Derivation of Backpropagation in Convolutional Neural Network (CNN)

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October 18, 2016

Abstract— Derivation of backpropagation in convolutional neural network (CNN) is conducted based on an example with two convolutional layers. The step-by-step derivation is

All bias, b_p^1, b_q^2, and b, are initialize to zero. The others are drew randomly from a uniform distribution defined based on the kernel size and number of input and output maps on corresponding layers [2] (see section 4.6 of [2]).

$$k_{1,p}^1 \sim U\left(\pm\sqrt{\frac{6}{(1+6)\times 5^2}}\right) \tag{1}$$

$$k_{p,q}^2 \sim U\left(\pm\sqrt{\frac{6}{(6+12)\times 5^2}}\right) \tag{2}$$

$$W \sim U\left(\pm\sqrt{\frac{6}{192+10}}\right) \tag{3}$$

where $U(\pm x)$ denotes a uniform distribution with upper and lower bounds of $\pm x$. Totally, the number of parameters is $(5\times 5+1)\times 6+(5\times 5\times 6+1)\times 12+10\times 192+10 = 3898$.

1.2 Convolution Layer C1

$$C_p^1 = \sigma(I * k_{1,p}^1 + b_p^1), \text{ where } \sigma(x) = \frac{1}{1+\exp^{-x}} \tag{4}$$

$$C_p^1(i,j) = \sigma\left(\sum_{u=-2}^{2}\sum_{v=-2}^{2} I(i-u, j-v)\cdot k_{1,p}^1(u,v) + b_p^1\right) \tag{5}$$

where $p = 1, 2, \cdots, 6$ because there are 6 feature maps on C1 layer, $*$ denotes the convolution, and i, j are row and column indices of the feature map. Only keeping those parts of the convolution that are computed without the zero-padded edges, the size of C_p^1 is 24×24, rather than 28×28 like I.

1.3 Pooling Layer S1

$$S_p^1(i,j) = \frac{1}{4}\sum_{u=0}^{1}\sum_{v=0}^{1} C_p^1(2i-u, 2j-v),\ i,j = 1,2,\cdots,12 \tag{6}$$

1.4 Convolution Layer C2

$$C_q^2 = \sigma\left(\sum_{p=1}^{6} S_p^1 * k_{p,q}^2 + b_q^2\right) \tag{7}$$

$$C_q^2(i,j) = \sigma\left(\sum_{p=1}^{6}\sum_{u=-2}^{2}\sum_{v=-2}^{2} S_p^1(i-u, j-v)\cdot k_{p,q}^2(u,v) + b_q^2\right) \tag{8}$$

where $q = 1, 2, \cdots, 12$ because there are 12 feature maps on C2 layer. Only keeping those parts of the convolution that are computed without the zero-padded edges, the size of C_q^2 is 8×8, rather than 12×12 like S_p^1.

1.5 Pooling Layer S2

$$S_q^2(i,j) = \frac{1}{4}\sum_{u=0}^{1}\sum_{v=0}^{1} C_q^2(2i-u, 2j-v),\ i,j = 1,2,\cdots,4 \tag{9}$$

1.6 Vectorization and Concatenation

Each S_q^2 is a 4×4 matrix, and there are 12 such matrices on the S2 layer. First, each S_q^2 is vectorized by column scan, then all 12 vectors are concatenated to form a long vector with the length of $4 \times 4 \times 12 = 192$. We denote this process by

$$f = F\left(\{S_q^2\}_{q=1,2,\cdots,12}\right), \tag{10}$$

and the reverse process is

$$\{S_q^2\}_{q=1,2,\cdots,12} = F^{-1}(f). \tag{11}$$

1.7 Fully Connection Layer FC

$$\hat{y} = \sigma(W \times f + b) \tag{12}$$

1.8 Loss Function

Assuming the true label is y, the loss function is express by

$$L = \frac{1}{2}\sum_{i=1}^{10}\left(\hat{y}(i) - y(i)\right)^2 \tag{13}$$

2 Backpropagation

In the backpropagation, we'll update the parameters from the back to start, namely W and b, $k_{p,q}^2$ and b_q^2, $k_{1,p}^1$ and b_p^1.

