Mathematical Olympiads Discord Server

2020 August Intermediate Contest

Saturday and Sunday, 8–9 August 2020

Problem 1. For a positive integer n, let m_n be the number formed by writing the digits of n in ascending order, and M_n be the number formed by writing the digits in descending order. For example, $m_{121} = 112, m_{420} = 24, M_{1337} = 7331$.

Do there exist infinitely many positive integers n such that $M_n - m_n = n$?

Problem 2. Let $\mathbb{R}^+$ denote the set of positive real numbers. Find all functions $f : \mathbb{R}^+ \to \mathbb{R}^+$ such that for all $x, y, z \in \mathbb{R}^+$ satisfying $x + y + z = xyz$,

$$f(xy) + f(xz) + f(yz) = 3.$$

Problem 3. Let S_0 be a set of 2020 real numbers. The sequence of sets $S_0, S_1, \ldots, S_n$ is called *culled* if for all $i \in \{0, 1, \ldots, n-1\}$,

- $S_{i+1} \subset S_i$
- No element $x \in S_i$ with $x \notin S_{i+1}$ is closer to the average of all elements in S_i than any element in S_{i+1}.

An element $b \in S_0$ is called *basic* if there exists a culled sequence of sets such that $S_n = \{b\}$. What is the least possible number of basic elements in S_0?

Problem 4. Let Γ_1 and Γ_2 be two disjoint circles whose interiors do not intersect. Points X_1 and X_2 are on Γ_1 and Γ_2 respectively such that X_1X_2 is an internal common tangent to Γ_1 and Γ_2. The circle Ω with diameter X_1X_2 intersects Γ_1 and Γ_2 again at $Y_1 \neq X_1$ and $Y_2 \neq X_2$ respectively. Suppose that X_1Y_1 and X_2Y_2 intersect at P. Prove that the perpendiculars from P to the common external tangents of Γ_1 and Γ_2 are tangent to Ω.

Language: English

Time: 4 hours
Each problem is worth 7 points