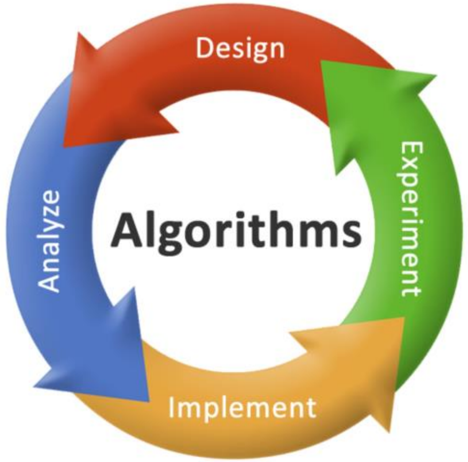


Algorithm Design and Analysis

算法设计与分析

Chapter 11: NP完全性

张乾坤

不可计算性

- 多项式时间规约
- Packing和covering问题
- 约束可满足性问题
- 一些经典组合问题

不可计算性

- 多项式时间规约
- Packing和covering问题
- 约束可满足性问题
- 一些经典组合问题

截至现在。。。

- 算法设计思路
  - 贪心
  - 分治
  - 动态规划
  - 对偶性Duality
  - 规约 Reductions

- 算法设计反向思路
  - NP完全性

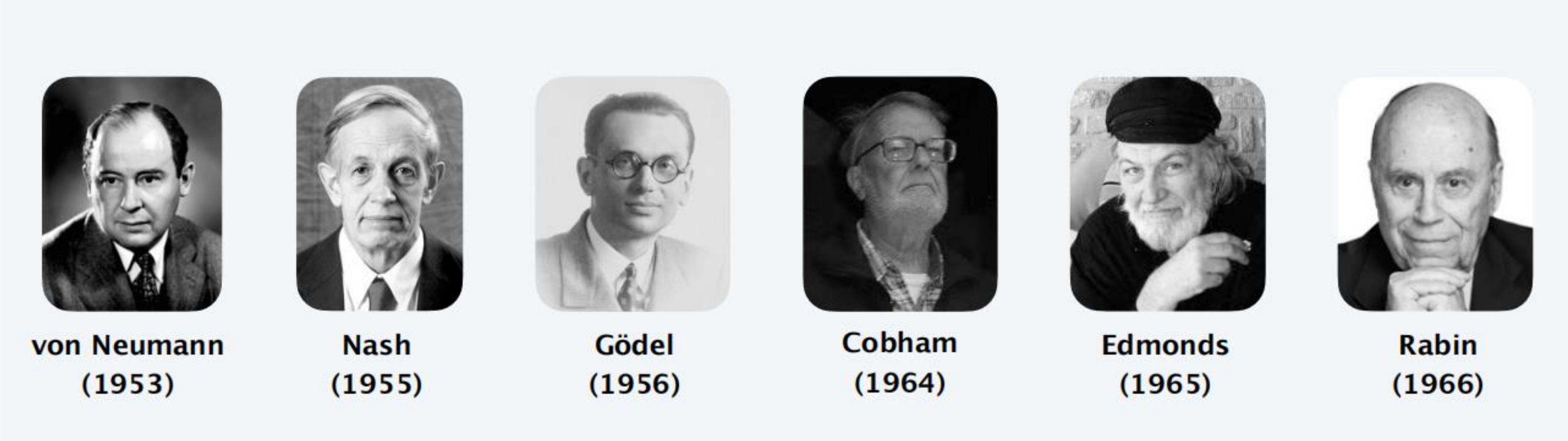
    $O(n^k)$算法不太可能
  - PSPACE完全性

    $O(n^k)$认证算法不太可能
  - 不可确定性

    不存在任何算法

根据计算要求对问题分类

- 问：现实中哪些计算问题是可以解决的？
- 当前的答案：多项式时间算法



根据计算要求对问题分类

- 问： 现实中哪些计算问题是可以解决的？
- 当前的答案： 多项式时间算法

yes	probably no
shortest path	longest path
min cut	max cut
2-satisfiability	3-satisfiability
planar 4-colorability	planar 3-colorability
bipartite vertex cover	vertex cover
matching	3d-matching
primality testing	factoring
linear programming	integer linear programming

“根据能否在多项式时间内求解分类”

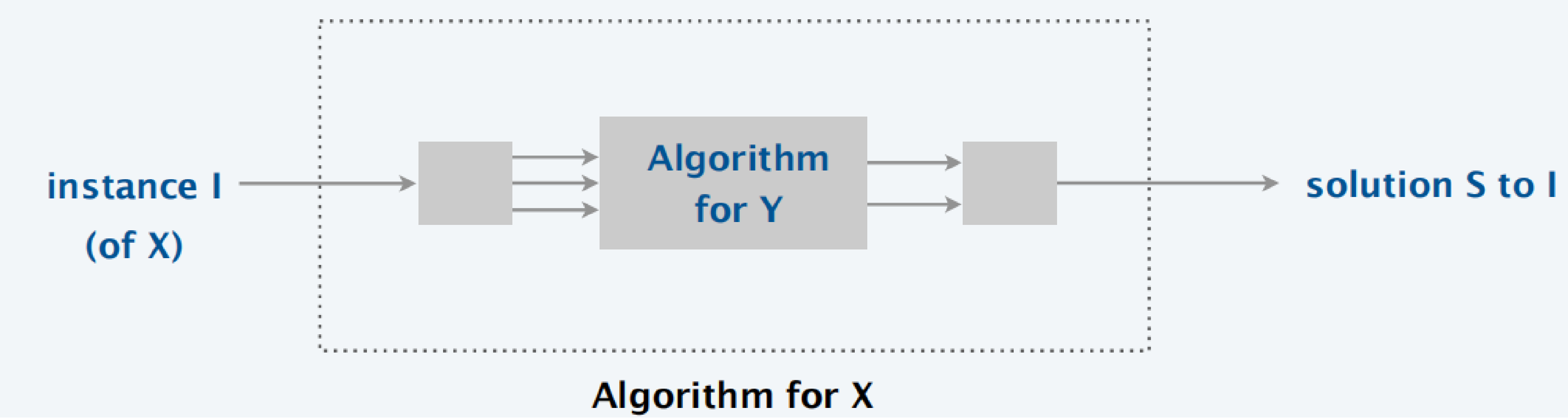
多项式时间规约 Poly-time reductions

- 假设我们可以在多项式时间内求解 Y 问题，我们还能在多项式时间内求解哪些问题?
- 规约Reduction：如果对任意 X 问题的输入实例可通过以下方式求解：

（1）使用多项式个数的简单计算步骤；

（2）使用多项式个数的对求解 Y 问题的函数调用

，则称 X 问题可在多项式时间规约到 Y 问题，记为$X \leq_P Y$ （注意不是$Y \leq_P X$ ）



Quiz 1

- 设$X \le_P Y$，下列说法正确的是?

