Mathematical Model

0.1 Problem 2: Travel Restrictions Only $(w_{max} = 0, u_i = 0)$

In Problem 2, we consider travel restrictions as the only control in the model; our model now becomes

$$\frac{dS}{dt} = -\beta S(I + u_\tau \tau) \tag{1}$$
$$\frac{dI}{dt} = \beta S(I + u_\tau \tau) - \mu I \tag{2}$$

Our objective is;

$$\min J = \int_{t_0}^{T} \beta S_{[u_\tau]} I_{[u_\tau]} \, dt \tag{3}$$

subject to equations (1)-(2), $T = \inf\{t | I_{[u_\tau]}(t) = 0.5\}, u_\tau(t) \in [0, u_{max}]$ for all $t \in [0, T]$ and subject to the resource constraint;

$$\int_{t_0}^{T} u_\tau \tau \, dt \leq z_{max} \tag{4}$$

Theorem 0.1.1. ***(Optimal Travel Restrictions Policy)*** *There exists $t_k \in [t_0, T]$ such that the optimal travel restrictions policy for problem 2 is*

$$u_\tau^*(t) = \begin{cases} u_{max}, & \text{if} \quad t \in [t_0, t_k) \\ 0, & \text{if} \quad t \in (t_k, T] \end{cases} \tag{5}$$

where $\int_{t_0}^{t_k} u_{max} \tau dt = z_{max}$ *if* $t_k < T$.

The conclusion drawn from Theorem 0.1.1 is that the most effective approach for travel restrictions is to exert maximum effort at the initial stages of the pandemic, maintaining these restrictions for as long as feasible, either until all resources are depleted or the pandemic comes to an end. An intriguing insight from the theorem is that, in specific situations, stringent restrictions might be inefficient, leading to unnecessarily high costs [65]. This is exemplified by scenarios where the impact of travel restrictions is minimal due to the limited contribution of imported cases to local transmission [58]. Hence, it is significant that policymakers consider the local incidence of the disease, the growth of local epidemics, and the volume of travel before implementing such restrictions.

Proof: The travel restrictions-only model with limited resources is described by the system of ordinary differential equations:

$$\frac{dS}{dt} = -\beta S(I + u_\tau \tau) \tag{6}$$

$$\frac{dI}{dt} = \beta S(I + u_\tau \tau) - \mu I \tag{7}$$

$$\frac{dz}{dt} = u_\tau \tau \tag{8}$$

Next, we formulate Problem 2 Sec.0.1 as a maximization problem and apply the Pontryagin's Maximum Principle (PMP). Our objective now becomes;

$$\max J = -\int_{t_0}^{T} \beta S_{[u_\tau]} I_{[u_\tau]} \; dt \tag{9}$$

We derive the necessary conditions for optimality and the associated adjoint variables. The Hamiltonian is

$$\begin{aligned} H(t) &= -\lambda_0 \beta S I - \lambda_S \beta S (I + u_\tau \tau) + \lambda_I \beta S (I + u_\tau \tau) - \lambda_I \mu I + \lambda_z u_\tau \tau & (10) \\ &= -\dot{\lambda}_I I + (\lambda_I - \lambda_S) \beta S u_\tau \tau = -\dot{\lambda}_S S - \lambda_I \mu = 0 & (11) \end{aligned}$$

There are associated adjoint variables, $\lambda_S, \lambda_I, \lambda_z$, which correspond to the states S, I, and z respectively such that;

$$\dot{\lambda}_S = -(\lambda_I - \lambda_0 - \lambda_S)\beta I + (\lambda_S - \lambda_I)\beta u_\tau \tau \tag{12}$$

$$\dot{\lambda}_I = -(\lambda_I - \lambda_0 - \lambda_S)\beta S + \lambda_I \mu \tag{13}$$

$$\dot{\lambda}_z = 0 \tag{14}$$

and the optimality condition is obtained as:

$$\frac{\partial H}{\partial u_\tau} = \psi_\tau(t) = (\lambda_I - \lambda_S)\beta S \tau + \lambda_z \tau \;\text{ at }\; u_\tau^* \tag{15}$$

The transversality conditions are $(\lambda_0, \lambda_S(T), \lambda_I(T), \lambda_z) = (\lambda_0, 0, \lambda_I(T), p)$ where $p \leq 0$.

