Strategic Timing and Resource Allocation for Optimal Isolation and Travel Restrictions in Infectious Disease Control

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Problem description and assumptions I

- We incorporate the widely used epidemiological compartmental model, SI model. In this model, births and deaths are neglected, and the recovered population is assumed to no longer infect others and cannot be reinfected.
- Here, the model captures the importation (infected non-resident travellers). We address epidemics with no vaccination, where the possible controls are isolation and travel restrictions.

Problem description and assumptions II

- We model these non-pharmaceutical interventions via finite time controls $(u_i(t), u_T(t)) \in [0, u_{max}] \times [0, u_{max}]$, where 0 corresponds to no control and u_{max} corresponds to when we have max control on.
- We denote S, I_1, I_2 the number of susceptibles, community infections, and infected non-resident travellers (imported cases), respectively, and their evolution is governed by the following system of nonlinear ordinary differential equations. Our model and analysis follow from [Hansen and Day, 2011].

Problem description and assumptions III

$$\frac{dS}{dt} = -\beta S(I_1 + I_2) \tag{1}$$

$$\frac{dI_1}{dt} = \beta S(I_1 + I_2) - (\mu + u_i)I_1 \tag{2}$$

$$\frac{dI_2}{dt} = \theta - (\gamma + u_\tau)I_2 \tag{3}$$

with $S(t_0) > 0, I_1(t_0) \geq 0, I_2(t_0) \geq 0, \ \beta, \mu, \theta, \gamma \geq 0$, where β is the transmission rate, μ is the per capita loss rate of infected community members through both mortality and recovery, θ is the baseline number of infected non-resident travellers per unit time, γ is the removal rate of non-resident travellers.

Problem description and assumptions IV

- We assume that non-resident travellers who are infected have contracted the infection before entering the community. These individuals have a temporary stay and are promptly either recovered or removed from the community; therefore, $\gamma > \mu$.

Objectives and contributions I

- To understand the effects of NPIs on an entire population with limited resources and without vaccines.
- Our primary objective is to understand the timing and best control strategies for implementing isolation and travel restrictions.
- To this end, we also investigate solutions in specific scenarios (such as isolation-only (no importation), isolation-only (with importations), travel restrictions only, and mixed policies).
- One novel aspect of this work is that the problem is posed in terms of the infinite time limit but formulated in a way that only requires a solution over a finite time interval.

Objectives and contributions II

- We assume that the interventions last a finite time $\tilde{t} \leq T$, the maximum time that the population will adhere since it is unrealistic that interventions can be sustained indefinitely due to limited resources. The selection of the terminal time T is one possible issue with optimal control [Hansen and Day, 2011].
- The concept behind our choice of T is straightforward. Given the limited resources at our disposal, a large selection of T may result in varying infection peaks since we are likely to run out of resources and be unable to control the disease. Another argument is that since we cannot predict when an epidemic will cease, we have established a threshold

Objectives and contributions III

$I(t) = I_{min} = 0.5$ to verify that the disease has died out. In other words, our terminal time is the instant we can get from a given initial state to a specified final state in the shortest possible time using the controls and resources available.

Now, there are several interesting questions related to the implementation of these measures:

1. When should the intervention begin in $[0, T]$?
2. Is it optimal to split the maximum time $\tilde{t}$ into different intervals?

- The answers clearly can depend on the goal, which, in our case, is to minimize the number of new infections.

- We shall prove that the optimal policies are of bang-bang type; that is, we can turn them on and off, and they must be turned off after the maximum allowed time $\tilde{t}$.

General Problem (Optimal Control Problem)

Let z denote the number of restricted non-resident infected travellers, and w denote the number of isolated individuals.

$S_{[u_i,u_\tau]}, I_{1[u_i,u_\tau]}, I_{2[u_i,u_\tau]}, z_{[u_i,u_\tau]}, w_{[u_i,u_\tau]}$ denote that the actual number of the state variables (S, I_1, I_2, z, w) which depend on the choice of the controls u_i and u_τ.

