Chapter1_Part 2 (End)

Chapter 1
Introduction to probability

4. Conditional Probability and Independence

In this section we introduce the important notion of *conditional probability*. The idea behind this concept is that the value of a probability can change if we get additional information. In the other words, the conditional probability answers the question 'how does the probability of an event change if we have extra information'. We'll illustrate with an example

Example 1. Toss a fair coin 3 times.
(a) What is the probability of 3 heads?
answer: Sample space $\Omega = \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}$.
All outcomes are equally likely, so $P(3 \text{ heads}) = 1/8$.

(b) Suppose we are told that the first toss was heads. Given this information how should we compute the probability of 3 heads?
answer: We have a new (reduced) sample space: $\Omega' = \{HHH, HHT, HTH, HTT\}$.
All outcomes are equally likely, so

$$P(3 \text{ heads given that the first toss is heads}) = 1/4.$$

This is called conditional probability, since it takes into account additional conditions. To develop the notation, we rephrase (b) in terms of *events*.

Rephrased (b) Let A be the event 'all three tosses are heads' $= \{HHH\}$.
Let B be the event 'the first toss is heads' $= \{HHH, HHT, HTH, HTT\}$.

The conditional probability of A knowing that B occurred is written

$$P(A|B)$$

This is read as

'the conditional probability of A given B'

or

'the probability of A conditioned on B'

or simply

'the probability of A given B'.

4.1. Formal definition of conditional probability

Let A and B be events. We define the conditional probability of A given B as

$$P(A|B) = \frac{P(A \cap B)}{P(B)}, \text{ provided } P(B) \neq 0.$$

Example 2: Let's redo the coin tossing example using the definition in above Equation.

Recall:

A = '3 heads'. Hence, A = {HHH}

B = 'first toss is heads'. Hence B = {HHH, HHT, HT H, HT T }.

Sample space $\Omega = \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}$.

Answer:

We have $P(A) = 1/8$ and $P(B) = 1/2$.

Since $A \cap B = A$, we also have $P(A \cap B) = 1/8$.

Now according to the equation of fomale defition, $P(A|B) = \frac{1/8}{1/2} = 1/4$, which agrees with our answer in Example 1b.

Multiplication Rule

The following formula is called the multiplication rule.

$$P(A \cap B) \quad = P(A|B) \cdot P(B).$$

This is simply a rewriting of the definition in Equation of conditional probability.

Law of Total Probability

Let B1, B2... be a sequence of events such that

(a) $P(B_k) > 0$ for $k = 1, 2, ...$

(b) B_i and B_j are disjoint whenever $i \neq j$

(c) $S = \bigcup_{k=1}^{\infty} B_k$

Then, for any event A, we have

$$P(A) = \sum_{k=1}^{\infty} P(A|B_k)P(B_k)$$

Bayes' Formula

We next turn to the situation when we know conditional probabilities in one direction but want to compute conditional probabilities "backwards." The following result is helpful.

Under the same assumptions as in the law of total probability and if $P(A) > 0$, then for any event B_j, we have

$$P(B_j|A) = \frac{P(A|B_j)P(B_j)}{\sum_{k=1}^{\infty} P(A|B_k)P(B_k)}$$

5. Independent Events

Two events A and B are independent if

$$P(A \cap B) = P(A) \cdot P(B)$$