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Forced Vibration Analysis of Isotropic Thin Rectangular SSSS Plate

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Abstract
This work used the general shape function assumed by Szilard (2004) to formulate the solution to the forced vibration equation of an isotropic thin rectangular plate. By applying the appropriate boundary conditions on dimensionless co-ordinates (ζ,η)it obtainedthe shape function of an SSSS plate, in terms of a deflection constant, A. It converted the forced vibration equation to an energy equation by multiplying it with a deflection term, w, and integrating over the whole surface of the plate. By substituting the shape function and aspect ratio of the plate into the energy equation, the value of A is obtained, from which the full deflection equation is derived. Then the shear force, bending and twisting moments were obtained from the deflection equation. The values of deflections and bending moments obtained satisfied the natural and geometric boundary conditions of the plate.
Keywords: *isotropic, thin, free and forced vibration, fundamental frequency, frequency factor, dynamic load factor.*

Introduction

The bodies of vehicles, ships and aircrafts are made of plates. They are subjected to vibration arising from the engines, rough roads and airstrips. The floors and walls of structures are made of plates. These components of a building suffer from dynamic impacts and vibrations from earthquakes, seismic loads, industrial equipment, and extreme winds particularly for High-rise structures. Hence, it is necessary to understand the response of these critical elements under different dynamic loadings. Ventsel and Krauthammer[1] considered two kinds of motion of plates, namely, free and forced vibrations. *Free or natural vibration* occurs in the absence of applied loads, but may be triggered off by applying initial conditions to the plate. Free vibration analysis produces the *natural or fundamental frequency* of the plate which depends on the geometry and material properties of the plate. *Forced vibration* results from an application of time-dependent loads, which are called dynamic loads. Forced vibration analysis produces the deflection equation, bending moments and shear forces resulting from dynamic loads. Many researchers have used various methods to determine the fundamental frequency of an SSSS plate (see References below). Much work has not been done in the area of forced vibration. Szilard[2] presented a forced vibration analysis of an SSSS plate by assuming a forcing function, but he did not relate it to the fundamental natural frequency of the plate.

Methodology

Formulation of an Exact Solution to the Governing Differential Equation

Ventsel and Krauthammer (2001) obtained the governing differential equation of motion of thin plate under *forced*, *undamped* motion as follows:

$$D\left(\frac{\partial^4}{\partial x^4} + \frac{2\,\partial^4}{\partial x^2\,\partial y^2} + \frac{\partial^4}{\partial y^4}\right)w_{(x,\,y,\,t)} = q_{(x,\,y,\,t)} - m\left(\frac{\partial^2 w}{\partial t^2}\right)_{(x,\,y,\,t)} \qquad (3.1)$$

q = plate loading per unit area
$w_{(x, y, t)}$ = displacement function (positive downwards)
m = ρh = mass of plate per unit area
ρ = mass density of plate material
h = plate thickness
t = time
x, y = co-ordinates.
a, b = plate dimensions in x and y axes respectively.

$$D = \frac{Eh^3}{12(1-v^2)} \qquad (3.2)$$

D = flexural rigidity of the plate material
E = modulus of elasticity of plate material
v = Poisson's ratio of plate material

According to Szilard[2], for a harmonic vibration, the load and deflection functions can be assumed to be Equations (3.3) and (3.4) respectively.

$$q_{(x,\,y,\,t)} = q_{(x,\,y)}\mathrm{Sin}\,\omega t \qquad (3.3)$$

$$w_{(x,\,y,\,t)} = w_{(x,\,y)}\mathrm{Sin}\,\omega t \qquad (3.4)$$

where ω is the fundamental natural frequency of the plate, and $w_{(x, y)}$ is the shape function of the plate.

For convenience let us express $w_{(x, y)}$ in terms of dimensionless co-ordinates, ζ and η, (see Figure 3.1), that is:

$$\zeta = x/a \qquad (3.5)$$

$$\eta = y/b \qquad (3.6)$$

where: a and b = plate dimensions, as shown in Figure 3.1.

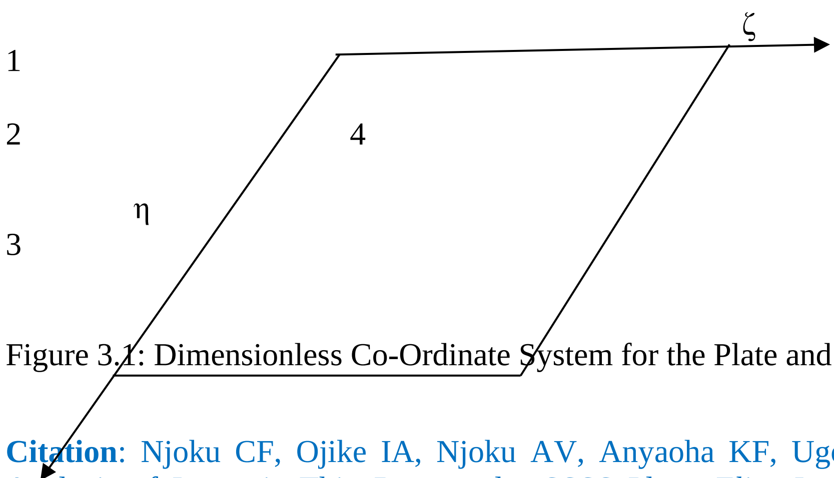

Figure 3.1: Dimensionless Co-Ordinate System for the Plate and Edge Numbering.

