Optimal Control

Midterm Exam

Spring 2022

Date: 14 March 2021

Materials Allowed During Examination

You are permitted to use any and all resources you want for this exam, but you are not permitted to communicate with anyone except myself during the course of completing this exam. All solutions must be typed; no handwritten solutions are permitted.

Guidelines for Solutions

As I have stressed in class, communication is an extremely important part of demonstrating that you understand the material. To this end, the following guidelines are in effect for all problems on the examination:

- Your solutions must be clearly explained. Any unclear explanations will be marked as incorrect.
- You must be crystal clear with every step of your solution. In other words, any step in a derivation or statement you write must be unambiguous (that is, every step must have one and only one meaning). If it is ambiguous as to what you mean in a step, then I will assume the step is incorrect.

In short, please write your solutions in a orderly fashion so that somebody else can make sense of what you are doing and saying.

Point Distribution

The exam consists of two questions and the value of each question clearly indicated. Unless otherwise stated, full credit will be given for a proper application of a relevant concept (for example, proper description of kinematics and kinetics, understanding how to compute a transfer function). Contrariwise, no credit will be given for a concept applied incorrectly, ***even if the final answer is correct***.

University of Florida Honor Code

On your exam you must state and sign the University of Florida honor pledge as follows:

I pledge on my honor that I did not violate University of Florida honor code during any portion of this exam.

Name: **UF-ID:**

Signature: **Date:**

Thought Before Starting Exam

You say : "'Ere thrice the sun done salutation to the dawn"
And you claim these words as your own
But I've read well, and I've heard them said
A hundred times (maybe less, maybe more)
If you must write prose/poems
The words you use should be your own
Don't plagiarise or take "on loan"
'Cause there's always someone, somewhere
With a big nose, who knows
And who trips you up and laughs
When you fall
Who'll trip you up and laugh
When you fall

1 Question 1: 50 Points

Objective

The goal of this problem is to formulate and solve an optimal control problem arising in spacecraft dynamics. Each section below provides a description of those aspects of the problem that are required both for the formulation and the solution.

1.1 Differential Equations of Motion

Consider a spacecraft, modeled as a point mass P of mass m, moving relative to an inertial reference frame $\mathcal{I}$. Furthermore, assume that the particle is moving in a plane that is itself fixed in $\mathcal{I}$. Next, let $\left\{\mathbf{e}_z, \mathbf{e}_y, \mathbf{e}_z\right\}$ be a right-handed orthonormal basis fxed in $\mathcal{I}$.

Assume now that the position of the spacecraft is denoted $\mathbf{r}_{P/O}$, where O is fixed in $\mathcal{I}$ and denotes the location of the Sun (that is, the position of the spacecraft is measured relative to the Sun). Suppose now that $\mathbf{r}_{P/O}$ is parameterized in terms of a basis $\{\mathbf{u}_r, \mathbf{u}_\theta, \mathbf{u}_z\}$, where $\{\mathbf{u}_r, \mathbf{u}_\theta, \mathbf{u}_z\}$ rotates about the $\mathbf{e}_z$–direction such that θ is the angle from $\mathbf{e}_x$ to $\mathbf{u}_r$. Finally, assume that the distance from O to P is denoted r. A schematic of the geometry of the problem is given in Fig. 1.

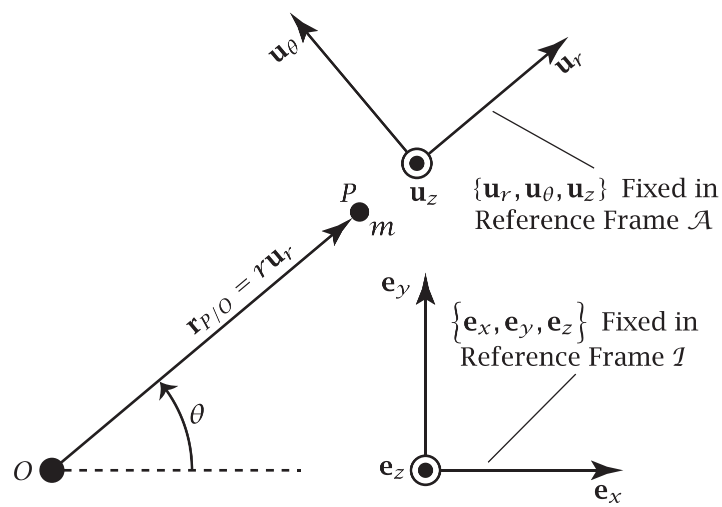

Figure 1: Schematic of particle moving in an inertially fixed plane.

