Numerical Method and Simulations

One challenging aspect of this work is finding the best numerical method to simulate our optimal control problem (**??**). Multiple attempts have been made using the forward-backward sweep method, yet the results have not been satisfactory. Hence, a hard-coded result is obtained for the isolation-only model. Since our model and that of Hansen and Day [1] are the same for the isolation-only model (**??**), it was expected to obtain the same results using the same parameter values defined in [1], but we observe not the exact solutions. No results have yet been obtained for the travel restriction-only model and the mixed-policy model since it is not clear which numerical approach (method) best suits our optimal control problem.

Numerical results of the Isolation-only model

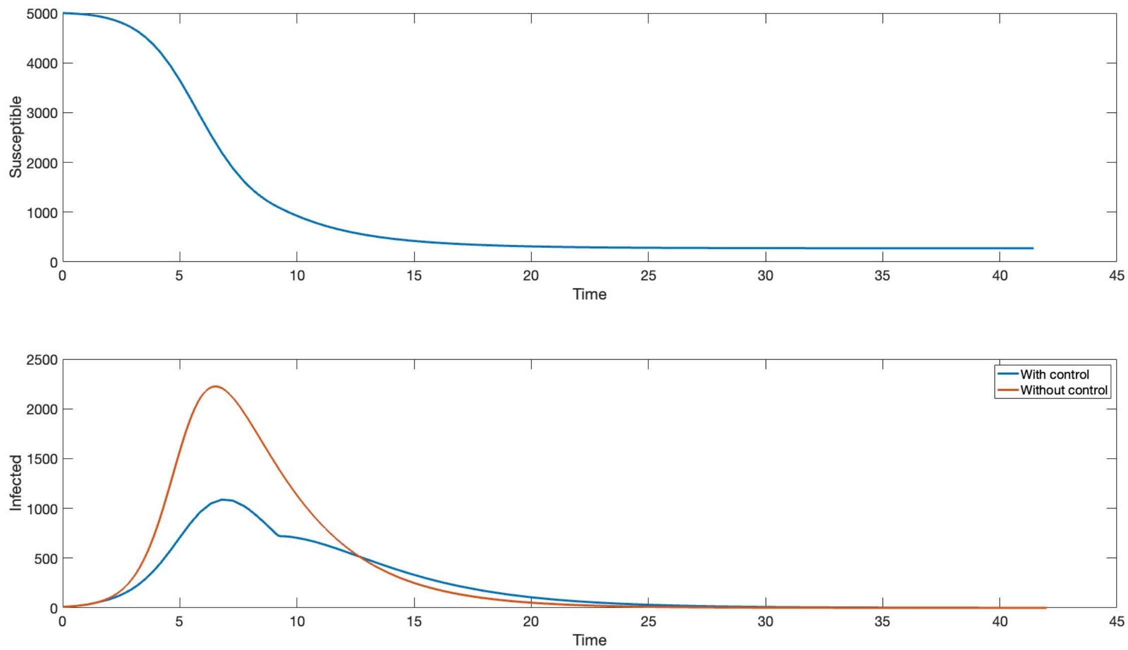


Figure 1: Dynamics of Isolation-only model

For the state's trajectory of the isolation-only model (**??**), we observe similar susceptible and infection dynamics as compared to Hansen and Day [1] (refer to the solid plot in fig 2 of [1]). We observe that by having the isolation control in place, we can minimize infections by 50%.