Fixing $w_{max} \geq 0$ and $z_{max} \geq 0$, our optimal control for the general problem is formulated as:

$$\min J = \int_0^T \beta S_{[u_i,u_\tau]}(I_{1[u_i,u_\tau]} + I_{2[u_i,u_\tau]})\ dt \tag{4}$$

subject to equations (1)-(3),

$T = \inf\{t | I_{1[u_i,u_\tau]}(t) = 0.5\}, (u_i(t), u_\tau(t)) \in [0, u_{max}] \times [0, u_{max}]$ for all $t \in [0, T]$, $u_{max} \in (0, \infty)$,

and subject to the resource constraints;

$$\int_0^T u_i I_{1[u_i,u_\tau]}\ dt \leq w_{max} \tag{5}$$

and

$$\int_0^T u_\tau I_{2[u_i, u_\tau]} \, dt \leq z_{max} \tag{6}$$

Existence and necessary conditions for an optimal control

- We provide a result stating the existence of at least one optimal solution to the optimal control problem (4)-(6) under some appropriate compactness and convexity assumptions. Precisely, we follow the standard Filippov's approach.

Theorem (1)

(Filippov's existence theorem) *Consider an optimal control problem defined by a differential inclusion $\dot{x} \in F(t, x, u)$, where $F : [t_0, T] \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is a set-valued mapping representing the dynamics, t is time, x is the state variable and u is the control input. Assume that the set-valued map F is upper semicontinuous in x and continuous in u for each fixed t. If the optimal control problem has nonempty, compact, and convex solution sets for all t, then an optimal control exists for almost every initial point in $\mathbb{R}^n$.*

To establish the existence of the optimal control, we address the boundedness of the state variables in the system (1)-(3). By summing up all the equations in the model (1)-(3), we obtain $N(t) \leq N(t_0)$.

Considering the characteristics of the infectious disease model, for $\mu \geq 0$, it is evident that $0 \leq S(t), I_1(t), I_2(t) \leq N(t_0)$. In other words, the state variables of the system are bounded.

The assurance of the existence of an optimal control solution is ensured by satisfying the following conditions.

(a) The set of control variables and corresponding state variables is not empty.

(b) The admissible control set is compact and bounded.

(c) The state variables are continuously differentiable.

By examining the control set (u_i, u_τ), it becomes apparent that the system state equation's solution remains continuous and bounded for every permissible control. Therefore, condition (a) and (b) are satisfied. The system, as described in (1)-(3), adheres to the Lipschitz condition concerning the state variables, ensuring the existence of the model's solution. The system's (1)-(**??**) solution is continuous, thereby fulfilling condition (c).

Pontryagin Maximum/Minimum Principle (PMP)

The Pontryagin Maximum Principle provides the necessary conditions that an optimal control and corresponding state trajectory must satisfy for a wide class of optimal control problems.

The first step is forming the Hamiltonian to solve the optimal control problem above. The Hamiltonian is defined as

$$H(t, x(t), u(t), \lambda(t)) = \lambda_0 L(t, x(t), u(t) + \lambda f(t, x(t), u(t)) \quad (7)$$

Theorem (2)

(PMP): *If $u^*(t)$ and $x^*(t)$ are the optimal solution of the control problem, then there exist piecewise differentiable adjoint variables $\lambda(t)$ such that*

$$H(t, x^*(t), u(t), \lambda(t)) \leq H(t, x^*(t), u^*(t), \lambda(t)) \tag{8}$$

for all controls u at each time t, where H is the Hamiltonian and

$$\dot{\lambda}(t) = \frac{\partial H(t, x^*(t), u^*(t)}{\partial x} \tag{9}$$

$$\lambda(T) = 0 \tag{10}$$

are the costate and transversality conditions, respectively.

