Fully Positional Linear Models

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Introduction

- Previous Work - Spatial Storm Surge model
- Extension to Previous model
- General Positional Modeling Framework

Spatial Storm Surge model

- Model the 100-year return period storm along the entire coast of Florida
- Approach - Model the yearly surge maxima as GEV distributed
- Regress GEV parameters at each location, assuming spatial correlation weighted by distance.
- Same model used in Scott and Huang [2025]

Data

- Storm surges (non-tidal residuals), at gauges provided by Dr. Thomas Wahl's team at the Coastal Risks and Engineering lab
- ERA5-Interim - wind speed, pressure, precipitation at gauge, 1979 - present, Hersbach et al. [2020]

Notation

- Anywhere we see an s represents a specific positiion in space
- S then represents a list of all of the locations we use
- i represents the different times the observations occur at

GEV

$$p(y_{is}) = \frac{1}{\sigma_s}\left(1 + \xi_s\left(\frac{y_{is} - \mu_s}{\sigma_s}\right)\right)^{-(1+\frac{1}{\xi_s})}.$$
$$\exp\left(-\left(1 + \xi_s\left(\frac{y_{is} - \mu_s}{\sigma_s}\right)\right)^{-\frac{1}{\xi_s}}\right)$$

- Models maxima of iid samples
- Assume $\xi_s = \frac{\sigma_s}{\mu_s}$
  - i.e. the distribution has a minimum value of 0

Distributional Assumption

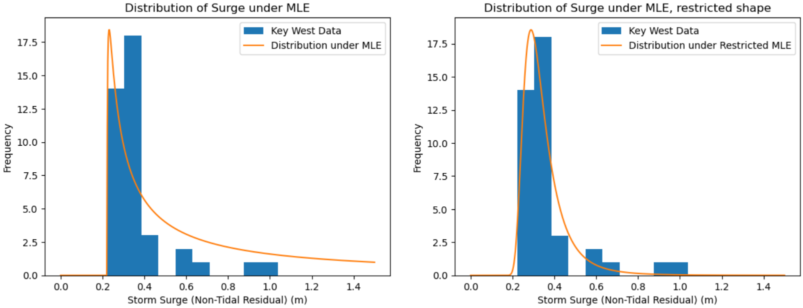


Figure: Left: Unrestricted GEV MLE for Key West data. Right: MLE, assuming that $\xi_s = \frac{\sigma_s}{\mu_s}$.

- GEV needs additional assumptions

Regression

$$\ln(\sigma_s) = \beta^\top X_s + a_s + \epsilon_{\sigma,s} \qquad \epsilon_{\sigma,s} \sim N(0, \sigma_\sigma^2)$$
$$\mu_s = \gamma^\top X_s + b_s + \epsilon_{\mu,s} \qquad \epsilon_{\mu,s} \sim N(0, \sigma_\mu^2)$$

- Assumes these parameters do not change over time
- At location, one X_s is associated to one σ_s and μ_s, which are in turn associated with many actual measurements y_{is}
- Use Gaussian process prior for overall a, b vectors
- Approach inspired by [Boumis et al., 2023] and [He and Huang, 2024].

Spatial Random Effects

$$a \sim N(0, \sigma_a^2 K_{\phi_a}(S, S)) \qquad \sigma_a^2 \sim \mathrm{Gamma}(\alpha, \theta)$$

$$K_\phi(s, s') = e^{-\left(\mathrm{dist}(s,s')^2\right)\phi}$$

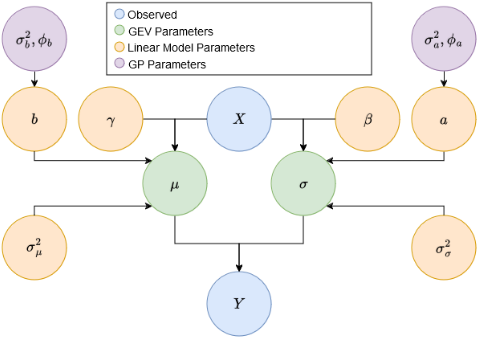
- Assumes that spatial correlations fall off according to the right half of some gaussian curve
  - Height of gaussian curve controlled by σ_a^2
  - Spread controlled by ϕ
- b (random effect for μ) will be modeled similarly