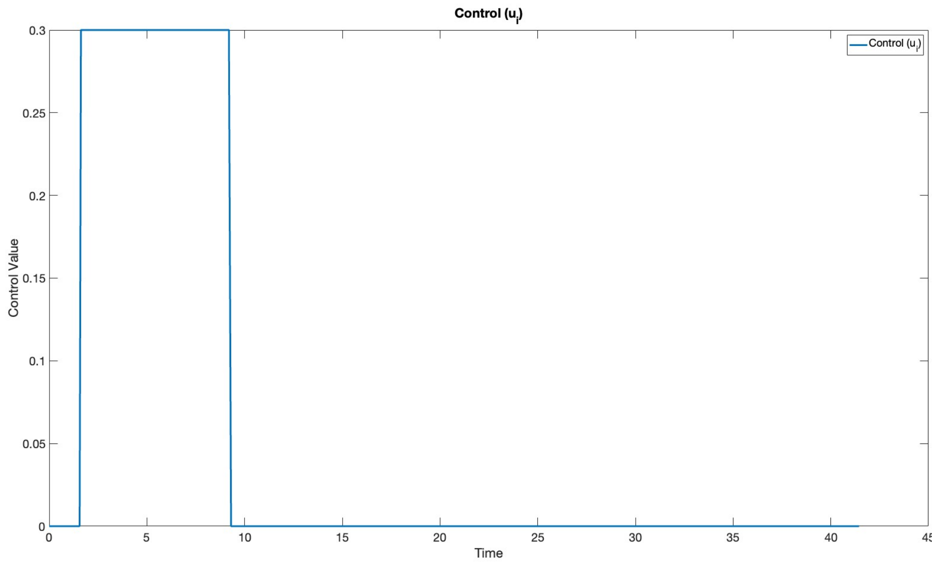


Figure 2: Control strategy for the isolation-only model

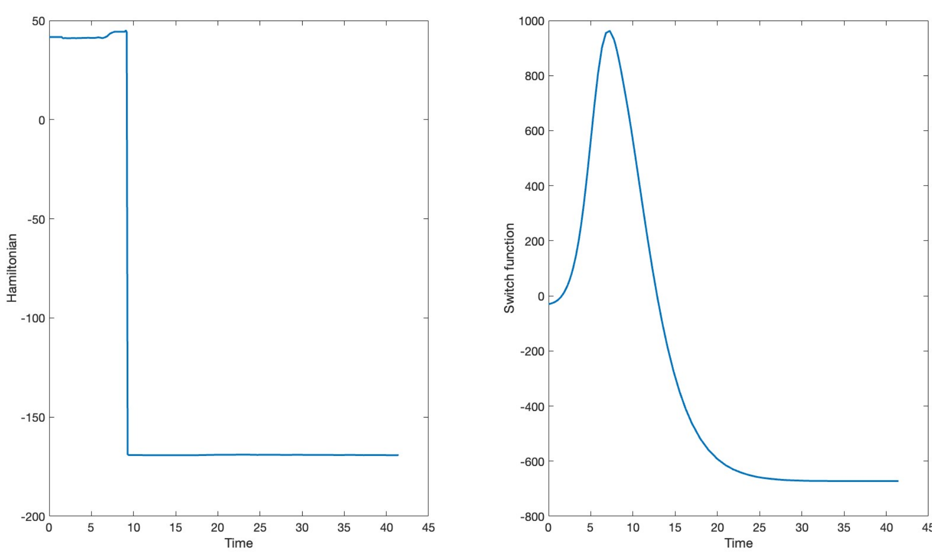


Figure 3: A plot of the Hamiltonian $H(t)$ and the switch function $\psi_i(t)$ against time

Although we observe a bang-bang control which switches between the values $u_{min} = 0$ and $u_{max} = 0.3$ within the initial time period, we observed a different control strategy as compared to that of Hansen and Day [1].

Numerical results of the Travel Restriction-Only model

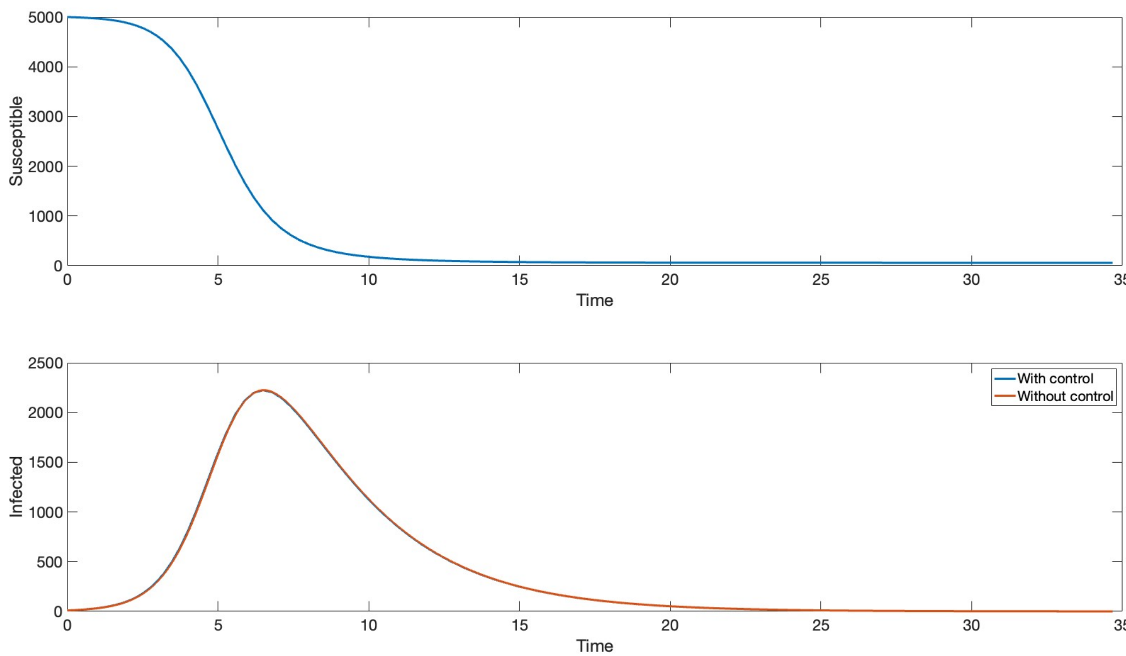


Figure 4: Dynamics of Travel Restrictions-only model

With the travel restrictions-only model, even though we have control in place almost throughout the time period, we observe no differences between the state of infection with no control and the state with control of the infection.

