Mathematical Model

This study's chief goal is to create a mathematical model capable of simulating the propagation of COVID-19 and assessing the most effective management of control measures across various scenarios. The model will account for the virus transmission dynamics and how control measures influence its spread.

We incorporate the widely used epidemiological compartmental model, Susceptibles Infected (SI) model. It will consider the movement of individuals between the compartments, encompassing those who are susceptible and infected.

We formulate the compartmental model for the period of the control phase (optimal control model).

0.1 Disease Dynamics: SI Model

During the control phase, our compartmental model is the SI model given below. Here, the SI model captures the import cases. Within the system, each of the two compartments represents a specific population group, and as time progresses, individuals transition through each category on the path toward recovery.

$$\frac{dS}{dt} = -\beta S(I + u_\tau \tau) \tag{1}$$
$$\frac{dI}{dt} = \beta S(I + u_\tau \tau) - (\mu + u_i)I \tag{2}$$

with $S(t_0) > 0, I(t_0) \geq 0$, where S is the number of susceptibles, I is the number of infected hosts, β is the transmission rate, μ is the per capita loss rate of infected individuals through both mortality and recovery, τ is the baseline number of infected non-resident travellers, u_τ is the rate of travel restrictions and u_i is the rate of isolation.

The set of admissible controls is given by
$U_{ad} = \{u = (u_i, u_\tau)$ such that (u_i, u_τ) measurable; $(u_i(t), u_\tau(t)) \in [0, u_{max}] \times [0, u_{max}]\}$; a measurable control set and the controls are bounded and Lebesque measurable.
Let $x = (x_1, x_2)$ represent the states.

Theorem 0.1.1. ***(The Gronwall Inequality)*** *Let $\mathbb{X}$ be a Banach space and $U \subset \mathbb{X}$ an open set in $\mathbb{X}$. Let $f, g : [t_0, T] \times U \to \mathbb{X}$ be continuous functions and let $y, z : [t_0, T] \to U$ satisfy the initial value problems*

$$S'(t) = f(t, S(t), u); \quad S(t_0) = S_0 \tag{3}$$
$$I'(t) = g(t, I(t), u); \quad I(t_0) = I_0 \tag{4}$$

Assume there is a constant $C \geq 0$ such that

$$\|g(t, x_2, u) - g(t, x_1, u)\| \leq C\|x_2 - x_1\| \tag{5}$$

and a continuous function $\psi : [t_0, T] \to [0, \infty)$ *so that*

$$\|f(t, x, u) - g(t, x, u)\| \leq \psi(t) \tag{6}$$

Then for $t \in [t_0, T]$

$$\|S(t) - I(t)\| \leq e^{C|t-t_0|}\|S_0 - I_0\| + e^{C|t-t_0|}\int_{t_0}^{T} e^{-C|s-t_0|}\psi(s)ds \tag{7}$$

Proof: We will utilize the inequality $\frac{d}{dt}\|x(t)\| \leq \|x^{'}(t)\|$, a relationship that is easily demonstrated to be valid for C^1 functions $x : [t_0, T] \to \mathbb{X}$. From Equations (5) and (6)

$$\begin{aligned} \frac{d}{dt}\|S(t) - I(t)\| &\leq \|S^{'}(t) - I^{'}(t)\| \\ &= \|f(t, S(t)) - g(t, I(t))\| \\ &\leq \|f(t, S(t)) - g(t, S(t))\| + \|g(t, S(t)) - g(t, I(t))\| \\ &\leq \psi(t) + C\|S(t) - I(t)\| \end{aligned}$$

This implies that

$$\frac{d}{dt}\|S(t) - I(t)\| - C\|S(t) - I(t)\| \leq \psi(t) \tag{8}$$

Multiplying Eq. (8) by the integrating factor e^{-Ct}, we get

$$\frac{d}{dt}(e^{-Ct}\|S(t) - I(t)\|) \leq e^{-Ct}\psi(t) \tag{9}$$

Integrating Eq. (9) from t_0 to T yields,

$$e^{-Ct}\|S(t) - I(t)\| - e^{-Ct_0}\|S_0 - I_0\| \leq \int_{t_0}^{T} e^{-Cs}\psi(s)ds \tag{10}$$

which is equivalent to Eq. (7).

