EML6934 Optimal Control
Final Project

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2 Differential Equations of Motion

The differential equations of motion were derived using both Newton's second law. The schematic for the problem can be seen Figure 1. The spacecraft is modeled as point P of mass m. The spacecraft moves relative to an inertial reference frame l. The reference frame fixed in l is expressed as $\{e_x, e_y, e_z\}$. The position of the spacecraft is denoted as $r_{P/O}$, where O is modeled as the sun, fixed in l. The spacecraft is parameterized in the basis $\{u_r, u_\theta, u_z\}$, where the rotation is about $u_z = e_z$. The rotation creates an angle θ between e_x and u_r, which can be seen in Figure 2. Two forces are said to act on the spacecraft. The first is the gravitational force which is given as

$$G = -m\mu \frac{r_{P/O}}{||r_{P/O}||^3}, \tag{1}$$

while the second is the thrust force given as

$$T = Tw, \tag{2}$$

where w is the unit vector that lies an angle β from the direction u_θ as seen in Figure 3.

2.1 Position, Velocity and Acceleration of the Spacecraft

As seen in Figure 1, the position of the spacecraft represented in reference frame A, is given as

$$^{A}\vec{r}_{P/O} = r u_r. \tag{3}$$

The velocity can then be represented in the inertial frame l by using equation 4, where $^{l}\vec{\omega}^{A}$ is the angular velocity between reference frame l and A.

$$\frac{^{l}d\vec{r}_{P/O}}{dt} = \frac{^{A}d\vec{r}_{P/O}}{dt} + \ ^{l}\vec{\omega}^{A} \ \times \ ^{A}\vec{r}_{P/O} \tag{4}$$

Using Equation 4, the velocity of the spacecraft in the inertial frame is then formulated as

$$\begin{aligned} ^{l}\vec{v_p} &= \frac{^{l}d\vec{r}_{P/O}}{dt} \\ ^{l}\vec{v_p} &= \dot{r}u_r + \ \dot{\theta}u_z \ \times \ ru_r \\ ^{l}\vec{v_p} &= \dot{r}u_r + \ \dot{\theta}ru_\theta \end{aligned} \tag{5}$$

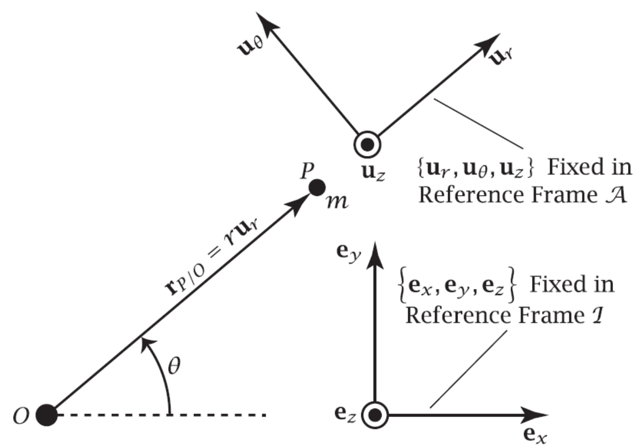

Figure 1: Schematic of particle moving in an inertially fixed plane

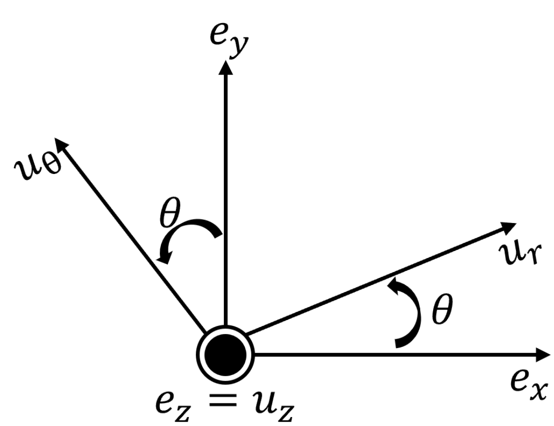

Figure 2: Reference frame rotation

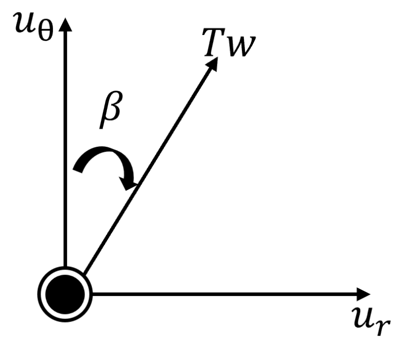

Figure 3: Thrust Force at Angle β

The acceleration of the particle can then be formulated as

$$
\begin{aligned}
{}^{l}\vec{a_p} &= \frac{{}^{l}d\vec{v_P}}{dt} \\
{}^{l}\vec{a_p} &= \frac{{}^{A}d\vec{v_P}}{dt} + \ {}^{l}\vec{\omega}^{A} \ \times \ {}^{A}\vec{v_P} \\
{}^{l}\vec{a_p} &= \ddot{r}u_r + \ (\ddot{\theta}r + \dot{\theta}\dot{r})u_\theta + \ \dot{\theta}\dot{r}u_\theta - \ \dot{\theta}^2 r u_r \\
{}^{l}\vec{a_p} &= (\ddot{r} - \ \dot{\theta}^2 r)u_r + \ (\ddot{\theta}r + 2\dot{\theta}\dot{r})u_\theta
\end{aligned} \tag{6}
$$

