Statistical Theory: Exercise Sheet 2

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Exercise 1. Give a counterexample to show that neither of $X_n \xrightarrow{P} X$ or $X_n \xrightarrow{d} X$ ensures that $\mathsf{E}\, X_n \to \mathsf{E}\, X$ as $n \to \infty$.

Exercise 2. Find the limit in distribution (as $n \to \infty$) for the sequence $\{X_n\}_{n\in\mathbb{N}}$ defined as:

(a) $\{E_k\}_{k\in\mathbb{N}}$ iid, $E_k \sim \text{Exp}(1)$ for every $k \in \mathbb{N}$,

$$X_n = \frac{1}{\sqrt{n}} \sum_{k=1}^{n} E_k - \sqrt{n},$$

(b) $\{U_k\}_{k\in\mathbb{N}}$ iid, U_k uniform on $(0,1)$ for every $k \in \mathbb{N}$,

$$X_n = n \times \min(U_1, \dots U_n),$$

(c) $X_n \sim \text{Bi}(n, p_n)$ such that $\lim_{n\to\infty} p_n = 0$ and $\lim_{n\to\infty} np_n = \lambda$.

Exercise 3. Let $h : [0,1] \to [0,1]$ be a continuous function. We are interested in computing its integral $\int_0^1 h(t)dt$ by Monte Carlo simulation.

(i) Let $\xi_1, \xi_2, \dots, \eta_1, \eta_2, \dots$ be independent random variables uniformly distributed on $[0,1]$ and let $X_k = 1_{[\eta_k \le h(\xi_k)]}$. Show that $\bar{X}_n = \frac{1}{n}\sum_{k=1}^n X_k$ converges almost surely to $\int_0^1 h(t)dt$.

(ii) Let $\xi_1, \xi_2, \dots$ be independent random variables uniformly distributed on $[0,1]$ and let $Y_k = h(\xi_k)$. Show that $\bar{Y}_n = \frac{1}{n}\sum_{k=1}^n Y_k$ converges almost surely to $\int_0^1 h(t)dt$.

(iii) Compute and compare $\operatorname{var} \bar{X}_n$ and $\operatorname{var} \bar{Y}_n$.

Exercise 4. Let $\{X_i\}_{i\in\mathbb{N}}$ be a collection of iid random variables from a probability distribution with finite second moment. Define $\bar{X}_n = n^{-1}\sum_{i=1}^n X_i$ and $s_n^2 = (n-1)^{-1}\sum_{i=1}^n (X_i - \bar{X}_n)^2$. Find the limit in distribution of $n^{1/2}(\bar{X}_n - \mathsf{E}\, X_1)/s_n$ as $n \to \infty$.

Exercise 5.

(a) Let $X_1, \dots, X_n$ be a sample of exponentially distributed variables with intensity $\lambda > 0$. We are interested in estimating λ. The sample mean $\hat{\mu} = n^{-1}\sum_{i=1}^n X_i$ is a meaningful estimator of $\mu = \mathsf{E}\, X$. Then $\lambda = 1/\mu$ is naturally estimated by $\hat{\lambda} = 1/\hat{\mu}$. Find the asymptotic distribution of $\hat{\lambda}$, that is, investigate the convergence in distribution of $n^{1/2}(\hat{\lambda} - \lambda)$.

(b) Let $X_1, \ldots, X_n$ be a sample from a Poisson distribution with intensity $\lambda > 0$. We might be interested in estimating $\pi = P(X = 0) = e^{-\lambda}$. The sample mean $\hat{\lambda} = n^{-1} \sum_{i=1}^{n} X_i$ is a meaningful estimator of $\lambda = \mathsf{E}\, X$. Then π is naturally estimated by $\hat{\pi} = e^{-\hat{\lambda}}$. Find the asymptotic distribution of $\hat{\pi}$, that is, investigate the convergence in distribution of $n^{1/2}(\hat{\pi} - \pi)$.

(c) Let $X_1, \ldots, X_n$ be a sample from a geometric distribution with success probability $p \in (0, 1)$. We might be interested in estimating $\pi = P(X > 0) = 1 - p$. The sample mean $\hat{\mu} = n^{-1} \sum_{i=1}^{n} X_i$ is a meaningful estimator of $\mu = \mathsf{E}\, X = (1-p)/p$. Then $p = 1/(\mu + 1)$ could be estimated by $1/(\hat{\mu} + 1)$, and thus $\pi = \mu/(\mu + 1)$ by $\hat{\pi} = \hat{\mu}/(\hat{\mu} + 1)$. Find the asymptotic distribution of $\hat{\pi}$, that is, investigate the convergence in distribution of $n^{1/2}(\hat{\pi} - \pi)$.

(d) Let $X_1, \ldots, X_n$ be a sample of Bernoulli distributed variables with success probability p. We are interested in estimating the odds defined as $r = \frac{p}{1-p}$. The sample mean $\hat{p} = n^{-1} \sum_{i=1}^{n} X_i$ is a meaningful estimator of $p = \mathsf{E}\, X$. Then r is naturally estimated by $\hat{r} = \frac{\hat{p}}{1-\hat{p}}$. Find the asymptotic distribution of $\hat{r}$, that is, investigate the convergence in distribution of $n^{1/2}(\hat{r} - r)$ as $n \to \infty$.