existence and Continuity Properties of Optimal Controls. In Wendell Fleming and Raymond Rishel, editors, *Deterministic and Stochastic Optimal Control*, Applications of Mathematics, pages 60–79. Springer, New York, NY, 1975.

[13] John Gibson and Carroll Johnson. Singular Solutions in Problems of Optimal Control. *Department of Electrical and Computer Engineering Technical Reports*, August 1963.

[14] Elsa Hansen and Troy Day. Optimal control of epidemics with limited resources. *Journal of Mathematical Biology*, 62(3):423–451, March 2011.

[15] Ralph Howard. THE GRONWALL INEQUALITY.

[16] Andrew D Lewis. The Maximum Principle of Pontryagin in control and in optimal control.

[17] Ruofei Lin, Shanlang Lin, Na Yan, and Junpei Huang. Do prevention and control measures work? Evidence from the outbreak of COVID-19 in China. *Cities*, 118:103347, November 2021.

[18] DL LUKES. DIFFERENTIAL EQUATIONS: CLASSICAL TO CONTROLLED. *DIFFERENTIAL EQUATIONS: CLASSICAL TO CONTROLLED*, 1982.

[19] M. Gameiro, J.P. Lessard, J. Mireles James, and K. Mischaikow. Gronwall Inequality - an overview | ScienceDirect Topics.

[20] Maria M. Martignoni, Julien Arino, and Amy Hurford. Is SARS-CoV-2 elimination or mitigation best? Regional and disease characteristics determine the recommended strategy, February 2024. Pages: 2024.02.01.24302169.

[21] G McKendrick. A contribution to the mathematical theory of epidemics.

[22] Imad A. Moosa. The effectiveness of social distancing in containing Covid-19. *Applied Economics*, 52(58):6292–6305, December 2020. Publisher: Routledge _eprint: https://doi.org/10.1080/00036846.2020.1789061.

[23] R. Morton and K. H. Wickwire. On the Optimal Control of a Deterministic Epidemic. *Advances in Applied Probability*, 6(4):622–635, 1974. Publisher: Applied Probability Trust.

[24] Lev Semenovich Pontryagin. *L.S. Pontryagin selected works / Volume 4, The mathematical theory of optimal processes ; [with the collab. of] V.G. Boltyanskii, R.V. Gamkrelidze, and E.F. Mishchenko ; transl. from the Russian by K.N. Trirogoff ..*

L.S. Pontryagin selected works. Gordon and Breach, New York, facsim. ed., english ed. by l.w. neustadt edition, 1986. OCLC: 467924324.

[25] Wenjie Qin, Sanyi Tang, Changcheng Xiang, and Yali Yang. Effects of limited medical resource on a Filippov infectious disease model induced by selection pressure. *Applied Mathematics and Computation*, 283:339–354, June 2016.

[26] Timothy W Russell, Joseph T Wu, Sam Clifford, W John Edmunds, Adam J Kucharski, and Mark Jit. Effect of internationally imported cases on internal spread of COVID-19: a mathematical modelling study. *The Lancet Public Health*, 6(1):e12–e20, January 2021.

[27] Chunhua Shan, Yingfei Yi, and Huaiping Zhu. Nilpotent singularities and dynamics in an SIR type of compartmental model with hospital resources. *Journal of Differential Equations*, 260(5):4339–4365, March 2016.

[28] Chunhua Shan and Huaiping Zhu. Bifurcations and complex dynamics of an SIR model with the impact of the number of hospital beds. *Journal of Differential Equations*, 257(5):1662–1688, September 2014.

[29] Wei Shi, Yun Qiu, Pei Yu, and Xi Chen. Optimal Travel Restrictions in Epidemics. 2022.

[30] Gerald Teschl. Ordinary Differential Equations and Dynamical Systems.

[31] H. E. Tillett. Infectious Diseases of Humans: Dynamics and Control. R. M. Anderson, R. M. May, Pp. 757. Oxford University Press; 1991 (£50.00). *Epidemiology & Infection*, 108(1):211–211, February 1992.

[32] Aili Wang, Yanni Xiao, and Robert A. Cheke. Global dynamics of a piece-wise epidemic model with switching vaccination strategy. *Discrete and Continuous Dynamical Systems - B*, 19(9):2915–2940, August 2014. Publisher: Discrete and Continuous Dynamical Systems - B.

[33] Aili Wang, Yanni Xiao, and Huaiping Zhu. Dynamics of a Filippov epidemic model with limited hospital beds. *Mathematical biosciences and engineering: MBE*, 15(3):739–764, June 2018.

[34] Wendi Wang. Backward bifurcation of an epidemic model with treatment. *Mathematical Biosciences*, 201(1):58–71, May 2006.

[35] Wendi Wang and Shigui Ruan. Bifurcations in an epidemic model with constant removal rate of the infectives. *Journal of Mathematical Analysis and Applications*, 291(2):775–793, March 2004.

[36] Suzanne Lenhart Workman, John T. *Optimal Control Applied to Biological Models*. Chapman and Hall/CRC, New York, May 2007.

[37] Xu Zhang and Xianning Liu. Backward bifurcation of an epidemic model with saturated treatment function. *Journal of Mathematical Analysis and Applications*, 348(1):433–443, December 2008.

[38] Linhua Zhou and Meng Fan. Dynamics of an SIR epidemic model with limited medical resources revisited. *Nonlinear Analysis: Real World Applications*, 13:312–324, February 2012.