Sunday, December 15, 2019

Problem 4. In an acute triangle ABC, the feet of the altitudes from B to AC and C to AB are X and Y respectively. Suppose that P is a point inside triangle ABC satisfying $\angle PBC = \angle PXY$ and $\angle PCB = \angle PYX$. Prove that AP bisects BC.

Problem 5. Consider the set $A = \{1, 2, ..., 2019\}$ and a function $f : A \to A$. For a subset $B \subseteq A$, we say that f *shrinks* B if and only if $f(x) \in B$ for all $x \in B$. (We consider the empty set to be shrunk by any function f.)

Is there a function $f : A \to A$ that shrinks exactly $2019^{60} - 1$ subsets?

Problem 6. Let $\mathbb{R}$ denote the set of real numbers, and let a function f be called *linear* if and only if it can be written in the form $f(x) = mx + d$ for some $m, d \in \mathbb{R}$. Find all positive real numbers r such that, if a function $f : \mathbb{R} \to \mathbb{R}$ satisfies

$$bf(x) + cf(x - a) + af(x + c) < 2019 + |a|^r + |b|^r + |c|^r$$

for all a, b, c, and $x \in \mathbb{R}$ with $a + b + c = 0$, then f is linear.

Language: English

Time: 4 hours and 30 minutes
Each problem is worth 7 points