0.2 Pontryagin's Maximum Principle

The Pontryagin Maximum Principle is a fundamental mathematical principle in the field of optimal control theory. It provides necessary conditions that an optimal control and corresponding state trajectory must satisfy for a wide class of optimal control problems. The principle is named after the Russian mathematician Lev Pontryagin, who played a key role in its development [37].

In its classical form, the Pontryagin Maximum Principle is applied to problems where the objective is to maximize a certain criterion, typically expressed as the integral of a given performance index over a specified time interval. The principle states that, under certain regularity conditions, an optimal control strategy and the corresponding state trajectory must satisfy a set of differential equations known as the canonical equations.

The canonical equations involve the system dynamics, the costate variables (Lagrange multipliers), and the partial derivatives of the Hamiltonian, which is a function combining the system dynamics and the cost function. The optimal control is determined by maximizing the Hamiltonian over the set of feasible controls.

The Pontryagin Maximum Principle is widely used to analyze and solve optimization problems, where the goal is to find the best control strategy for a dynamic system.

Theorem 0.2.1. *(Pontryagin's Maximum Principle (PMP)) If $u^*(t)$ and $x^*(t)$ are the optimal solution of the control problem, then there exist piecewise differentiable adjoint variables $\lambda(t)$ such that*

$$H(t, x^*(t), u(t), \lambda(t)) \leq H(t, x^*(t), u^*(t), \lambda(t)) \tag{11}$$

for all controls u at each time t, where H is the Hamiltonian and

$$\lambda^{'}(t) = \frac{\partial H(t, x^*(t), u^*(t)}{\partial x} \tag{12}$$

$$\lambda(T) = 0 \tag{13}$$

are the costate and transversality conditions, respectively.

Theorem 0.2.2. *Suppose that $f(t, x, u)$ and $g(t, x, u)$ are continuously differentiable functions in their three arguments and concave in u. Suppose u^* is an optimal control with associated state x^*, and λ a piecewise differentiable function with $\lambda(t) \geq 0 \; \forall \; t$. Suppose for all $t_0 \leq t \leq T$*

$$0 = H_u(t, x^*(t), u^*(t), \lambda(t)). \tag{14}$$

Then for all controls u and each $t_0 \leq t \leq T$, we have

$$H(t, x^*(t), u(t), \lambda(t)) \leq H(t, x^*(t), u^*(t), \lambda(t)) \tag{15}$$

Proof: Fix a control u and $t_0 \leq t \leq T$. Then,

$$\begin{aligned} H(t, x^*(t), u^*(t), \lambda(t)) - H(t, x^*(t), u^*(t), \lambda(t)) = {} & [f(t, x^*(t), u^*(t)) + \lambda(t) g(t, x^*(t), u^*(t))] \\ & - [f(t, x^*(t), u(t)) + \lambda(t) g(t, x^*(t), u(t))] \end{aligned}$$

$$= [f(t, x^*(t), u^*(t)) - f(t, x^*(t), u(t))] + \lambda(t)[g(t, x^*(t), u^*(t)) - g(t, x^*(t), u(t))] \tag{16}$$

$$\geq (u^*(t) - u(t)) f_u(t, x^*(t), u^*(t)) + \lambda(t)(u^*(t) - u(t)) g_u(t, x^*(t), u^*(t)) \tag{17}$$

$$= (u^*(t) - u(t)) H_u(t, x^*(t), u^*(t), \lambda(t)) = 0 \tag{18}$$

We get Eq. (17) by applying the tangent line property to f and g and because $\lambda(t) \geq 0$.