  A：如果X能在多项式时间求解，Y也可

  B：X能在多项式时间求解当且仅当Y能在多项式时间求解

  C：如果X不能在多项式时间求解，Y也不可

  D：如果Y不能在多项式时间求解，X也不可

Quiz 2

- 以下哪些多项式规约存在？

  A：最大流$\leq_P$最小割

  B：最小割 $\leq_P$最大流

  C：A和B都对

  D：A和B都不对

多项式时间规约——应用

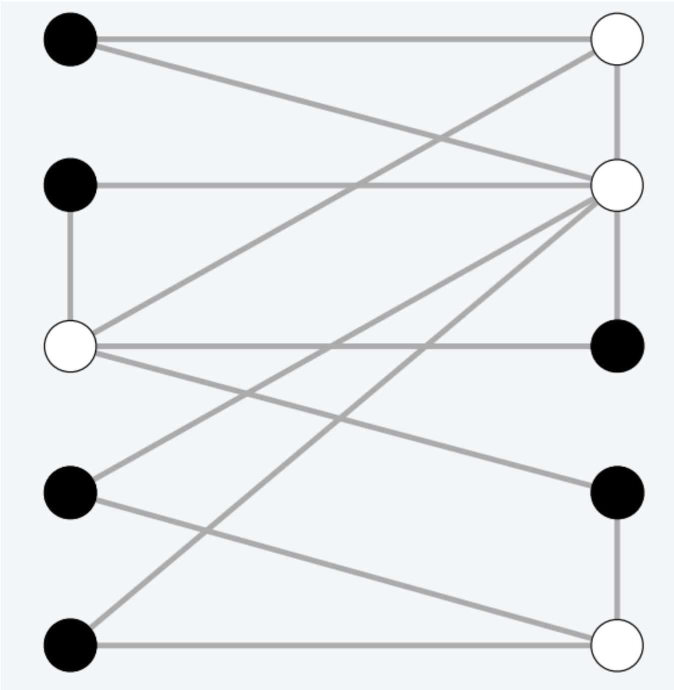
- 设计算法：若$X \leq_P Y$且Y可在多项式时间解决，则X也存在多项式时间算法
- 证明不可计算性：若$Y \leq_P X$且X无法在多项式时间求解，则Y也不能
- 证明等价性：若且，则X和Y可相互规约，记为$X \equiv_P Y$
- 规约是根据问题的相对困难程度来对问题分类

不可计算性

- 多项式时间规约
- Packing和covering问题
- 约束可满足性问题
- 一些经典组合问题

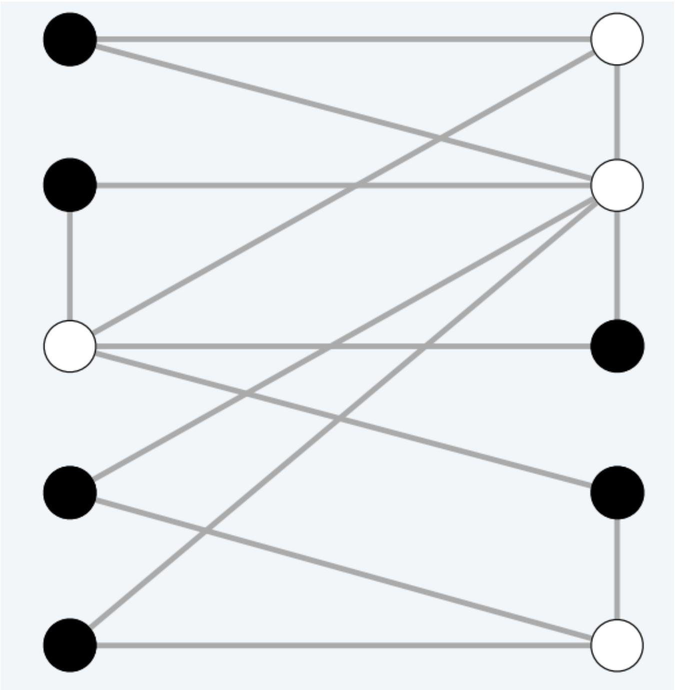
独立集问题 Independent set

- 定义：给定图$G = (V, E)$ 和整数k，是否存在一个大小至少为k的顶点子集满足任意两个顶点不相邻（无边连接）？
- 是否存在大小至少是6的独立集？Yes
- 是否存在大小至少是7的独立集？No

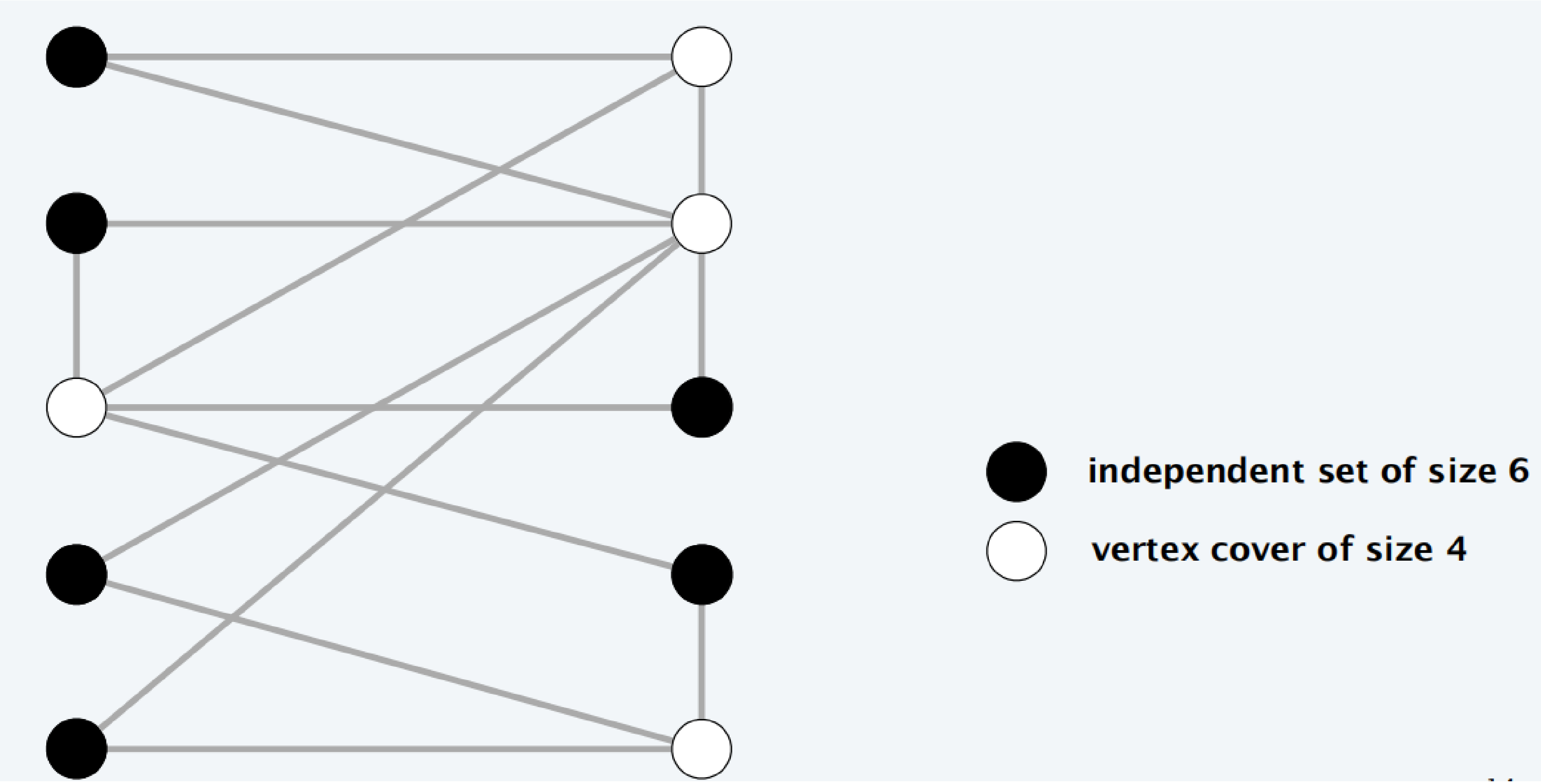


顶点覆盖问题 Vertex cover

- 定义：给定图$G = (V, E)$ 和整数k，是否存在一个大小至多为k的顶点子集满足任意所有边都和集合中至少一点相邻?
- 是否存在大小至多是4的独立集？Yes
- 是否存在大小至多是3的独立集？No



