B.3 Spherical Coordinates

We first choose an origin. Then we choose a coordinate, r, that measures the radial distance from the origin to the point P. The coordinate r ranges in value from $0 \le r < \infty$. The set of points that have constant value for r are spheres ("level surfaces").

Any point on the sphere can be defined by two angles (θ, ϕ) and r. We will define these angles with respect to a choice of Cartesian coordinates (x, y, z). The angle θ is defined to be the angle between the positive z-axis and the ray from the origin to the point P. Note that the values of θ only range from $0 \le \theta \le \pi$. The angle ϕ is defined (in a similar fashion to polar coordinates) as the angle in the between the positive x-axis and the projection in the x-y plane of the ray from the origin to the point P. The coordinate angle ϕ can take on values from $0 \le \phi < 2\pi$.

The spherical coordinates (r, θ, ϕ) for the point P are shown in Figure B.3.1. We choose the unit vectors $(\hat{\mathbf{r}}, \hat{\boldsymbol{\theta}}, \hat{\boldsymbol{\varphi}})$ at the point P as follows. Let $\hat{\mathbf{r}}$ point radially away from the origin, and $\hat{\boldsymbol{\theta}}$ point tangent to a circle in the positive θ direction in the plane formed by the z-axis and the ray from the origin to the point P. Note that $\hat{\boldsymbol{\theta}}$ points in the direction of increasing θ. We choose $\hat{\boldsymbol{\varphi}}$ to point in the direction of increasing ϕ. This unit vector points tangent to a circle in the xy – plane centered on the z-axis. These unit vectors are also shown in Figure B.3.1.

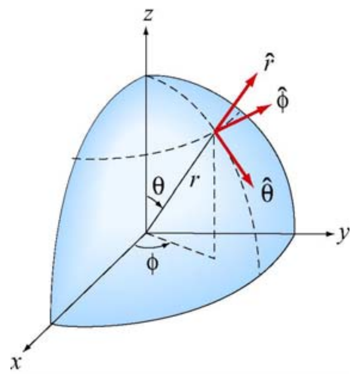

Figure B.3.1 Spherical coordinates

If you are given spherical coordinates (r, θ, ϕ) of a point in the plane, the Cartesian coordinates (x, y, z) can be determined from the coordinate transformations

$$
\begin{aligned}
x &= r\sin\theta\cos\phi \\
y &= r\sin\theta\sin\phi \\
z &= r\cos\theta
\end{aligned}
\tag{B.3.1}
$$

Conversely, if you are given the Cartesian coordinates (x, y, z), the spherical coordinates (r, θ, ϕ) can be determined from the coordinate transformations

$$
\begin{aligned}
r &= +(x^2 + y^2 + z^2)^{1/2} \\
\theta &= \cos^{-1}\left(\frac{z}{(x^2 + y^2 + z^2)^{1/2}}\right) \\
\phi &= \tan^{-1}(y/x)
\end{aligned}
\tag{B.3.2}
$$

The unit vectors also are related by the coordinate transformations

$$
\begin{aligned}
\hat{\mathbf{r}} &= \sin\theta\cos\phi\,\hat{\mathbf{i}} + \sin\theta\sin\phi\,\hat{\mathbf{j}} + \cos\theta\,\hat{\mathbf{k}} \\
\hat{\boldsymbol{\theta}} &= \cos\theta\cos\phi\,\hat{\mathbf{i}} + \cos\theta\sin\phi\,\hat{\mathbf{j}} - \sin\theta\,\hat{\mathbf{k}} \\
\hat{\boldsymbol{\varphi}} &= -\sin\phi\,\hat{\mathbf{i}} + \cos\phi\,\hat{\mathbf{j}}
\end{aligned}
\tag{B.3.3}
$$

These results can be understood by considering the projection of $(\hat{\mathbf{r}}, \hat{\boldsymbol{\theta}})$ into the unit vectors $(\hat{\boldsymbol{\rho}}, \hat{\mathbf{k}})$, where $\hat{\boldsymbol{\rho}}$ is the unit vector from cylindrical coordinates (Figure B.3.2),