The same essential conditions are derived through similar reasoning when the problem involves minimizing rather than maximizing. In a minimization problem, we are minimizing the Hamiltonian pointwise, and the inequality in PMP is reversed [53]. Indeed, for a minimization problem with f and g being convex in u, we can derive

$$H(t, x^*(t), u(t), \lambda(t)) \geq H(t, x^*(t), u^*(t), \lambda(t)) \tag{19}$$

by the same argument as in Theorem 0.2.2

0.3 General Problem (Optimal Control Problem)

Let $z(t)$ denote the total number of restricted non-resident infected travellers at every time t, and $w(t)$ denote the total number of isolated individuals at every time t.
$S_{[u_i,u_\tau]}, I_{[u_i,u_\tau]}, z_{[u_i,u_\tau]}, w_{[u_i,u_\tau]}$ denote that the actual number of the state variables (S, I, z, w) depends on the choice of the controls u_i and u_τ.

The aim is to reduce the overall cost of infections over a specified duration while adhering to the epidemic dynamics delineated by the system of differential equations. The variables we seek to optimize are denoted as u_τ, u_i, the control variables.
The current optimal control challenge within our SI model is to determine the values of u_τ and u_i that minimize the cumulative infections.

Fixing $w_{max} \geq 0$ and $z_{max} \geq 0$, and following the work of [21], our optimal control model for the general problem is formulated as:

$$\min_{u_\tau, u_i} \int_{t_0}^{T} \beta S_{[u_i,u_\tau]} I_{[u_i,u_\tau]} \; dt \tag{20}$$

subject to the SI Model, $T = \inf\{t | I_{[u_i,u_\tau]}(t) = 0.5\}, (u_i(t), u_\tau(t)) \in [0, u_{m_i}] \times [0, u_{m_\tau}]$ for all $t \in [0, T]$ and subject to the resource constraints;

$$\int_{t_0}^{T} u_i I_{[u_i,u_\tau]} \; dt \leq w_{max} \tag{21}$$

and

$$\int_{t_0}^{T} u_\tau \tau_{[u_i,u_\tau]} \; dt \leq z_{max} \tag{22}$$

$u_{m_i} \in (0, \infty)$ and $u_{m_i} \in (0, \infty)$ but for simply notation, we assume that $u_{m_i} = u_{m_\tau} = u_{max}$.

0.4 Existence of Optimal Controls

The PMP only provides necessary conditions for optimality, and the fulfilment of necessary conditions alone does not guarantee optimality. Application of necessary conditions for optimality to identify a set of candidates to the optimal solutions only makes sense if the optimal solution exists. Tonelli (1915) introduced the first theorem of the existence of a solution for the calculus of variations problem. For an optimal control to exist, we want to have compactness of feasible solution sets. We provide a result stating the existence of at least one optimal solution to the Optimal Control Problem under some appropriate compactness and convexity assumptions. Precisely, we follow the standard Filippov's approach. Filippov's existence theorem is a result of the theory of differential inclusions, which are generalizations of ordinary differential equations that allow for multiple possible trajectories at a single point in the state space. Filippov's existence theorem addresses the existence of solutions for differential inclusions [14], [15].

Theorem 0.4.1. ***(Filippov's existence theorem)*** *Consider an optimal control problem defined by a differential inclusion* $x' \in F(t, x, u)$*, where* $F : [t_0, T] \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ *is a set-valued mapping representing the dynamics,* t *is time,* x *is the state variable and* u *is the control input. Assume that the set-valued map* F *is upper semicontinuous in* x *and continuous in* u *for each fixed* t*. If the optimal control problem has nonempty, compact, and convex solution sets for all* t*, then an optimal control exists for almost every initial point in* $\mathbb{R}^n$*.*

To establish the existence of optimal control, we rely on findings presented in [16] and [32]. Initially, we address the boundedness of the state variables in the system. By summing up all the equations in the model (1)-(2), we obtain $N(t) \leq N(t_0)$. Considering the characteristics of the infectious disease model, it is evident that $0 \leq S(t), I(t) \leq N(t_0)$. In other words, the state variables of the system are bounded. The assurance of the existence of an optimal control solution is ensured by satisfying the following conditions.

