Statistical Theory: Exercise Sheet 2 — Corrections

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Exercise 1. Give a counterexample to show that neither of $X_n \xrightarrow{P} X$ or $X_n \xrightarrow{d} X$ ensures that $\mathsf{E}\, X_n \to \mathsf{E}\, X$ as $n \to \infty$.

Solution to Exercise 1. Let X_n satisfy $P(X_n = n) = 1/(n+1)$, $P(X_n = 1/n) = n/(n+1)$. Then $X_n \xrightarrow{P} 0$ and $X_n \xrightarrow{d} 0$ but $\mathsf{E}\, X_n = 1 \to 1 \neq \mathsf{E}\, 0$.

Exercise 2. Find the limit in distribution (as $n \to \infty$) for the sequence $\{X_n\}_{n\in\mathbb{N}}$ defined as:

(a) $\{E_k\}_{k\in\mathbb{N}}$ iid, $E_k \sim \text{Exp}(1)$ for every $k \in \mathbb{N}$,

$$X_n = \frac{1}{\sqrt{n}} \sum_{k=1}^{n} E_k - \sqrt{n},$$

(b) $\{U_k\}_{k\in\mathbb{N}}$ iid, U_k uniform on $(0,1)$ for every $k \in \mathbb{N}$,

$$X_n = n \times \min(U_1, \dots U_n),$$

(c) $X_n \sim \text{Bi}(n, p_n)$ such that $\lim_{n\to\infty} p_n = 0$ and $\lim_{n\to\infty} np_n = \lambda$.

Solution to Exercise 2.

(a) $X \sim \text{N}(0,1)$ (use the CLT).

(b) $X \sim \text{Exp}(1)$ (find of the distribution of X_n and take limits).

(c) $X \sim \text{Po}(\lambda)$ (use Scheffe's theorem, and note that $n!p_n^k/(n-k)! = \prod_{i=0}^{k-1}((n-i)p_n) \to \lambda^k$ and $(1-p_n)^{n-k} = (1-np_n/n)^{n-k} \to e^{-\lambda}$ as $n \to \infty$).

Exercise 3. Let $h : [0,1] \to [0,1]$ be a continuous function. We are interested in computing its integral $\int_0^1 h(t)dt$ by Monte Carlo simulation.

(i) Let $\xi_1, \xi_2, \dots, \eta_1, \eta_2, \dots$ be independent random variables uniformly distributed on $[0,1]$ and let $X_k = 1_{[\eta_k \leq h(\xi_k)]}$. Show that $\bar{X}_n = \frac{1}{n}\sum_{k=1}^{n} X_k$ converges almost surely to $\int_0^1 h(t)dt$.

(ii) Let $\xi_1, \xi_2, \dots$ be independent random variables uniformly distributed on $[0,1]$ and let $Y_k = h(\xi_k)$. Show that $\bar{Y}_n = \frac{1}{n}\sum_{k=1}^{n} Y_k$ converges almost surely to $\int_0^1 h(t)dt$.

(iii) Compute and compare $\operatorname{var}\bar{X}_n$ and $\operatorname{var}\bar{Y}_n$.

Solution to Exercise 3.

(i) Compute $\mathsf{E}\,X_1 = \mathsf{E}\,1_{[\eta_1 \le h(\xi_1)]} = P(\eta_1 \le h(\xi_1)) = \int_{\{(\xi,\eta):\eta \le h(\xi)\}} d\xi\, d\eta = \int_0^1 h(t)dt$, or alternatively $\mathsf{E}\,X_1 = \mathsf{E}\,1_{[\eta_1 \le h(\xi_1)]} = P(\eta_1 \le h(\xi_1)) = \mathsf{E}\,P(\eta_1 \le h(\xi_1)|\xi_1) = \mathsf{E}\,h(\xi_1) = \int_0^1 h(t)dt$. Therefore, by the Strong Law of Large Numbers, $\bar{X}_n \xrightarrow{\text{a.s.}} \int_0^1 h(t)dt$.

(ii) Similarly, compute $\mathsf{E}\,Y_1 = \mathsf{E}\,h(\xi_1) = \int_0^1 h(t)dt$ and conlude that by the Strong Law of Large Numbers, $\bar{Y}_n \xrightarrow{\text{a.s.}} \int_0^1 h(t)dt$.

(iii) The variable X_1 is Bernoulli distributed with success probability $\mathsf{E}\,X_1 = \int_0^1 h(t)dt$, thus its variance is $\big(\int_0^1 h(t)dt\big)\big(1 - \int_0^1 h(t)dt\big)$. The variance of Y_1 is $\mathsf{E}\,Y_1^2 - (\mathsf{E}\,Y_1)^2 = \int_0^1 h(t)^2 dt - \big(\int_0^1 h(t)dt\big)^2$. Now $\operatorname{var}\bar{X}_n - \operatorname{var}\bar{Y}_n = \frac{1}{n}\int_0^1 (h(t) - h(t)^2)dt \ge 0$. Hence $\bar{Y}_n$ is more accurate as an estimator of $\int_0^1 h(t)dt$.

Exercise 4. Let $\{X_i\}_{i\in\mathbb{N}}$ be a collection of iid random variables from a probability distribution with finite second moment. Define $\bar{X}_n = n^{-1}\sum_{i=1}^n X_i$ and $s_n^2 = (n-1)^{-1}\sum_{i=1}^n (X_i - \bar{X}_n)^2$. Find the limit in distribution of $n^{1/2}(\bar{X}_n - \mathsf{E}\,X_1)/s_n$ as $n \to \infty$.

