Model cont...

0.1 Problem 2: Travel Restrictions Only ($w_{max} = 0, u_i = 0$)

In Problem 2, we consider travel restrictions as the only control in the model; our Susceptibles Infected (SI) model now becomes

$$\frac{dS}{dt} = -\beta S(I + u_\tau I) \tag{1}$$

$$\frac{dI}{dt} = \beta S(I + u_\tau I) - \mu I \tag{2}$$

Our objective is to;

$$\min_{u_\tau} \int_{t_0}^{T} \beta S_{[u_\tau]} I_{[u_\tau]} \, dt \tag{3}$$

subject to the SI Model (1)-(2), $T = \inf\{t | I_{[u_\tau]}(t) = 0.5\}, u_\tau(t) \in [0, u_{max}]$ for all $t \in [0, T]$ and subject to the resource constraint;

$$\int_{t_0}^{T} u_\tau I_{[u_\tau]} \, dt \leq z_{max} \tag{4}$$

Theorem 0.1.1. ***(Optimal Travel Restrictions Policy)*** *There exists $\hat{t} \in [t_0, T]$ such that the optimal travel restrictions policy for problem 2 is*

$$u_\tau^*(t) = \begin{cases} u_{max}, & if \quad t \in [t_0, \hat{t}) \\ 0, & if \quad t \in [\hat{t}, T] \end{cases} \tag{5}$$

where $\int_{t_0}^{\hat{t}} u_{max} \tau dt = z_{max}$ *if* $\hat{t} < T$.

The conclusion drawn from Theorem 0.1.1 is that the most effective approach for travel restrictions is to exert maximum effort at the initial stages of the pandemic, maintaining these restrictions for as long as feasible, either until all resources are depleted, or the pandemic comes to an end. An intriguing insight from the theorem is that, in specific situations, stringent restrictions might be inefficient, leading to unnecessarily high costs [51]. This is exemplified by scenarios where the impact of travel restrictions is minimal due to the limited contribution of imported cases to local transmission [46]. Hence, it is significant that policymakers should consider the local incidence of COVID-19, the growth of local epidemics, and the volume of travel before implementing such restrictions.

Proof: The travel restrictions-only model with limited resources is described by the system of ordinary differential equations:

$$\frac{dS}{dt} = -\beta S(I + u_\tau I) \tag{6}$$

$$\frac{dI}{dt} = \beta S(I + u_\tau I) - \mu I \tag{7}$$

$$\frac{dz}{dt} = u_\tau \tau \tag{8}$$

Next, we formulate Problem 2 Sec.0.1 as a maximization problem and apply the Pontryagin's Maximum Principle (PMP); we derive the necessary conditions for the optimal control model and the associated adjoint variables. The Hamiltonian is

$$H(t) = -\lambda_0 \beta SI - \lambda_S \beta S(I + u_\tau \tau) + \lambda_I \beta S(I + u_\tau \tau) - \lambda_I \mu I + \lambda_z u_\tau \tau \tag{9}$$

$$= -\lambda_I^{'} I + (\lambda_I - \lambda_S)\beta S u_\tau \tau = -\lambda_S^{'} S - \lambda_I \mu = 0 \tag{10}$$

There are associated adjoint variables, $\lambda_S, \lambda_I, \lambda_z$, which correspond to the states S, I, and z respectively such that;

$$\lambda_S^{'} = -(\lambda_I - \lambda_0 - \lambda_S)\beta I + (\lambda_S - \lambda_I)\beta u_\tau \tau \tag{11}$$

$$\lambda_I^{'} = -(\lambda_I - \lambda_0 - \lambda_S)\beta S + \lambda_I \mu \tag{12}$$

$$\lambda_z^{'} = 0 \tag{13}$$

The transversality conditions are $(\lambda_0, \lambda_S(T), \lambda_I(T), \lambda_z) = (\lambda_0, 0, \lambda_I(T), p)$ where $p \leq 0$.

The control characterization is given as:

$$u_\tau^* = \begin{cases} u_{max}, & \text{if} \quad \lambda_I > \lambda_S \\ ?, & \text{if} \quad \lambda_I = \lambda_S \\ 0, & \text{if} \quad \lambda_I < \lambda_S \end{cases} \tag{14}$$

From (14), we show that if $\lambda_I = \lambda_S$ cannot be sustained over an interval of time but occurs only at finitely many points, then the control u_τ^* is purely bang-bang. If $\lambda_I = \lambda_S$ on some interval of time, then u_τ^* is singular. We show that u_τ^* is not singular.