Theorem (3)

Suppose that $f(t, x, u)$ is a continuously differentiable function in its three arguments and concave in u. Suppose u^* is an optimal control with associated state x^*, and λ a piecewise differentiable function with $\lambda(t) \geq 0 \ \forall \ t$. Suppose for all $t_0 \leq t \leq T$

$$0 = H_u(t, x^*(t), u^*(t), \lambda(t)). \tag{11}$$

Then for all controls u and each $t_0 \leq t \leq T$, we have

$$H(t, x^*(t), u(t), \lambda(t)) \leq H(t, x^*(t), u^*(t), \lambda(t)) \tag{12}$$

The same essential conditions are derived through similar reasoning when the problem involves minimizing rather than maximizing. In a minimization problem, we minimize the Hamiltonian pointwise and the inequality in PMP is reversed. Indeed, for a minimization problem with f being convex in u, we can derive

$$H(t, x^*(t), u(t), \lambda(t)) \geq H(t, x^*(t), u^*(t), \lambda(t)) \tag{13}$$

by the same argument as in Theorem 3

Theorem (4)

Given an optimal control pair (u_i^*, u_T^*) *and solutions* $S^*, I_1^*, I_2^*, w^*, z^*$, *of the corresponding state system that minimizes the objective function* J *over the set of admissible controls. Then there exists adjoint variables* $\lambda_S, \lambda_{I_1}, \lambda_{I_2}, \lambda_w, \lambda_z$ *satisfying* $\frac{d\lambda_i}{dt} = -\frac{\partial H}{\partial i}$ *and with the transversality conditions* $\lambda_i(T)$ *where* $i = S, I_1, I_2, w, z$.

Proof:

Proof: We apply the pmp on our optimal control problem (4)-(6). We begin by forming the system state equations for the general problem.

$$\frac{dS}{dt} = -\beta S(I_1 + I_2) \tag{14}$$

$$\frac{dI_1}{dt} = \beta S(I_1 + I_2) - (\mu + u_i)I_1 \tag{15}$$

$$\frac{dI_2}{dt} = \theta - (\gamma + u_\tau)I_2 \tag{16}$$

$$\frac{dw}{dt} = u_i I_1 \tag{17}$$

$$\frac{dz}{dt} = u_\tau I_2 \tag{18}$$

Next, we form the Hamiltonian H

$$H(t) = (\lambda_0 - \lambda_S + \lambda_{I_1})\beta S(I_1 + I_2) - \lambda_{I_1}(\mu + u_i)I_1 + \lambda_{I_2}(\theta - (\gamma + u_\tau))I_2 + \lambda_w u_i I_1 + \lambda_z u_\tau I_2 \tag{19}$$

The adjoint variables, $\lambda_S, \lambda_{I_1}, \lambda_{I_2}, \lambda_w, \lambda_z$ correspond to the states S, I_1, I_2, w and z respectively such that;

$$\dot{\lambda}_S = -(\lambda_0 - \lambda_S + \lambda_{I_1})\beta(I_1 + I_2) \tag{20}$$

$$\dot{\lambda}_{I_1} = -(\lambda_0 - \lambda_S + \lambda_{I_1})\beta S + (\lambda_{I_1} - \lambda_z)u_i + \lambda_{I_1}\mu \tag{21}$$

$$\dot{\lambda}_{I_2} = -(\lambda_0 - \lambda_S + \lambda_{I_1})\beta S + (\gamma - \theta)\lambda_{I_2} + (\lambda_{I_2} - \lambda_z)u_\tau \tag{22}$$

$$\dot{\lambda}_w = 0 \tag{23}$$

$$\dot{\lambda}_z = 0 \tag{24}$$

The adjoint variables are the marginal variations in the objective function for the respective state variables at every time t, and this helps to determine what sign (positive or negative) to expect from an adjoint variable.