From Chakraverty[3], Szilard [2], and Ibearugbulem *et al.*[4] the exact solution can be assumed to be a polynomial in x and y (or, ζ and η, for the dimensionless co-ordinate system). So, we have:

$w_{(x,y)} = w_{(\zeta,\eta)} = AS_p$ (3.7)

where: A = Deflection constant

S_p = a polynomial in ζ and η

By substituting Equation (3.7) into Equation (3.4), we get the exact solution to Equation (3.1) in the form of Equation (3.8).

$w_{(x,y,t)} = w_{(\zeta,\eta,t)} = AS_p \mathrm{Sin}\omega t$ (3.8)

Shape Function of SSSS Plate

By applying the boundary conditions to the deflection functions of a two-dimensional plate, Njoku (2018) obtained Equation (3.9) as the shape function of an SSSS plate.

$w = A(\zeta - 2\zeta^3 + \zeta^4)(\eta - 2\eta^3 + \eta^4)$ (3.9)

so, $S_p = (\zeta - 2\zeta^3 + \zeta^4)(\eta - 2\eta^3 + \eta^4)$ (3.10)

Fundamental Natural Frequencies of SSSS Plate

Njoku[5] obtained Equation (3.11) as the fundamental natural frequency of a thin rectangular isotropic plate.

$$\omega = \frac{H_{b\beta}}{b^2}\sqrt{\frac{D}{m}} \quad (3.11)$$

where $H_{b\beta}$ is a numerical coefficient called the non-dimensional frequency parameter, expressed in terms of the dimension 'b', for a plate of aspect ratio $\beta = \frac{a}{b}$.

$$H_{b\beta} = \sqrt{\left(\frac{C_2}{B_2}\right)} \quad (3.12)$$

where $B_2 = \int_0^1 \int_0^1 {S_p}^2\, \partial\zeta\, \partial\eta$ (3.13)

$C_2 = \int_0^1 \int_0^1 (K_2\, S_p)\, \partial\zeta\, \partial\eta$ (3.14)

$K_2 = [\frac{1}{\beta^4}(\frac{\partial^4 S_p}{\partial\zeta^4}) + \frac{2}{\beta^2}(\frac{\partial^4 S_p}{\partial\zeta^2\, \partial\eta^2}) + (\frac{\partial^4 S_p}{\partial\eta^4})]$ (3.15)

and $\beta = \frac{a}{b}$ (3.16)

For an SSSS plate $H_{b\beta}$ was evaluated as:

$$H_{b\beta} = \sqrt{\left(\frac{97.54677239}{\beta^4} + \frac{194.8688721}{\beta^2} + 97.54677239\right)} \quad (3.17)$$

Determination of the Deflection Constant A

Before collapse can occur, the external work done by the dynamic load, q, must be equal to the internal strain energy of the plate. This satisfies the principle of conservation of energy.

Substituting Equations (3.3) and (3.4) into Equation (3.1) we get Equation (3.18).

$$D(\mathrm{Sin}\omega t)\left(\frac{\partial^4}{\partial x^4} + \frac{2\,\partial^4}{\partial x^2\, \partial y^2} + \frac{\partial^4}{\partial y^4}\right)w_{(x,y)} = q_{(x,y)}\mathrm{Sin}\omega t + m\omega^2 w\mathrm{Sin}\omega t \quad (3.18)$$

By dividing Equation (3.46)3.18 with DSinωt we obtain Equation (3.19).

$$\frac{\partial^4 w}{\partial x^4} + \frac{2\,\partial^4 w}{\partial x^2\,\partial y^2} + \frac{\partial^4 w}{\partial y^4} - \frac{q}{D} - \frac{m\omega^2 w}{D} = 0 \quad (3.19)$$

where w and q represent $w_{(x,\,y)}$ and $q_{(x,\,y)}$ respectively.
Rearranging yields:

$$q = D\left(\frac{\partial^4 w}{\partial x^4} + \frac{2\,\partial^4 w}{\partial x^2\,\partial y^2} + \frac{\partial^4 w}{\partial y^4}\right) - m\omega^2 w \quad (3.20)$$

In this form, both sides of the equation represent force per unit area. The left-hand side represents external forces, while the right-hand side represents internal forces.

