Mathematical Olympiads Discord Server

2019 November Intermediate Contest

Time: 4 hours *Each problem is worth 7 points*

Calculators and protractors are not allowed. Do not write your name on your working. After your timeslot finishes, please read the instructions in `#how-to-submit-scripts`*. Do not discuss the contents of this paper outside the text channel* `#finished-contestants` *and the voice channel* `Post-Contest Banter` *until notified by staff.*

Problem 1. Let S be the set of all points with integer coordinates (x, y) satisfying $x \geq 0$, $y \geq 0$, and $x + y \leq 2019$. Let n points in S be red. Andy the ant walks on the plane, always moving up, down, left, or right. He can only turn at a red point or on the line $x + y = 2020$, turning 90 degrees clockwise or anticlockwise if he so chooses. Andy wishes to choose a red point to start on such that it is possible for him to come back to where he started. Find the minimum n such that this is always possible.

Problem 2. Let ABC be a triangle, and let Γ be a circle through A and B with centre O. Suppose that the internal and external angle bisectors of $\angle ABC$ meet Γ again at X and Y. Prove that X, Y, and C are collinear if and only if A, B, C, and O lie on a circle.

Problem 3. Let $\mathbb{R}$ denote the set of real numbers. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that for any function $g : \mathbb{R} \to \mathbb{R}$, there exists a constant $c \in \mathbb{R}$ such that for all for all $x \in \mathbb{R}$,

$$f(g(x)) - g(f(x)) = c.$$

Problem 4. Let $\mathbb{Z}^+$ denote the set of positive integers. Let $n \in \mathbb{Z}^+$ such that for each $m \in \mathbb{Z}^+$, there exists an $a \in \mathbb{Z}^+$ such that $a^a \equiv m \pmod{n}$. Prove that the number of positive divisors of n is a power of 2.