(a) The set of control variables and corresponding state variables is not empty.

(b) The admissible control set U_{ad} is compact and bounded.

c) The vector function $f(t, x, u)$ formed by the right side of the system state equation is continuous.

By examining the definition of the control set, it becomes apparent that for every permissible control function, the system state equation's solution remains continuous and bounded. The function on the model's right side, as described in (1)-(2), adheres to the Lipschitz condition concerning the state variables, ensuring the existence of the model's solution. Therefore, condition (a) and (b) are satisfied. The expression on the model's right-hand side in (1)-(2) is evidently continuous, thereby fulfilling condition (c).

0.5 Bang-Bang Optimal Controls

We shift our focus to a particular scenario frequently encountered in practical applications. More precisely, we concentrate on scenarios characterized by linearity in the controls. In these cases, optimal solutions often incorporate discontinuities in the control variables. Notice that equations (1), (2) and the integrand functions in (21) and (22) are both linear functions of the controls u_i, u_τ. Thus, the Hamiltonian is also a linear function of the controls; hence, the optimality condition contains no information on the controls. The consequence of this problem is that we are not able to find a characterization of the optimal controls. We define a switching function $\psi(t)$, and then our controls are characterized by the control input switching between two extreme values, typically denoted as "on" (maximum) and "off" (minimum). This binary or on-off control is often used in systems where continuous control is not practical or necessary. The control law causes the system's behaviour to exhibit switching dynamics.

$$u_i^*(t) = \begin{cases} u_{max}, & \text{if} \quad \psi_1(t) > 0 \\ ?, & \text{if} \quad \psi_1(t) = 0 \\ 0, & \text{if} \quad \psi_1(t) < 0 \end{cases} \tag{23}$$

If $\psi_1 = 0$ cannot be sustained over an interval of time but occurs only at the finite many points, then we refer to the control as bang-bang control.

If $\psi_1(t) \equiv 0$ on some interval of time, we say the control u_i^* is singular on that interval.

0.6 Problem 1: Isolation Only ($z_{max=0,u_\tau=0}$)

Considering isolation as the only control in the model, our SI model now becomes;

$$\frac{dS}{dt} = -\beta SI \tag{24}$$
$$\frac{dI}{dt} = \beta SI - (\mu + u_i)I \tag{25}$$

Our objective is to;

$$\min_{u_i} \int_{t_0}^{T} \beta S_{[u_i]} I_{[u_i]} \; dt \tag{26}$$

subject to the SI Model, $T = \inf\{t | I_{[u_i]}(t) = 0.5\}, u_i(t) \in [0, u_{max}]$ for all $t \in [0, T]$ and subject to the resource constraint;

$$\int_{t_0}^{T} u_i I_{[u_i]} \; dt \leq w_{max} \tag{27}$$

From Eq. (24); we have

$$ds = \beta SI \; dt \tag{28}$$

integrating both sides, we get

$$\int_{t_0}^{T} dS = -\int_{t_0}^{T} \beta SI \; dt \tag{29}$$
$$S(T) - S(t_0) = -\int_{t_0}^{T} \beta SI \; dt \tag{30}$$
$$S_0 - S(T) = \int_{t_0}^{T} \beta SI \; dt \tag{31}$$

on the other hand, rearranging Eq. (24), we get

$$\frac{1}{S}\,dS = -\beta I\,dt \tag{32}$$

Taking integral on both sides, we have

$$\int_{t_0}^{T} \frac{1}{S}\,dS = -\beta \int_{t_0}^{T} I\,dt \tag{33}$$

$$-\frac{1}{\beta}\ln\left(\frac{S(T)}{S_0}\right) = \int_{t_0}^{T} I\,dt \tag{34}$$

We observe from equations (31) and (34) that the terms on the right-hand side are both minimized by maximizing $S(T)$ since S_0 is a fixed quantity.