The transversality conditions are $(\lambda_0, \lambda_S(T), \lambda_{I_1}(T), \lambda_{I_2}(T), \lambda_w, \lambda_z) = (\lambda_0, 0, \lambda_{I_1}(T), \lambda_{I_2}(T), q, p)$ where $p, q \leq 0$. The optimality conditions are

$$\frac{\partial H}{\partial u_i} = \psi_i(t) = (\lambda_w - \lambda_{I_1}) I_1 \text{ at } u_i^* \tag{25}$$

$$\frac{\partial H}{\partial u_\tau} = \psi_\tau(t) = (\lambda_z - \lambda_{I_2}) I_2 \text{ at } u_\tau^* \tag{26}$$

When $\frac{\partial H}{\partial u_i} = 0$ and $\frac{\partial H}{\partial u_\tau} = 0$, we are not able to find a characterization of the controls u_i, u_τ respectively. Therefore, we define $\psi_i(t)$ and $\psi_\tau(t)$ called the switching functions.

The optimal controls are bang-bang

- The PMP, when applied to bounded control problems that are linear in the control variables, explicitly defines the bang-bang controls.
- However, the bang-bang control is undefined when the switching function is identically zero. The consequence of this problem is that we cannot find a characterization of the optimal controls.
- We define switching functions $\psi_i(t)$, $\psi_\tau(t)$, and then our controls are characterized by the control input switching between two extreme values, typically denoted as u_{max} and 0.

Lemma (1)

Let $(u_i^*(t), u_\tau^*(t))$ *be a pair of optimal controls for* (3). *Then*

$$u_i^*(t) = \begin{cases} u_{max}, & when \quad \lambda_w < \lambda_{I_1} \\ 0, & when \quad \lambda_w > \lambda_{I_1} \end{cases} \tag{27}$$

$$u_\tau^*(t) = \begin{cases} u_{max}, & when \quad \lambda_z < \lambda_{I_2} \\ 0, & when \quad \lambda_z > \lambda_{I_1} \end{cases} \tag{28}$$

where $\lambda_{I_1}, \lambda_{I_2}, \lambda_w, \lambda_z$ *are given by* (20)-(24).

Proof: From (25)-(26), we have

$$\frac{\partial H}{\partial u_i} = (\lambda_w - \lambda_{I_1}) I_1 \tag{29}$$

$$\frac{\partial H}{\partial u_\tau} = (\lambda_z - \lambda_{I_2}) I_2 \tag{30}$$

The optimality condition then implies (27)-(28) except at points where $\frac{\partial H}{\partial u_i} = 0$ and $\frac{\partial H}{\partial u_\tau} = 0$. It remains to show that there are no singular arcs. Since $S(t), I_1(t), I_2(t) > 0$ for $t < \infty$, we have that $\frac{\partial H}{\partial u_i} = 0$ and $\frac{\partial H}{\partial u_\tau} = 0$ if and only if $\lambda_w = \lambda_{I_1}$ and $\lambda_z = \lambda_{I_2}$ respectively. Suppose by contradiction that the latter conditions hold on an open interval. Then, on that interval, we can differentiate both sides,

$$\dot{\lambda}_w = \dot{\lambda}_{I_1} = -(\lambda_0 - \lambda_S + \lambda_{I_1})\beta S + \lambda_{I_1}\mu = 0 \tag{31}$$

$$\dot{\lambda}_z = \dot{\lambda}_{I_2} = -(\lambda_0 - \lambda_S + \lambda_{I_1})\beta S + (\gamma - \theta)\lambda_{I_2} = 0 \tag{32}$$

By continuity, this would imply that $\lambda_w(t) = \lambda_{I_1}(t)$ and $\lambda_z(t) = \lambda_{I_2}(t)$ over the whole interval $[0, T]$, and in particular at time T. But then from (21) and (22), $\dot{\lambda}_{I_1} \neq 0$ and $\dot{\lambda}_{I_2} \neq 0$, so this is a contradiction.

