q1

November 23, 2020

1 question 1

1.1

$$\textbf{u is } C^2 \textbf{ on both } \Omega \textbf{ and } \partial\Omega \textbf{ and v is } C^1 \textbf{ on both } \Omega \textbf{ and } \partial\Omega$$

$$\int_\Omega (u_t v - [\nabla k(\overrightarrow{x})\nabla u)]v = \int_\Omega f(\overrightarrow{x}, t)v$$

$$\textbf{with First Green identity} \int_\Omega (u_t v + k(\overrightarrow{x})\nabla u \nabla v) - \int_{\partial\Omega} k(\overrightarrow{x})\frac{\partial u}{\partial n} v = \int_\Omega f(\overrightarrow{x}, t)v$$

$$-\int_{\partial\Omega} k(\overrightarrow{x})\frac{\partial u}{\partial n} v = -\int_{\partial\Omega} n(k(\overrightarrow{x})\nabla u)v = \int_{\partial\Omega} \alpha(u - u_0)v$$

$$\int_\Omega (u_t v + k(\overrightarrow{x})\nabla u \nabla v) + \int_{\partial\Omega} \alpha u v = \int_\Omega f v + \int_{\partial\Omega} \alpha u_0 v$$

1.2

$$u_t = u_0 \frac{\partial u_*}{\partial t_*}\frac{\partial t_*}{\partial t_*} = u_0 \frac{\alpha}{R}\frac{\partial u_*}{\partial t_*}$$

$$u_p = u_0 \frac{\partial u_*}{\partial \rho_*}\frac{\partial \rho_*}{\partial \rho} = \frac{u_0}{R}\frac{\partial u_*}{\partial \rho_*}$$

$$f = \frac{\alpha u_0}{R} f_* \quad \frac{\partial \rho *}{\partial \rho} = 1/R$$

$$u(R, t) = u_0 u_*(R_*, t_*) = u_0 u_*(1, t_*)$$

$$v(R) = u_0 v_*(R_*) = u_0 v_*(1)$$

$$\begin{aligned}
LHS &= \int_0^R (u_t v + k u_\rho v_\rho)(\rho)^2 \, d\rho \ + \alpha R^2 u(R,t)v(R) + \alpha R^2 u(R,t)v(R) \\
&= \int_0^1 (u_0 \frac{\alpha}{R}\frac{\partial u_*}{\partial t_*} u_0 v_* + \alpha R k_* \frac{u_0}{R}\frac{\partial u_*}{\partial \rho_*}\frac{u_0}{R}\frac{\partial v_*}{\partial \rho_*})(R\rho_*)^2 R \, d\rho_* \ + \alpha R^2 u_0 u_*(1, t_*) u_0 v_*(1) \\
&= \int_0^1 (\alpha {u_0}^2 R^2 \frac{\partial u_*}{\partial t_*} v_* + \alpha {u_0}^2 R^2 k_* \frac{\partial u_*}{\partial \rho_*}\frac{\partial v_*}{\partial \rho_*})(\rho_*)^2 \, d\rho_* + \alpha R^2 {u_0}^2 u_*(1, t_*) v_*(1) \\
&= \alpha {u_0}^2 R^2 \int_0^1 ((u_*)_{t*} v_* + k_* u_\rho v_\rho)(\rho)^2 \, d\rho_* + \alpha {u_0}^2 R^2 u_*(1, t_*) v_*(1)
\end{aligned}$$

$$\begin{aligned}
RHS &= \int_0^R f v(\rho)^2 \, d\rho \ + \alpha R^2 u_0 v(R) \\
&= \int_0^1 u_0 f_* \frac{\alpha}{R} u_0 v(\rho_*) R^2 (\rho)^2 R \, d\rho_* \ + \alpha R^2 {u_0}^2 v_*(1) \\
&= \alpha {u_0}^2 R^2 \int_0^1 f_* v_*(\rho_*)^2 \, d\rho_* \ + \alpha R^2 {u_0}^2 v_*(1)
\end{aligned}$$

$$LHS = RHS = \int_0^1 ((u_*)_{t*} v_* + k_* u_\rho v_\rho)\rho^2 \, d\rho_* + u_*(1, t_*) v_*(1) = \int_0^1 f_* v_* \rho_*^2 \, d\rho_* \ + \alpha R^2 (u_0)^2 v_*(1)$$

$$\overline{u}(t*) = \frac{\overline{u}(t)}{u_0} = \frac{3}{R^3 u_0}\int_0^R u(\rho,t)\rho^2\,d\rho \;= \frac{3}{R^3 u_0}\int_0^1 u_0 u_*(\rho_*,t_*)R^2\rho_*^2 R\,d\rho_* \;= 3\int_0^1 u_0 u_*(\rho_*,t_*)\rho_*^2\,d\rho_* \tag{1}$$

1.3

$\overline{U}^n$