Generalization to Nearby Positions

- Multivariate normal conditioning rule:

$$\begin{bmatrix} \boldsymbol{f} \\ \boldsymbol{f^*} \end{bmatrix} \sim N\left(\mathbf{0}, \begin{bmatrix} \boldsymbol{\Sigma}_{11} & \boldsymbol{\Sigma}_{12} \\ \boldsymbol{\Sigma}_{21} & \boldsymbol{\Sigma}_{22} \end{bmatrix}\right)$$
$$\boldsymbol{f^*}|\boldsymbol{f} \sim N(\boldsymbol{\Sigma}_{21}\boldsymbol{\Sigma}_{11}^{-1}\boldsymbol{f}, \boldsymbol{\Sigma}_{22} - \boldsymbol{\Sigma}_{21}\boldsymbol{\Sigma}_{11}^{-1}\boldsymbol{\Sigma}_{12})$$

Model Visualization



$$p(. \mid y) \propto \underbrace{p(y|\mu_s, \sigma_s)}_{\text{GEV likelihood}}$$

$$\cdot \underbrace{p(\sigma_s|\beta, a, \sigma_\sigma^2) \cdot p(\beta, \sigma_\sigma^2) \cdot p(a|\phi_a, \sigma_a^2) \cdot p(\phi_a, \sigma_a^2)}_{\text{Model for } \sigma_s}$$

$$\cdot \underbrace{p(\mu_s|\gamma, b, \sigma_{\mu_s}^2) \cdot p(\gamma, \sigma_\mu^2) \cdot p(b|\phi_b, \sigma_b^2) \cdot p(\phi_b, \sigma_b^2)}_{\text{Model for } \mu_s}$$

Gibbs Sampling Details

- GEV likelihood has no conjugate prior, so each μ_s and σ_s was sampled with Metropolis-Hastings.
- Gaussian Process prior is conjugate for mean of Normal Distribution, this part causes no issues
- Length scale parameters ϕ are used to create the covariance matrix $K_\phi(S, S)$, also non-conjugate and sampled via MH.

Results

Regression Coefficient	95 % C.I.
Intercept	$[0.178, 0.773]$
Sea Level Pressure	$[-0.199, -0.038]$
Wind	$[-0.060, 0.016]$
Precipitation	$[-0.066, 0.030]$

Table: 95% credible intervals for the model coefficients for μ.

- For every 146.7 pascal decrease in the mean annual minimum sea level pressure at a location, we expect somewhere from a 0.038 meter to 0.199 meter increase in the GEV location parameter μ.

Results

Parameter	95 % C.I.
ϕ	$[3.03, 133.7]$
σ_b^2	$[0.0061, 0.34]$

Table: 95% credible intervals for the GP parameters for μ.

- Intercept correlation at locations less than 50 miles apart are 0.2 or greater.
- The correlation decays to 0.2 at a distance of 50 to 300 miles.
- The error variance for μ is between 0.001 and 0.009; spatial correlations may* explain 1.027 to 141 times as much variance in μ between locations than the error term.
  - * Simulations show that this estimate is not always reliable.

Results

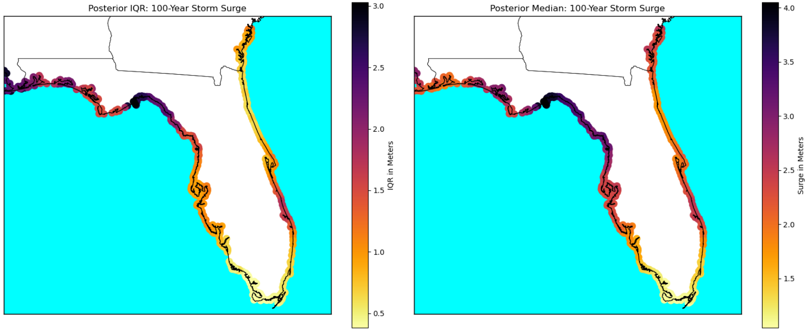


Figure: Left: Posterior IQR of 100-year storm surges. Right: Posterior median for the same.