2.1 ΔW (size 10×192)

$$\begin{aligned}
\Delta W(i,j) &= \frac{\partial L}{\partial W(i,j)} && (14)\\
&= \frac{\partial L}{\partial \hat{y}(i)} \cdot \frac{\partial \hat{y}(i)}{\partial W(i,j)} && (15)\\
&= (\hat{y}(i) - y(i)) \cdot \frac{\partial}{\partial W(i,j)}\sigma\left(\sum_{j=1}^{192} W(i,j) \times f(j) + b(i)\right) && (16)\\
&= (\hat{y}(i) - y(i)) \cdot \hat{y}(i)(1 - \hat{y}(i)) \cdot f(j) && (17)
\end{aligned}$$

Let $\Delta\hat{y}(i) = (\hat{y}(i) - y(i)) \cdot \hat{y}(i)(1 - \hat{y}(i))$, whose size is 10×1, then

$$\begin{aligned}
&\Delta W(i,j) = \Delta\hat{y}(i) \cdot f(j) && (18)\\
&\Longrightarrow \Delta W = \Delta\hat{y} \times f^T && (19)
\end{aligned}$$

2.2 Δb (size 10×1)

$$
\begin{aligned}
\Delta b(i) &= \frac{\partial L}{\partial b(i)} && (20)\\
&= \frac{\partial L}{\partial \hat{y}(i)} \cdot \frac{\partial \hat{y}(i)}{\partial b(i)} && (21)\\
&= (\hat{y}(i) - y(i)) \cdot \frac{\partial}{\partial b(i)} \sigma \left(\sum_{j=1}^{192} W(i,j) \times f(j) + b(i) \right) && (22)\\
&= (\hat{y}(i) - y(i)) \cdot \hat{y}(i)(1 - \hat{y}(i)) && (23)\\
\Longrightarrow \Delta b &= \Delta \hat{y} && (24)
\end{aligned}
$$

2.3 $\Delta k^2_{p,q}$ (size 5×5)

Because of concatenation, vectorization, and pooling, we need to compute the back-propagation error ΔC^2_q on C2 layer before calculating $\Delta k^2_{p,q}$.

$$
\begin{aligned}
\Delta f(j) &= \frac{\partial L}{\partial f} && (25)\\
&= \sum_{i=1}^{10} \frac{\partial L}{\partial \hat{y}(i)} \cdot \frac{\partial \hat{y}(i)}{\partial f(j)} && (26)\\
&= (\hat{y}(i) - y(i)) \cdot \frac{\partial}{\partial f(j)} \sigma \left(\sum_{j=1}^{192} W(i,j) \times f(j) + b(i) \right) && (27)\\
&= \sum_{i=1}^{10} (\hat{y}(i) - y(i)) \cdot \hat{y}(i)(1 - \hat{y}(i)) \cdot W(i,j) && (28)\\
&= \sum_{i=1}^{10} \Delta \hat{y}(i) \cdot W(i,j) && (29)\\
\Longrightarrow \Delta f &= W^T \times \Delta \hat{y} && (30)
\end{aligned}
$$

From section 1.6, we reshape the long error vector Δf (size 192×1) by

$$
\{\Delta S^2_q\}_{q=1,2,\cdots,12} = F^{-1}(\Delta f), \tag{31}
$$

which gets the error on S2 layer (twelve 4×4 error maps). Because there is no parameters on S2 layer, we do not need to do any derivative stuff. Then, upsampling is performed to obtain the error on C2 layer.

$$
\Delta C^2_q(i,j) = \frac{1}{4} \Delta S^2_q \left(\lceil i/2 \rceil, \lceil j/2 \rceil \right), \; i,j = 1,2,\cdots,8 \tag{32}
$$

where $\lceil\cdot\rceil$ denotes the ceiling function. Note that the size of ΔS_q^2 and ΔC_q^2 are 4×4 and 8×8, respectively. Now, we are ready to derive $\Delta k_{p,q}^2$.

$$\begin{aligned}
\Delta k_{p,q}^2(u,v) &= \frac{\partial L}{\partial k_{p,q}^2(u,v)} && (33)\\
&= \sum_{i=1}^{8}\sum_{j=1}^{8}\frac{\partial L}{\partial C_q^2(i,j)}\cdot\frac{\partial C_q^2(i,j)}{\partial k_{p,q}^2(u,v)} && (34)\\
&= \sum_{i=1}^{8}\sum_{j=1}^{8}\Delta C_q^2(i,j)\cdot\frac{\partial}{\partial k_{p,q}^2(u,v)}\sigma\left(\sum_{p=1}^{6}\sum_{u=-2}^{2}\sum_{v=-2}^{2}S_p^1(i-u,j-v)\cdot k_{p,q}^2(u,v)+b_q^2\right) && (35)\\
&= \sum_{i=1}^{8}\sum_{j=1}^{8}\Delta C_q^2(i,j)\cdot C_q^2(i,j)\left(1-C_q^2(i,j)\right)\cdot S_p^1(i-u,j-v) && (36)
\end{aligned}$$

