【DS】Day2(2)

Tags	
Date	@May 21, 2022
Summary	Mathematical Model

【Week1】Analysis of Algorithms

1.7 Mathematical Model

Total running time: sum of cost * frequency for all operations

Cost of Basic Operations

operation	example	nanoseconds †
integer add	a + b	2.1
integer multiply	a * b	2.4
integer divide	a / b	5.4
floating-point add	a + b	4.6
floating-point multiply	a * b	4.2
floating-point divide	a / b	13.5
sine	Math.sin(theta)	91.3
arctangent	Math.atan2(y, x)	129.0
...	...	...

† Running OS X on Macbook Pro 2.2GHz with 2GB RAM

Example: 1-sum

```
int count = 0;
for (int i = 0; i < N; ++i) {
  if (a[i] == 0)
```

```
    count++;
}
```

operation	frequency
variable declaration	2
assignment statement	2
less than compare	N + 1
equal to compare	N
array access	N
increment	N to 2 N

Example: 2-sum

```
int count = 0;
for (int i = 0; i < N; i++)
   for (int j = i+1; j < N; j++)
      if (a[i] + a[j] == 0)
         count++;
```

$$0 + 1 + 2 + \ldots + (N-1) = \frac{1}{2} N(N-1) = \binom{N}{2}$$

operation	frequency
variable declaration	N + 2
assignment statement	N + 2
less than compare	½ (N + 1) (N + 2)
equal to compare	½ N (N − 1)
array access	N (N − 1)
increment	½ N (N − 1) to N (N − 1)

tedious to count exactly

Simplification 1: cost model

Use some basic operation as a proxy for running time.

Simplification 2: Tilde notation

Ignore lower order terms.

- When N is large, terms are negligible
- When N is small, we don't care.

operation	frequency	tilde notation
variable declaration	$N + 2$	$\sim N$
assignment statement	$N + 2$	$\sim N$
less than compare	$\frac{1}{2} (N + 1) (N + 2)$	$\sim \frac{1}{2} N^2$
equal to compare	$\frac{1}{2} N (N - 1)$	$\sim \frac{1}{2} N^2$
array access	$N (N - 1)$	$\sim N^2$
increment	$\frac{1}{2} N (N - 1)$ to $N (N - 1)$	$\sim \frac{1}{2} N^2$ to $\sim N^2$

Estimating a Discrete Sum

Replace the sum with an integral, and use calculus.

Ex 1. $1 + 2 + \ldots + N$.

$$\sum_{i=1}^{N} i \sim \int_{x=1}^{N} x\,dx \sim \frac{1}{2} N^2$$

Ex 2. $1 + 1/2 + 1/3 + \ldots + 1/N$.

$$\sum_{i=1}^{N} \frac{1}{i} \sim \int_{x=1}^{N} \frac{1}{x} dx = \ln N$$

Ex 3. 3-sum triple loop.

$$\sum_{i=1}^{N} \sum_{j=i}^{N} \sum_{k=j}^{N} 1 \sim \int_{x=1}^{N} \int_{y=x}^{N} \int_{z=y}^{N} dz\,dy\,dx \sim \frac{1}{6} N^3$$