Moving Forward

- I find this model difficult to interpret, as neither μ_s nor σ_s are directly related to typical values of interest.
- Justification for assumption that $\xi_s = \frac{\sigma_s}{\mu_s}$ is not clear theoretically.
- In this section, bolded lowercase letters indicate vectors, and bolded uppercase indicates matrices.

Frechet Distribution

$$X \sim GEV(\xi, \sigma, \mu) \to 1 + \frac{\xi}{\sigma}(x - \mu) = Y \sim Fr\left(\alpha = \frac{1}{\xi}, \sigma = 1\right)$$

$$f(y) = \frac{\alpha}{\sigma}\left(\frac{y}{\sigma}\right)^{-1-\alpha} e^{-\left(\frac{y}{\sigma}\right)^{-\alpha}}$$

Frechet Distribution

- This distribution has a relatively nice analytical median, we will use μ to denote the Frechet median:

$$\mu = \frac{\sigma}{\ln(2)^{\frac{1}{\alpha}}}$$

- Reparameterize distribution in terms of α and μ.

Reparameterized Frechet

$$f(y_{is}) = \frac{\alpha_s}{\mu_s \ln(2)^{\frac{1}{\alpha_s}}} \left(\frac{y_{is}}{\mu_s \ln(2)^{\frac{1}{\alpha_s}}} \right)^{-1-\alpha_s} \exp\left(- \left(\frac{y_{is}}{\mu_s \ln(2)^{\frac{1}{\alpha_s}}} \right)^{-\alpha_s} \right)$$

- Likelihood at a location simplifies nicely, $\tilde{Y}_s$ represents the geometric mean of all observations at location s.

$$L_s = \left(\frac{\alpha_s \mu_s^{\alpha_s} \ln(2)}{\tilde{Y}_s^{\,1+\alpha}} \right)^{n_s} \exp\left(- \ln(2) \mu_s^{\alpha_s} \sum_{i=1}^{n_s} y_{is}^{-\alpha_s} \right)$$

Hierarchical Model

$$
\begin{aligned}
\boldsymbol{Y} &\sim Fr(\boldsymbol{\alpha}, \boldsymbol{\mu}) \\
\boldsymbol{\alpha} &= \alpha_0 + \boldsymbol{a} + \boldsymbol{\epsilon_{\alpha}} \\
\boldsymbol{\mu} &= \mu_0 + \boldsymbol{X}\beta + \boldsymbol{m} + \boldsymbol{\epsilon_{\mu}} \\
\boldsymbol{\epsilon_{\alpha}} &\sim N(0, \sigma_{\alpha}^2 \boldsymbol{I}) \\
\boldsymbol{\epsilon_{\mu}} &\sim N(0, \sigma_{\mu}^2 \boldsymbol{I}) \\
\boldsymbol{a} &\sim N(0, \sigma_a^2 \boldsymbol{\Sigma_{\phi_a}}) \\
\boldsymbol{m} &\sim N(0, \sigma_m^2 \boldsymbol{\Sigma_{\phi_m}})
\end{aligned}
$$

Interpretation of Parameters

- α_s represents the spread of the distribution about the median
- μ_s represents the median of the distribution
- No need for tricky assumptions

Model Differences

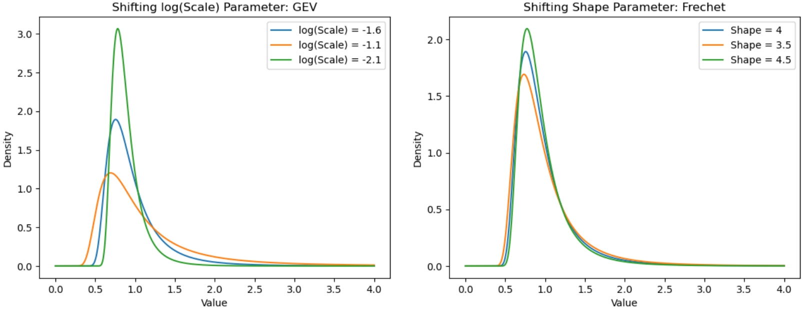


Figure: Left: Effect of shifting $\ln(\sigma)$ in GEV, Right: Effect of shifting α in median-parameterized Frechet

