COMPUTATIONAL FLUID DYNAMICS

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Problem Statement

Develop a numerical code for obtaining steady state variation of a temperature in a 2-D channel subjected to a parabolic velocity profile.
Generate solution for different combination of maximum velocity and thermal diffusivity.

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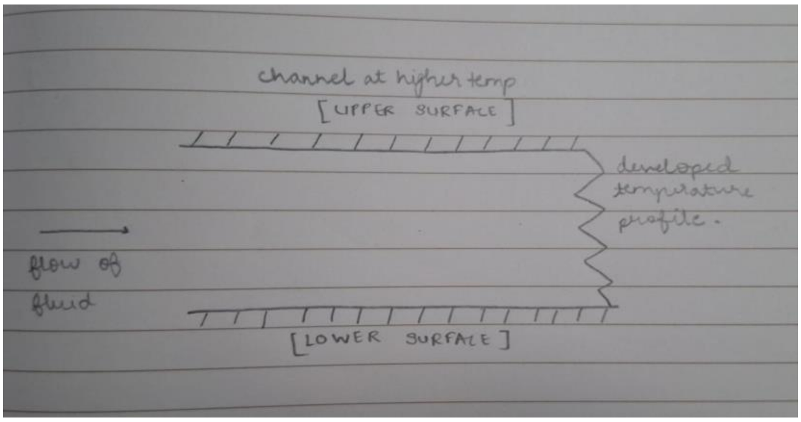
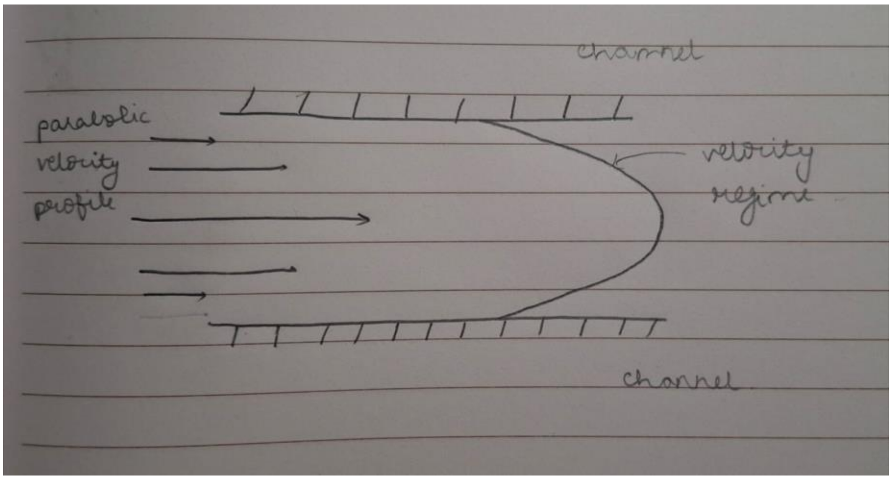
Diagram and Understanding



In the above diagram, we see the fluid with a velocity (in x direction), and a particular temperature flows to a channel, which has higher temperature. The fluid exits with a developed temperature regime. The fluid has a parabolic velocity profile.



The velocity profile of the fluid is parabolic. The velocity at the center is maximum which reduces parabolically and is zero at the edges. The velocity profile is symmetrical about the centerline.

Identifying Variables

We have to find the developed Temperature regime which is dependent on x and y position, time, thermal diffusivity and velocity of the fluid.

Thus Temperature (T) becomes a dependent variable dependent on X position (x), Y position (y), time (t), thermal diffusivity of the fluid (α) and the velocity of the fluid (u).

$$T = f(x, y, t, u, \alpha)$$

But since we are considering steady state only

$$T = f(x, y, u, \alpha)$$

Model Transport Equation

$$\frac{\partial}{\partial t} \iiint_{c\forall} \rho\, C\, T \; dV \; + \; \iint_{cs} \rho\, C\, T\, \vec{v} \cdot \overrightarrow{dA} = \iint_{cs} K\, \vec{\nabla} T \cdot \overrightarrow{dA}$$

But since Steady State is considered

$$\frac{\partial}{\partial t} \iiint_{c\forall} \rho\, C\, T \; dV = 0$$

$$\therefore \; \iint_{cs} \rho\, C\, T\, \vec{v} \cdot \overrightarrow{dA} = \iint_{cs} K\, \vec{\nabla} T \cdot \overrightarrow{dA}$$

Here:

$\frac{\partial}{\partial t} \iiint_{c\forall} \rho\, C\, T \; dV$ signifies the rise in internal energy with time.