Multiplying both sides of Equation (3.20) by $\frac{1}{2}$w, gives:

$$\frac{1}{2}qw = \frac{1}{2}D\left(\frac{\partial^4 w}{\partial x^4} + \frac{2\,\partial^4 w}{\partial x^2\,\partial y^2} + \frac{\partial^4 w}{\partial y^4}\right)w - \frac{1}{2}m\omega^2 w^2 \quad (3.21)$$

The left-hand side represents external work done while the right-hand side represents internal strain energy, all per unit area of the plate.

Integrating both sides of the Equation (3.21) yields the total external work and the total internal strain energy for the whole plate. Thus, integrating the equation with respect to x and y:

$$\int_0^b \int_0^a \left(\left(\frac{\partial^4 w}{\partial x^4}\right)w + \left(\frac{2\,\partial^4 w}{\partial x^2\,\partial y^2}\right)w + \left(\frac{\partial^4 w}{\partial y^4}\right)w - \frac{qW}{D} - \frac{m\omega^2 w^2}{D}\right)\partial x\partial y = 0 \quad (3.22)$$

The Equation (3.22) can be called the Energy equation.

After substituting the expressions ζ, η and β (from Equations (3.5), (3.6) and (3.16)) the Equation (3.22) becomes:

$$\int_0^1 \int_0^1 \left(\left(\frac{\partial^4 w}{a^4\,\partial \zeta^4}\right)w + \frac{2}{a^2 b^2}\left(\frac{\partial^4 w}{\partial \zeta^2\,\partial \eta^2}\right)w + \frac{1}{b^4}\left(\frac{\partial^4 w}{\partial \eta^4}\right)w - \frac{qw}{D} - \frac{m\omega^2 w^2}{D}\right)ab\,\partial\zeta\partial\eta = 0 \quad (3.23)$$

where $\partial x = a\partial\zeta$ and $\partial y = b\partial\eta$.
Multiplying Equation (3.23) by b^4 we obtain Equation (3.24).

$$\frac{ab}{b^4}\int_0^1 \int_0^1 \left(\frac{b^4}{a^4}\left(\frac{\partial^4 w}{\partial \zeta^4}\right)w + \frac{2b^2}{a^2}\left(\frac{\partial^4 w}{\partial \zeta^2\,\partial \eta^2}\right)w + \left(\frac{\partial^4 w}{\partial \eta^4}\right)w - \frac{b^4 qw}{D} - \frac{b^4 m\omega^2 w^2}{D}\right)\partial\zeta\partial\eta = 0 \quad (3.24)$$

Putting $\beta = \frac{a}{b}$ and substituting in Equation (3.24) we obtain Equation (3.25).

$$\int_0^1 \int_0^1 \left(\frac{1}{\beta^4}\left(\frac{\partial^4 w}{\partial \zeta^4}\right)w + \frac{2}{\beta^2}\left(\frac{\partial^4 w}{\partial \zeta^2\,\partial \eta^2}\right)w + \left(\frac{\partial^4 w}{\partial \eta^4}\right)w - \frac{b^4 qw}{D} - \frac{b^4 m\omega^2 w^2}{D}\right)\partial\zeta\partial\eta = 0 \quad (3.25)$$

$$\int_0^1 \int_0^1 \left(\frac{1}{\beta^4}\left(\frac{\partial^4 w}{\partial \zeta^4}\right)w + \frac{2}{\beta^2}\left(\frac{\partial^4 w}{\partial \zeta^2\,\partial \eta^2}\right)w + \left(\frac{\partial^4 w}{\partial \eta^4}\right)w\right)\partial\zeta\,\partial\eta - \int_0^1 \int_0^1 \frac{b^4 qw}{D}\,\partial\zeta\partial\eta - \int_0^1 \int_0^1 \frac{b^4 m\omega^2 w^2}{D}\,\partial\zeta\,\partial\eta = 0 \quad (3.26)$$

For free vibration, the load in Equation (3.26) is set to zero (i.e. q = 0), so the Equation (3.26) becomes:

$$\int_0^1 \int_0^1 \left(\frac{1}{\beta^4}\left(\frac{\partial^4 W}{\partial \zeta^4}\right)w + \frac{2}{\beta^2}\left(\frac{\partial^4 W}{\partial \zeta^2\,\partial \eta^2}\right)w + \left(\frac{\partial^4 W}{\partial \eta^4}\right)w\right)\partial\zeta\,\partial\eta - \int_0^1 \int_0^1 \frac{b^4 m\omega^2 W^2}{D}\,\partial\zeta\,\partial\eta = 0 \quad (3.27)$$

Substituting the Equation (3.7) into the free vibration Equation (3.27), yields Equation (3.28).

