Statistical Theory:
Exercise Sheet 3

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Remark. When we say that $X_1, \ldots, X_n$ is a sample from the distribution F it means that $X_1, \ldots, X_n$ are iid with the distribution F.

Exercise 1. Let $X_1, \ldots, X_n$ be a sample from the uniform distribution on $(0, \theta)$ $(\theta > 0)$.

(a) Show that $S = X_{(n)}$ is a complete sufficient statistic for θ.

(b) Show that $T = X_{(1)}/X_{(n)}$ is ancillary for θ.

(c) Use Basu's theorem to show that S and T are independent.

Exercise 2. Let $X_1, \ldots, X_n$ be a sample from the uniform distribution on $(-\theta, \theta)$ $(\theta > 0)$.

(a) Show that $S = (X_{(1)}, X_{(n)})^\mathsf{T}$ is a sufficient statistic for θ.

(b) Show that S is not a complete statistics.

(c) Suggest a different statistics that is complete.

Exercise 3. Let $X_1, \ldots, X_n$ be a sample from the distribution with density $f(x; \mu) = e^{-(x-\mu)} 1_{(\mu,\infty)}(x)$ with a parameter $\mu \in \mathbb{R}$.

(a) Show that $X_{(1)}$ is a complete sufficient statistic for μ.

(b) Use Basu's theorem to show that $X_{(1)}$ and $\frac{1}{n}\sum_{i=1}^n (X_i - \bar{X})^2$ are independent.

Exercise 4. Let $Y_1, \ldots, Y_n$ follow the normal linear regression model, that is, they are independent with distribution $\mathrm{N}(\beta_0 + \beta_1 x_i, \sigma^2)$, where $\sigma > 0$ is known and x_i are some fixed known constants.

Find a minimal sufficient statistic for the parameter $(\beta_0, \beta_1)^\mathsf{T}$ (find a sufficient statistic using the factorization lemma and then prove its minimality).

Exercise 5. Let $X_1, \ldots, X_n$ be a sample from the exponential distribution with intensity $\lambda > 0$. Show that $T = \sum_{i=1}^n X_i$ is a complete and sufficient statistic for λ. Show that $R = X_1/(\sum_{i=1}^n X_i)$ is ancillary for λ. Conclude (using Basu's theorem) that T and R are independent. (*Useful fact:* If $X_1 \sim \Gamma(a, p_1)$, $X_2 \sim \Gamma(a, p_2)$ are independent, then $X_1 + X_2 \sim \Gamma(a, p_1 + p_2)$.)

Exercise 6. Suppose that $X_1, \ldots, X_n$ are independent random variables, each uniformly distributed on $(0, \theta)$, where $\theta > 0$ is a parameter. For each of the variables $T_1, \ldots, T_6$ below, decide whether it is a sufficient statistic or not.

$$\begin{aligned} T_1 &= (X_1, \ldots, X_n), \\ T_2 &= \big(X_{(1)}, \ldots, X_{(n)}\big), \\ T_3 &= \big(X_{(1)}, X_{(n)}\big), \\ T_4 &= X_{(1)} \\ T_5 &= X_{(n)} \\ T_6 &= \frac{n+1}{n} \times X_{(n)} \end{aligned}$$

Is any of them minimal sufficient? If so, is it also complete? (Notice the dimensions.)

Exercise 7. Suppose that $X_1, \ldots, X_n$ are independent random variables, each with normal distribution $\mathrm{N}(\mu, \sigma^2)$, where σ^2 is a known constant and μ is an unknown parameter. Decide whether any of the variables $T_1, \ldots, T_4$ below is an ancillary statistic.

$$\begin{aligned} T_1 &= (X_1, \ldots, X_n), \\ T_2 &= (X_1 - X_2, X_2 - X_3, \ldots, X_{n-1} - X_n), \\ T_3 &= \frac{1/n \sum_{i=1}^n X_i - \mu}{\sigma/\sqrt{n}}. \\ T_4 &= \frac{1/n \sum_{i=1}^{n-1} (X_i - X_{i+1})^2}{\operatorname{var}\big(1/n \sum_{i=1}^{n-1} (X_i - X_{i+1})^2\big)}. \end{aligned}$$