$\iint_{cs} \rho\, C\, T\, \vec{v} \cdot \overrightarrow{dA}$ signifies the advection term of heat transfer.

$\iint_{cs} K\, \vec{\nabla} T \cdot \overrightarrow{dA}$ signifies the conductive term of heat transfer.

Assumptions made while solving the problem

We are assuming the problem to be taken into a 3 * 3 grid, with fluid inlet temperature 100 units, wall temperature 300 units, length (l) as 1unit, thermal diffusivity and maximum velocity different for different cases.

$n = 3$, $l = H = 1$, $U = D = 300$, $L = 300$.

1 The grid is supposed to be considered 2 Dimensional.

2 The flow is supposed to be considered fully developed before it enters the channel.

3 The walls have a fixed temperature higher than that of the fluid.

4 While the fluid enters the channel, it has a constant temperature profile.

5 Uniform Grid size is considered.

6 No heat Generation takes place in the entirety of the problem.

7 No pressure gradient exists at all, thus having no external forces.

8 The velocity is one dimensional that too in the x axis.

Boundary Condition

Boundary 1 (Left Boundary): The left boundary shows inlet fluid at temperature 100 units. This is the Dirichlet type of boundary condition where the Temp is a constant.

$$T = constant = f()$$

$$\mathrm{T} = 100 \text{ units}$$

Boundary 2 (Right Boundary): The right boundary assumes that the temperature regime is fully developed. This makes it a Newman's Homogenous boundary condition.

$$\frac{\partial T}{\partial x} = 0$$

Boundary 3 (Top Boundary): The top boundary shows channel wall at temperature 300 units. This is the Dirichlet type of boundary condition where the Temp is a constant.

$$T = constant = f()$$

$$\mathrm{T} = 300 \text{ units}$$

Boundary 4 (Bottom Boundary): The bottom boundary shows channel wall at temperature 300 units. This is the Dirichlet type of boundary condition where the Temp is a constant.

$$T = constant = f()$$

$$\mathrm{T} = 300 \text{ units}$$

Initial Condition

This is a problem where steady state is to considered. So, we will run iterations until we reach stages with less error. So, for the initial iteration (n=0) we take the temperature of the grids as the temperature of what the water would flow in with; that is 100 units.

Domain Discretization

Domain discretization refers to the grids which are to be used in the problem, which is 3 * 3 in this case. Δx is the value of grids horizontal distance. Since we assume the grid structure to be symmetrical and of 9 grids, n here refers to the number of horizontal or vertical grids:

$$\Delta x = \Delta y = {}^{l}/_{n} = {}^{1}/_{3} = 0.33$$

1,3	2,3	3,3
1,2	2,2	3,2
1,1	2,1	3,1

Velocity Profile

Velocity profile given to us for this problem is parabolic in nature, which means it will have maximum value at the center and zero at the edge. The equation it will follow is:

$$u = \vartheta\left(1 - \frac{4y^2}{H^2}\right)$$

Here,

ϑ is the maximum velocity or the velocity at the centerline.

u is the velocity at the point we wish to find dependent on y.

H is the Total width of the channel.

y is the Distance of the cell/point from the centerline, where the velocity is maximum.

In the 3*3 grid we have taken, for H = 1;

For center grid: $u = \vartheta\left(1 - 4\,{}^{0^2}/_{1^2}\right) = \vartheta$

For up and down grids: $u = \vartheta\left(1 - 4\,{}^{0.33^2}/_{1^2}\right) = \vartheta * \left({}^{5}/_{9}\right) = \vartheta * (0.556)$

Consider this as the velocities for all the grid cells, they are not in relation with x, but are a function of y.