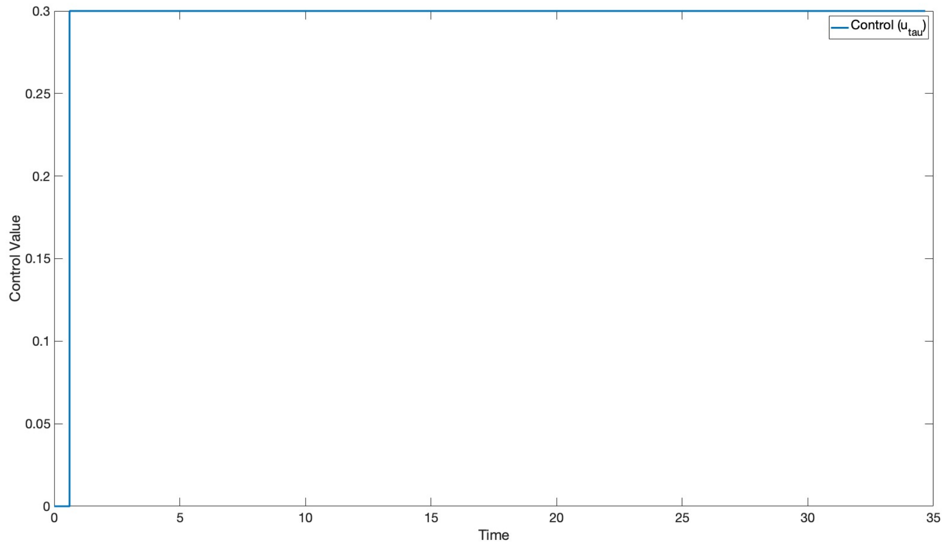


Figure 5: Control strategy for the Travel Restriction-Only model

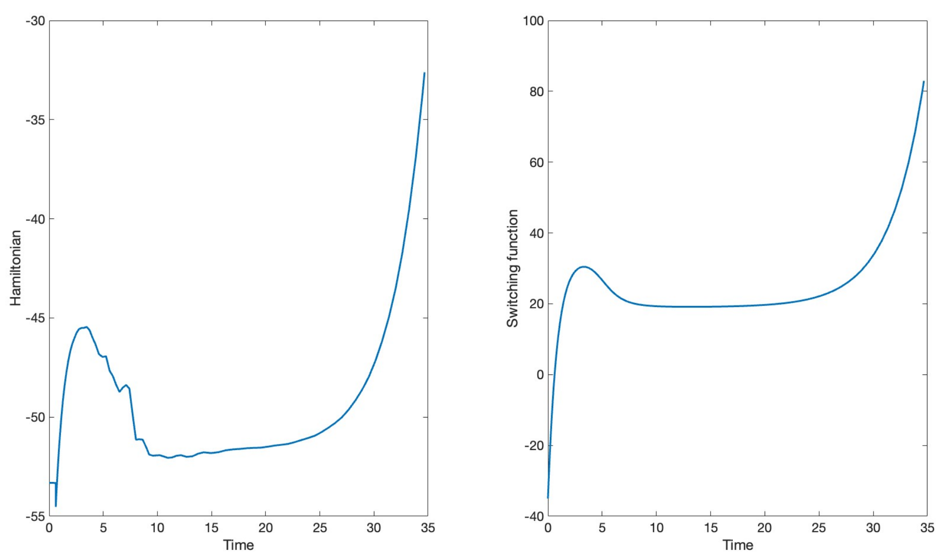


Figure 6: A plot of the Hamiltonian $H(t)$ and the switch function $\psi_{tau}(t)$ against time

0.1 Modified Version of Our model

Similar to our already existing model discussed above, we are also looking into a different way to model importation resulting from infected non-resident travellers. The mixed policy model with limited resources is described by the system of ordinary differential equations:

$$\frac{dS_1}{dt} = -\beta S_1(I_1 + I_2) \tag{1}$$
$$\frac{dI_1}{dt} = \beta S_1(I_1 + I_2) - (\mu + u_i)I_1 \tag{2}$$
$$\frac{dI_2}{dt} = \theta - u_\tau I_2 - \gamma I_2 \tag{3}$$
$$\tag{4}$$

Our objective is to

$$\min J = \int_{t_0}^{T} \beta S_1(I_1 + I_2)\ dt \tag{5}$$

subject to equations (1)-(3), $T = \inf\{t|I_{[u_i,u_\tau]}(t) = 0.5\}$, $(u_i(t), u_\tau(t)) \in [0, u_{max}] \times [0, u_{max}]$ for all $t \in [0, T]$,
and subject to the resource constraints;

$$\int_{t_0}^{T} u_i I_1\ dt \leq w_{max} \tag{6}$$

and

$$\int_{t_0}^{T} u_\tau I_2\ dt \leq z_{max} \tag{7}$$

Here, S_1 is the susceptible class (community residents), I_1 denotes the community infection class (Infected residents) and I_2 is the non-resident infected class (Infected travellers), β is the transmission rate, θ is the recruitment rate of infected non-resident travellers, μ is the recovery/ removal rate of infected residents, γ is the recovery/removal rate of infected non-resident travellers, and u_i, u_τ are the controls corresponding to the isolation of infected residents and travel restriction of infected non-resident travellers respectively. We assume infected non-resident travellers have temporal stay and are quickly removed from the community; therefore, $\gamma > \mu$.