Let

$$\Delta C_{q,\sigma}^2(i,j) = \Delta C_q^2(i,j)\cdot C_q^2(i,j)\left(1-C_q^2(i,j)\right), \tag{37}$$

which is actually the error before sigmoid function on C2 layer. Therefore,

$$C_{q,\sigma}^2(i,j) = \sum_{p=1}^{6}\sum_{u=-2}^{2}\sum_{v=-2}^{2}S_p^1(i-u,j-v)\cdot k_{p,q}^2(u,v)+b_q^2 \tag{38}$$

Rotating S_p^1 180 degrees, we get $S_{p,rot180}^1$, thus $S_{p,rot180}^1(u-i,v-j) = S_p^1(i-u,j-v)$. Therefore, $\Delta k_{p,q}^2$ can be expressed by

$$\begin{aligned}
\Delta k_{p,q}^2(u,v) &= \sum_{i=1}^{8}\sum_{j=1}^{8}S_{p,rot180}^1(u-i,v-j)\cdot\Delta C_{q,\sigma}^2(i,j) && (39)\\
\Longrightarrow \Delta k_{p,q}^2 &= S_{p,rot180}^1 * \Delta C_{q,\sigma}^2 && (40)
\end{aligned}$$

2.4 Δb_q^2 (size 1×1)

$$\begin{aligned}
\Delta b_q^2 &= \frac{\partial L}{\partial b_q^2} && (41)\\
&= \sum_{i=1}^{8}\sum_{j=1}^{8}\frac{\partial L}{\partial C_q^2(i,j)}\cdot\frac{\partial C_q^2(i,j)}{\partial b_q^2} && (42)\\
&= \sum_{i=1}^{8}\sum_{j=1}^{8}\Delta C_q^2(i,j)\cdot\frac{\partial}{\partial b_q^2}\sigma\left(\sum_{p=1}^{6}\sum_{u=-2}^{2}\sum_{v=-2}^{2}S_p^1(i-u,j-v)\cdot k_{p,q}^2(u,v)+b_q^2\right) && (43)\\
&= \sum_{i=1}^{8}\sum_{j=1}^{8}\Delta C_q^2(i,j)\cdot C_q^2(i,j)\left(1-C_q^2(i,j)\right) && (44)\\
&= \sum_{i=1}^{8}\sum_{j=1}^{8}\Delta C_{q,\sigma}^2(i,j) && (45)
\end{aligned}$$

2.5 $\Delta k_{1,p}^1$ (size 5×5)

Similar to the derivation of $\Delta k_{p,q}^2$, we should first obtain ΔS_p^1, the error on S1 layer. Then, upsampling will be performed to get ΔC_p^1, the error on C1 layer. Finally, following the same way, we can calculate $\Delta k^1_{1,p}$.

$$\Delta S^1_p(i,j) = \frac{\partial L}{\partial S^1_p(i,j)} \tag{46}$$

$$= \sum_{q=1}^{12}\sum_{u=-2}^{2}\sum_{v=-2}^{2} \frac{\partial L}{\partial C^2_{q,\sigma}(i+u,j+v)} \cdot \frac{\partial C^2_{q,\sigma}(i+u,j+v)}{\partial S^1_p(i,j)} \tag{47}$$

$$= \sum_{q=1}^{12}\sum_{u=-2}^{2}\sum_{v=-2}^{2} \Delta C^2_{q,\sigma}(i+u,j+v) \cdot \frac{\partial}{\partial S^1_p(i,j)} \left(\sum_{p=1}^{6}\sum_{u=-2}^{2}\sum_{v=-2}^{2} S^1_p(i,j)\cdot k^2_{p,q}(u,v) + b^2_q\right) \tag{48}$$

$$= \sum_{q=1}^{12}\sum_{u=-2}^{2}\sum_{v=-2}^{2} \Delta C^2_{q,\sigma}(i+u,j+v) \cdot k^2_{p,q}(u,v) \tag{49}$$

Rotating $k^2_{p,q}$ 180 degrees, we get $k^2_{p,q,rot180}(-u,-v) = k^2_{p,q}(u,v)$. Therefore,

$$\Delta S^1_p(i,j) = \sum_{q=1}^{12}\sum_{u=-2}^{2}\sum_{v=-2}^{2} \Delta C^2_{q,\sigma}(i-(-u),j-(-v)) \cdot k^2_{p,q,rot180}(-u,-v) \tag{50}$$

$$\Longrightarrow \Delta S^1_p = \sum_{q=1}^{12} \Delta C^2_{q,\sigma} * k^2_{p,q,rot180} \tag{51}$$

