1 Functions and Permutations

A function f from a set D into a set C is a map with first input from D and output from C such that each element of D has a unique output.

More formally, a function $f : D \to C$ can be defined as a set of ordered pairs

$$f \subseteq D \times C = \{(d, c) : d \in D, c \in C\}$$

such that

$$\text{if } (d, c_1) \in f \text{ and } (d, c_2) \in f \text{ then } c_1 = c_2.$$

If $f : D \to C$ is a function then D is the **domain** of f and C is the **codomain** of f.

We may write a function in a number of ways. We can write it as a collection of ordered pairs. We *may* be able to provide a formula for the outputs $f(d)$. We can also write the function using **two-row** notation in which the first row is the domain and the second row is the output of the domain elements.

For example, suppose that our domain is $\mathbb{Z}_7 := \{0, 1, 2, 3, 4, 5, 6\}$, the set of integers modulo 7. (That is, our domain is the remainders which appear upon dividing integers by 7.)

Let

$$f := \{(0, 0), (1, 1), (2, 4), (3, 2), (4, 2), (5, 4), (6, 1)\}.$$

This set of ordered pairs is the function which maps 0 to 0, 1 to 1, 2 to 4, 3 to 2, ... and so on.

We could write this function using two-row notation as follows:

$$f = \begin{bmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 \\ 0 & 1 & 4 & 2 & 2 & 4 & 1 \end{bmatrix}.$$

We might also write this function in the form $f : \mathbb{Z}_7 \to \mathbb{Z}_7$ where

$$f(z) \equiv z^2 \mod 7.$$

(For example, $f(3) = 3^2 = 9 \equiv 2 \mod 7$.)

The composition function $g \circ f$ will be (in set notation)

1.1 Composition of functions

If $f : D \to C$ is a function and $g : C \to B$ is a function then we can create the composition function $g \circ f : D \to B$ by defining $(g \circ f)(d) := g(f(d))$.

For example, if $g : \mathbb{Z}_7 \to \mathbb{Z}$ is defined by $g(x) = 3x + 1$ then in set notation,

$$g = \{(0, 1), (1, 4), (2, 7), (3, 10), (4, 13), (5, 16), (6, 19)\}$$

and in two row notation

$$g = \begin{bmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 4 & 7 & 10 & 13 & 16 & 19 \end{bmatrix}$$

then the composition function $g \circ f$ maps 2 to 13 since $f(2) = 4$ and $g(4) = 13$.

The composition function $g \circ f$ looks like this in set notation:

$$g \circ f = \{(0, 1), (1, 4), (2, 13), (3, 7), (4, 7), (5, 13), (6, 4)\}$$

and in two row notation

$$g \circ f = \begin{bmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 4 & 13 & 7 & 7 & 13 & 4 \end{bmatrix}.$$

1.2 Permutations

A function $f : D \to C$ is **one-to-one** if

$$f(x) = f(y) \implies x = y.$$

The function is **onto** if for each element $c \in C$, it is true that there is a $d \in D$ with $f(d) = c$.

A function which is both one-to-one and onto is called a **bijection** or a one-to-one correspondence.

If, for a bijection f, both the domain D and the codomain C are the same then f is called a **permutation** of D.

A set of permutations which are **closed** under function composition is called a **group.** We will look at examples of all of these.

One type of group is the set of all permutations from a set to itself.

A permutation of a set X is a one-to-one function from X onto itself. The set of all permutations of X will be written Sym_X. If $X = \{1, 2, 3, \ldots, n\}$ we will write S_n for Sym_X.

If X is a finite set, then we can explicitly write out the permutations. We will use **two-row** notation first. Write out the domain on the top row and the images on the bottom row. For example, let α be the permutation $\begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 6 & 4 & 5 & 8 & 7 & 2 & 3 & 1 \end{bmatrix}$. This mean that $\alpha(1) = 6; \alpha(2) = 4$, etc.

There is a simpler notation, **cycle notation**, where we write a permutation as a collection of cycles. For example, α, above, sends 1 to 6 and then also sends 6 to 2, so we might write $\alpha = (1, 6, 2, \ldots.)$ Filling this out, $\alpha = (16248)(357)$.

Worksheet on Functions & Permutations

1. (Composition of relations) Consider the functions

$$f = \{(1,3),(2,3),(3,3),(4,3)\}, g = \{(1,1),(2,2),(3,3),(4,4)\} \text{ and } h = \{(1,2),(2,3),(3,4),(4,1)\}$$

with domain $A := \{1,2,3,4\}$

   (a) Write out the two row notation for the functions f, g and h.
   (b) Write out the two row notation for the functions $f \circ f$, $f \circ g$ and $g \circ f$.
   (c) Write out the two row notation for the functions $f \circ h$, $h \circ f$.
   (d) For each of the functions f, g, and h, determine if the function has an inverse. If it does, write out the two row notation for the inverse function.

2. Suppose $f := \{(1,1),(2,1),(3,4),(4,3)\}$ and $g := \{(1,2),(2,3),(3,4),(4,1)\}$ are functions on the set $X = \{1,2,3,4\}$.

   Compute:

   (a) $f \circ f$
   (b) $g \circ f$
   (c) $f \circ g$
   (d) $g \circ g$

   (Note – if it helps – one can write the functions f and g in "two-row" notation as follows:

$$f := \begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 4 & 3 \end{bmatrix} \text{ and } g := \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{bmatrix}$$

   You may use two-row notation (*if* you wish) *or* write your functions as ordered pairs.)

3. Suppose $f := \begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 4 & 3 \end{bmatrix}$ and $g := \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{bmatrix}$ are functions on the set $X = \{1,2,3,4\}$.

   Compute $f \circ f$, $g \circ f$, $f \circ g$ and $g \circ g$.