Independent set vs. Vertex cover



Quiz 3

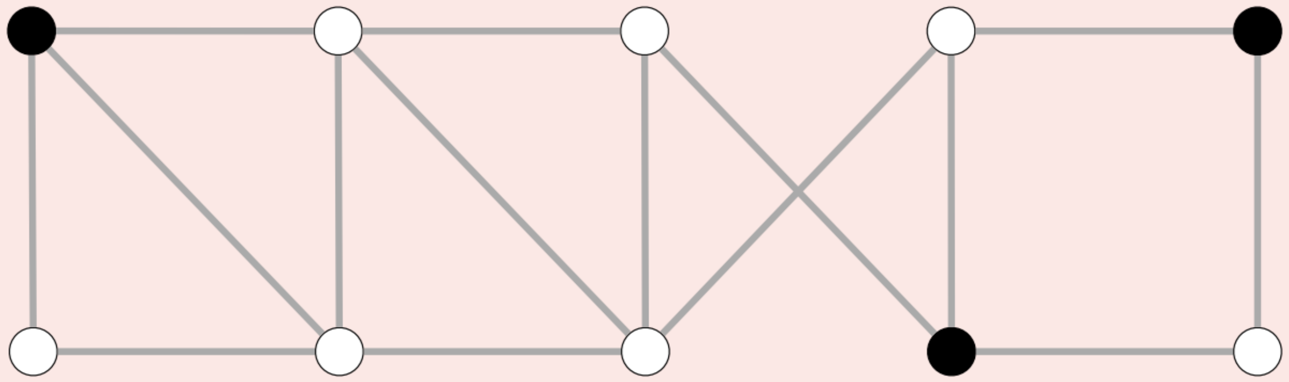
Consider the following graph G. Which are true?

A. The white vertices are a vertex cover of size 7.

B. The black vertices are an independent set of size 3.

C. Both A and B.

D. Neither A nor B.



Independent set和Vertex cover可相互规约

- 定理：独立集 $\equiv_P$ 顶点覆盖
- 证明思路：假设共有n个顶点。需证S是一个大小为k的独立集当且仅当V-S是个大小为n-k的顶点覆盖

$\Rightarrow$

- Let S be any independent set of size k.
- $V - S$ is of size $n - k$.
- Consider an arbitrary edge $(u, v) \in E$.
- S independent $\Rightarrow$ either $u \notin S$, or $v \notin S$, or both.
  $\Rightarrow$ either $u \in V - S$, or $v \in V - S$, or both.
- Thus, $V - S$ covers (u, v). ▪

Independent set和Vertex cover可相互规约

- 定理：独立集 $\equiv_P$ 顶点覆盖
- 证明思路：假设共有n个顶点。需证S是一个大小为k的独立集当且仅当V-S是个大小为n-k的顶点覆盖

$\Leftarrow$

- Let $V - S$ be any vertex cover of size $n - k$.
- S is of size k.
- Consider an arbitrary edge $(u, v) \in E$.
- $V - S$ is a vertex cover $\Rightarrow$ either $u \in V - S$, or $v \in V - S$, or both.
  $\Rightarrow$ either $u \notin S$, or $v \notin S$, or both.
- Thus, S is an independent set. ▪

集合覆盖 Set cover

- 定义：给定全集U，一个U的若干子集构成的集合S，和一个整数k，问是否存在至多k个子集的并等于全集U？

$$U = \{1, 2, 3, 4, 5, 6, 7\}$$

$$S_a = \{3, 7\} \qquad S_b = \{2, 4\}$$

$$S_c = \{3, 4, 5, 6\} \qquad S_d = \{5\}$$

$$S_e = \{1\} \qquad S_f = \{1, 2, 6, 7\}$$

$$k = 2$$

a set cover instance

Quiz 4

Given the universe U = { 1, 2, 3, 4, 5, 6, 7 } and the following sets, which is the minimum size of a set cover?

A. 1

B. 2

C. 3

D. None of the above.

$U = \{ 1, 2, 3, 4, 5, 6, 7 \}$

$S_a = \{ 1, 4, 6 \}$ $\quad S_b = \{ 1, 6, 7 \}$

$S_c = \{ 1, 2, 3, 6 \}$ $\quad S_d = \{ 1, 3, 5, 7 \}$

$S_e = \{ 2, 6, 7 \}$ $\quad S_f = \{ 3, 4, 5 \}$

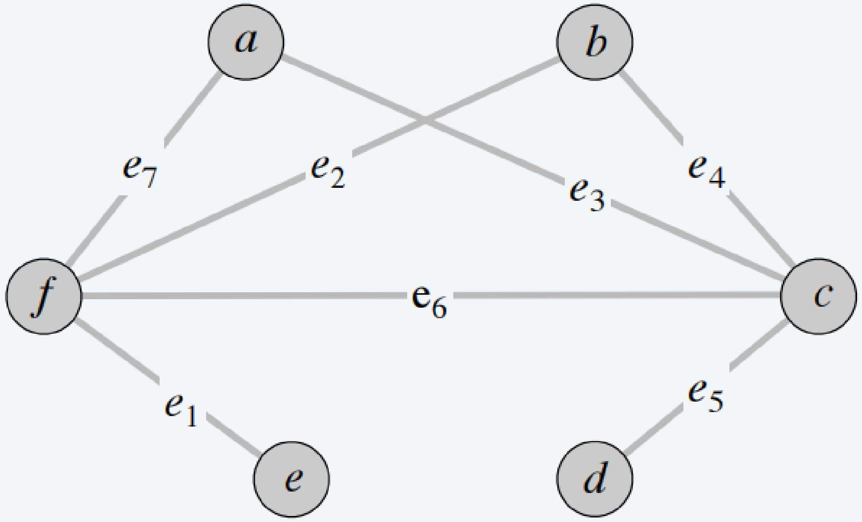
顶点覆盖规约到集合覆盖

- 定理：顶点覆盖$\leq_P$ 集合覆盖
- 证明思路：给定一个顶点覆盖实例G = (V, E) 和整数k，构造一个集合覆盖实例（U, S, k）使得，存在大小为k的集合覆盖当且仅当存在一个大小为k顶点覆盖

顶点覆盖规约到集合覆盖

Construction.