2.2 Newton's Second Law for A Particle

Newton's second law for a particle is represented by

$$\sum F_P = m_P * a_P. \tag{7}$$

Figure 4 represents the free body diagram of the particle system. F_G, represented by equation 1, is the gravitational force and acts along the u_r direction. F_T, represented by equation 2, is the thrust force and acts in the direction w. It can be seen in Figure 3 that the thrust force can be re-written as

$$F_T = Tsin(\beta)u_r + T\cos(\beta)u_\theta, \tag{8}$$

while the gravitational force can be written as

$$F_G = -m\mu\frac{1}{r^2}u_r. \tag{9}$$

We can now substitute equations 6, 8, and 9 into equation 7 to obtain

$$-m\mu\frac{1}{r^2}u_r + Tsin(\beta)u_r + T\cos(\beta)u_\theta = m[(\ddot{r} - \ \dot{\theta}^2 r)u_r + \ (\ddot{\theta}r + 2\dot{\theta}\dot{r})u_\theta].$$

After equating terms, the two equations of motion using Newtons Seconds become:

$$(u_r) \qquad \ddot{r} = \dot{\theta}^2 r - \frac{\mu}{r^2} + \frac{Tsin(\beta)}{m} \tag{10}$$

$$(u_\theta) \qquad \ddot{\theta} = -\frac{2\dot{r}\dot{\theta}}{r} + \frac{T\cos(\beta)}{mr} \tag{11}$$

2.3 Conversion to First-Order Equations

To re-write the two second-order equations into first four first-order equations, the following substitutions can be made:

$$\dot{r} = v_r, \tag{12}$$

$$r\dot{\theta} = v_\theta,$$

$$\dot{\theta} = \frac{v_\theta}{r}. \tag{13}$$

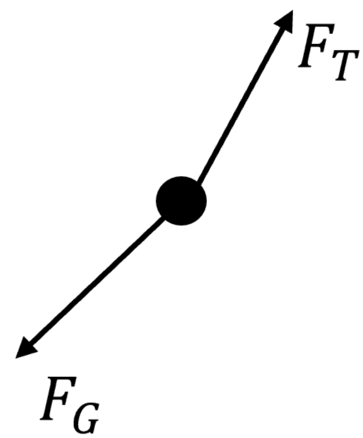

Figure 4: Free Body Diagram

Then, taking the derivatives of r and θ of equations 12 and 13, respectively, we obtain:

$$\ddot{r} = \dot{v}_r, \tag{14}$$

$$\ddot{\theta} = \frac{\dot{v}_\theta r - v_\theta \dot{r}}{r^2}. \tag{15}$$

After substituting equations 12, 13, 14, and 15 into equations 10 and 11, two first-order equations are derived as:

$$\begin{aligned}\dot{v}_r &= \frac{v_\theta^2 r}{r^2} - \frac{\mu}{r^2} + \frac{T\sin(\beta)}{m},\\ \dot{v}_r &= \frac{v_\theta^2}{r} - \frac{\mu}{r^2} + \frac{T\sin(\beta)}{m},\end{aligned} \tag{16}$$

and,

$$\begin{aligned}\frac{\dot{v}_\theta r - v_\theta v_r}{r^2} &= -\frac{2v_\theta v_r}{r^2} + \frac{T\cos(\beta)}{mr},\\ \dot{v}_\theta r - v_\theta v_r &= -\frac{2v_\theta v_r r^2}{r^2} + \frac{T\cos(\beta) r^2}{mr},\\ \dot{v}_\theta &= -\frac{2v_\theta v_r}{r} + \frac{T\cos(\beta)}{m} + \frac{v_\theta v_r}{r},\\ \dot{v}_\theta &= -\frac{v_\theta v_r}{r} + \frac{T\cos(\beta)}{m}.\end{aligned} \tag{17}$$

Next, a fifth first-order equation is given as

$$\dot{m} = -\frac{T}{v_e}. \tag{18}$$

The five first-order differential equations are listed below:

$$\begin{aligned}
\dot{r} &= v_r, \\
\dot{\theta} &= \frac{v_\theta}{r}, \\
\dot{v}_r &= \frac{v_\theta^2}{r} - \frac{\mu}{r^2} + \frac{T\sin(\beta)}{m}, \\
\dot{v}_\theta &= -\frac{v_\theta v_r}{r} + \frac{T\cos(\beta)}{m}, \\
\dot{m} &= -\frac{T}{v_e}.
\end{aligned}$$

3 Formulation of Optimal Control Problem

There are two objectives for the optimal control problem. The first objective is to minimize the time to transfer from an initial circular orbit to a final circular orbit. The second objective is to maximize the terminal mass when transferring from an initial circular orbit to a final circular orbit. Below is an overview of the problem:

$$\begin{aligned}
\text{Objective 1}: &\quad \min\ t_f \\
\text{Objective 2}: &\quad \max\ m(t_f) \\
\text{State}: &\quad r(t), \theta(t), v_r(t), v_\theta(t), m(t) \\
\text{Control}: &\quad \beta(t), T(t)
\end{aligned}$$