Solution to Exercise 4. The limit distribution is $\mathrm{N}(0,1)$ (use the CLT to show that $n^{1/2}(\bar{X}_n - \mathsf{E}\,X_1)/[\operatorname{var}X_1]^{1/2}$ has a limiting $N(0,1)$ distribution, then show that s_n^2 converges in probability to $\operatorname{var}X_1$ by using the WLLN for the iid sequence $\{X_i^2\}_{i\in\mathbb{N}}$ along with the convergence in probability of $\bar{X}_n$ to $\mathsf{E}\,X_1$, and finally use Slutsky's theorem).

Exercise 5.

(a) Let $X_1, \ldots, X_n$ be a sample of exponentially distributed variables with intensity $\lambda > 0$. We are interested in estimating λ. The sample mean $\hat{\mu} = n^{-1}\sum_{i=1}^n X_i$ is a meaningful estimator of $\mu = \mathsf{E}\,X$. Then $\lambda = 1/\mu$ is naturally estimated by $\hat{\lambda} = 1/\hat{\mu}$. Find the asymptotic distribution of $\hat{\lambda}$, that is, investigate the convergence in distribution of $n^{1/2}(\hat{\lambda} - \lambda)$.

(b) Let $X_1, \ldots, X_n$ be a sample from a Poisson distribution with intensity $\lambda > 0$. We might be interested in estimating $\pi = P(X = 0) = e^{-\lambda}$. The sample mean $\hat{\lambda} = n^{-1}\sum_{i=1}^n X_i$ is a meaningful estimator of $\lambda = \mathsf{E}\,X$. Then π is naturally estimated by $\hat{\pi} = e^{-\hat{\lambda}}$. Find the asymptotic distribution of $\hat{\pi}$, that is, investigate the convergence in distribution of $n^{1/2}(\hat{\pi} - \pi)$.

(c) Let $X_1, \ldots, X_n$ be a sample from a geometric distribution with success probability $p \in (0,1)$. We might be interested in estimating $\pi = P(X > 0) = 1 - p$. The sample mean $\hat{\mu} = n^{-1}\sum_{i=1}^n X_i$ is a meaningful estimator of $\mu = \mathsf{E}\,X = (1-p)/p$. Then $p = 1/(\mu + 1)$ could be estimated by $1/(\hat{\mu} + 1)$, and thus $\pi = \mu/(\mu + 1)$ by $\hat{\pi} = \hat{\mu}/(\hat{\mu} + 1)$. Find the asymptotic distribution of $\hat{\pi}$, that is, investigate the convergence in distribution of $n^{1/2}(\hat{\pi} - \pi)$.

(d) Let $X_1, \ldots, X_n$ be a sample of Bernoulli distributed variables with success probability p. We are interested in estimating the odds defined as $r = \frac{p}{1-p}$. The sample mean

$\hat{p} = n^{-1}\sum_{i=1}^{n} X_i$ is a meaningful estimator of $p = \mathsf{E}\, X$. Then r is naturally estimated by $\hat{r} = \frac{\hat{p}}{1-\hat{p}}$. Find the asymptotic distribution of $\hat{r}$, that is, investigate the convergence in distribution of $n^{1/2}(\hat{r} - r)$ as $n \to \infty$.

Solution to Exercise 5.

(a) $n^{1/2}(\hat{\mu} - \mu) \xrightarrow{d} \mathrm{N}(0, \mu^2)$ by CLT, $\lambda = g(\mu) = 1/\mu$, $g'(\mu) = -1/\mu^2$, $n^{1/2}(\hat{\lambda} - \lambda) \xrightarrow{d} -1/\mu^2 \mathrm{N}(0, \mu^2) = \mathrm{N}(0, 1/\mu^2)$ by the delta method.

(b) $n^{1/2}(\hat{\lambda} - \lambda) \xrightarrow{d} \mathrm{N}(0, \lambda)$ by CLT, $\pi = g(\lambda) = e^{-\lambda}$, $g'(\lambda) = -e^{-\lambda}$, $n^{1/2}(\hat{\pi} - \pi) \xrightarrow{d} -e^{-\lambda}\mathrm{N}(0, \lambda) = \mathrm{N}(0, \lambda e^{-2\lambda})$ by the delta method.

(c) $n^{1/2}(\hat{\mu} - \mu) \xrightarrow{d} \mathrm{N}(0, \mu(\mu + 1))$ by CLT, $\pi = g(\mu) = \mu/(\mu + 1)$, $g'(\mu) = 1/(\mu + 1)^2$, $n^{1/2}(\hat{\pi} - \pi) \xrightarrow{d} 1/(\mu + 1)^2 \mathrm{N}(0, \mu(\mu + 1)) = \mathrm{N}(0, \mu/(\mu + 1)^3)$ by the delta method.

(d) $n^{1/2}(\hat{p} - p) \xrightarrow{d} \mathrm{N}(0, p(1-p))$ by CLT, $r = g(p) = p/(1-p)$, $g'(\mu) = 1/(1-p)^2$, $n^{1/2}(\hat{r} - r) \xrightarrow{d} 1/(1 - p)^2 \mathrm{N}(0, p(1 - p)) = \mathrm{N}(0, p/(1 - p)^3)$ by the delta method.