- Universe $U = E$.
- Include one subset for each node $v \in V$: $S_v = \{e \in E : e \text{ incident to } v\}$.



vertex cover instance
(k = 2)

$U = \{ 1, 2, 3, 4, 5, 6, 7 \}$

$S_a = \{ 3, 7 \}$ $\quad$ $S_b = \{ 2, 4 \}$

$S_c = \{ 3, 4, 5, 6 \}$ $\quad$ $S_d = \{ 5 \}$

$S_e = \{ 1 \}$ $\quad$ $S_f = \{ 1, 2, 6, 7 \}$

set cover instance
(k = 2)

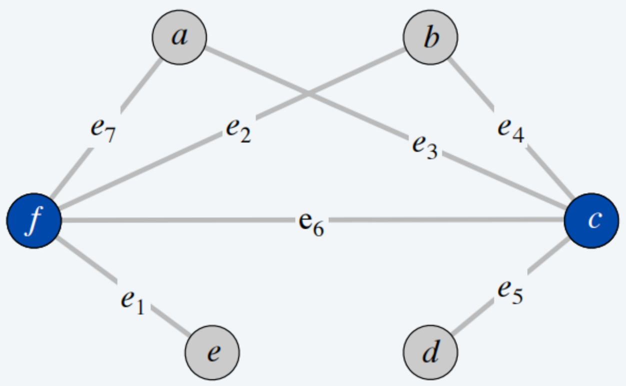
顶点覆盖规约到集合覆盖

定理：顶G = (V, E) 中存在一个大小为k顶点覆盖当且仅当（U, S, k）中存在大小为k的集合覆盖

Pf. $\Rightarrow$ Let $X \subseteq V$ be a vertex cover of size k in G.

- Then $Y = \{ S_v : v \in X \}$ is a set cover of size k. ▪

"yes" instances of VERTEX-COVER are solved correctly



vertex cover instance
(k = 2)

$U = \{ 1, 2, 3, 4, 5, 6, 7 \}$

$S_a = \{ 3, 7 \}$ $\quad$ $S_b = \{ 2, 4 \}$

$S_c = \{ 3, 4, 5, 6 \}$ $\quad$ $S_d = \{ 5 \}$

$S_e = \{ 1 \}$ $\quad$ $S_f = \{ 1, 2, 6, 7 \}$

set cover instance
(k = 2)

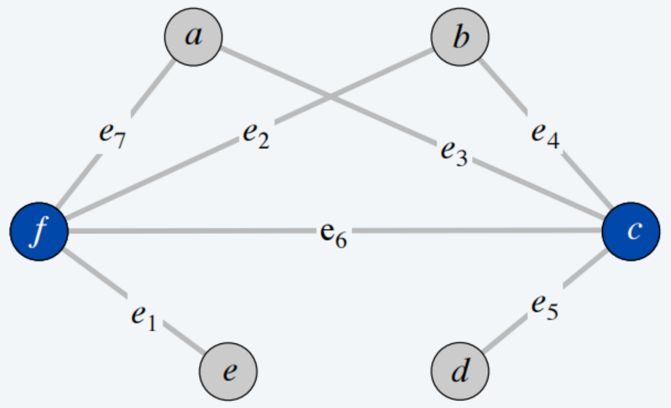
顶点覆盖规约到集合覆盖

定理：顶G = (V, E) 中存在一个大小为k顶点覆盖当且仅当（U, S, k）中存在大小为k的集合覆盖

Pf. $\Leftarrow$ Let $Y \subseteq S$ be a set cover of size k in (U, S, k).

- Then $X = \{ v : S_v \in Y \}$ is a vertex cover of size k in G. ▪

"no" instances of VERTEX-COVER are solved correctly



vertex cover instance
(k = 2)

$U = \{ 1, 2, 3, 4, 5, 6, 7 \}$

$S_a = \{ 3, 7 \}$ $\quad S_b = \{ 2, 4 \}$

$S_c = \{ 3, 4, 5, 6 \}$ $\quad S_d = \{ 5 \}$

$S_e = \{ 1 \}$ $\quad S_f = \{ 1, 2, 6, 7 \}$

set cover instance
(k = 2)

不可计算性

- 多项式时间规约
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- 约束可满足性问题
- 一些经典组合问题

可满足性问题

字　Literal. A Boolean variable or its negation. x_i or $\overline{x_i}$

句子　Clause. A disjunction of literals. $C_j = x_1 \vee \overline{x_2} \vee x_3$

合取范式　Conjunctive normal form (CNF). A propositional formula Φ that is a conjunction of clauses. $\Phi = C_1 \wedge C_2 \wedge C_3 \wedge C_4$

问题定义　SAT. Given a CNF formula Φ, does it have a satisfying truth assignment?

3-SAT. SAT where each clause contains exactly 3 literals (and each literal corresponds to a different variable).

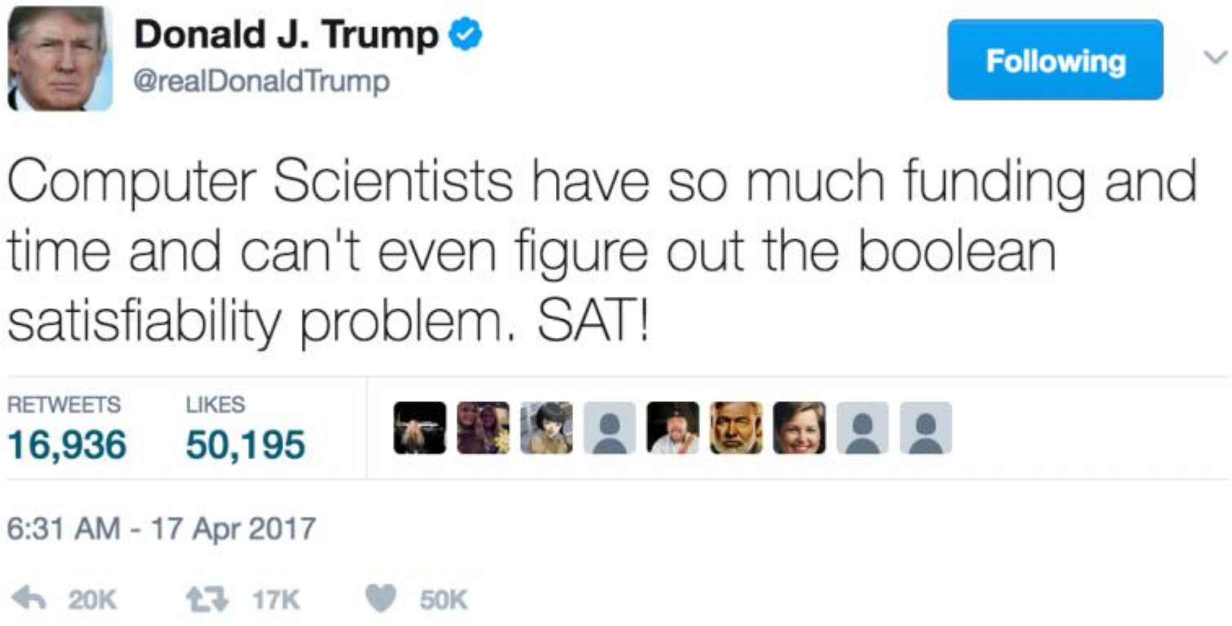
$$\boxed{\Phi = (\overline{x_1} \vee x_2 \vee x_3) \wedge (x_1 \vee \overline{x_2} \vee x_3) \wedge (\overline{x_1} \vee x_2 \vee x_4)}$$

yes instance: x_1 = true, x_2 = true, x_3 = false, x_4 = false

应用　Key application. Electronic design automation (EDA).