Suppose by contradiction that $\lambda_I = \lambda_S$ on some interval $\mathbb{J}$. The adjoint equation (11) now becomes

$$\lambda_S^{'} = \lambda_0 \beta I \tag{15}$$

This implies that

$$\lambda_S(T) = \lambda_S(t_0) + \lambda_0\beta \int_{t_0}^{T} I \, dt \tag{16}$$

Moreover, according to equations (11) - (12), once u_τ^* reaches singularity, it remains singular throughout the entire duration, meaning $T \in \mathbb{J}$. Given that $T \in \mathbb{J}$, the expression on the right-hand side of equation (16) is obligated to fulfill the transversality condition $\lambda_S(T) = 0$. However, it is observed that $\lambda_0\beta \int_{t_0}^{T} I \, dt$ does not equal zero, leading to a contradiction. As the transversality condition cannot be met, the optimal control must follow a bang-bang pattern [22]. Therefore, the control characterization is given as

$$u_\tau^*(t) = \begin{cases} u_{max}, & t \in [t_0, \hat{t}) \\ 0, & t \in [\hat{t}, T] \end{cases} \tag{17}$$

Remarks 0.1.2. *For a function to be identically zero on some measurable interval, it is necessary that the function and all its derivatives be zero on that interval.*

Claim 0.1.3. *The control $u_\tau \equiv 0$ is not an optimal solution for Problem 2.*

Proof: If $u_\tau^* \equiv 0$, then λ_I is constrained by $\lambda_I \leq \lambda_S$. Additionally, when $u_\tau^* \equiv 0$, it signifies that none of the available resources are utilized for maximum travel restrictions, implying $\lambda_z = 0$. Consequently, this indicates that $\lambda_S \geq 0$. Furthermore, according to equation (10), if $u_\tau^* \equiv 0$, it implies $\lambda_I^{'} \equiv 0$. By rearranging equation (11) and substituting $t = T$, we obtain $\lambda_I(\beta S(T) - \mu) = \lambda_0 \beta S(T)$. This results in two possible scenarios:

1. If $\lambda_0 = 0$, it follows that $\lambda_I \equiv 0$. Since $\lambda_S(T) = 0$ as well, this implies that we have a zero adjoint vector at $t = T$. Consequently, this scenario is not feasible.

2. If $\lambda_0 = 1$, then $\lambda_I < 0$ due to the condition $S(T) < \frac{\mu}{\beta}$. Consequently, this implies, according to equation (10), that $\lambda_S^{'} > 0$. Therefore, as $\lambda_S(T) = 0$, it necessitates that $\lambda_S(t) < 0$ for all $t < T$. However, this contradicts the fact that when $u_\tau^* \equiv 0$, it implies $\lambda_S \geq 0$.

0.2 Problem 3: Mixed Policy (General Problem)

Theorem 0.2.1. ***(Optimal Mixed Policy)*** *There exists $\hat{t} \in [t_0, T]$ such that the optimal mixed policy for problem 3 has one of the following forms:*

1.

$$(u_i^*(t), u_\tau^*(t)) = \begin{cases} (u_{max}, u_{max}), & \text{if} \quad t \in [t_0, \hat{t}] \\ (0, u_\tau^*(t)), & \text{if} \quad t \in (\hat{t}, T] \end{cases} \tag{18}$$

where $\int_{t_0}^{\hat{t}} u_{max} I_{[u_{max}, u_{max}]} dt = w_{max}$ if $\hat{t} < T$ or

2.

$$(u_i^*(t), u_\tau^*(t)) = \begin{cases} (u_{max}, u_{max}), & \text{if} \quad t \in [t_0, \hat{t}] \\ (u_i^*(t), 0), & \text{if} \quad t \in (\hat{t}, T] \end{cases} \tag{19}$$

where $\int_{t_0}^{\hat{t}} u_{max} \tau dt = z_{max}$ if $\hat{t} < T$.

Remarks 0.2.2. *In Equation (18), when $t > \hat{t}$, $u_\tau^*(t)$ signifies the optimal control specific to the travel restriction-only model. In other words, for $t > \hat{t}$, the solution to Problem 2 is represented by $u_\tau^*(t)$. Similarly, in Equation (19), if $t > \hat{t}$, $u_i^*(t)$ denotes the optimal control for the isolation-only model. Hence, for $t > \hat{t}$, the solution to Problem 1 is given by $u_i^*(t)$.*

Proof: The model for the mixed policy with limited resources is described by the system of ordinary differential equations:

$$\frac{dS}{dt} = -\beta S(I + u_\tau I) \tag{20}$$
$$\frac{dI}{dt} = \beta S(I + u_\tau I) - (\mu + u_i)I \tag{21}$$
$$\frac{dw}{dt} = u_i I \tag{22}$$
$$\frac{dz}{dt} = u_\tau \tau \tag{23}$$

Next, we formulate Problem 3 Sec.0.2 as a maximization problem and apply the PMP; we derive the necessary conditions for the optimal control model and the associated adjoint variables. The Hamiltonian is

$$H(t) = -\lambda_0 \beta SI - \lambda_S \beta S(I + u_\tau \tau) + \lambda_I \beta S(I + u_\tau \tau) - \lambda_I(\mu + u_i)I + \lambda_w u_i I + \lambda_z u_\tau \tau \tag{24}$$
$$= -\lambda_I^{'} I + (\lambda_I - \lambda_S)\beta S u_\tau \tau = -\lambda_S^{'} S + (\lambda_w - \lambda_I)u_i I - \lambda_I \mu I = 0 \tag{25}$$