$$\int_0^1 \int_0^1 \left(\frac{1}{\beta^4}\left(\frac{\partial^4 (AS_p)}{\partial \zeta^4}\right)AS_p + \frac{2}{\beta^2}\left(\frac{\partial^4 (AS_p)}{\partial \zeta^2 \, \partial \eta^2}\right)AS_p + \left(\frac{\partial^4 (AS_p)}{\partial \eta^4}\right)AS_p\right) \partial\zeta \, \partial\eta - \int_0^1 \int_0^1 \frac{b^4 m\omega^2 (AS_p)^2}{D}) \, \partial\zeta \, \partial\eta = 0 \quad (3.28)$$

Since A is a constant, the Equation (3.28), can be re-written as:

$$A^2\int_0^1 \int_0^1 \left(\frac{1}{\beta^4}\left(\frac{\partial^4 S_p}{\partial \zeta^4}\right)S_p + \frac{2}{\beta^2}\left(\frac{\partial^4 S_p}{\partial \zeta^2 \, \partial \eta^2}\right)S_p + \left(\frac{\partial^4 SS_p}{\partial \eta^4}\right)S_p\right) \partial\zeta \, \partial\eta - A^2 \int_0^1 \int_0^1 \frac{b^4 m\omega^2 {S_p}^2}{D}) \, \partial\zeta \, \partial\eta = 0 \quad (3.29)$$

Let $K_2 = \left[\frac{1}{\beta^4}\left(\frac{\partial^4 S_p}{\partial \zeta^4}\right) + \frac{2}{\beta^2}\left(\frac{\partial^4 S_p}{\partial \zeta^2 \, \partial \eta^2}\right) + \left(\frac{\partial^4 S_p}{\partial \eta^4}\right)\right]$ (3.30)

Substituting the Equation (3.30) into the Equation (3.29) gives Equation (3.31).

$$\int_0^1 \int_0^1 (K_2 \, S_p) \, \partial\zeta \, \partial\eta = \frac{b^4 m\omega^2}{D} \int_0^1 \int_0^1 {S_p}^2 \, \partial\zeta \, \partial\eta \quad (3.31)$$

Let $B_2 = \int_0^1 \int_0^1 {S_p}^2 \, \partial\zeta \, \partial\eta$ (3.32)

and $C_2 = \int_0^1 \int_0^1 (K_2 \, S_p) \, \partial\zeta \, \partial\eta$ (3.33)

Substituting the Equations (3.32) and (3.33) into the Equation (3.31), we get:

$$C_2 = \frac{b^4 m\omega^2 B_2}{D}$$

and rearranging, yields Equation (3.34).

$$\omega^2 = \frac{C_2}{b^4 m} \frac{D}{B_2} \quad (3.34)$$

For forced vibration, ω is replaced with Ω, the forcing frequency on the plate, so the Equation (3.26) becomes:

$$\int_0^1 \int_0^1 \left(\frac{1}{\beta^4}\left(\frac{\partial^4 w}{\partial \zeta^4}\right)w + \frac{2}{\beta^2}\left(\frac{\partial^4 w}{\partial \zeta^2 \, \partial \eta^2}\right)w + \left(\frac{\partial^4 w}{\partial \eta^4}\right)w\right) \partial\zeta \, \partial\eta - \int_0^1 \int_0^1 \frac{b^4 qw}{D} \, \partial\zeta\partial\eta - \int_0^1 \int_0^1 \frac{b^4 m\Omega^2 w^2}{D} \partial\zeta \, \partial\eta = 0 \quad (3.35)$$

Substituting the Equation (3.7) into the forced vibration Equation (3.35), gives Equation (3.36).

$$\int_0^1 \int_0^1 \left(\frac{1}{\beta^4}\left(\frac{\partial^4 (AS_p)}{\partial \zeta^4}\right)AS_p + \frac{2}{\beta^2}\left(\frac{\partial^4 (AS_p)}{\partial \zeta^2 \, \partial \eta^2}\right)AS_p + \left(\frac{\partial^4 (AS_p)}{\partial \eta^4}\right)AS_p\right) \partial\zeta \, \partial\eta - \int_0^1 \int_0^1 \frac{b^4 qAS_p}{D} \, \partial\zeta\partial\eta - \int_0^1 \int_0^1 \frac{b^4 m\Omega^2 (AS_p)^2}{D}) \, \partial\zeta \, \partial\eta = 0 \quad (3.36)$$

The Equation (3.36) can be re-written as follows:

$$A^2\int_0^1 \int_0^1 \left(\frac{1}{\beta^4}\left(\frac{\partial^4 S_p}{\partial \zeta^4}\right)S_p + \frac{2}{\beta^2}\left(\frac{\partial^4 S_p}{\partial \zeta^2 \, \partial \eta^2}\right)S_p + \left(\frac{\partial^4 S_p}{\partial \eta^4}\right)S_p\right) \partial\zeta \, \partial\eta - A^2 \int_0^1 \int_0^1 \frac{b^4 m\Omega^2 {S_p}^2}{D} \partial\zeta \, \partial\eta = A \int_0^1 \int_0^1 \frac{b^4 qS_p}{D} \, \partial\zeta\partial\eta \quad (3.37)$$