可满足性问题很难

- 科学假说：3-SAT问题不存在多项式时间算法
- P vs. NP：以上假说成立当且仅当**P≠NP**



3-SAT规约到独立集

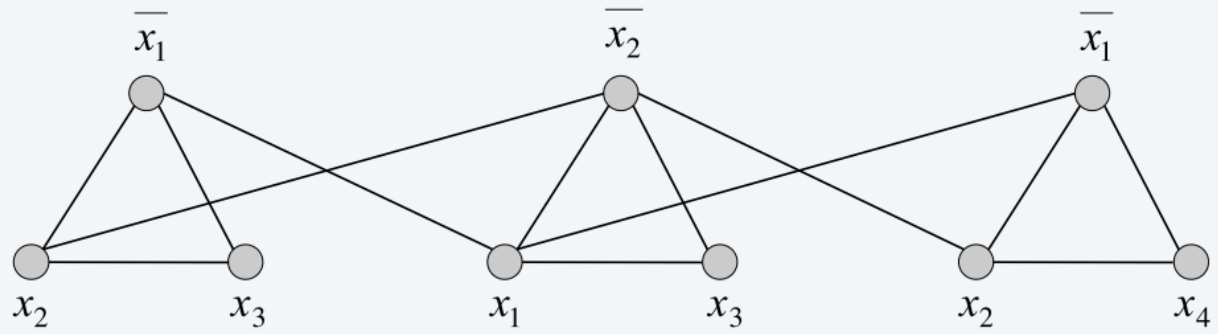
Theorem. 3-SAT $\le_P$ INDEPENDENT-SET.

Pf. Given an instance Φ of 3-SAT, we construct an instance (G, k) of INDEPENDENT-SET that has an independent set of size $k = |\Phi|$ iff Φ is satisfiable.

Construction.

- G contains 3 nodes for each clause, one for each literal.
- Connect 3 literals in a clause in a triangle.
- Connect literal to each of its negations.

G



k = 3

$$\Phi = (\overline{x_1} \vee x_2 \vee x_3) \wedge (x_1 \vee \overline{x_2} \vee x_3) \wedge (\overline{x_1} \vee x_2 \vee x_4)$$

3-SAT规约到独立集

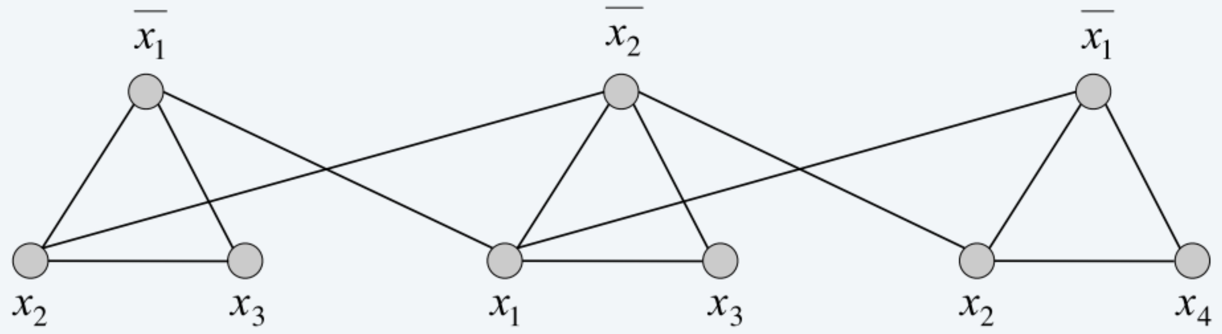
Lemma. Φ is satisfiable iff G contains an independent set of size $k = |\Phi|$.

Pf. $\Rightarrow$ Consider any satisfying assignment for Φ.

- Select one true literal from each clause/triangle.
- This is an independent set of size $k = |\Phi|$. ▪

"yes" instances of 3-SAT are solved correctly

G



k = 3

$$\Phi = \left(\overline{x_1} \vee x_2 \vee x_3\right) \wedge \left(x_1 \vee \overline{x_2} \vee x_3\right) \wedge \left(\overline{x_1} \vee x_2 \vee x_4\right)$$

3-SAT规约到独立集

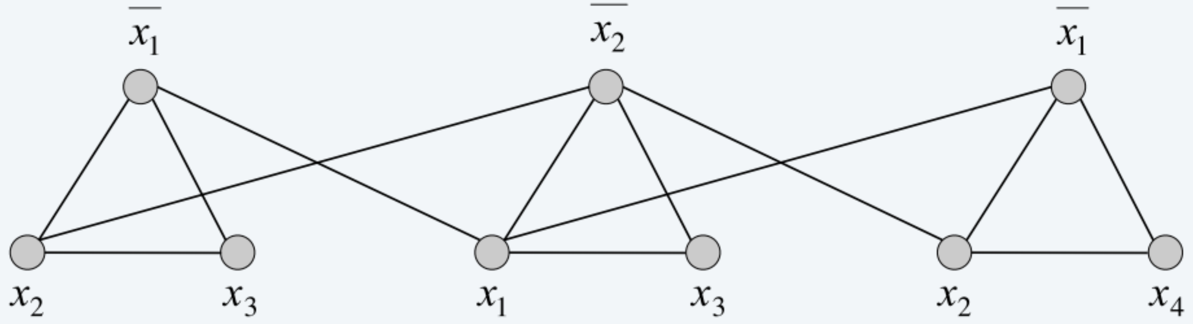
Lemma. Φ is satisfiable iff G contains an independent set of size $k = |\Phi|$.

Pf. $\Leftarrow$ Let S be independent set of size k.

- S must contain exactly one node in each triangle.
- Set these literals to *true* (and remaining literals consistently).
- All clauses in Φ are satisfied. ▪

"no" instances of 3-SAT are solved correctly

G



k = 3

$$\Phi = (\overline{x_1} \lor x_2 \lor x_3) \land (x_1 \lor \overline{x_2} \lor x_3) \land (\overline{x_1} \lor x_2 \lor x_4)$$

小结

- 入门级规约技巧
  - 简单的等价性：独立集$\equiv_P$顶点覆盖
  - 特殊情况规约到一般情况：顶点覆盖$\leq_P$ 集合覆盖
  - 利用小工具编码：3-SAT$\leq_P$ 独立集
- 规约的传递性：若X$\leq_P$ Y且Y$\leq_P$ Z，则X$\leq_P$ Z
- 例如 3-SAT $\leq_P$ INDEPENDENT-SET $\leq_P$ VERTEX-COVER $\leq_P$ SET-COVER.