0.556 ϑ	0.556 ϑ	0.556 ϑ
ϑ	ϑ	ϑ
0.556 ϑ	0.556 ϑ	0.556 ϑ

Discretizing the Model Transport Equation.

$$\therefore \iint_{cs} \rho\, C\, T\, \vec{v} \cdot \overrightarrow{dA} = \iint_{cs} K\, \vec{\nabla} T \cdot \overrightarrow{dA}$$

$$\rho\, C\, [\iint_{cs} T\, \vec{v} \cdot \vec{A}\,]_{e,w,n,s} = K \iint_{cs} \vec{\nabla} T \cdot \overrightarrow{dA}$$

$$u_e T_e\, dy - u_w T_w\, dy + u_n T_n\, dx - u_s T_s\, dx = \alpha \left(\frac{\partial T}{\partial x_E}\, dy - \frac{\partial T}{\partial x_W}\, dy + \frac{\partial T}{\partial y_N}\, dx - \frac{\partial T}{\partial y_S}\, dx \right)$$

- Here the significance of (e,w,n,s) and (E,W,N,S,P) is important. (e,w,n,s) shows the surface of the respective boundary of the cells, whereas (E,W,N,S,P) shows the respective center of the grids cells.
- Now consider, $u_n \cdot dx = 0$ and $u_s \cdot dx = 0$ as the velocity is only in x direction ($\hat{\imath}$).
- $\frac{\partial T}{\partial x_E} = \frac{T_E - T_P}{\Delta x}$, $\frac{\partial T}{\partial x_W} = \frac{T_P - T_W}{\Delta x}$, $\frac{\partial T}{\partial y_N} = \frac{T_N - T_P}{\Delta y}$, $\frac{\partial T}{\partial y_S} = \frac{T_P - T_S}{\Delta y}$
- Since we are taking Central Difference Advection Scheme:

$$T_e = \frac{T_E + T_P}{2}, T_w = \frac{T_W + T_P}{2}$$

$$\left({u * \Delta x}/{2 * \alpha} \right)(T_E - T_W) = (T_E + T_W + T_N + T_S - 4 * T_P)$$

$$T_P = \frac{1}{4} \left\{ T_E \left(1 - \frac{u * \Delta x}{2 * \alpha} \right) + T_W \left(1 + \frac{u * \Delta x}{2 * \alpha} \right) + T_N + T_S \right\}$$

Limiting values of u and α.

Notice the fact the effect of T_E and T_W on T_P is dependent on $u, \alpha, \Delta x$. But an effect of an adjacent cell can not be negative, as in the effect has always to be greater than 1.

$$\left(1 - \frac{u * \Delta x}{2 * \alpha}\right) > 0$$

$$u * \Delta x / \alpha < 2$$

$$u < \frac{2 * \alpha}{\Delta x}$$

$$\left(1 + \frac{u * \Delta x}{2 * \alpha}\right) > 0$$

$$u * \Delta x / \alpha > -2$$

$$u > -\frac{2 * \alpha}{\Delta x}$$

$$-\frac{2 * \alpha}{\Delta x} < u < \frac{2 * \alpha}{\Delta x}$$

In this case, the way we have taken values of all the other variables,

$$-6 < u < 6$$

As $\Delta x = \frac{1}{3}$, and $\alpha = 1$.

We will obtain results for various values of u, maximum of which will be 6 (limiting value). Other values will include $u = 4$, and $u = 2$.