By putting the Equation (3.30) into the Equation (3.37), Equation (3.38) is obtained.

$$A^2\int_0^1 \int_0^1 (K_2 \, S_p) \, \partial\zeta \, \partial\eta - A^2 \frac{b^4 m\Omega^2}{D} \int_0^1 \int_0^1 {S_p}^2 \, \partial\zeta \, \partial\eta = A \frac{b^4 q}{D} \int_0^1 \int_0^1 S_p \, \partial\zeta\partial\eta \quad (3.38)$$

Putting the Equations (3.32) and (3.33) into the Equation (3.208)3.38 results into Equation (3.39).

$$A\left(C_2 - \frac{b^4 m\Omega^2}{D} B_2\right) = \frac{b^4 q}{D} \int_0^1 \int_0^1 S_p \, \partial\zeta\partial\eta \quad (3.39)$$

Rearranging Equation (3.39) gives:

$$A = \frac{\frac{b^4 q}{D} \int_0^1 \int_0^1 S_p \, \partial\zeta \, \partial\eta}{\left(C_2 - \frac{b^4 m\Omega^2}{D} B_2\right)} \quad (3.40)$$

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The Equation (3.40) gives the deflection constant A of forced vibration, while $S_p$is the deflection polynomial in ζ and η.

Determination of the Dynamic Load Factor DLF

The dynamic load factor DLF relates the dynamic response of a point (x, y) on the plate to the static response of the same point under the same loading. It is the ratio of dynamic deflection to static deflection at a point in the plate.

Under static loading conditions (pure bending), there is no vibration ($\Omega = 0$), so Equation (3.40) becomes:

$$A = \frac{\frac{b^4 q}{D}\int_0^1\int_0^1 S_p\ \partial\zeta\,\partial\eta}{C_2} \qquad (3.41)$$

So, the dynamic load factor (DLF) can be obtained by dividing Equation (3.40) with Equation (3.41), thus:

$$DLF = \frac{C_2}{(C_2 - \frac{b^4 m\Omega^2}{D}B_2)} \qquad (3.42)$$

After dividing numerator and denominator by C_2, Equation (3.42) becomes:

$$DLF = \frac{1}{(1 - \frac{b^4 m\Omega^2}{D}\frac{B_2}{C_2})} \qquad (3.43)$$

But, substituting $\omega^2 = \frac{C_2}{b^4 m}\frac{D}{B_2}$ from Equation (3.34) in the denominator of Equation (3.43), we get:

$$DLF = \frac{1}{(1 - \frac{\Omega^2}{\omega^2})} \qquad (3.44)$$

Equation (3.44) can be re-written as:

$$DLF = \frac{1}{(1 - N^2)} \qquad (3.45)$$

where N is called the frequency factor, representing the ratio of forcing frequency (for forced vibration) to fundamental natural frequency (for free vibration).

$$N = \frac{\Omega}{\omega}, \qquad (0 \le N^2 \le 1) \qquad (3.46)$$

Determination of the Numerical Deflection Coefficient G

Rearranging Equation (3.34), we obtain Equation (3.47).

$$\frac{b^4 m B_2}{D} = \frac{C_2}{\omega^2} \qquad (3.47)$$

By substituting Equation (3.47) into Equation (3.40) we get Equation (3.48).

$$A = \frac{\frac{b^4 q}{D}\int_0^1\int_0^1 S_p\,\partial\zeta\,\partial\eta}{(C_2 - \frac{\Omega^2}{\omega^2}C_2)} \qquad (3.48)$$

Substituting Equation (3.45) into Equation (3.48) and simplifying we get Equation (3.49).

$$A = \frac{\frac{b^4 q}{D}\int_0^1\int_0^1 S_p\,\partial\zeta\,\partial\eta}{C_2(1 - N^2)} \qquad (3.49)$$

Substituting Equation (3.45) into Equation (3.49) we get Equation (3.50).

$$A = \left(\frac{\frac{b^4 q}{D}\int_0^1\int_0^1 S_p \,\partial\zeta\,\partial\eta}{C_2}\right)(DLF) \tag{3.50}$$

Equation (3.49) can be re-written as:

$$A = Gb^4q/D \tag{3.51}$$

Where G is a numerical deflection coefficient.

$$G = \frac{\int_0^1\int_0^1 S_p \,\partial\zeta\,\partial\eta}{C_2(1 - N^2)} \tag{3.52}$$

$$\text{or,}\ \ G = \frac{\int_0^1\int_0^1 S_p \,\partial\zeta\,\partial\eta}{C_2}(DLF) \tag{3.53}$$