- Use inverse link instead of log link for σ in GEV model

Proposed Work

- Apply this model to skew-surge maxima with extended ERA5 data
- Add a predictor that increases yearly to check for increasing storm severity
  - Difference between daily maximum water level and daily maximum predicted tide
- Evaluate scalability of various posterior inference methods:
  - Hamiltonian Monte Carlo (e.g. Gelman et al. [2014])
  - Variational Inference (e.g. David M. Blei and McAuliffe [2017])
  - Integrated Nested Laplace Approximations (e.g. Rue et al. [2009])

Variational Inference

- Relatively new, quick explanation:
- Assuming some approximation to the posterior, what is the best possible distribution for this approximation?
- Minimize Kullback-Leibler divergence between posterior P and "Variational Distribution" Q

$$
\begin{aligned}
KL(Q,P) &= \int_x q(x)\ln\left(\frac{q(x)}{p(x)}\right)dx \\
&= E_Q(\ln(q(x)) - \ln(p(x)))
\end{aligned}
$$

Variational Inference

- How to choose posterior approximation?
- Assume independence of parameters, called mean field approximation
- Analytically best distributions Q are available when a conjugate prior is used for a given parameter in a model
- Otherwise, approximate with something, e.g. another Gaussian process

$$Q(\ldots) = Q_{\boldsymbol{\mu}}(\boldsymbol{\mu}) Q_{\boldsymbol{\alpha}}(\boldsymbol{\alpha}) Q_{\boldsymbol{m}}(\boldsymbol{m}) Q_{\boldsymbol{a}}(\boldsymbol{a}) Q_{\sigma^2_\alpha}(\sigma^2_\alpha) Q_{\sigma^2_a}(\sigma^2_a) Q_{\phi_a}(\phi_a) Q_{\alpha_0}(\alpha_0) \cdot \\ Q_{\sigma^2_\mu}(\sigma^2_\mu) Q_{\sigma^2_m}(\sigma^2_m) Q_{\phi_m}(\phi_m) Q_{\mu_0}(\mu_0)$$

Proposesd Work

- Generalize method to all models of the following form:

$$
\begin{aligned}
\boldsymbol{Y} &\sim D(\boldsymbol{\Theta}) \\
g_i(\boldsymbol{\Theta_i}) &\sim N(g_{i,0} + \boldsymbol{X_i}\beta_{\boldsymbol{i}} + \boldsymbol{a_i}, \sigma_i^2 \boldsymbol{I}) \\
\boldsymbol{a_i} &\sim GP(\phi_i, \sigma_{a_i}^2)
\end{aligned}
$$

- Write software to sample from posterior of all models of this form.

Preliminary Results

- I ran a quick MAP estimation for the parameters under the Frechet model without the regression coefficients β

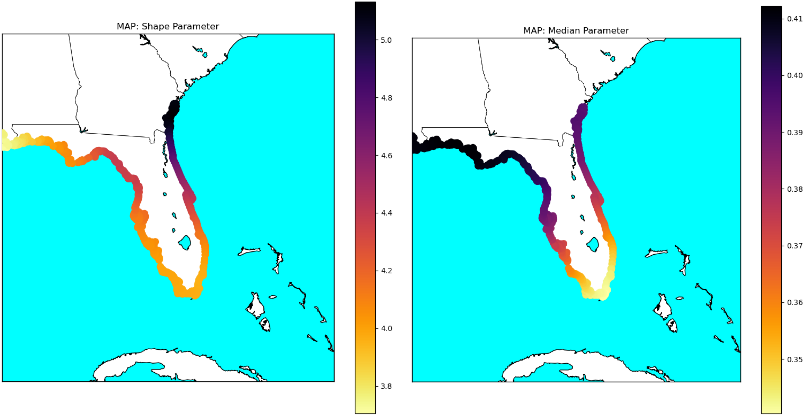


Figure: Left: MAP Shape parameter under proposed model. Right: MAP Median parameter.

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