决策、搜索、优化问题

- 决策问题：是否存在大小至多是k的顶点覆盖？
- 搜索问题：找到一个大小是k的顶点覆盖
- 优化问题：找到最小的顶点覆盖

- 结论：上述三种问题均可在多项式时间相互规约

不可计算性

- 多项式时间规约
- Packing和covering问题
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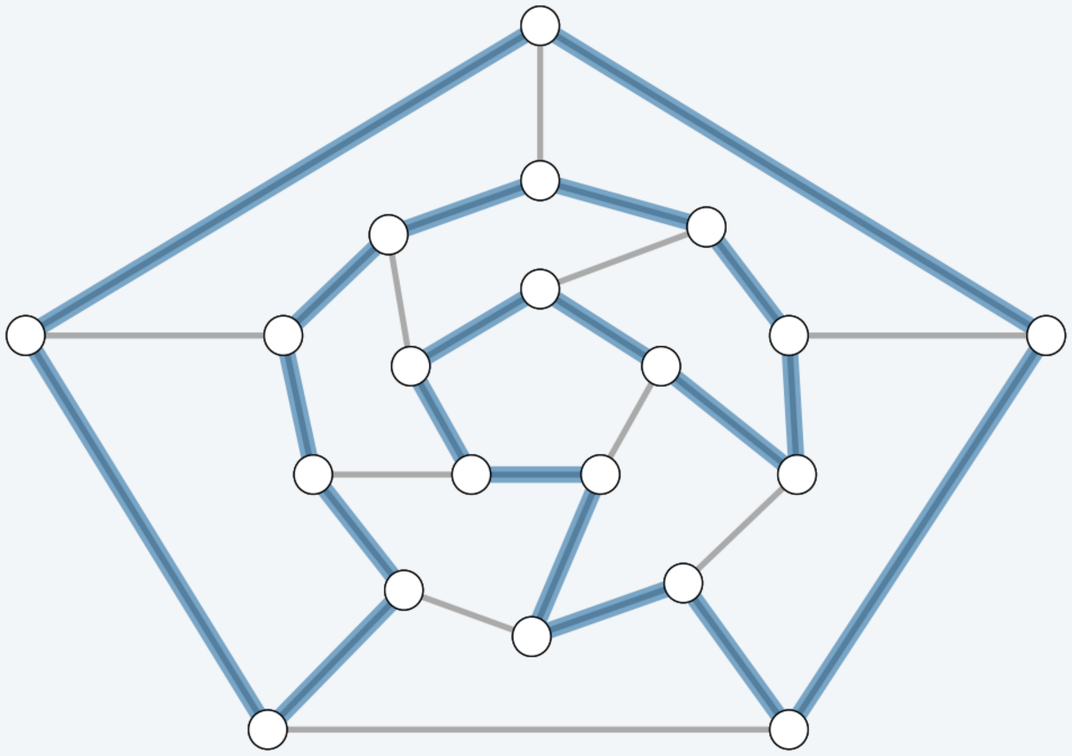
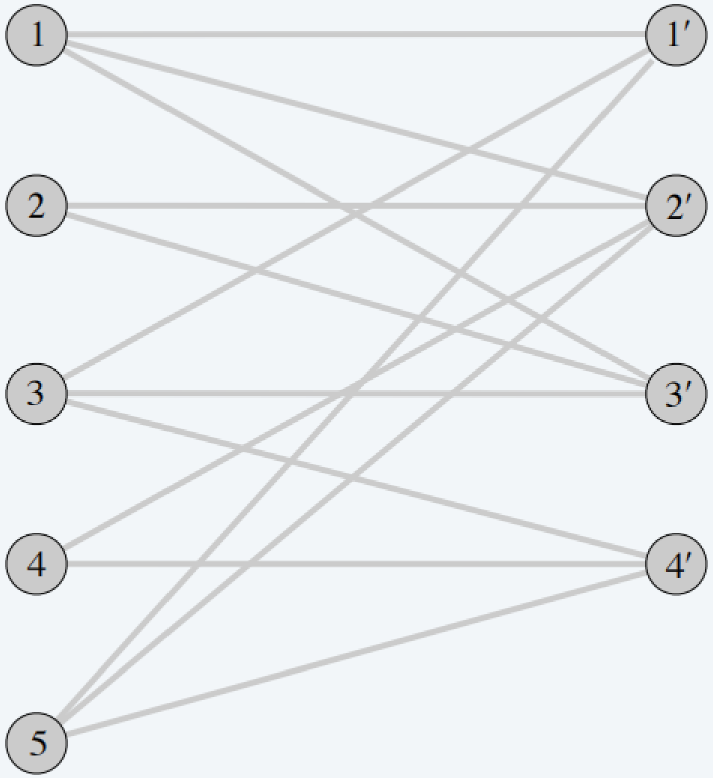
哈密尔顿回路问题

HAMILTON-CYCLE. Given an undirected graph $G = (V, E)$, does there exist a cycle Γ that visits every node exactly once?



yes

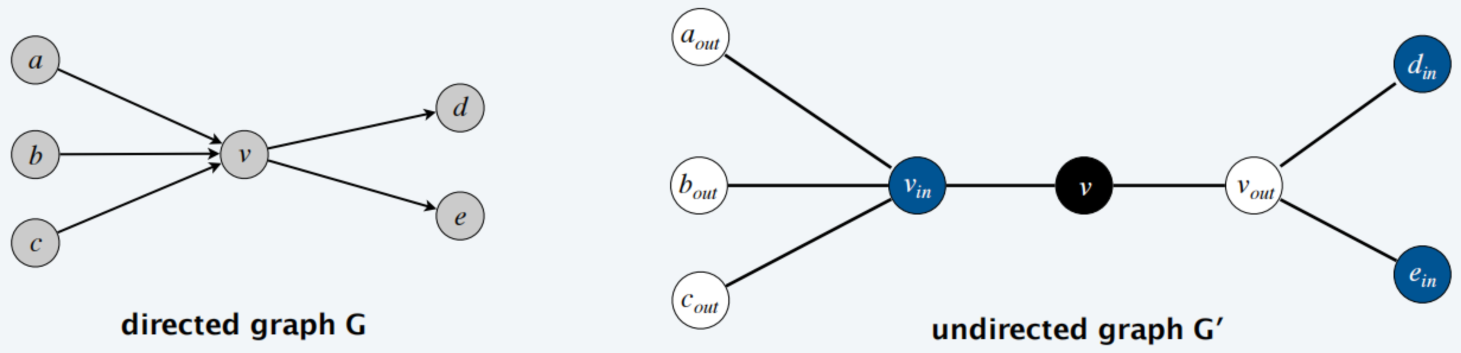


no

（有向）哈密尔顿回路问题

- 有向哈密尔顿回路问题$\leq_P$ 哈密尔顿回路问题

Pf. Given a directed graph $G = (V, E)$, construct a graph G' with $3n$ nodes.



- 下一步：3-SAT $\leq_P$ 有向哈密尔顿回路问题

  证明：略

分割问题

- 3维匹配

3D-MATCHING. Given n instructors, n courses, and n times, and a list of the possible courses and times each instructor is willing to teach, is it possible to make an assignment so that all courses are taught at different times?

instructor	course	time
Wayne	COS 226	TTh 11–12:20
Wayne	COS 423	MW 11–12:20
Wayne	COS 423	TTh 11–12:20
Tardos	COS 423	TTh 3–4:20
Tardos	COS 523	TTh 3–4:20
Kleinberg	COS 226	TTh 3–4:20
Kleinberg	COS 226	MW 11–12:20
Kleinberg	COS 423	MW 11–12:20

分割问题

- 3维匹配

3D-MATCHING. Given 3 disjoint sets X, Y, and Z, each of size n and a set $T \subseteq X \times Y \times Z$ of triples, does there exist a set of n triples in T such that each element of $X \cup Y \cup Z$ is in exactly one of these triples?