Now applying the Pontryagin's Maximum Principle (PMP), our system of Ordinary Differential Equations (ODEs) is given as;

$$\frac{dS_1}{dt} = -\beta S_1(I_1 + I_2) \tag{8}$$
$$\frac{dI_1}{dt} = \beta S_1(I_1 + I_2) - (\mu + u_i)I_1 \tag{9}$$
$$\frac{dI_2}{dt} = \theta - u_\tau I_2 - \gamma I_2 \tag{10}$$
$$\frac{dw}{dt} = u_i I_1 \tag{11}$$
$$\frac{dz}{dt} = u_\tau I_2 \tag{12}$$

Next, we derive the necessary conditions for the optimal control model and the associated adjoint variables. The Hamiltonian is

$$H(t) = -\lambda_0 \beta S_1(I_1 + I_2) - \lambda_{S_1}\frac{dS_1}{dt} + \lambda_{I_1}\frac{dI_1}{dt} + \lambda_{I_2}\frac{dI_2}{dt} + \lambda_w \frac{dw}{dt} + \lambda_z \frac{dz}{dt} \tag{13}$$

There are associated adjoint variables, $\lambda_{S_1}, \lambda_{I_1}, \lambda_{I_2}, \lambda_w, \lambda_z$, which correspond to the states

S_1, I_1, I_2, w, and z respectively such that;

$$\dot{\lambda}_{S_1} = -(\lambda_{I_1} - \lambda_0 - \lambda_{S_1})\beta(I_1 + I_2) \tag{14}$$

$$\dot{\lambda}_{I_1} = -(\lambda_{I_1} - \lambda_0 - \lambda_{S_1})\beta S_1 - (\lambda_z - \lambda_{I_1})u_i + \lambda_{I_1}\mu \tag{15}$$

$$\dot{\lambda}_{I_2} = -(\lambda_{I_1} - \lambda_0 - \lambda_{S_1})\beta S_1 - (\lambda_z - \lambda_{I_2})u_\tau + \lambda_{I_2}\gamma \tag{16}$$

$$\dot{\lambda}_w = 0 \tag{17}$$

$$\dot{\lambda}_z = 0 \tag{18}$$

The transversality conditions are $(\lambda_0, \lambda_{S_1}(T), \lambda_{I_1}(T), \lambda_{I_2}(T), \lambda_z, \lambda_w) = (\lambda_0, 0, \lambda_{I_1}(T), \lambda_{I_2}(T), p, q)$ where $p, q \leq 0$.

and the optimality conditions:

$$\frac{\partial H}{\partial u_i} = \psi_i(t) = (\lambda_w - \lambda_{I_1})I_1 \text{ at } u_i^* \tag{19}$$

$$\frac{\partial H}{\partial u_\tau} = \psi_\tau(t) = (\lambda_z - \lambda_{I_2})I_2 \text{ at } u_\tau^* \tag{20}$$

The controls characterization are given as:

$$u_i^* = \begin{cases} u_{max}, & \text{if } \lambda_w > \lambda_{I_1} \\ ?, & \text{if } \lambda_w = \lambda_{I_1} \\ 0, & \text{if } \lambda_w < \lambda_{I_1} \end{cases} \tag{21}$$

and

$$u_\tau^* = \begin{cases} u_{max}, & \text{if } \lambda_z > \lambda_{I_2} \\ ?, & \text{if } \lambda_z = \lambda_{I_2} \\ 0, & \text{if } \lambda_z < \lambda_{I_2} \end{cases} \tag{22}$$

0.1.1 Travel Restrictions-only model

From our modified model above, we study the travel restrictions-only model and observe the results. [Note that the isolation-only model in this modified version is the same as that in our previous model].

$$\frac{dS_1}{dt} = -\beta S_1(I_1 + I_2) \tag{23}$$
$$\frac{dI_1}{dt} = \beta S_1(I_1 + I_2) - \mu I_1 \tag{24}$$
$$\frac{dI_2}{dt} = \theta - u_\tau I_2 - \gamma I_2 \tag{25}$$
$$\frac{dz}{dt} = u_\tau I_2 \tag{26}$$

The Hamiltonian is obtained as

$$H(t) = -\lambda_0 \beta S_1(I_1 + I_2) - \lambda_{S_1}\frac{dS_1}{dt} + \lambda_{I_1}\frac{dI_1}{dt} + \lambda_{I_2}\frac{dI_2}{dt} + \lambda_z\frac{dz}{dt} \tag{27}$$