The C language sample code for the 3 by 3 grid cell with $u = u_{max}$

```
#include<stdio.h>
#include<math.h>
int main()
{
  int n, j, i, a = 5000;
   double T[5][5][5000];
   double err = 0.000001;
   double Pe[3];
   int U = 300, D = 300, L = 100;
   // Initial values
   for (i = 1; i <= 3; i++) {
      for (j = 1; j <= 3; j++) {
         T[i][j][0] = 100;
      }    }
   // Running code and printing values
   for(n = 0; n < a; n++)
   {
      // Boundary conditions
      for (i = 1; i <= 3; i++)
      {
         T[i][0][n] = 300; // Bottom boundary
         T[i][4][n] = 300; // Up boundary
         T[0][i][n] = 100; // Left boundary
         T[4][i][n] = T[3][i][n]; // Right Boundary
      }
      // Calculation
      for (i = 1; i <= 3; i++)
      {
         for (j = 1; j <= 3; j++) {
            if (j == 2) {
               T[i][j][n+1] = (T[i-1][j][n] * 0.4167) + (T[i+1][j][n] * 0.0833) + (0.25 * (T[i][j+1][n] + T[i][j-1][n]));
            } else {
               T[i][j][n+1] = (T[i-1][j][n] * 0.3426) + (T[i+1][j][n] * 0.1574) + (0.25 * (T[i][j+1][n] + T[i][j-1][n]));
            }
         }
      }
      // Printing values
      for (i = 1; i <= 3; i++)
      {
         for (j = 1; j <= 3; j++){
      printf("%f   ", T[i][j][n+1]);}
      }
      printf("\n");
      // Convergence check (start from second iteration)
      if (n > 0) {
         int converged = 1;
         for (i = 1; i <= 3; i++) {
            for (j = 1; j <= 3; j++) {
               if (fabs(T[i][j][n] - T[i][j][n-1]) > err) {
                  converged = 0;
                  break;
               }
            }
            if (!converged) break;
         }
         if (converged) {
            printf("Converged at iteration %d\n", n);

            break;
         }
      }
   }
   return 0;
}
```

Hand calculations while assuming $\alpha = 1$, and $u = 4$ unit(s) respectively, in a 3 by 3 grid

$$T_P = \frac{1}{4}\left\{T_E\left(1 - \frac{u * \Delta x}{2 * \alpha}\right) + T_W\left(1 + \frac{u * \Delta x}{2 * \alpha}\right) + T_N + T_S\right\}$$

For middle row, $y = 2$:

$$T_P = \frac{1}{4}\left\{T_E\left(1 - \frac{4 * \frac{1}{3}}{2 * 1}\right) + T_W\left(1 + \frac{4 * \frac{1}{3}}{2 * 1}\right) + T_N + T_S\right\}$$

$$\therefore T_P = \frac{1}{4}\left\{T_E\left(\frac{1}{3}\right) + T_W\left(\frac{4}{3}\right) + T_N + T_S\right\}$$

$$\therefore T_P = \{T_E(0.0833) + T_W(0.4167) + T_N(0.25) + T_S(0.25)\}$$

For first and third row, $y = 1,3$:

$$T_P = \frac{1}{4}\left\{T_E\left(1 - \frac{(0.5556 * 4) * \frac{1}{3}}{2 * 1}\right) + T_W\left(1 + \frac{(0.5556 * 4) * \frac{1}{3}}{2 * 1}\right) + T_N + T_S\right\}$$

$$\therefore T_P = \frac{1}{4}\{T_E(0.6296) + T_W(1.3704) + T_N + T_S\}$$

$$\therefore T_P = \{T_E(0.1574) + T_W(0.3426) + T_N(0.25) + T_S(0.25)\}$$

Showing the values of first five iterations:

First Iteration:

150	150	150
100	100	100
150	150	150

Second Iteration:

157.87	175	175
125	125	125
157.87	175	175

Third Iteration:

168.06	187.88	193.75
131.02	150	150
168.06	187.88	193.75

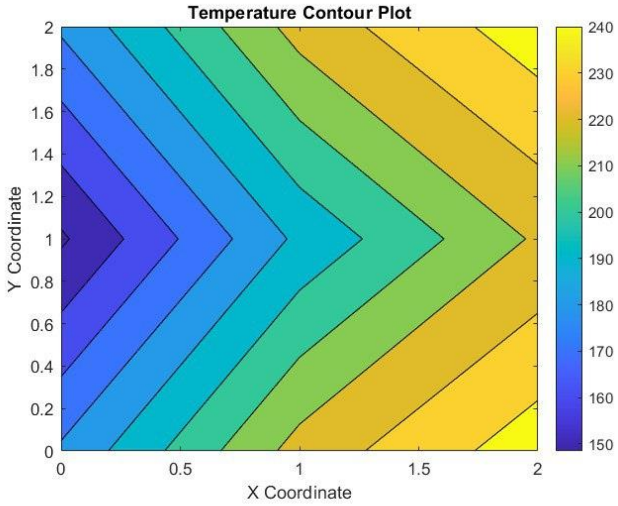
Fourth Iteration:

