EML6934 Optimal Control
Final Project

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The equations of motion were derived using both Newton's second law and Lagrange's equations. The schematic for the problem can be seen Figure 1. The spacecraft is modeled as point P of mass m. The spacecraft moves relative to an inertial reference frame l. The reference frame fixed in l is expressed as $\{e_x, e_y, e_z\}$. The position of the spacecraft is denoted as $r_{P/O}$, where O is modeled as the sun, fixed in l. The spacecraft is parameterized in the basis $\{u_r, u_\theta, u_z\}$, where the rotation is about $u_z = e_z$. The rotation creates an angle θ between e_x and u_r, which can be seen in Figure 2. Two forces are said to act on the spacecraft. The first is the gravitational force which is given as

$$G = -m\mu \frac{r_{P/O}}{||r_{P/O}||^3}, \tag{1}$$

while the second is the thrust force given as

$$T = Tw, \tag{2}$$

where w is the unit vector that lies an angle β from the direction u_θ as seen in Figure 3.

2.1 Position, Velocity and Acceleration of the Spacecraft

As seen in Figure 1, the position of the spacecraft represented in reference frame A, is given as

$$^A\vec{r}_{P/O} = ru_r. \tag{3}$$

The velocity can then be represented in the inertial frame l by using equation 4, where $^l\vec{\omega}^A$ is the angular velocity between reference frame l and A.

$$\frac{^l d\vec{r}_{P/O}}{dt} = \frac{^A d\vec{r}_{P/O}}{dt} + \ ^l\vec{\omega}^A \ \times \ ^A\vec{r}_{P/O} \tag{4}$$

Using Equation 4, the velocity of the spacecraft in the inertial frame is then formulated as

$$\begin{aligned}
^l\vec{v_p} &= \frac{^l d\vec{r}_{P/O}}{dt} \\
^l\vec{v_p} &= \dot{r}u_r + \ \dot{\theta}u_z \ \times \ ru_r \\
^l\vec{v_p} &= \dot{r}u_r + \ \dot{\theta}ru_\theta
\end{aligned} \tag{5}$$

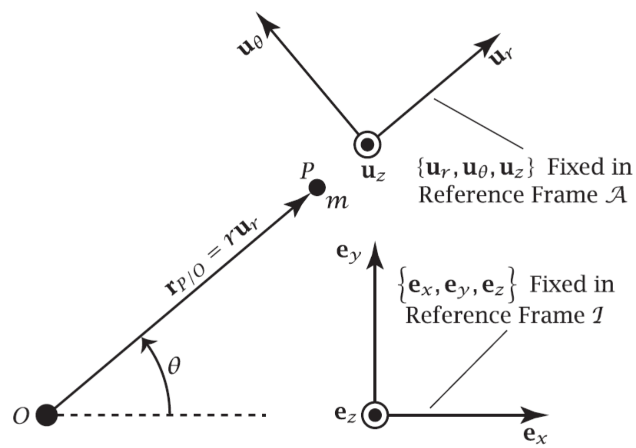

Figure 1: Schematic of particle moving in an inertially fixed plane

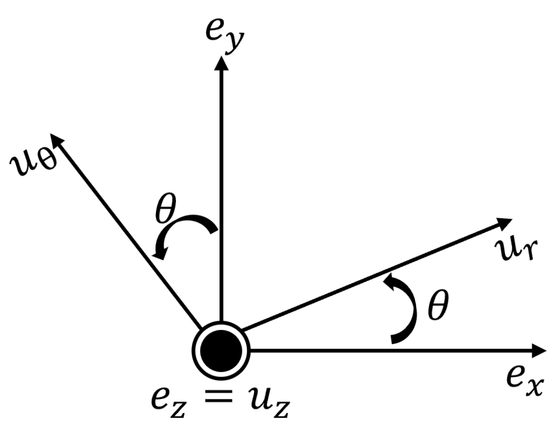

Figure 2: Reference frame rotation

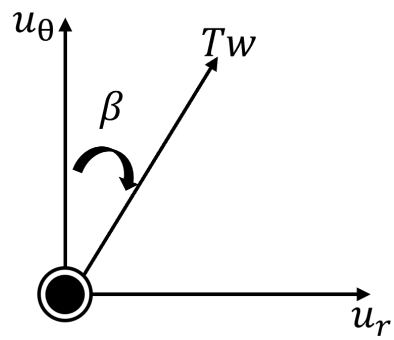

Figure 3: Thrust Force at Angle β

The acceleration of the particle can then be formulated as

$$
\begin{aligned}
{}^{l}\vec{a_p} &= \frac{{}^{l}d\vec{v_P}}{dt} \\
{}^{l}\vec{a_p} &= \frac{{}^{A}d\vec{v_P}}{dt} + \ {}^{l}\vec{\omega}^{A} \ \times \ {}^{A}\vec{v_P} \\
{}^{l}\vec{a_p} &= \ddot{r}u_r + \ (\ddot{\theta}r + \dot{\theta}\dot{r})u_\theta + \ \dot{\theta}\dot{r}u_\theta - \ \dot{\theta}^2 r u_r \\
{}^{l}\vec{a_p} &= (\ddot{r} - \ \dot{\theta}^2 r)u_r + \ (\ddot{\theta}r + 2\dot{\theta}\dot{r})u_\theta \qquad (6)
\end{aligned}
$$

2.2 Newton's Second Law for A Particle

The equations of motion are first derived using Newton's Second Law for a Particle. Newton's Second Law for a particle is represented by

$$\sum F_P = m_P * a_P. \qquad (7)$$

Figure 4 represents the free body diagram of the particle system. F_G, represented by equation 1, is the gravitational force and acts along the u_r direction. F_T, represented by equation 2, is the thrust force and acts in the direction w. It can be seen in Figure 3 that the thrust force can be re-written as

$$F_T = Tsin(\beta)u_r + T\cos(\beta)u_\theta, \qquad (8)$$

while the gravitational force can be written as

$$F_G = -m\mu\frac{1}{r^2}u_r. \qquad (9)$$

We can now substitute equations 6, 8, and 9 into equation 7 to obtain

$$-m\mu\frac{1}{r^2}u_r + Tsin(\beta)u_r + T\cos(\beta)u_\theta = m[(\ddot{r} - \ \dot{\theta}^2 r)u_r + \ (\ddot{\theta}r + 2\dot{\theta}\dot{r})u_\theta].$$

After equating terms, the two equations of motion using Newtons Seconds become:

$$(u_r) \qquad \ddot{r} = \dot{\theta}^2 r - \frac{\mu}{r^2} + \frac{Tsin(\beta)}{m} \qquad (10)$$

$$(u_\theta) \qquad \ddot{\theta} = -\frac{2\dot{r}\dot{\theta}}{r} + \frac{T\cos(\beta)}{mr} \qquad (11)$$

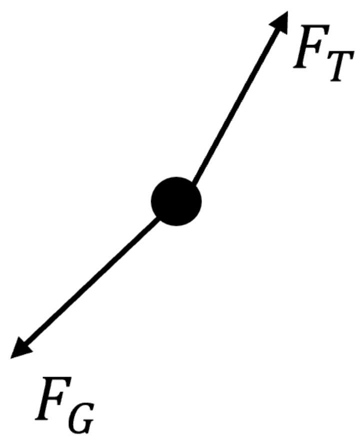

Figure 4: Free Body Diagram