Determination of the Dynamic Deflection Equation

From Equations (3.8) and (3.51), the deflection is given by Equation (3.54).

$$w_{(x,\,y,\,t)} = w_{(\zeta,\,\eta,\,t)} = Gb^4qS_p(\mathrm{Sin}\Omega t)/D \tag{3.54}$$

Putting Equation (3.46) into Equation (3.54) yields the final deflection equation as:

$$w_{(x,\,y,\,t)} = w_{(\zeta,\,\eta,\,t)} = Gb^4qS_p(\mathrm{Sin}N\omega t)/D \tag{3.55}$$

Determination of the Slopes

The slopes in the ζ and η directions are given by the following derivatives: $\frac{\partial w_{(\zeta,\eta,t)}}{\partial\zeta}$ and $\frac{\partial w_{(\zeta,\eta,t)}}{\partial\eta}$, respectively.

Determination of Bending and Twisting Moments

The equations derived by Szilard[2] for moments, M_x, M_y and M_{xy}, on the plate were used in this work. They are presented in Equations (3.56) to (3.58).

$$M_x = -D\left(\frac{\partial^2 w}{\partial x^2} + v\frac{\partial^2 w}{\partial y^2}\right) \tag{3.56}$$

$$M_y = -D\left(\frac{\partial^2 w}{\partial y^2} + v\frac{\partial^2 w}{\partial x^2}\right) \tag{3.57}$$

$$M_{xy} = M_{yx} = -D(1-v)\left(\frac{\partial^2 w}{\partial x\,\partial y}\right) \tag{3.58}$$

Expressing Equations (3.56), (3.57) and (3.58) in dimensionless form we get Equations (3.59), (3.60) and (3.61) for M_x, M_y and M_{xy}, respectively.

The bending moment, M_x, about x-axis is given by Equation (3.59).

$$M_x = -D\left(\frac{\partial^2 w}{a^2\,\partial\zeta^2} + \frac{v}{b^2}\frac{\partial^2 w}{\partial\eta^2}\right) = -\frac{D}{a^2}\left(\frac{\partial^2 w}{\partial\zeta^2} + v\beta^2\frac{\partial^2 w}{\partial\eta^2}\right) \tag{3.59}$$

The bending moment, M_y, about y-axis is given by Equation (3.60).

$$M_y = -D\left(\frac{v\,\partial^2 w}{a^2\,\partial\zeta^2} + \frac{1}{b^2}\frac{\partial^2 w}{\partial\eta^2}\right) = -\frac{D}{a^2}\left(\frac{v\,\partial^2 w}{\partial\zeta^2} + \beta^2\frac{\partial^2 w}{\partial\eta^2}\right) \tag{3.60}$$

The twisting moment, M_{xy}, can be obtained from Equation (3.61).

$$M_{xy} = M_{yx} = -\frac{D\beta(1-v)}{a^2}\frac{\partial^2 w}{\partial\zeta\,\partial\eta} \tag{3.61}$$

Values of Bending and Twisting Moment Coefficients

For each value of aspect ratio $\beta = \frac{a}{b}$, the expressions for $w_{(x,\ y,\ t)}$, and the values of ζ and η, are substituted into the Equations (3.59) to (3.61) to obtain the bending and twisting moments at midspan and at the supports, in the form:

$$M_x = Zqb^2 \tag{3.62}$$

$$M_y = Jqb^2 \tag{3.63}$$

$$M_{yx} = Iqb^2 \tag{3.64}$$

$$M_{xy} = Kqb^2 \tag{3.65}$$

where: Z, J, I and K are numerical coefficients.

Determination of Shear Forces

The equations derived by Szilard[2] for shears forces,$Q_x$andQ_y,on the plate were used in this work. They are presented in Equations (3.66) and (3.67).

$$Q_x = \frac{\partial M_x}{\partial X} + \frac{\partial M_{xy}}{\partial Y} = -\ D\left(\frac{\partial^3 w}{\partial x^3} + \frac{\partial^3 w}{\partial x\, \partial y^2}\right) \tag{3.66}$$

$$Q_y = \frac{\partial M_y}{\partial y} + \frac{\partial M_{xy}}{\partial x} = -\ D\left(\frac{\partial^3 w}{\partial x^2\, \partial y} + \frac{\partial^3 w}{\partial y^3}\right) \tag{3.67}$$

The shear forces, Q_x and Q_y, for plates are obtained by expressing Equations (3.66) and (3.67) in a dimensionless form.