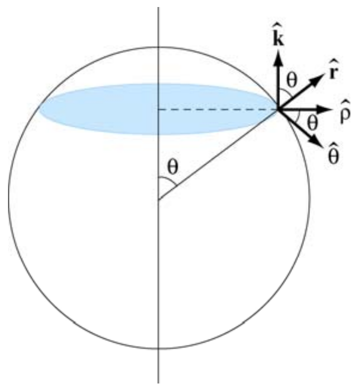

Figure B.3.2 Cylindrical and spherical unit vectors

$$
\begin{aligned}
\hat{\mathbf{r}} &= \sin\theta\,\hat{\boldsymbol{\rho}} + \cos\theta\,\hat{\mathbf{k}} \\
\hat{\boldsymbol{\theta}} &= \cos\theta\,\hat{\boldsymbol{\rho}} - \sin\theta\,\hat{\mathbf{k}}
\end{aligned}
\tag{B.3.4}
$$

We can use the vector decomposition of $\hat{\boldsymbol{\rho}}$ into the Cartesian unit vectors $(\hat{\mathbf{i}}, \hat{\mathbf{j}})$:

$$
\hat{\boldsymbol{\rho}} = \cos\phi\,\hat{\mathbf{i}} + \sin\phi\,\hat{\mathbf{j}} \tag{B.3.5}
$$

To find the inverse transformations we can use the fact that

$$\hat{\boldsymbol{\rho}} = \sin\theta\,\hat{\mathbf{r}} + \cos\theta\,\hat{\boldsymbol{\theta}} \tag{B.3.6}$$

to express

$$\begin{aligned} \hat{\mathbf{i}} &= \cos\phi\,\hat{\boldsymbol{\rho}} - \sin\phi\,\hat{\boldsymbol{\varphi}} \\ \hat{\mathbf{j}} &= \sin\phi\,\hat{\boldsymbol{\rho}} + \cos\phi\,\hat{\boldsymbol{\varphi}} \end{aligned} \tag{B.3.7}$$

as

$$\begin{aligned} \hat{\mathbf{i}} &= \cos\phi\sin\theta\,\hat{\mathbf{r}} + \cos\phi\cos\theta\,\hat{\boldsymbol{\theta}} - \sin\phi\,\hat{\boldsymbol{\varphi}} \\ \hat{\mathbf{j}} &= \sin\phi\sin\theta\,\hat{\mathbf{r}} + \sin\phi\cos\theta\,\hat{\boldsymbol{\theta}} + \cos\phi\,\hat{\boldsymbol{\varphi}} \end{aligned} \tag{B.3.8}$$

The unit vector $\hat{\mathbf{k}}$ can be decomposed directly into $(\hat{\mathbf{r}}, \hat{\boldsymbol{\theta}})$ with the result that

$$\hat{\mathbf{k}} = \cos\theta\,\hat{\mathbf{r}} - \sin\theta\,\hat{\boldsymbol{\theta}} \tag{B.3.9}$$

B.3.1 Infinitesimal Line Element

Consider a small infinitesimal displacement $d\,\vec{\mathbf{s}}$ between two points (Figure B.3.3). This vector can be decomposed into

$$d\,\vec{\mathbf{s}} = dr\hat{\mathbf{r}} + rd\theta\hat{\boldsymbol{\theta}} + r\sin\theta d\phi\hat{\boldsymbol{\varphi}} \tag{B.3.10}$$

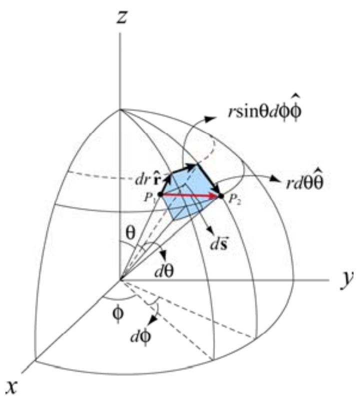

Figure B.3.3 Infinitesimal displacement vector $d\,\vec{\mathbf{s}}$.