The associated adjoint variables, $\lambda_{S_1}, \lambda_{I_1}, \lambda_{I_2}, \lambda_w, \lambda_z$, which correspond to the states S_1, I_1, I_2, w, and z respectively such that;

$$\dot{\lambda}_{S_1} = -(\lambda_{I_1} - \lambda_0 - \lambda_{S_1})\beta(I_1 + I_2) \tag{28}$$
$$\dot{\lambda}_{I_1} = -(\lambda_{I_1} - \lambda_0 - \lambda_{S_1})\beta S_1 + \lambda_{I_1}\mu \tag{29}$$
$$\dot{\lambda}_{I_2} = -(\lambda_{I_1} - \lambda_0 - \lambda_{S_1})\beta S_1 + (\lambda_{I_2} - \lambda_z)u_\tau + \lambda_{I_2}\gamma \tag{30}$$
$$\dot{\lambda}_z = 0 \tag{31}$$

and the optimality condition:

$$\frac{\partial H}{\partial u_\tau} = \psi_\tau(t) = (\lambda_z - \lambda_{I_2})I_2 \;\text{ at }\; u_\tau^* \tag{32}$$

The control characterization is given as:

$$u_\tau^* = \begin{cases} u_{max}, & \text{if} \quad \lambda_z > \lambda_{I_2} \\ ?, & \text{if} \quad \lambda_z = \lambda_{I_2} \\ 0, & \text{if} \quad \lambda_z < \lambda_{I_2} \end{cases} \tag{33}$$

Numerical results of the Travel Restrictions-only model

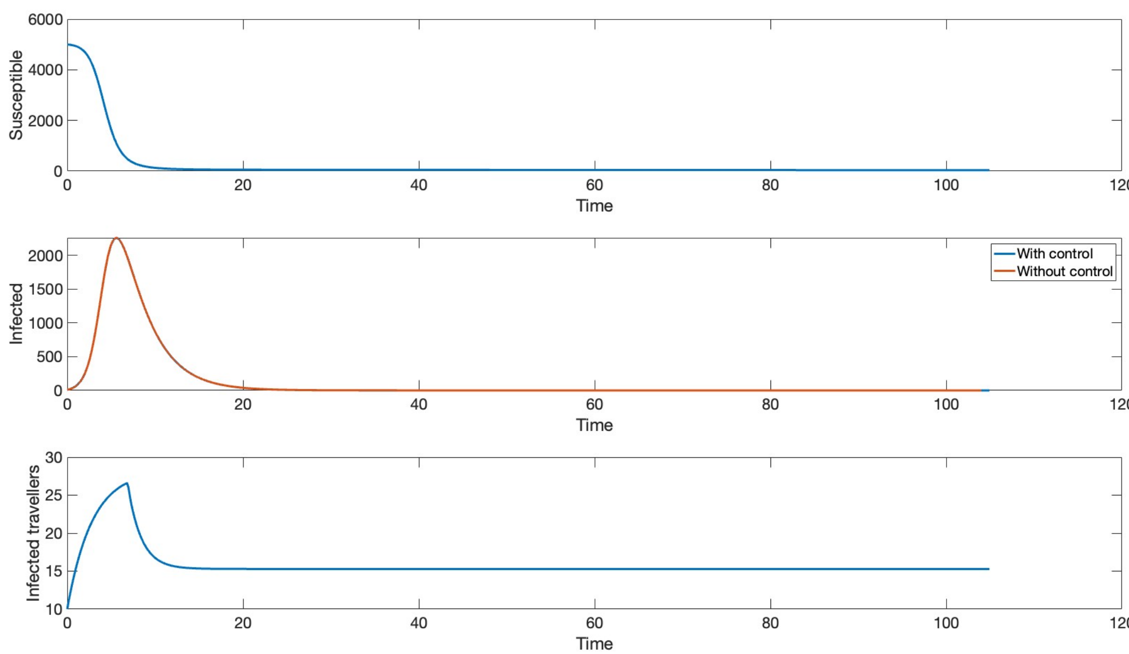


Figure 7: Dynamics of Travel Restrictions-only model for $\theta = 10$ and $z_{max} = 1500$

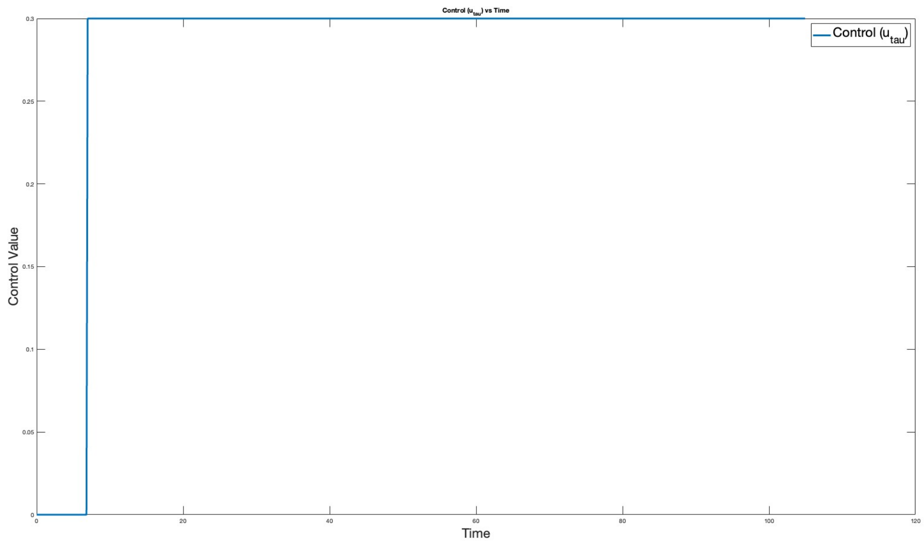


Figure 8: Control strategy for the Travel Restriction-Only model for $\theta = 10$ and $z_{max} = 1500$