The control characterization is given as:

$$u_\tau^* = \begin{cases} u_{max}, & \text{if} \quad \lambda_z > (\lambda_S - \lambda_I)\beta S \\ ?, & \text{if} \quad \lambda_z = (\lambda_S - \lambda_I)\beta S \\ 0, & \text{if} \quad \lambda_z < (\lambda_S - \lambda_I)\beta S \end{cases} \tag{16}$$

Next, we prove that if $\psi_\tau = 0$ (i.e. $\lambda_z = (\lambda_S - \lambda_I)\beta S$) cannot be sustained over an interval of time but occurs only at finitely many points, then the control u_τ^* is purely bang-bang. If $\psi_\tau = 0$ on some interval of time, then u_τ^* is singular. We prove that u_τ is not singular. Suppose, on the contrary, that u_τ is singular on some interval. Then $\lambda_z = (\lambda_S - \lambda_I)\beta S$. As this holds on an interval, we can differentiate both sides,

$$\dot{\lambda}_z = (\lambda_S - \lambda_I)\beta\dot{S} + (\dot{\lambda}_S - \dot{\lambda}_I)\beta S \tag{17}$$

From (14), we have that $\dot{\lambda}_z = 0$, which implies that,

$$(\lambda_S - \lambda_I)\beta\dot{S} + (\dot{\lambda}_S - \dot{\lambda}_I)\beta S = 0 \tag{18}$$

$$(\lambda_S - \lambda_I)\dot{S} = (\dot{\lambda}_I - \dot{\lambda}_S)S \tag{19}$$

$$\dot{\lambda}_I = (\lambda_S - \lambda_I)\frac{\dot{S}}{S} + \dot{\lambda}_S \tag{20}$$

Substituting equations (6) and (12) into (20), we obtain

$$\dot{\lambda}_I = \lambda_0\beta I \tag{21}$$

Equation (21) clearly contradicts equation (13), therefore u_τ is not singular, hence bang-bang.

$$u_\tau^*(t) = \begin{cases} u_{max}, & t \in [t_0, t_k) \\ 0, & t \in (t_k, T] \end{cases} \tag{22}$$

Claim 0.1.2. *The control $u_\tau \equiv 0$ on $[t_0, T]$ is not an optimal solution for Problem 2.*

Proof: If $u_\tau^* \equiv 0$, then $\lambda_z \leq (\lambda_S - \lambda_I)\beta S$. Additionally, when $u_\tau^* \equiv 0$, it signifies that none of the available resources are utilized for maximum travel restrictions, implying

$\lambda_z = 0$. Consequently, this implies that $\lambda_S \geq \lambda_I$. Furthermore, according to equation (11), if $u_\tau^* \equiv 0$, then $\dot{\lambda}_I \equiv 0$. By rearranging equation (13) and substituting $t = T$, we obtain $\lambda_I(\beta S(T) - \mu) = \lambda_0 \beta S(T)$ for $(\lambda_S(T) = 0)$. This results in two possible scenarios:

1. If $\lambda_0 = 0$, it follows that $\lambda_I \equiv 0$. Since $\lambda_S(T) = 0$ as well, this implies that we have a zero adjoint vector at $t = T$. Consequently, this scenario is not feasible.

2. If $\lambda_0 = 1$, then $\lambda_I < 0$ due to the condition $S(T) < \frac{\mu}{\beta}$. Consequently, this implies, according to equation (11), that $\dot{\lambda}_S > 0$. Therefore, as $\lambda_S(T) = 0$, it necessitates that $\lambda_S(t) < 0$ for all $t < T$. However, this contradicts the fact that when $u_\tau^* \equiv 0$, it implies $\lambda_S \geq 0$.

0.2 Problem 3: Mixed Policy

Theorem 0.2.1. ***(Optimal Mixed Policy)*** *There exists $t_k \in [t_0, T]$ such that the optimal mixed policy for problem 3 has one of the following forms:*

1.

$$(u_i^*(t), u_\tau^*(t)) = \begin{cases} (u_{max}, u_{max}), & \text{if} \quad t \in [t_0, t_k) \\ (0, u_\tau^*(t)), & \text{if} \quad t \in (t_k, T] \end{cases} \tag{23}$$

where $\int_{t_0}^{t_k} u_{max} I_{[u_{max}, u_{max}]} dt = w_{max}$ if $t_k < T$ or

2.

$$(u_i^*(t), u_\tau^*(t)) = \begin{cases} (u_{max}, u_{max}), & \text{if} \quad t \in [t_0, t_k) \\ (u_i^*(t), 0), & \text{if} \quad t \in (t_k, T] \end{cases} \tag{24}$$

where $\int_{t_0}^{\hat{t}} u_{max} \tau dt = z_{max}$ if $t_k < T$.