Problem 1: Isolation Only (no importation)

The derivation and proof of the theorems of Problem 1 is a direct result of the work of Hansen and Day 2011 [Hansen and Day, 2011]. Considering isolation as the only control in the model, our model now becomes;

$$\frac{dS}{dt} = -\beta S I_1 \tag{33}$$

$$\frac{dI_1}{dt} = \beta S I_1 - (\mu + u_i) I_1 \tag{34}$$

Our objective is;

$$\min J = \int_{t_0}^{T} \beta S_{[u_i]} I_{1[u_i]} \, dt \tag{35}$$

subject to equations (33)-(34), $T = \inf\{t | I_{1[u_i]}(t) = 0.5\}, u_i(t) \in [0, u_{max}]$ for all $t \in [0, T]$ and subject to the resource constraint;

$$\int_0^T u_i I_{1[u_i]} \, dt \leq w_{max} \tag{36}$$

From Eq. (33), we have

$$dS = \beta S I_1 \, dt \tag{37}$$

integrating both sides, we get

$$\int_0^T dS = -\int_0^T \beta S I_1 \, dt \tag{38}$$

$$S(T) - S(0) = -\int_0^T \beta S I_1 \, dt \tag{39}$$

$$S_0 - S(T) = \int_0^T \beta S I_1 \, dt \tag{40}$$

on the other hand, rearranging Eq. (6), we get

$$\frac{1}{S} \, dS = -\beta I_1 \, dt \tag{41}$$

Taking integral on both sides, we have

$$\int_0^T \frac{1}{S}\, dS = -\beta \int_0^T I_1 \, dt \tag{42}$$

$$-\frac{1}{\beta} \ln\left(\frac{S(T)}{S_0}\right) = \int_0^T I_1 \, dt \tag{43}$$

- We observe from equations (40) and (43) that the terms on the right-hand side are both minimized by maximizing $S(T)$ since S_0 is a fixed quantity. Next, we state and prove the main result of the isolation policy.

Theorem (5)

(Optimal Isolation Policy): *If* $w_{u_{max}}(T) \leq w_{max}$, *then the optimal isolation policy for Problem 1 is* $u_i^* = u_{max}$. *If* $w_{u_{max}}(T) > w_{max}$, *then the optimal policy* u_i^* *is any control* u_i *such that* $w_{u_{max}}(T) = w_{max}$.

Problem 2: Isolation Only (with importation)

In this model, we assume there are case importations, yet the only control measure is isolation.

$$\frac{dS}{dt} = -\beta S(I_1 + I_2) \tag{44}$$

$$\frac{dI_1}{dt} = \beta S(I_1 + I_2) - (\mu + u_i)I_1 \tag{45}$$

$$\frac{dI_2}{dt} = \theta - \gamma I_2 \tag{46}$$

$$\frac{dw}{dt} = u_i I_1 \tag{47}$$

We derive the necessary conditions for optimality and the associated adjoint variables. The Hamiltonian is

$$H(t) = (\lambda_0 - \lambda_S + \lambda_{I_1})\beta S(I_1 + I_2) + (\lambda_w - \lambda_{I_1})u_i I_1 - \lambda_{I_1}\mu I_1 + \lambda_{I_2}(\theta - \gamma I_2) \tag{48}$$

There are associated adjoint variables, $\lambda_S, \lambda_{I_1}, \lambda_{I_2}, \lambda_w$, which correspond to the states S, I_1, I_2, and w respectively such that;

$$\dot{\lambda}_S = -(\lambda_0 - \lambda_S + \lambda_{I_1})\beta(I_1 + I_2) \tag{49}$$

$$\dot{\lambda}_{I_1} = -(\lambda_0 - \lambda_S + \lambda_{I_1})\beta S - (\lambda_w - \lambda_{I_1})u_i + \lambda_{I_1} \tag{50}$$

$$\dot{\lambda}_{I_1} = -(\lambda_0 - \lambda_S + \lambda_{I_1})\beta S + \lambda_{I_2}\gamma \tag{51}$$

$$\dot{\lambda}_w = 0 \tag{52}$$

and the optimality condition is obtained as follows:

$$\frac{\partial H}{\partial u_i} = \psi_i(t) = (\lambda_w - \lambda_{I_1})I_1 \ \text{ at } \ u_i^* \tag{53}$$

with the boundary conditions

$(\lambda_0, \lambda_S(T), \lambda_{I_1}(T), \lambda_{I_2}(T), \lambda_w) = (\lambda_0, 0, \lambda_{I_1}(T), 0, q)$ known as the transversality conditions, where $q \leq 0$.