Theorem 0.6.1. ***(Optimal Isolation Policy)*** *If $w_{u_{max}}(T) \leq w_{max}$, then the optimal isolation policy for Problem 1 is $u_i^* = u_{max}$. If $w_{u_{max}}(T) > w_{max}$, then the optimal policy u_i^* is any control u_i such that $w_{u_{max}}(T) = w_{max}$.*

Proof: Following equations (24), (25) and (27), the isolation model with limited resources is described by the system of ordinary differential equations:

$$\frac{dS}{dt} = -\beta S I \tag{35}$$

$$\frac{dI}{dt} = \beta S I - (\mu + u_i) I \tag{36}$$

$$\frac{dw}{dt} = u_i I \tag{37}$$

Next, we formulate Problem 1 Sec.0.6 as a maximization problem and apply the PMP; we derive the necessary conditions for the optimal control model and the associated adjoint variables. The Hamiltonian is

$$H(t) = -\lambda_0\beta SI - \lambda_S\beta SI + \lambda_I\beta SI - \lambda_I(\mu + u_i)I + \lambda_w u_i I \tag{38}$$

$$= -\lambda_I^{'} I = \lambda_S^{'} S - \lambda_I\mu + (\lambda_w - \lambda_I)u_i I = 0 \tag{39}$$

There are associated adjoint variables, $\lambda_S, \lambda_I, \lambda_w$, which correspond to the states S, I, and w respectively such that;

$$\lambda_S^{'} = -\frac{\partial H}{\partial S} = -(\lambda_I - \lambda_0 -_S)\beta I \tag{40}$$

$$\lambda_I^{'} = -\frac{\partial H}{\partial I} = -(\lambda_I - \lambda_0 -_S)\beta S - (_w - \lambda_I)u_i + \lambda_I\mu \tag{41}$$

$$\lambda_w^{'} = -\frac{\partial H}{\partial w} = 0 \tag{42}$$

and the optimality condition is obtained as:

$$\frac{\partial H}{\partial u_i} = -\lambda_I I = 0 \text{ at } u_i^* \tag{43}$$

with the boundary conditions $(\lambda_0, \lambda_S(T), \lambda_I(T), \lambda_w) = (\lambda_0, 0, \lambda_I(T), q)$ known as the transversality conditions, where $q \leq 0$. Equations (40)-(42) form the necessary conditions. The adjoint variables are the marginal variations in the objective function with respect to the respective state variables at every time t, and this helps to determine what sign (positive or negative) to expect from an adjoint variable.

We now summarize the control characterization as:

$$u_i^* = \begin{cases} u_{max}, & \text{if} \quad \lambda_w > \lambda_I \\ ?, & \text{if} \quad \lambda_w = \lambda_I \\ 0, & \text{if} \quad \lambda_w < \lambda_I \end{cases} \tag{44}$$

From Eq.(39), we observe that $\lambda_I^{'} = 0$ and therefore the optimal control is either $u_i^* \equiv 0, u_i^* \equiv u_{max}$ or u_i^* is singular.

We observe that without the constraint Eq. (27), Problem 1 (0.6) becomes an unconstrained optimal control problem, and its solution is $u_i^* \equiv u_{max}$.

Claim 0.6.2. *The Optimal control for Problem 1 with* $w_{max} = \infty$ *is* $u_i^* \equiv u_{max}$.