By upsampling, we get the error on C1 layer,

$$\Delta C^1_p(i,j) = \frac{1}{4}\Delta S^1_p\left(\lceil i/2\rceil, \lceil j/2\rceil\right),\ i,j = 1,2,\cdots,24 \tag{52}$$

Now, we are ready to calculate $\Delta k^1_{1,p}$,

$$\Delta k^1_{1,p}(u,v) = \frac{\partial L}{\partial k^1_{1,p}(u,v)} \tag{53}$$

$$= \sum_{i=1}^{24}\sum_{j=1}^{24} \frac{\partial L}{\partial C^1_p(i,j)} \cdot \frac{\partial C^1_p(i,j)}{\partial k^1_{1,p}(u,v)} \tag{54}$$

$$= \sum_{i=1}^{24}\sum_{j=1}^{24} \Delta C^1_p(i,j) \cdot \frac{\partial}{\partial k^1_{1,p}(u,v)} \sigma\left(\sum_{u=-2}^{2}\sum_{v=-2}^{2} I(i-u,j-v)\cdot k^1_{1,p}(u,v) + b^1_p\right) \tag{55}$$

$$= \sum_{i=1}^{24}\sum_{j=1}^{24} \Delta C^1_p(i,j) \cdot C^1_p(i,j)\left(1 - C^1_p(i,j)\right) \cdot I(i-u,j-v) \tag{56}$$

By the same token, rotate I 180 degrees, and let

$$\Delta C^1_{p,\sigma}(i,j) = \Delta C^1_p(i,j) \cdot C^1_p(i,j)\left(1 - C^1_p(i,j)\right). \tag{57}$$

Finally,

$$\Delta k^1_{1,p}(u,v) = \sum_{i=1}^{24}\sum_{j=1}^{24} I_{rot180}(u-i,v-j) \cdot \Delta C^1_{p,\sigma}(i,j) \tag{58}$$

$$\Longrightarrow \Delta k^1_{1,p} = I_{rot180} * \Delta C^1_{p,\sigma} \tag{59}$$

2.6 Δb_p^1 (size 1×1)

$$\Delta b_p^1 = \frac{\partial L}{\partial b_p^1} \tag{60}$$

$$= \sum_{i=1}^{24} \sum_{j=1}^{24} \frac{\partial L}{\partial C_p^1(i,j)} \cdot \frac{\partial C_p^1(i,j)}{\partial b_p^1} \tag{61}$$

$$= \sum_{i=1}^{24} \sum_{j=1}^{24} \Delta C_p^1(i,j) \cdot \frac{\partial}{\partial b_p^1} \sigma \left(\sum_{u=-2}^{2} \sum_{v=-2}^{2} I(i-u, j-v) \cdot k_{1,p}^1(u,v) + b_p^1 \right) \tag{62}$$

$$= \sum_{i=1}^{24} \sum_{j=1}^{24} \Delta C_p^1(i,j) \cdot C_p^1(i,j) \left(1 - C_p^1(i,j)\right) \tag{63}$$

$$= \sum_{i=1}^{24} \sum_{j=1}^{24} \Delta C_{p,\sigma}^1(i,j) \tag{64}$$

3 Parameter Update

We need to set a learning rate $\alpha \in (0, 1]$.

$$k_{1,p}^1 \leftarrow k_{1,p}^1 - \alpha \cdot \Delta k_{1,p}^1 \tag{65}$$

$$b_p^1 \leftarrow b_p^1 - \alpha \cdot \Delta b_p^1 \tag{66}$$

$$k_{p,q}^2 \leftarrow k_{p,q}^2 - \alpha \cdot \Delta k_{p,q}^2 \tag{67}$$

$$b_q^2 \leftarrow b_q^2 - \alpha \cdot \Delta b_q^2 \tag{68}$$

$$W \leftarrow W - \alpha \cdot \Delta W \tag{69}$$

$$b \leftarrow b - \alpha \cdot \Delta b \tag{70}$$

References

[1] Palm, Rasmus Berg. "Prediction as a candidate for learning deep hierarchical models of data." *Technical University of Denmark* 5 (2012).

[2] LeCun, Yann A., Leon Bottou, Genevieve B. Orr, and Klaus-Robert MÃijller. "Efficient backprop." In *Neural networks: Tricks of the trade*, pp. 9-48. Springer Berlin Heidelberg, 2012.