The optimal control is therefore characterized as:

$$u_i^*(t) = \begin{cases} u_{max}, & \text{when} \quad \lambda_w < \lambda_{I_1} \\ 0, & \text{when} \quad \lambda_w > \lambda_{I_1} \end{cases} \tag{54}$$

Problem 3: Travel Restrictions Only

We study the travel restrictions-only model from our model above and observe the results.

$$\frac{dS_1}{dt} = -\beta S_1(I_1 + I_2) \tag{55}$$

$$\frac{dI_1}{dt} = \beta S_1(I_1 + I_2) - \mu I_1 \tag{56}$$

$$\frac{dI_2}{dt} = \theta - (u_\tau + \gamma) I_2 \tag{57}$$

$$\frac{dz}{dt} = u_\tau I_2 \tag{58}$$

The Hamiltonian is obtained as

$$H(t) = (\lambda_0 - \lambda_S + \lambda_{I_1})\beta S(I_1 + I_2) - \lambda_{I_1}\mu I_1 + (\lambda_z - \lambda_{I_2})u_\tau I_2 + \lambda_{I_2}\theta - \lambda_{I_2}\gamma I_2 \tag{59}$$

The associated adjoint variables, $\lambda_S, \lambda_{I_1}, \lambda_{I_2}, \lambda_z$, which correspond to the states S, I_1, I_2, and z respectively such that;

$$\dot{\lambda}_S = -(\lambda_0 - \lambda_S + \lambda_{I_1})\beta(I_1 + I_2) \tag{60}$$

$$\dot{\lambda}_{I_1} = -(\lambda_0 - \lambda_S + \lambda_{I_1})\beta S + \lambda_{I_1}\mu \tag{61}$$

$$\dot{\lambda}_{I_2} = -(\lambda_0 - \lambda_S + \lambda_{I_1})\beta S - (\lambda_z - \lambda_{I_2})u_\tau + \lambda_{I_2}\gamma \tag{62}$$

$$\dot{\lambda}_z = 0 \tag{63}$$

The transversality conditions are $(\lambda_0, \lambda_{S_1}(T), \lambda_{I_1}(T), \lambda_{I_2}(T), \lambda_z, \lambda_w) = (\lambda_0, 0, \lambda_{I_1}(T), \lambda_{I_2}(T), p, q)$ where $p, q \leq 0$.

and the optimality condition:

$$\frac{\partial H}{\partial u_\tau} = \psi_\tau(t) = (\lambda_z - \lambda_{I_2}) I_2 \text{ at } u_\tau^* \tag{64}$$

The control characterization is given as follows:

$$u_\tau^* = \begin{cases} u_{max}, & \text{when} \quad \lambda_z < \lambda_{I_2} \\ 0, & \text{when} \quad \lambda_z > \lambda_{I_2} \end{cases} \tag{65}$$

Theorem (6)

(Optimal Travel Restrictions Policy) *There exists* $\tilde{t} \in [0, T]$ *such that the optimal travel restrictions policy for problem 3 is*

$$u_{\tau}^{*}(t) = \begin{cases} u_{max}, & \text{if} \quad t \in [0, \tilde{t}) \\ 0, & \text{if} \quad t \in (\tilde{t}, T] \end{cases} \qquad (66)$$

where $\int_0^{\tilde{t}} u_{max} \tau dt = z_{max}$ *if* $\tilde{t} < T$.