$$X = \{ x_1, x_2, x_3 \}, \quad Y = \{ y_1, y_2, y_3 \}, \quad Z = \{ z_1, z_2, z_3 \}$$

$$T_1 = \{ x_1, y_1, z_2 \}, \quad T_2 = \{ x_1, y_2, z_1 \}, \quad \boxed{T_3 = \{ x_1, y_2, z_2 \}}$$

$$T_4 = \{ x_2, y_2, z_3 \}, \quad \boxed{T_5 = \{ x_2, y_3, z_3 \}},$$

$$T_7 = \{ x_3, y_1, z_3 \}, \quad \boxed{T_8 = \{ x_3, y_1, z_1 \}}, \quad T_9 = \{ x_3, y_2, z_1 \}$$

an instance of 3d-matching (with n = 3)

分割问题

- 3维匹配

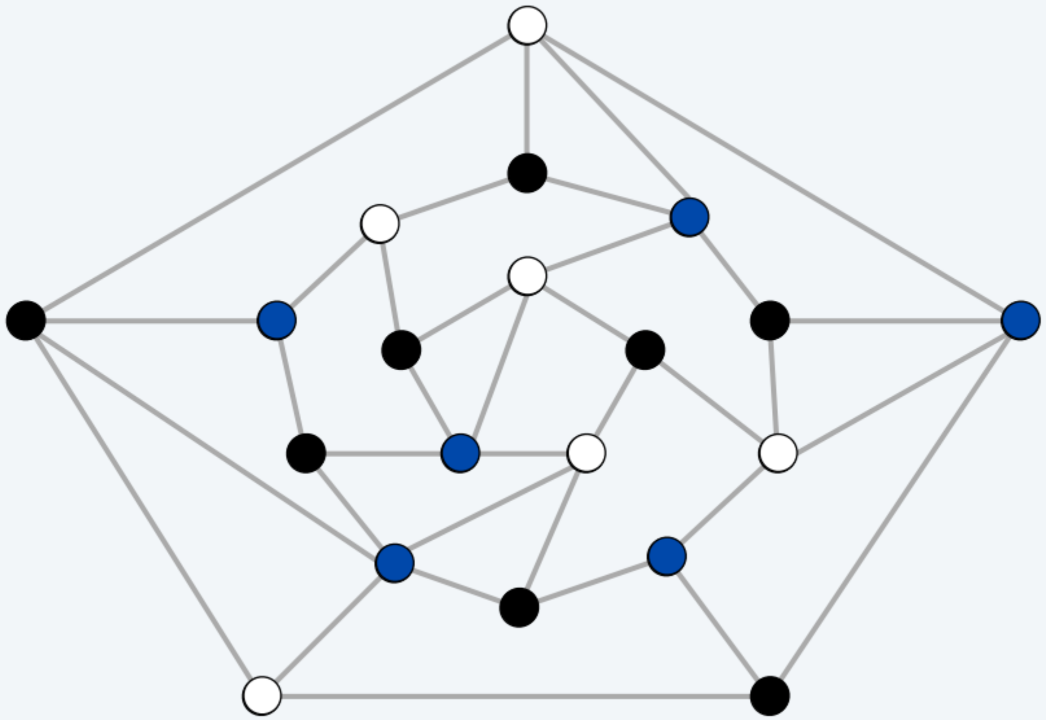
- 结论： 3-SAT $\leq_P$ 分割问题

图染色问题

- 3着色

- 结论
- 3-SAT $\leq_P$ 3着色问题

3-COLOR. Given an undirected graph G, can the nodes be colored black, white, and blue so that no adjacent nodes have the same color?



yes instance

Quiz 6

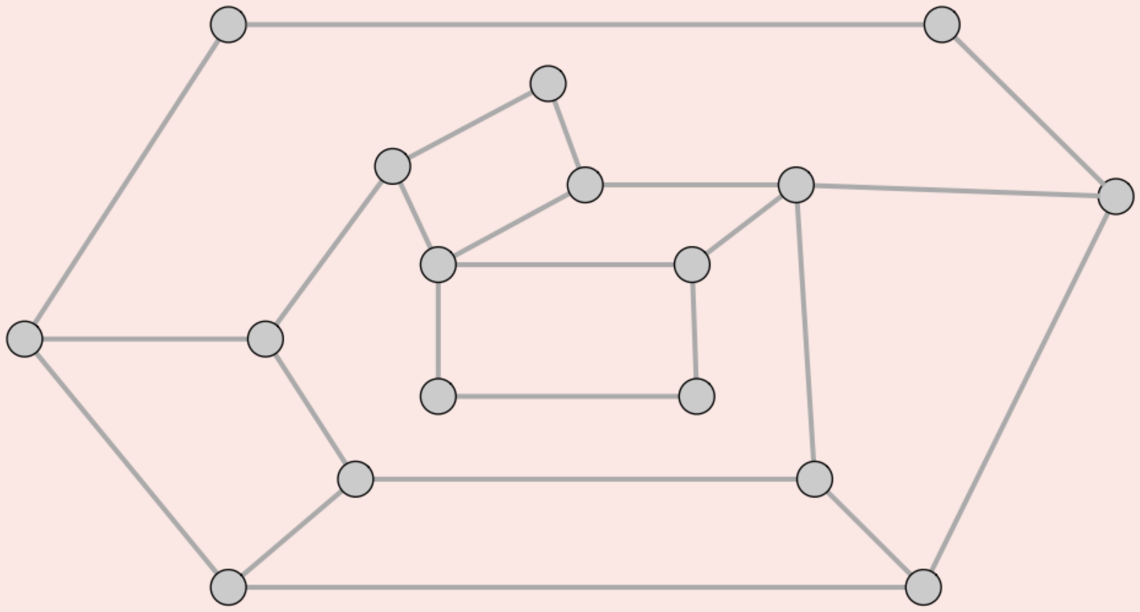
How difficult to solve 2-COLOR?

A. $O(m + n)$ using BFS or DFS.

B. $O(mn)$ using maximum flow.

C. $\Omega(2^n)$ using brute force.

D. Not even Tarjan knows.



数值组合优化

- 子集和问题

SUBSET-SUM. Given n natural numbers $w_1, \ldots, w_n$ and an integer W, is there a subset that adds up to exactly W?

Ex. $\{ 215, 215, 275, 275, 355, 355, 420, 420, 580, 580, 655, 655 \}$, $W = 1505$.

Yes. $215 + 355 + 355 + 580 = 1505$.