There are associated adjoint variables, $\lambda_S, \lambda_I, \lambda_w, \lambda_z$, which correspond to the states S, I, w, and z respectively such that;

$$\lambda_S^{'} = -(\lambda_I - \lambda_0 - \lambda_S)\beta I - (\lambda_I - \lambda_S)\beta u_\tau \tau \tag{26}$$
$$\lambda_I^{'} = -(\lambda_I - \lambda_0 - \lambda_S)\beta S - (\lambda_w - \lambda_I)u_i + \lambda_I \mu \tag{27}$$
$$\lambda_w^{'} = 0 \tag{28}$$
$$\lambda_z^{'} = 0 \tag{29}$$

The transversality conditions are $(\lambda_0, \lambda_S(T), \lambda_I(T), \lambda_z, \lambda_w) = (\lambda_0, 0, \lambda_I(T), p, q)$ where $p, q \leq 0$.

The control characterization is given as:

$$u_\tau^* = \begin{cases} u_{max}, & \text{if} \quad \lambda_I > \lambda_S \\ ?, & \text{if} \quad \lambda_I = \lambda_S \\ 0, & \text{if} \quad \lambda_I < \lambda_S \end{cases} \tag{30}$$

and

$$u_i^* = \begin{cases} u_{max}, & \text{if} \quad \lambda_w > \lambda_I \\ ?, & \text{if} \quad \lambda_w = \lambda_I \\ 0, & \text{if} \quad \lambda_w < \lambda_I \end{cases} \tag{31}$$

Claim 0.2.3. *If u_τ is singular on some interval $\mathbb{J}$, then u_i is not singular on $\mathbb{J}$ and $u_i = 0$.*

Proof: Let assume by contradiction that u_i is singular on $\mathbb{J}$. Then $\lambda_I^{'} = 0$, $\lambda_w = \lambda_I$ and consequently, $\lambda_I^{'} = \lambda_S^{'} = 0$. Now u_i and u_τ being singular implies from (30) and (31) that $\lambda_S = \lambda_I = \lambda_w$. Since $T \in \mathbb{J}$ and $\lambda_S(T) = 0$, it implies that $\lambda_S = \lambda_I = \lambda_w = 0$ on the interval. Furthermore, given that $\lambda_I^{'} = 0$, we have from (27) that $\lambda_0 = 0$ and obviously $\lambda_z = 0$. This solution leads to a zero adjoint vector on the interval $\mathbb{J}$. Since we cannot have a zero adjoint vector, it implies that u_i cannot be singular, a contradiction to our assumption.

On the other hand, if u_i is not singular, then $\lambda_S = \lambda_I \neq \lambda_w$. We have that $\lambda_I^{'} = 0$. From (25), we have $\lambda_I = \lambda_w \frac{u_i}{u_i + \mu}$. But $\lambda_w \neq 0$, if $\lambda_w = 0$, then $\lambda_I = 0$, which leads to the contradiction, hence it must be that $\lambda_w < 0$ which implies that $\lambda_w < \lambda_I$. Therefore, from (31), $u_i = 0$.

Claim 0.2.4. *If u_i is singular on $\mathbb{J}$, then $u_\tau = 0$ on $\mathbb{J}$.*

Proof: If u_i is singular on $\mathbb{J}$, then $\lambda_I^{'} = 0$.
From (25), this implies that $(\lambda_I - \lambda_S)\beta S u_\tau \tau = 0$. So either u_τ is singular ($\lambda_I = \lambda_S$) or $u_\tau = 0$. But we just proved in claim 0.2.3 that u_i and u_τ cannot be singular at the same time; therefore, it must be that $u_\tau = 0$ on $\mathbb{J}$.

Claim 0.2.5. *If there exists a $\hat{t} \geq 0$ such that $u_i(t) = 0$ is constant for all $t \in (\hat{t}, T]$, then $u_\tau(t) = u_\tau^*(t)$ for all $t \in (\hat{t}, T]$.*

Proof: Once $t > \hat{t}$ the mixed isolation–travel restrictions model becomes a travel restrictions-only model, and therefore, the optimal u_τ is the optimal control for the travel restrictions-only model with parameters $\tilde{t}_0 = \hat{t},\ \tilde{z}_{max} = z_{max} - \int_{t_0}^{\hat{t}} u_\tau \tau\ dt$.

Claim 0.2.6. *If there exists a $\hat{t} \geq 0$ such that $u_\tau(t) = 0$ is constant for all $t \in (\hat{t}, T]$, then $u_i(t) = u_i^*(t)$ for all $t \in (\hat{t}, T]$.*

Proof: Once $t > \hat{t}$ the mixed isolation–travel restrictions model becomes an isolation-only model, and therefore, the optimal u_i is the optimal control for the isolation-only model with parameters $\tilde{t}_0 = \hat{t},\ \tilde{w}_{max} = w_{max} - \int_{t_0}^{\hat{t}} u_i I\ dt$.

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