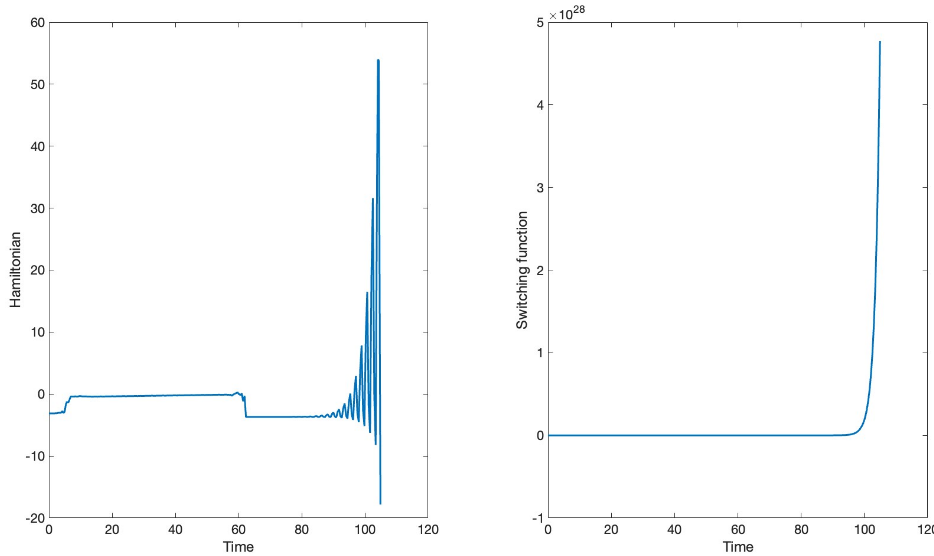


Figure 9: A plot of the Hamiltonian $H(t)$ and the switch function $\psi_{tau}(t)$ against time for $\theta = 10$ and $z_{max} = 1500$

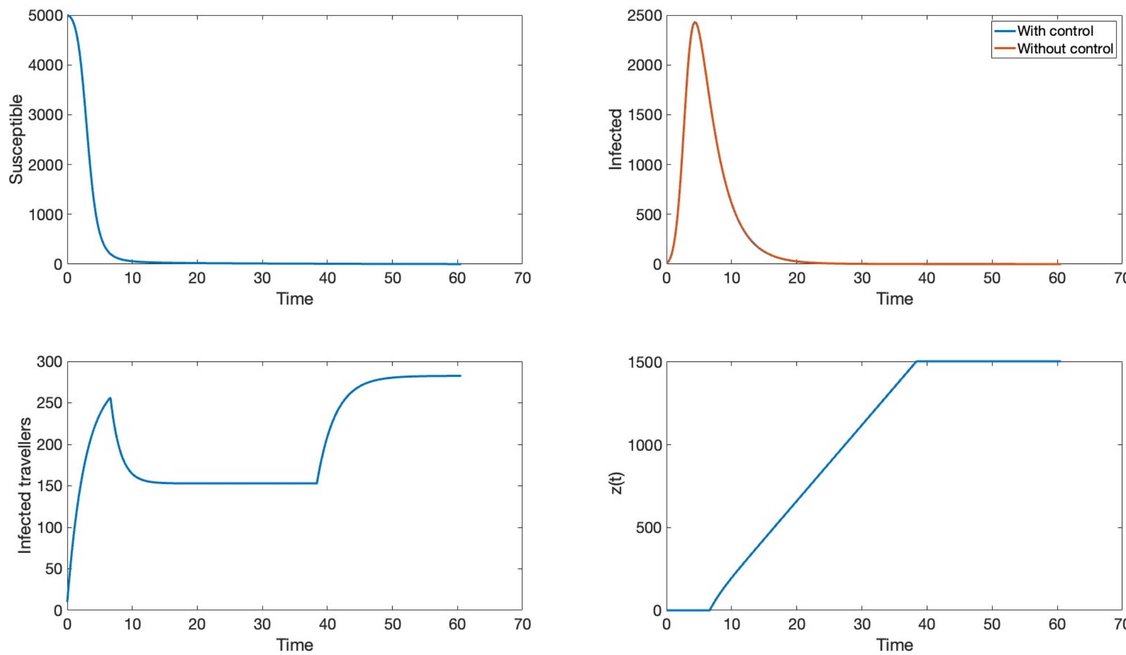


Figure 10: Dynamics of Travel Restrictions-only model for $\theta = 100$ and $z_{max} = 1500$