- 结论： 3-SAT $\leq_P$ 子集和问题

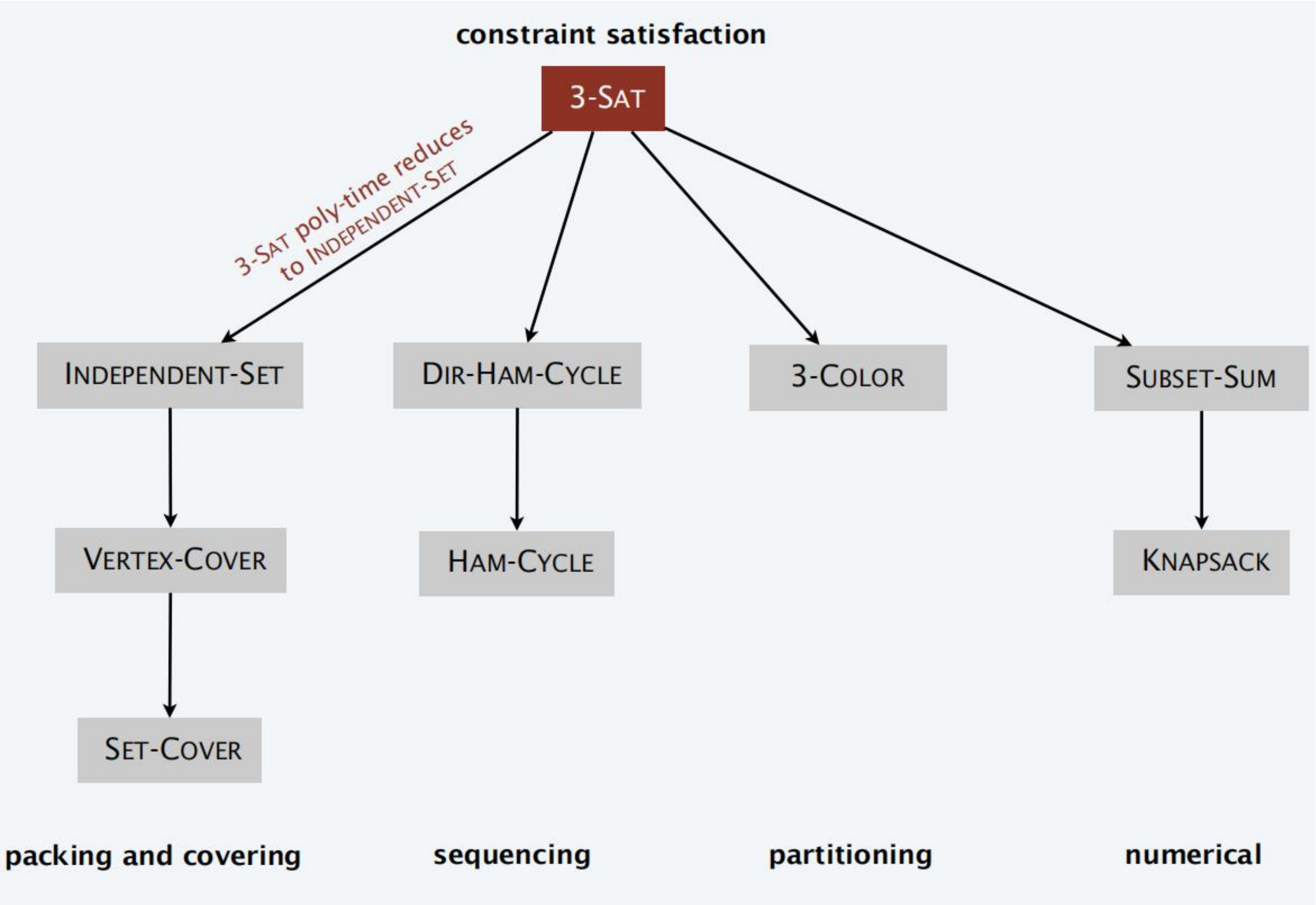
数值组合优化

- 子集和问题：已知一些自然数和一个整数W，问是否存在和恰为W的子集?

- 对比背包问题，定义?
- 回顾背包问题动态规划算法，时间复杂度?

- 结论： 子集和问题 $\le_P$ 背包问题

小结


constraint satisfaction
3-SAT
3-SAT poly-time reduces
to INDEPENDENT-SET
INDEPENDENT-SET
DIR-HAM-CYCLE
3-COLOR
SUBSET-SUM
VERTEX-COVER
HAM-CYCLE
KNAPSACK
SET-COVER
packing and covering
sequencing
partitioning
numerical

小结——20个从SAT规约的问题

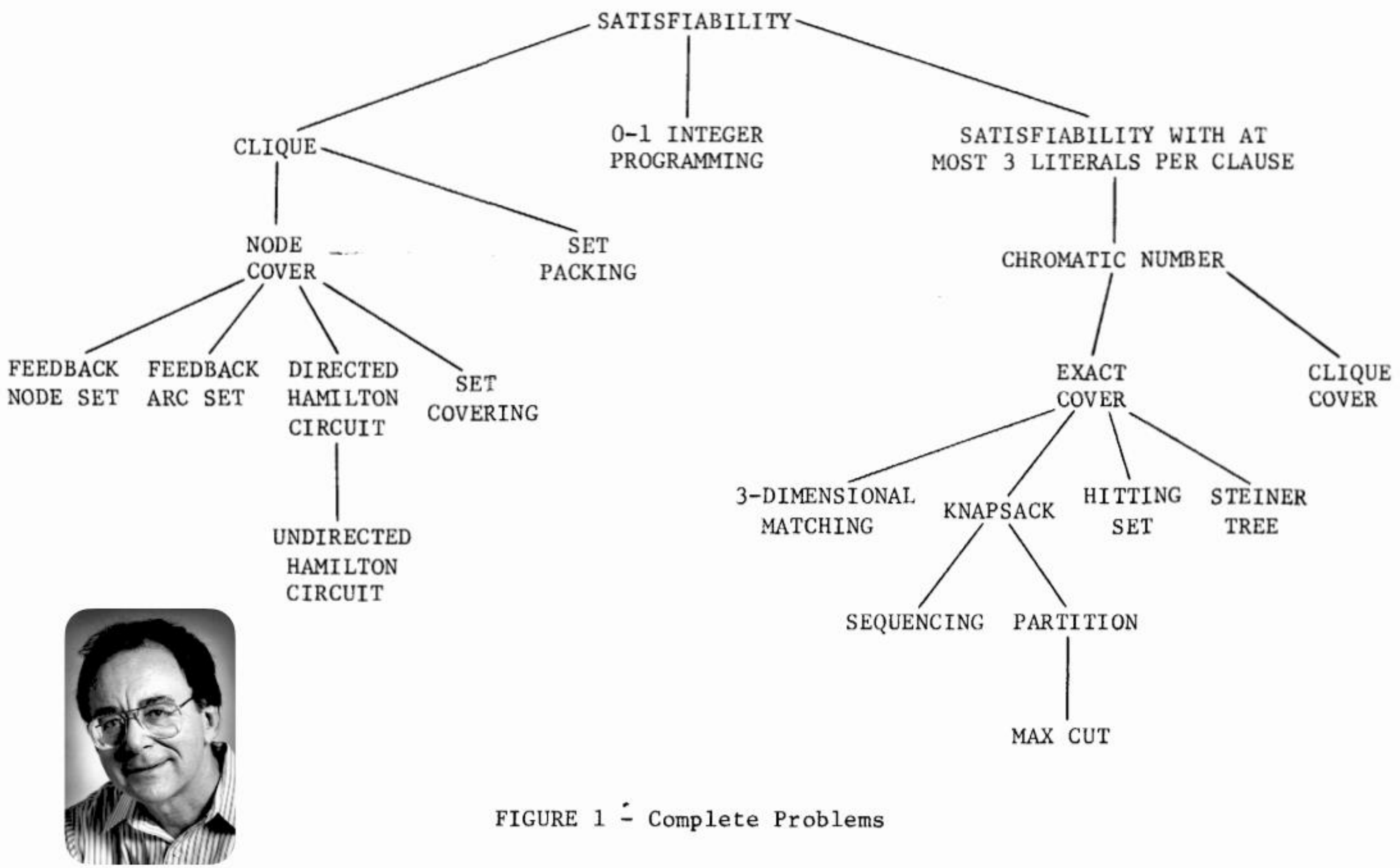


FIGURE 1 - Complete Problems

Dick Karp (1972)
1985 Turing Award