Mathematical Olympiads Discord Server

2020 September Advanced Contest

Saturday and Sunday, 12–13 September 2020

Problem 1. In three-dimensional space, a *lattice point* is a triple of integers (x, y, z). Find the maximum size of a set S of lattice points in space such that the taxicab distance between any two points is at most 2020.

(The *taxicab distance* between two lattice points (x_1, y_1, z_1) and (x_2, y_2, z_2) is defined to be $|x_1 - x_2| + |y_1 - y_2| + |z_1 - z_2|$.)

Problem 2. Let I be the incenter of a triangle ABC, and let D, E, and F be the intersections of the incircle of ABC with BC, CA, and AB respectively. Suppose that the circumcircles of ABC and AID intersect again at $P \neq A$, and that AP intersects EF at Q. Show that DQ is perpendicular to EF.

Problem 3. Let $\mathbb{R}_{>0}$ denote the set of positive real numbers. Find all functions $f : \mathbb{R}_{>0} \to \mathbb{R}_{>0}$ such that for all positive real numbers x and y,

$$\frac{f(x+y)}{f(x)+f(y)} \quad \text{is an integer.}$$

Problem 4. For a positive integer $n > 2$:

- Let A_n be the number of integers x satisfying $0 < x < n/2$ and $\gcd(x, 2n) = 1$.
- Let B_n be the number of integers y satisfying $n/2 < y < n$ and $\gcd(y, 2n) = 1$.

Find all integers m such that there exists a positive integer n satisfying $A_n - B_n = m$.

Language: English

Time: 4 hours
Each problem is worth 7 points