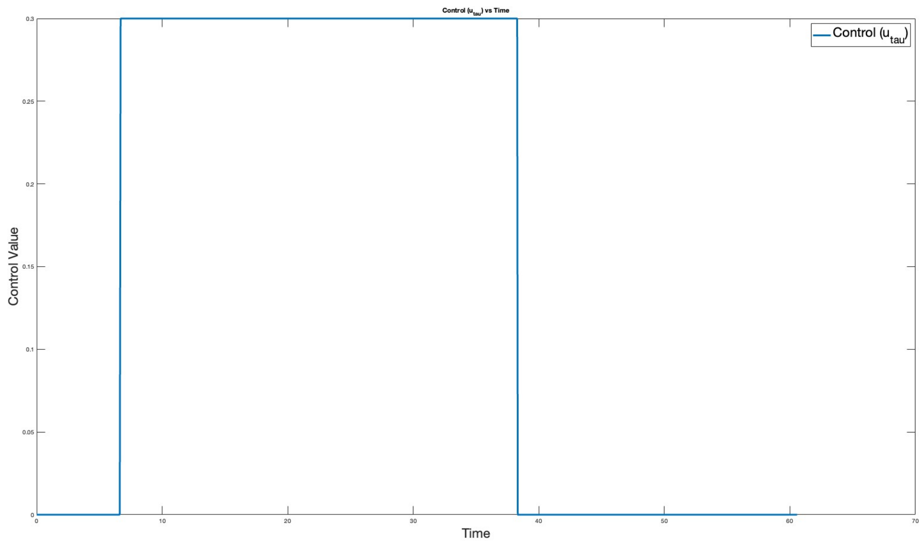
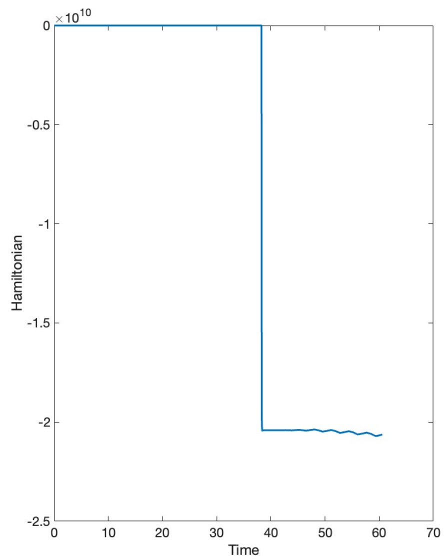


Figure 11: Control strategy for the Travel Restriction-Only model $\theta = 100$ and $z_{max} = 1500$



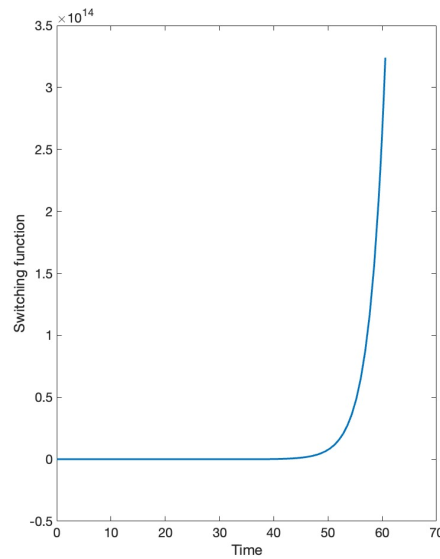


Figure 12: A plot of the Hamiltonian $H(t)$ and the switch function $\psi_{tau}(t)$ against time $\theta = 100$ and $z_{max} = 1500$