Proof: substituting equation (40) into (41) with $\lambda_w = 0$ and making $\lambda_S^{'}$ the subject gives,

$$\lambda_S^{'} = \lambda_I^{'} \frac{I}{S} - \lambda_I(\mu + u_i)\frac{I}{S} \tag{45}$$

We show that the optimal control is purely bang-bang (no singular components).
Since by Eq.(39), λ_I is a constant, if u_i is singular then it must be singular on the entire interval $[0, T]$. This implies, from Eq.(45), that λ_S is constant and since $\lambda_S(T) = 0$, it must be that $\lambda_S \equiv 0$. Equation (40) then gives $\lambda_0 = 0$. this contradicts the assumption that $(\lambda_0, \lambda_I(t), \lambda_S(t))$ must be nonzero for all $t \in [0, T]$. Therefore, u_i^* cannot be singular. The optimal control will be determined once the sign of λ_I is determined. To determine the sign of λ_I, we use the transversality condition $\lambda_S(T) = 0$. Since λ_I is a constant, Eq(41) gives

$$\lambda_I = \frac{(\lambda_0 + \lambda_S)\beta S}{\beta S - u_i - \mu} = \frac{(\lambda_0)\beta S(T)}{\beta S(T) - u_i(T) - \mu} \tag{46}$$

This implies that sign (λ_I) = sign $\left(S(T) - \frac{u_i(T)+\mu}{\beta}\right)$. From Eq.(1)-(2), λ_I is negative if and only if $I^{'}(T) < 0$. Since T is the smallest time that $I = 0.5$ and $I(0) > 0.5$, it must be that $I^{'}(T)$ is negative. Therefore, $u_i^* \equiv u_{max}$.

Our second observation is that the total number of isolated individuals can be calculated as;

From Eq(35), we can write $-S^{'} = \beta SI$ and $I = -\frac{S^{'}}{\beta S}$. Substituting these two expressions into Eq.(36) gives;

$$I^{'} = -S^{'} + \frac{\mu}{\beta}\frac{S^{'}}{S} - u_i I. \tag{47}$$

Rearranging Eq.(47) and integrating from t_0 to T, we obtain:

$$\int_{t_0}^{T} u_i I \, dt = S_0 - S(T) + I_0 - I(T) + \frac{\mu}{\beta}\ln\left(\frac{S(T)}{S_0}\right) \tag{48}$$

Equation (48) shows that the constraint value $w_{[u_i]}(T) = \int_{t_0}^{T} u_i I_{[u_i]} \, dt$ depends only on $S_{[u_i]}(T)$.

Again, the cost function can be rewritten as

$$\int_{t_0}^{T} \beta I_{[u_i]} S_{[u_i]} \, dt = S_0 - S_{[u_i]}(T) \tag{49}$$

and therefore minimizing the cost function is equivalent to maximizing $S_{[u_i]}(T)$.

Now to determine the optimal control when $w_{[u_{max}]}(T) > w_{max}$, we rewrite Eq(48) as

$$\frac{\mu}{\beta}\ln(S_{[u_i]}(T)) - S_{[u_i]}(T) = I_{[u_i]}(T) - I_0 - S_0 + \frac{\mu}{\beta}\ln(S_0) + w_{[u_i]}(T). \tag{50}$$

There are two possible scenarios:

1. If $w_{[u_{max}]}(T) > w_{max}$ and $S_{[u_{max}]}(T) < \frac{\mu}{\beta}$, then as long as $w_{[u_i]}(T) \leq w_{max} < w_{[u_{max}]}(T)$, the function $S_{[u_i]}(T)$ shows an upward trend concerning $w_{[u_i]}(T)$. This implies that any control strategy u_i^* utilizing the entire available resource set will be optimal.

2. If $w_{[u_{max}]}(T) > w_{max}$ and $S_{[u_{max}]}(T) > \frac{\mu}{\beta}$, considering the convex downward function $f(S) = \frac{\mu}{\beta}\ln(S) - S$ with a maximum at $S = \frac{\mu}{\beta}$, for any u_i with $w_{[u_i]}(T) < w_{max}$, it implies that $S_{[u_i]}(T) < \frac{\mu}{\beta}$. Consequently, $S_{[u_i]}(T)$ increases with $w_{[u_i]}(T)$ for any $w_{[u_i]}(T) < w_{max}$. Thus, u_i^* denotes any control strategy utilizing all available resources. This concludes the proof of Theorem 0.6.1.

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