Suppose now that the following two forces act on the spacecraft: (1) gravitation and (2) thrust. The gravitational force is given as

$$\mathbf{G} = -m\mu \frac{\mathbf{r}_{P/O}}{\|\mathbf{r}_{P/O}\|^3}. \tag{1}$$

while the thrust force is given as

$$\mathbf{T} = T\mathbf{w}, \tag{2}$$

where $\mathbf{w}$ is a unit vector that lies an angle β from the direction $\mathbf{u}_\theta$ (in other words, β is the angle between $\mathbf{u}_\theta$ and $\mathbf{w}$).

Using the above formulation, derive a system of two second-order differential equations using

(a) Newton's second law for a particle

(b) Lagrange's equations

When applying Lagrange's equations, the generalized coordinates should be r and θ and it is known that the generalized force associated $\mathbf{G}$ in Eq. (1) is derivable from the scalar potential function

$$U = -\frac{m\mu}{r}$$

The inputs to the process should be T and β.

Next, rewrite the two second-order differential equations that have been derived into a system of four first-order equations. When rewriting in the first-order form, make the following substitutions:

$$\begin{aligned} \dot{r} &= v_r, \\ r\dot{\theta} &= v_\theta \end{aligned} \tag{3}$$

Upon using the substitions given in Eq. (3), a system of four first-order differential equations should be obtained in terms of $(r, \theta, v_r, v_\theta)$ with T and β as inputs to the system and μ being a parameter of the system.

FInally, suppose that the mass flow rate of the engine is given by the equation

$$\dot{m} = -\frac{T}{v_e} \tag{4}$$

where v_e is the exhaust speed of the engine. Including Eq. (4), the system should now consist of *five* first-order differential equations with an additional parameter v_e included in the process.

1.2 Formulation of Optimal Control Problem

Using the dynamics derived in Section 1.1, the following optimal control problem is now formulated. First, the spacecraft starts in a heliocentric circular orbit of radius $r_0 = 1$ and terminates in a heliocentric circular orbit of radius $r_f = 1.5$. It is assumed that the initial longitude (where θ denotes the longitude) is zero while the terminal longitude is free. Furthermore, the initial mass is unity (that is, $m_0 = 1$) while the terminal mass is free. The objective is to minimize the time taken to transfer the spacecraft from the initial orbit to the terminal orbit. Using this information, determine quantitative descriptions of

(a) the boundary conditions

(b) the objective functional

Then, including the differential equations derived in Section 1.1, state the optimal control problem formally.

2 Numerical Solution of Optimal Control Problem

In this section various methods will be used to solve the optimal control problem formulated in Section 1. In particular, solve the optimal control problem using the following numerical methods:

(a) indirect shooting

(b) indirect multiple-shooting with $K = (2, 4, 8, 16)$ intervals

(c) direct shooting

(d) direct multiple-shooting with $K = (2, 4, 8, 16)$ intervals

When using direct shooting and direct multiple-shooting, parameterize the control using a polynomial of degree N in each interval, where N can take on the values $(2, \dots, 6)$ (that is, the degree of the polynomial can be set to between two and six). Perform a study that compares the solutions obtained using the two indirect methods with the solutions obtained uisng the two direct methods. Describe the differences obtained and analyze which methods provide the best solutions for this problem. When performing your study, use the following values for the constants:

$$\begin{aligned} \mu &= 1 \\ T &= 0.1405 \\ v_e &= 1.8758344 \end{aligned}$$

Note that, as a result of using the above constant, T is not used as a control but is set equal to a constant.