Problem 4: Mixed Policy

The formulation of the mixed policy model is the same as that of the general problem.

The Hamiltonian is given as

$$H(t) = (\lambda_0 - \lambda_S + \lambda_{I_1})\beta S(I_1 + I_2) - \lambda_{I_1}(\mu + u_i)I_1 + \lambda_{I_2}(\theta - (\gamma + u_\tau))I_2 + \lambda_w u_i I_1 + \lambda_z u_\tau I_2 \tag{67}$$

We find a characterization of the controls u_i, u_τ as:

$$u_i^*(t) = \begin{cases} u_{max}, & \text{when} \quad \lambda_w < \lambda_{I_1} \\ 0, & \text{when} \quad \lambda_w > \lambda_{I_1} \end{cases} \tag{68}$$

$$u_\tau^*(t) = \begin{cases} u_{max}, & \text{when} \quad \lambda_z < \lambda_{I_2} \\ 0, & \text{when} \quad \lambda_z > \lambda_{I_1} \end{cases} \tag{69}$$

Theorem

(Optimal Mixed Policy) *There exists $\tilde{t} \in [0, T]$ such that the optimal mixed policy for problem 3 has one of the following forms:*

1.

$$(u_i^*(t), u_T^*(t)) = \begin{cases} (u_{max}, u_{max}), & if \quad t \in [0, \tilde{t}) \\ (0, u_T^*(t)), & if \quad t \in (\tilde{t}, T] \end{cases} \tag{70}$$

where $\int_0^{\tilde{t}} u_{max} I_{[u_{max}, u_{max}]} dt = w_{max}$ *if* $\tilde{t} < T$ *or*

2.

$$(u_i^*(t), u_T^*(t)) = \begin{cases} (u_{max}, u_{max}), & if \quad t \in [0, \tilde{t}) \\ (u_i^*(t), 0), & if \quad t \in (\tilde{t}, T] \end{cases} \tag{71}$$

where $\int_0^{\tilde{t}} u_{max} \tau dt = z_{max}$ *if* $\tilde{t} < T$.

Remarks

In equation (70)*, when* $t > \tilde{t}$*,* $u_T^*(t)$ *signifies the optimal control specific to the travel restriction-only model. Similarly, in equation* (71)*, if* $t > \tilde{t}$*,* $u_i^*(t)$ *denotes the optimal control for the isolation-only model.*

Claim (1)

If there exists a $\tilde{t} \geq 0$ *such that* $u_i(t) = 0$ *is constant for all* $t \in (\tilde{t}, T]$*, then* $u_\tau(t) = u_\tau^*(t)$ *for all* $t \in (\tilde{t}, T]$*.*

Proof: Once $t > \tilde{t}$, the mixed isolation–travel restrictions model becomes a travel restrictions-only model, and therefore, the optimal u_τ is the optimal control for the travel restrictions-only model with parameters $\hat{t}_0 = \tilde{t},\ \hat{z}_{max} = z_{max} - \int_0^{\tilde{t}} u_\tau \tau \ dt$.

Claim (2)

If there exists a $\tilde{t} \geq 0$ *such that* $u_\tau(t) = 0$ *is constant for all* $t \in (\tilde{t}, T]$, *then* $u_i(t) = u_i^*(t)$ *for all* $t \in (\tilde{t}, T]$.

Proof: Once $t > \tilde{t}$, the mixed isolation–travel restrictions model becomes an isolation-only model, and therefore, the optimal u_i is the optimal control for the isolation-only model with parameters $\hat{t}_0 = \tilde{t}$, $\hat{w}_{max} = w_{max} - \int_0^{\tilde{t}} u_i I \, dt$.

Numerical Simulations

Figure 1: Dynamics of the isolation-only model (no importation)

Figure 2: Dynamics of the isolation-only model (with importations)

Figure 3: Dynamics of the travel restrictions-only model

Figure 4: Dynamics of the mixed model

Conclusion

References I

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