171.59	200.57	207.36
138.20	161.03	171.88
171.59	200.57	207.36

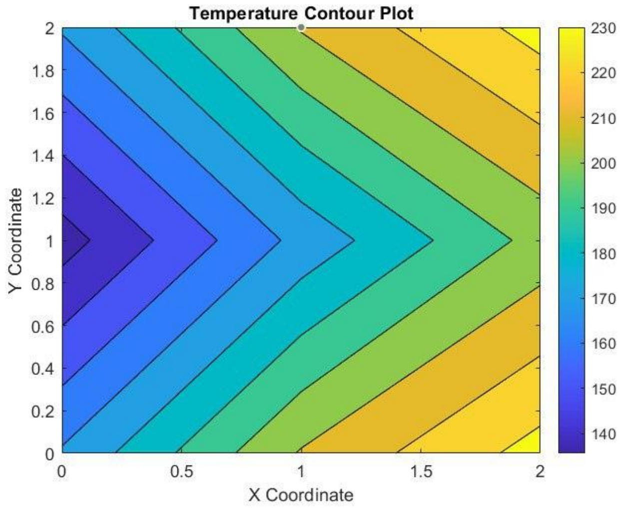
Fifth Iteration:

175.38	206.68	219.32
140.88	172.19	185.10
175.38	206.68	219.32

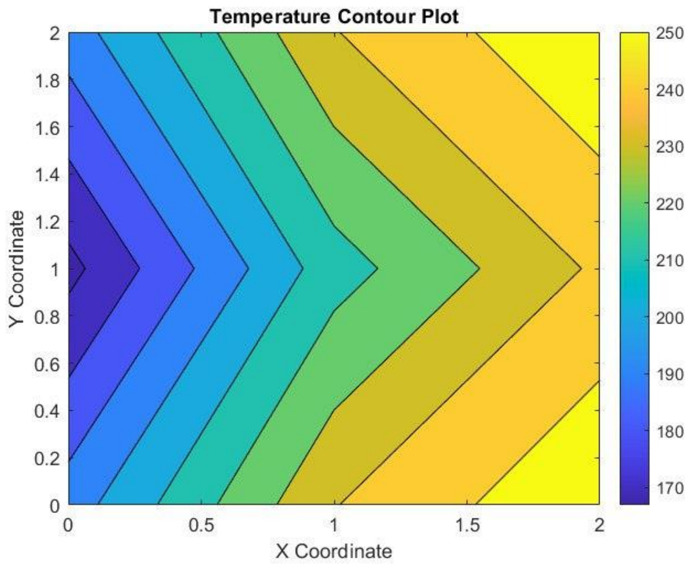
Results of a 3 by 3 grid with velocity $u = 6,4,2\ units$



Temp. Dist. when u=6 units



Temp. Dist. when u=4 units

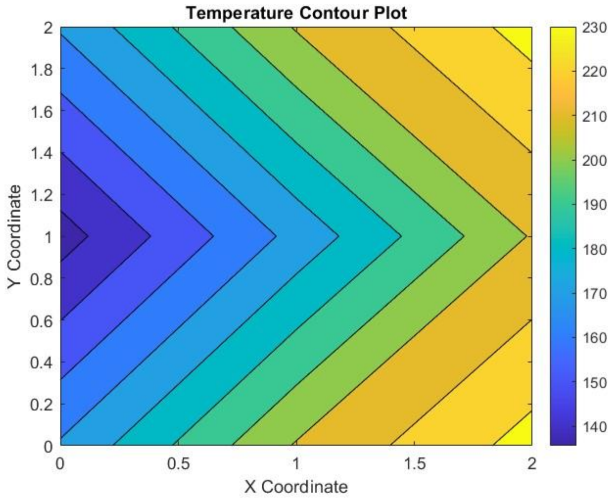


Temp. Dist. when u=2 units'

