Mathematical Induction

An essential property of the set $N = \{1, 2, 3 \ldots\}$ of positive integers follows:

Principle of Mathematical Induction I:

Let P be a proposition defined on the positive integers N; that is, $P(n)$ is either true or false for each $n \in N$. suppose P has the following two properties:

(I) $P(1)$ is true.

(II) $P(K+1)$ is true whenever $P(K)$ is true.

Then P is true for every positive integer $n \in N$.

Exp Let P be the proposition that the sum of the first n odd numbers is n^2.

ie $P(n) = 1 + 3 + 5 + \cdots + (2n-1) = n^2$

Proof: observe that $P(n)$ is true for $n = 1$

ie $P(1) = 1^2$

Assuming $P(K)$ is true, we add $(2K+1)$ to both sides of $P(K)$, obtaining

$1+3+5+\cdots+(2k-1)+(2k+1)=k^2+(2k+1)$

$$=(k+1)^2$$

which is $P(k+1)$.

i.e. $P(k+1)$ is true whenever $P(k)$ is true.

so, by the principle of mathematical induction, P is true for all n.

Principle of Mathematical Induction II:

Let P be a proposition defined on the positive integers N such that:

(i) $P(1)$ is true.

(ii) $P(k)$ is true whenever $P(j)$ is true for all $1 \le j < k$.

Then P is true for every positive integer $n \in N$.

Exp (1) Prove the proposition $P(n)$ that the sum of the first n positive integers is $\frac{1}{2}n(n+1)$.

i.e. $$P(n) = 1+2+3+\cdots+n = \frac{1}{2}n(n+1)$$

Proof: The proposition holds for $n=1$ since:

$$P(1) = 1 = \frac{1}{2}1\cdot(1+1)$$

Assuming $P(K)$ is true, we add $(K+1)$ to both sides of $P(K)$; obtaining

$$1+2+3+\cdots+K+(K+1)=\frac{1}{2}K(K+1)+(K+1)$$

$$=\frac{1}{2}[K(K+1)+2(K+1)]$$

$$=\frac{1}{2}[(K+1)(K+2)]$$

which is $P(K+1)$.

i.e. $P(K+1)$ is true whenever $P(K)$ is true.

By the Principle of Induction, P is true for all n.

Exp(2): Prove the following proposition (for $n\geq 0$).

$$P(n)=1+2+2^2+2^3+\cdots+2^n=2^{n+1}-1$$

Proof: $P(0)$ is true since $1=2^1-1$

Assuming $P(K)$ is true, we add 2^{K+1} to both sides of $P(K)$, obtaining

$$1+2+2^2+\cdots+2^K+2^{K+1}=2^{K+1}-1+2^{K+1}$$

$$=2(2^{K+1})-1$$

$$=2^{K+2}-1$$

which is $P(K+1)$.

ie $P(k+1)$ is true whenever $P(k)$ is true.

By the principle of induction, $P(n)$ is true for all n.

Exp ③: show that if n is an integer greater than 1, then n can be written as the product of primes.

solution: Let $P(n)$ be the proposition that n can be written as the product of primes.

For $n = 2$:
$P(2)$ is true, since 2 can be written as the product of one prime, itself.

For $n = k$: Assume that $P(k)$ is true for all positive integers k with $k \le n$.

it must be shown that $P(n+1)$ is true under this assumption:

There are two cases to consider, namely, when $(n+1)$ is prime and when $(n+1)$ is composite.

If $(n+1)$ is prime, we immediately see that $P(n+1)$ is true.

otherwise, $(n+1)$ is composite and can be written as the product of two positive integers a and b with $a \leq a \leq b < n+1$.

By the induction hypothesis, both a and b can be written as the product of primes.

Thus, if $(n+1)$ is composite, it can be written as the product of primes.

Recursive Definitions

Sometimes it is difficult to define an object explicitly. However, it may be easy to define this object in terms of itself. This process is called recursion.

Consequently, the given problem can be solved provided the simpler version can be solved.

This idea is pictorially represented as

The given complex problem $\xrightarrow[\text{if}]{\text{can be solved}}$ this simpler version can be solved $\xrightarrow[\text{if}]{\text{can be solved}}$; $\xrightarrow[\text{if}]{\text{can be solved}}$ this simpler version can be solved

This is solvable.

Recursive Definition of a Function :

Let $a \in W$ and $X = \{a, a+1, a+2, \ldots\}$. The recursive definition of a function f with domain X consists of three parts, where $k \geq 1$.

(1) Basic clause: A few initial values of the function $f(a), f(a+1), \ldots f(a+k-1)$ are specified. An equation that specifies such initial values is an initial conditions.

(2) Recursive clause: A formula to compute $f(n)$ from the k preceding functional values $f(n-1), f(n-2) \ldots f(n-k)$ is made. Such a formula is a recurrence relation (or recursion formula):

(3) Terminal clause: Only values thus obtained are valid functional values.

(For convenience, we drop this clause from our recursive definition.).

Theorem: Let $a \in W$, $X = \{a, a+1, a+2, \ldots\}$, and $k \in N$. Let $f: X \to R$ such that $f(a), f(a+1), \ldots, f(a+k-1)$ are known. Let n be any positive integer $\geq a+k$ such that $f(n)$ is defined in terms of $f(n-1), f(n-2) \ldots$ and $f(n-k)$. Then $f(n)$ is defined for every $n \geq a$.

Note: Recursive definitions are also known as inductive definitions.

Exp: 1 Suppose that f is defined recursively by

$$f(0) = 3$$
$$f(n+1) = 2f(n) + 3$$

Find $f(1)$, $f(2)$, $f(3)$, and $f(4)$.

Solution: From the recursive definition it follows that

$$f(1) = 2f(0) + 3 = 2 \cdot 3 + 3 = 9$$
$$f(2) = 2f(1) + 3 = 2 \cdot 9 + 3 = 21$$
$$f(3) = 2f(2) + 3 = 2 \cdot 21 + 3 = 45$$
$$f(4) = 2f(3) + 3 = 2 \cdot 45 + 3 = 93$$