Remarks 0.2.2. *In equation (23), when $t > t_k$, $u_\tau^*(t)$ signifies the optimal control specific to the travel restriction-only model. In other words, for $t > t_k$, the solution to Problem 2 is represented by $u_\tau^*(t)$. Similarly, in equation (24), if $t > t_k$, $u_i^*(t)$ denotes the optimal control for the isolation-only model. Hence, for $t > \hat{t}$, the solution to Problem 1 is given by $u_i^*(t)$.*

Proof: The model for the mixed policy with limited resources is described by the system of ordinary differential equations:

$$\frac{dS}{dt} = -\beta S(I + u_\tau \tau) \tag{25}$$

$$\frac{dI}{dt} = \beta S(I + u_\tau \tau) - (\mu + u_i)I \tag{26}$$

$$\frac{dw}{dt} = u_i I \tag{27}$$

$$\frac{dz}{dt} = u_\tau \tau \tag{28}$$

Next, we formulate Problem 3 Sec.0.2 as a maximization problem and apply the PMP; we derive the necessary conditions for the optimal control model and the associated adjoint variables. The Hamiltonian is

$$H(t) = -\lambda_0 \beta SI - \lambda_S \beta S(I + u_\tau \tau) + \lambda_I \beta S(I + u_\tau \tau) - \lambda_I(\mu + u_i)I + \lambda_w u_i I + \lambda_z u_\tau \tau \tag{29}$$

$$= -\dot{\lambda}_I I + (\lambda_I - \lambda_S)\beta S u_\tau \tau = -\dot{\lambda}_S S + (\lambda_w - \lambda_I)u_i I - \lambda_I \mu I = 0 \tag{30}$$

There are associated adjoint variables, $\lambda_S, \lambda_I, \lambda_w, \lambda_z$, which correspond to the states

S, I, w, and z respectively such that;

$$\dot{\lambda}_S = -(\lambda_I - \lambda_0 - \lambda_S)\beta I - (\lambda_I - \lambda_S)\beta u_\tau \tau \tag{31}$$

$$\dot{\lambda}_I = -(\lambda_I - \lambda_0 - \lambda_S)\beta S - (\lambda_w - \lambda_I)u_i + \lambda_I \mu \tag{32}$$

$$\dot{\lambda}_w = 0 \tag{33}$$

$$\dot{\lambda}_z = 0 \tag{34}$$

The transversality conditions are $(\lambda_0, \lambda_S(T), \lambda_I(T), \lambda_z, \lambda_w) = (\lambda_0, 0, \lambda_I(T), p, q)$ where $p, q \leq 0$.

and the optimality conditions:

$$\frac{\partial H}{\partial u_i} = \psi_i(t) = (\lambda_w - \lambda_I)I \;\text{ at }\; u_i^* \tag{35}$$

$$\frac{\partial H}{\partial u_\tau} = \psi_\tau(t) = (\lambda_I - \lambda_S)\beta S\tau + \lambda_z \tau \;\text{ at }\; u_\tau^* \tag{36}$$

The controls characterization are given as:

$$u_i^* = \begin{cases} u_{max}, & \text{if} \quad \lambda_w > \lambda_I \\ ?, & \text{if} \quad \lambda_w = \lambda_I \\ 0, & \text{if} \quad \lambda_w < \lambda_I \end{cases} \tag{37}$$

and

$$u_\tau^* = \begin{cases} u_{max}, & \text{if} \quad \lambda_z > (\lambda_S - \lambda_I)\beta S \\ ?, & \text{if} \quad \lambda_z = (\lambda_S - \lambda_I)\beta S \\ 0, & \text{if} \quad \lambda_z < (\lambda_S - \lambda_I)\beta S \end{cases} \tag{38}$$

Now we examine that, if there exists a time interval $\mathbb{I}$ during which $\lambda_w = \lambda_I$, then a singular control exists (i.e u_i^* is singular). Suppose that $\lambda_w = \lambda_I$, then it must be that $\dot{\lambda}_w = \dot{\lambda}_I$. The costate equations

$$\dot{\lambda}_I = -(\lambda_I - \lambda_0 - \lambda_S)\beta S - (\lambda_w - \lambda_I)u_i + \lambda_I \mu \tag{39}$$

$$\dot{\lambda}_w = 0 \tag{40}$$

contradict our assumption that $\lambda_w = \lambda_I$. We conclude that $\lambda_w \neq \lambda_I$ during a finite time interval and thus a singular control cannot exist.