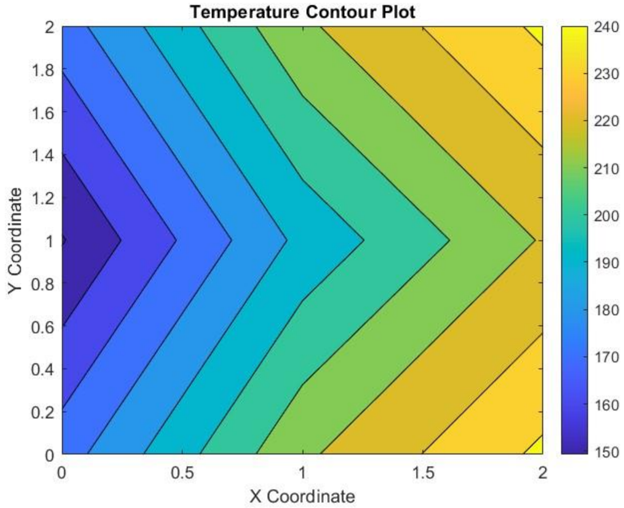
We observe that the transport of temperature information due to velocity reduces as the speed is reduced, but the conductive transfer of temperature information is same as the diffusivity constant isn't changed.

Thus, higher the velocity, lower exit temperatures are recorded, steeper temperature gradient are recorded and a flatter temperature profile across height is seen.

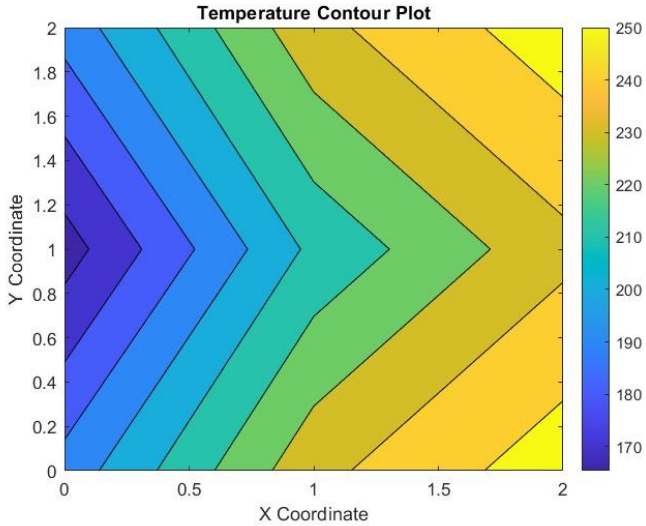
Results of a 3 by 3 grid with velocity $\alpha = 1,2,3\ units$



Temp. Dist. when alpha = 1 unit'



Temp. Dist. when alpha = 2 units'



Temp. Dist. when alpha = 3 units'

We observe that whilst the thermal diffusivity increases, the temperature gradient is reduced, peak temperatures also reduce and we observe that since conduction will take place over internal rise in temperature, the steady state is obtained early.

Higher the thermal diffusivity, higher exit temperature are seen, a less steep temp gradient is seen and a flatter temperature profile across the height is clearly visible.

Results

Linear Temperature Gradient in the Flow Direction (x): The temperature distribution along the channel length x typically exhibits a linear gradient if heat is being added or removed along the walls or at the inlet. This means the bulk temperature will increase (or decrease) along the flow direction.

Parabolic Temperature Profile Across the Height (y): Due to the parabolic velocity distribution, the temperature will also tend to exhibit a parabolic distribution across the height of the channel y, especially in cases of convective heat transfer dominated by advection.

In this case, the channel walls are held at a different higher temperature, the temperature profile across the channel will show a parabolic-like shape, with the maximum temperature at the walls and the temperature decreasing toward the channel centre.

Conclusion

The final steady-state temperature profile will show linear gradient along the channel length due to advection and a parabolic distribution across the channel height due to the parabolic velocity profile.

Consider that the velocity is increased, then the advection becomes more dominant than the diffusion term in the energy equation. This means that the fluid carries heat more quickly along the flow direction resulting in a steeper temperature gradient. The temperature profile becomes more flattened across the channel height because the fluid moves faster, reducing the time available for heat diffusion to cause significant temperature variations across the channel.

As the thermal diffusivity is increased, the diffusion of heat diffusion becomes ore efficient, meaning temperature variations across the height of the channel are reduced. The system will reach thermal equilibrium faster across the height of the channel, so the temperature profile will flatten out and the parabolic shape will become less pronounced. The temperature gradient along the channel length may become less steep since diffusion is counteracting the temperature changes caused by advection more effectively.