The shear force, Q_x, about x-axis is as follows:

$$Q_x = \frac{\partial M_x}{\partial x} + \frac{\partial M_{xy}}{\partial y} = -\ \frac{D}{a^3}\left(\frac{\partial^3 w}{\partial \zeta^3} + (2 - v)\beta^2 \frac{\partial^3 w}{\partial \zeta\, \partial \eta^2}\right) \tag{3.68}$$

The shear force, Q_y, about y-axis is given by Equation (3.69).

$$Q_y = \frac{\partial M_y}{\partial y} + \frac{\partial M_{xy}}{\partial x} = -\ \frac{D}{a^3}\left(\beta^3 \frac{\partial^3 w}{\partial \eta^3} + (2 - v)\beta \frac{\partial^3 w}{\partial \eta\, \partial \zeta^2}\right) \tag{3.69}$$

Values of Shear Forces Coefficients

For each value of the aspect ratio, β, the expressions for $w_{(x,\ y,\ t)}$ are substituted into Equations (3.68) and (3.69) to obtain, respectively, the shear forces (reactions) at the supports, in the form:

$$Q_x = Lqb \tag{3.70}$$

$$Q_y = Rqb \tag{3.71}$$

where: L and R are numerical coefficients.

Results and Discussions

Deflection Equation

From Equation (3.55) above the dynamic deflection equation is given by:

$$w_{(\zeta,\ \eta,\ t)} = Gb^4 q S_p(\mathrm{Sin} N\omega t)/D \tag{3.55}$$

By using Equation (3.52), G is obtained as in Equation (4.1).

$$G = \frac{0.04}{(1 - N^2)(0.236190476(\frac{1}{\beta^4}) + 0.471836734(\frac{1}{\beta^2}) + 0.236190476)} \tag{4.1}$$

So the final dynamic deflection is given by Equation (4.2).

$$w_{(\zeta,\ \eta,\ t)} = \frac{Gqb^4}{D}(\zeta - 2\zeta^3 + \zeta^4)(\eta - 2\eta^3 + \eta^4)\mathrm{Sin} N\omega t \tag{4.2}$$

The values of G for different values of N, β and φ ($=\frac{b}{a}=\frac{1}{\beta}$) are presented in Table 4.1.

The sine function in the deflection equation obtained in this work helps to satisfy the natural boundary conditions because vibrational deflections are usually sinusoidal. Also, sine is the function that has a zero value when the time t = 0, at the onset of the vibration. So we can say that the amplitude of the vibration is given by Equation (4.3).

Table 4.1: Values of Numerical Coefficient of Deflection, G, for SSSS Plate

β \ N	0.000	0.200	0.400	0.600	0.800	1.000	N / φ
0.500	0.006777	0.007059	0.008067	0.010589	0.018824	∞	**2.000**
0.526	0.007957	0.008289	0.009473	0.012433	0.022103	∞	**1.900**
0.556	0.009447	0.009841	0.011247	0.014761	0.026243	∞	**1.800**
0.588	0.011183	0.011649	0.013314	0.017474	0.031065	∞	**1.700**
0.625	0.013369	0.013926	0.015916	0.020889	0.037136	∞	**1.600**
0.666	0.015997	0.016664	0.019044	0.024995	0.044436	∞	**1.500**
0.714	0.019319	0.020124	0.022998	0.030185	0.053663	∞	**1.400**
0.769	0.023399	0.024374	0.027856	0.036561	0.064998	∞	**1.300**
0.833	0.028435	0.02962	0.033851	0.044429	0.078985	∞	**1.200**
0.909	0.034687	0.036132	0.041294	0.054198	0.096353	∞	**1.100**
1.000	0.042363	0.044128	0.050432	0.066192	0.117675	∞	**1.000**

$$\text{Amplitude} = (Gqb^4S_p)/D \qquad (4.3)$$

which occurs at the following values of time: $\frac{\pi}{2N\omega}, \frac{3\pi}{2N\omega}$. Since S_p is a polynomial in ζ and η, it means that the amplitude of vibration at any particular time varies with position (co-ordinates) on the plate.

4.2 Bending and Twisting Moment Coefficients

The coefficients Z, J, I and K, obtained for bending and twisting moments, M_x, M_y, M_{yx} and M_{xy}, are presented in Table 4.2.

Table 4.2: Coefficients of Bending and Twisting Moments for SSSS Plate

S/N	Moment Type	Coefficients
1	M_x	$Z = -\frac{G((-12\zeta+12\zeta^2)(\eta-2\eta^3+\eta^4)+v\beta^2(\zeta-2\zeta^3+\zeta^4)(-12\eta+12\eta^2))\mathrm{Sin}N\omega t}{\beta^2}$
2	M_y	$J = -\frac{G(v(-12\zeta+12\zeta^2)(\eta-2\eta^3+\eta^4)+\beta^2(\zeta-2\zeta^3+\zeta^4)(-12\eta+12\eta^2))\mathrm{Sin}N\omega t}{\beta^2}$
3	M_{yx}	$I = -\frac{G(1-v)(1-6\zeta^2+4\zeta^3)(1-6\eta^2+4\eta^3)\mathrm{Sin}N\omega t}{\beta}$

4	M_{xy}	K = - $\frac{G(1-v)(1-6\zeta^2+4\zeta^3)(1-6\eta^2+4\eta^3)SinN\omega t}{\beta}$

4.3 Shear Force Coefficients

The coefficients L and R, obtained for shear forces, $Q_x$andQ_y, are presented in Table 4.3.