Similarly, if u_τ^* is singular on a time interval $\mathbb{I}$, then $\lambda_z = (\lambda_S - \lambda_I)\beta S$. We have from equation (34) that $\lambda_z(t)$ is constant throughout the singular interval $\mathbb{I}$. But $(\lambda_S(t) - \lambda_I(t))\beta S(t)$ is not constant over the singular interval for $t \in \mathbb{I}$. Therefore we conclude that u_τ^* is not singular. The presence of singular arcs in the solution are thus ruled out.

Claim 0.2.3. *If there exists a $t_k \geq 0$ such that $u_i(t) = 0$ is constant for all $t \in (t_k, T]$, then $u_\tau(t) = u_\tau^*(t)$ for all $t \in (t_k, T]$.*

Proof: Once $t > t_k$ the mixed isolation–travel restrictions model becomes a travel restrictions-only model, and therefore, the optimal u_τ is the optimal control for the travel restrictions-only model with parameters $\tilde{t}_0 = t_k,\ \tilde{z}_{max} = z_{max} - \int_{t_0}^{t_k} u_\tau \tau \, dt$.

Claim 0.2.4. *If there exists a $t_k \geq 0$ such that $u_\tau(t) = 0$ is constant for all $t \in (t_k, T]$, then $u_i(t) = u_i^*(t)$ for all $t \in (t_k, T]$.*

Proof: Once $t > t_k$ the mixed isolation–travel restrictions model becomes an isolation-only model, and therefore, the optimal u_i is the optimal control for the isolation-only model with parameters $\tilde{t}_0 = t_k,\ \tilde{w}_{max} = w_{max} - \int_{t_0}^{t_k} u_i I \, dt$.

0.2.1 Switching times and Final time

Applying the PMP to our model, we observed that the results of the optimal controls are bang-bang which switches between the upper and lower bounds of the control inputs. Upon characterizing the controls as bang-bang, the problem of identifying all the required controls now becomes one of determining the switching times. To simplify the presentation, we make the following definitions, assumptions and denotions for the calculation of the switching and final times. Given initial conditions for our system, the state trajectory $x(t)$ is determined by the switching times.

Definition 0.2.5. *Let the kth switching time be denoted by* $t_k,\ k = 1, 2, \cdots, r-1$ *such that* $0 = t_0 < t_1 < t_2 < \cdots < t_{r-1} < t_r = T$. *The initial and final times are given by* $t_0 = 0$ *and* $t_r = T$. *The non-negative integer* $r - 1$ *is the number of times the control switches during the time horizon* T.

Definition 0.2.6. *The segment of the trajectory* $x(t)$ *where* $t_{k-1} \leq t \leq t_k,\ k = 1, 2, \cdots, r$ *is called the kth arc or the kth bang arc and denoted by* $x_k(t)$. *(Note: The state trajectory* $x(t)$ *for* $0 \leq t \leq T$*, is the concatenation of the* $x_k(t)$*).*

Definition 0.2.7. *The time spent on the kth arc is called the kth arc time given by* $\xi_k = t_k - t_{k-1}$. *We also define the kth arc times vector* $\xi = (\xi_1, \xi_2, \cdots, \xi_r)$.

Note that $t_k = \sum_{j=1}^{k} \xi_j$ and $T = \sum_{j=1}^{r} \xi_j$. We require that each arc time ξ_k must be non-negative for our system states trajectory (i.e. $\xi_k \geq 0$).

We assume that the control signal is at u_{max} at the beginning of the bang-bang control until switching time t_k and then the control signal switches to u_{min}.

In general, there is no analytic solution for (22), (23), (24) due to lack of sufficient boundary conditions on the adjoint systems (12)-(14), (31)-(34), so the switching function $\psi_i(t), \psi_\tau(t)$ in the bang-bang control law (16), (37),(38) cannot be solved analytically.

The switching and final time of the bang-bang controls can be calculated by solving the system numerically.

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