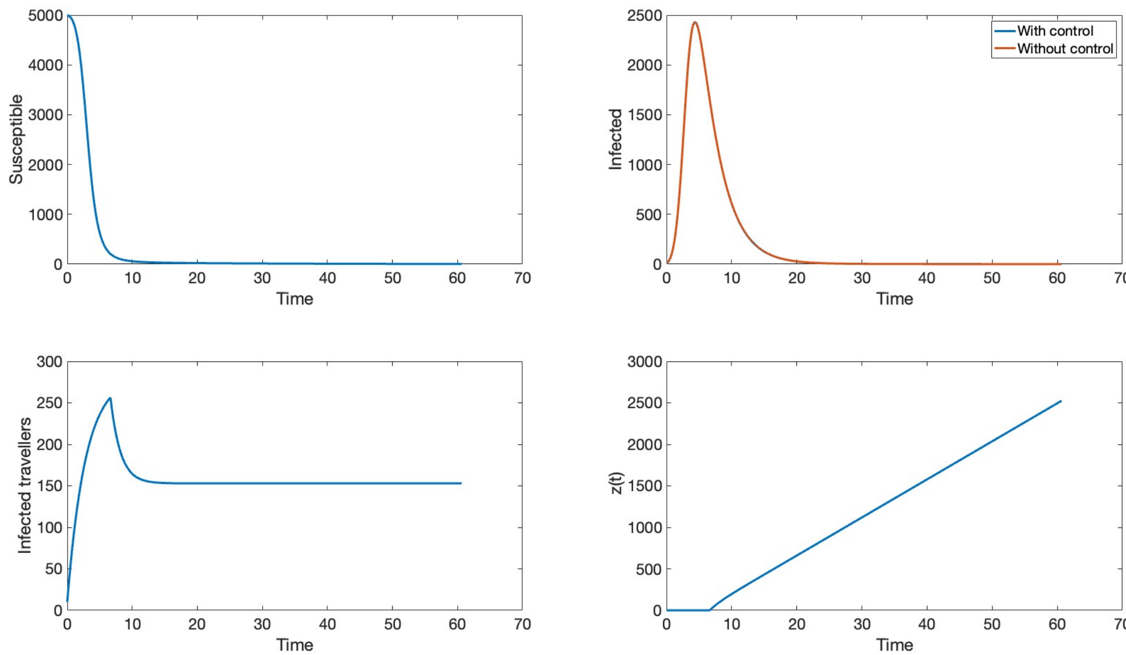


Figure 13: Dynamics of Travel Restrictions-only model for $\theta = 100$ and $z_{max} = 3500$

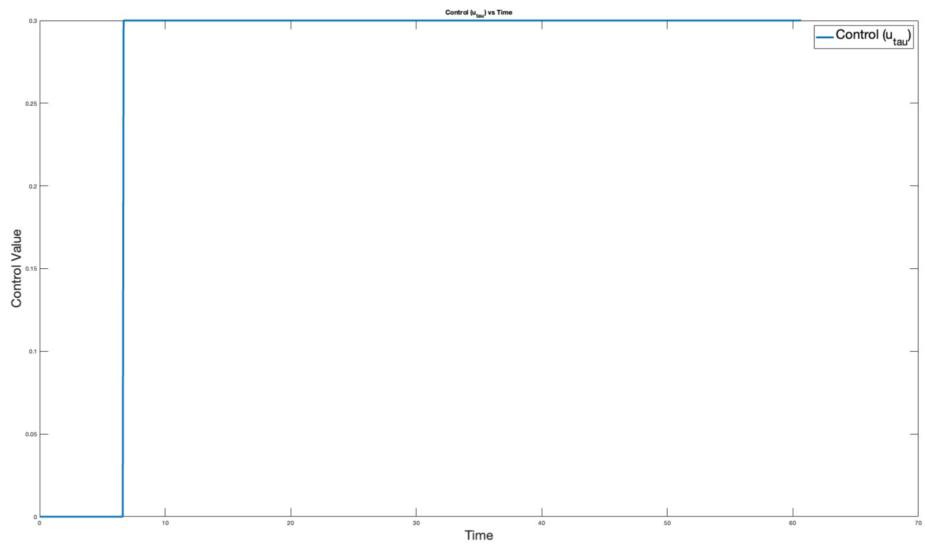


Figure 14: Control strategy for the Travel Restriction-Only model for $\theta = 100$ and $z_{max} = 3500$

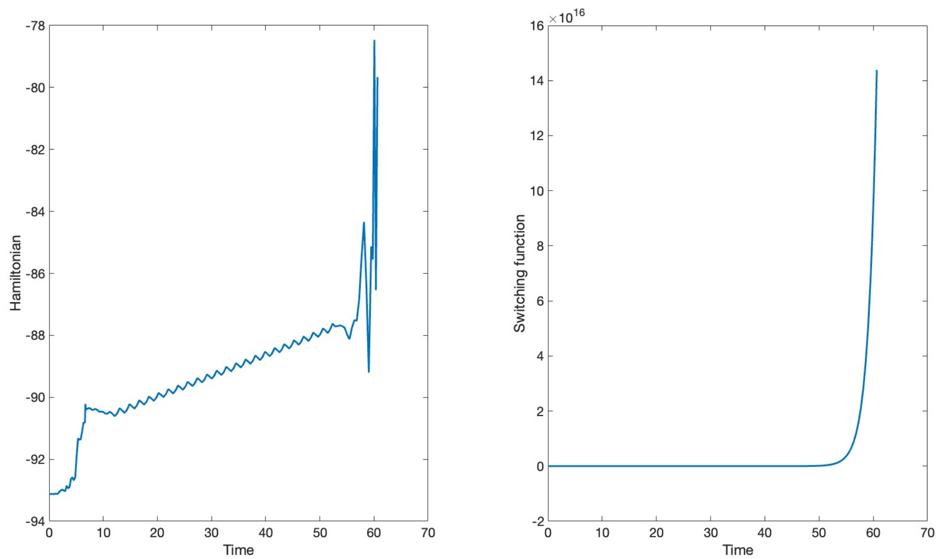


Figure 15: A plot of the Hamiltonian $H(t)$ and the switch function $\psi_{tau}(t)$ against time for $\theta = 100$ and $z_{max} = 3500$

With the travel restrictions-only model, we observe different control strategies corresponding to the different values of θ and z_{max}. Even though we have the control in place almost throughout the time period, we observe no differences between the state of infection with no control and the state with control of the infection.

References

[1] Elsa Hansen and Troy Day. Optimal control of epidemics with limited resources. *Journal of Mathematical Biology*, 62(3):423–451, March 2011.