Table 4.3: Coefficients of Shear Force for SSSS Plate

S/N	Moment Type	Coefficients
1	Q_x	L = - $\frac{G((-12+24\zeta)(\eta-2\eta^3+\eta^4)+(2-v)\beta^2(1-6\zeta^2+4\zeta^3)(-12\eta+12\eta^2))SinN\omega t}{\beta^3}$
2	Q_y	R = - $\frac{G(\beta^3(\zeta-2\zeta^3+\zeta^4)(-12\eta^3+24\eta)+\beta(2-v)(-12\eta+12\zeta^2)(1-6\eta^2+4\eta^3))SinN\omega t}{\beta^3}$

4.4 Checks on the Satisfaction of Boundary Conditions

Values of deflectionandbending moments at the edges of the plate must be zerosin order to satisfy the boundary conditions for an SSSS plate.

4.4.1 Deflection

At time t = 0, SinNωt = 0. So deflection for the plate will be zero. At other values of time, t, the deflection at the S boundaries must be zero.

So, from Equation (4.2):

At edge 1, $\zeta = \frac{1}{2}$ and $\eta = 0$. So, $w_{(\zeta,\eta,t)} = \frac{Gqb^4}{D}(\frac{1}{2} - \frac{1}{4} + \frac{1}{16})(0-0+0)SinN\omega t = 0$.

At edge 2, $\zeta = 0$ and $\eta = \frac{1}{2}$. So, $w_{(\zeta,\eta,t)} = \frac{Gqb^4}{D}(0-0+0)(\frac{1}{2} - \frac{1}{4} + \frac{1}{16})SinN\omega t = 0$.

At edge 3, $\zeta = \frac{1}{2}$ and $\eta = 1$. So, $w_{(\zeta,\eta,t)} = \frac{Gqb^4}{D}(\frac{1}{2} - \frac{1}{4} + \frac{1}{16})(1-2+1)SinN\omega t = 0$.

At edge 4, $\zeta = 1$ and $\eta = \frac{1}{2}$. So, $w_{(\zeta,\eta,t)} = \frac{Gqb^4}{D}(1-2+1)(\frac{1}{2} - \frac{1}{4} + \frac{1}{16})SinN\omega t = 0$.

4.4.2 Bending Moments

At time t = 0, SinNωt = 0. So bending moments for the plate will be zero. At other values of time, t, the bending moments coefficients Z and J at the S boundaries must be zero.

So, from Table (4.2):

At edge 1, $\zeta = \frac{1}{2}$ and $\eta = 0$. So, J = - $\frac{G(v(-6+3)(0-0+0)+\beta^2(\frac{1}{2}-\frac{1}{4}+\frac{1}{16})(-0+0))SinN\omega t}{\beta^2} = 0$.

At edge 2, $\zeta = 0$ and $\eta = \frac{1}{2}$. So, Z = - $\frac{G((-0+0)(\frac{1}{2}-\frac{1}{4}+\frac{1}{16})+v\beta^2(0-0+0)(-6+3))SinN\omega t}{\beta^2} = 0$.

At edge 3, $\zeta = \frac{1}{2}$ and $\eta = 1$. So, J = - $\frac{G(v(-6+3)(1-2+1)+\beta^2(\frac{1}{2}-\frac{1}{4}+\frac{1}{16})(-12+12))SinN\omega t}{\beta^2} = 0$.

At edge 4, $\zeta = 1$ and $\eta = \frac{1}{2}$. So, Z = - $\frac{G((-12+12)(\frac{1}{2}-\frac{1}{4}+\frac{1}{16})+v\beta^2(1-2+1)(-6+3))SinN\omega t}{\beta^2} = 0$.

The results of these checks show that the deflection and bending moments obtained in this work satisfy the natural and geometric boundary conditions of the SSSS plate.

Conclusion

The work has come up with the exact values of dynamic deflection and internal forces (shear forces, bending and twisting moments) induced on an SSSS plate by a uniform dynamic load. Using these exact values of these quantities in design will result in efficient, safe and economic designs of plates. By means of a frequency factor, N, it has related these quantitiesgenerated by any value of forcing frequency, to the plate's fundamental natural frequency, ω. All these quantities depend on a numerical deflection coefficient, G, the co-ordinates (ζ, η) of the point on the plate and also the aspect ratio,β,of the plate.

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