Introduction

This Mathematical Formaulae handbook has been prepared in response to a request from the Physics Consultative Committee, with the hope that it will be useful to those studying physics. It is to some extent modelled on a similar document issued by the Department of Engineering, but obviously reflects the particular interests of physicists. There was discussion as to whether it should also include physical formulae such as Maxwell's equations, etc., but a decision was taken against this, partly on the grounds that the book would become unduly bulky, but mainly because, in its present form, clean copies can be made available to candidates in exams.

There has been wide consultation among the staff about the contents of this document, but inevitably some users will seek in vain for a formula they feel strongly should be included. Please send suggestions for amendments to the Secretary of the Teaching Committee, and they will be considered for incorporation in the next edition. The Secretary will also be grateful to be informed of any (equally inevitable) errors which are found.

This book was compiled by Dr John Shakeshaft and typeset originally by Fergus Gallagher, and currently by Dr Dave Green, using the TEX typesetting package.

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Bibliography

Abramowitz, M. & Stegun, I.A., Handbook of Mathematical Functions, Dover, 1965.

Gradshteyn, I.S. & Ryzhik, I.M., Table of Integrals, Series and Products, Academic Press, 1980.

Jahnke, E. & Emde, F., Tables of Functions, Dover, 1986.

Nordling, C. & Österman, J., Physics Handbook, Chartwell-Bratt, Bromley, 1980.

Speigel, M.R., Mathematical Handbook of Formulas and Tables.
(Schaum's Outline Series, McGraw-Hill, 1968).

Physical Constants

Based on the *"Review of Particle Properties"*, Barnett et al., 1996, Physics Review D, **54**, p1, and *"The Fundamental Physical Constants"*, Cohen & Taylor, 1997, Physics Today, BG7. (The figures in parentheses give the 1-standard-deviation uncertainties in the last digits.)

speed of light in a vacuum	c	$2{\cdot}997\,924\,58 \times 10^8$ m s^{-1} (by definition)
permeability of a vacuum	μ_0	$4\pi \times 10^{-7}$ H m^{-1} (by definition)
permittivity of a vacuum	ϵ_0	$1/\mu_0 c^2 = 8{\cdot}854\,187\,817\ldots \times 10^{-12}$ F m^{-1}
elementary charge	e	$1{\cdot}602\,177\,33(49) \times 10^{-19}$ C
Planck constant	h	$6{\cdot}626\,075\,5(40) \times 10^{-34}$ J s
$h/2\pi$	$\hbar$	$1{\cdot}054\,572\,66(63) \times 10^{-34}$ J s
Avogadro constant	N_{A}	$6{\cdot}022\,136\,7(36) \times 10^{23}$ mol^{-1}
unified atomic mass constant	m_{u}	$1{\cdot}660\,540\,2(10) \times 10^{-27}$ kg
mass of electron	m_{e}	$9{\cdot}109\,389\,7(54) \times 10^{-31}$ kg
mass of proton	m_{p}	$1{\cdot}672\,623\,1(10) \times 10^{-27}$ kg
Bohr magneton $eh/4\pi m_{\mathrm{e}}$	μ_{B}	$9{\cdot}274\,015\,4(31) \times 10^{-24}$ J T^{-1}
molar gas constant	R	$8{\cdot}314\,510(70)$ J K^{-1} mol^{-1}
Boltzmann constant	k_{B}	$1{\cdot}380\,658(12) \times 10^{-23}$ J K^{-1}
Stefan–Boltzmann constant	σ	$5{\cdot}670\,51(19) \times 10^{-8}$ W m^{-2} K^{-4}
gravitational constant	G	$6{\cdot}672\,59(85) \times 10^{-11}$ N m^2 kg^{-2}
Other data		
acceleration of free fall	g	$9{\cdot}806\,65$ m s^{-2} (standard value at sea level)

1. Series

Arithmetic and Geometric progressions

A.P. $S_n = a + (a+d) + (a+2d) + \cdots + [a+(n-1)d] = \dfrac{n}{2}[2a+(n-1)d]$

G.P. $S_n = a + ar + ar^2 + \cdots + ar^{n-1} = a\dfrac{1-r^n}{1-r}$, $\left(S_\infty = \dfrac{a}{1-r} \quad \text{for } |r| < 1\right)$

(These results also hold for complex series.)

Convergence of series: the ratio test

$$S_n = u_1 + u_2 + u_3 + \cdots + u_n \quad \text{converges as} \quad n \to \infty \quad \text{if} \quad \lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| < 1$$

Convergence of series: the comparison test

If each term in a series of positive terms is less than the corresponding term in a series known to be convergent, then the given series is also convergent.

Binomial expansion

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \cdots$$

If n is a positive integer the series terminates and is valid for all x: the term in x^r is ${}^nC_r x^r$ or $\binom{n}{r}$ where ${}^nC_r \equiv \dfrac{n!}{r!(n-r)!}$ is the number of different ways in which an unordered sample of r objects can be selected from a set of n objects without replacement. When n is not a positive integer, the series does not terminate: the infinite series is convergent for $|x| < 1$.

Taylor and Maclaurin Series

If $y(x)$ is well-behaved in the vicinity of $x = a$ then it has a Taylor series,

$$y(x) = y(a+u) = y(a) + u\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{u^2}{2!}\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + \frac{u^3}{3!}\frac{\mathrm{d}^3y}{\mathrm{d}x^3} + \cdots$$

where $u = x - a$ and the differential coefficients are evaluated at $x = a$. A Maclaurin series is a Taylor series with $a = 0$,

$$y(x) = y(0) + x\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{x^2}{2!}\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + \frac{x^3}{3!}\frac{\mathrm{d}^3y}{\mathrm{d}x^3} + \cdots$$

Power series with real variables

$e^x = 1 + x + \dfrac{x^2}{2!} + \cdots + \dfrac{x^n}{n!} + \cdots$ valid for all x

$\ln(1+x) = x - \dfrac{x^2}{2} + \dfrac{x^3}{3} + \cdots + (-1)^{n+1}\dfrac{x^n}{n} + \cdots$ valid for $-1 < x \le 1$

$\cos x = \dfrac{e^{ix} + e^{-ix}}{2} = 1 - \dfrac{x^2}{2!} + \dfrac{x^4}{4!} - \dfrac{x^6}{6!} + \cdots$ valid for all values of x

$\sin x = \dfrac{e^{ix} - e^{-ix}}{2i} = x - \dfrac{x^3}{3!} + \dfrac{x^5}{5!} + \cdots$ valid for all values of x

$\tan x = x + \dfrac{1}{3}x^3 + \dfrac{2}{15}x^5 + \cdots$ valid for $-\dfrac{\pi}{2} < x < \dfrac{\pi}{2}$

$\tan^{-1} x = x - \dfrac{x^3}{3} + \dfrac{x^5}{5} - \cdots$ valid for $-1 \le x \le 1$

$\sin^{-1} x = x + \dfrac{1}{2}\dfrac{x^3}{3} + \dfrac{1.3}{2.4}\dfrac{x^5}{5} + \cdots$ valid for $-1 < x < 1$

Integer series

$$\sum_1^N n \;= 1+2+3+\cdots+N = \frac{N(N+1)}{2}$$

$$\sum_1^N n^2 = 1^2+2^2+3^2+\cdots+N^2 = \frac{N(N+1)(2N+1)}{6}$$

$$\sum_1^N n^3 = 1^3+2^3+3^3+\cdots+N^3 = [1+2+3+\cdots N]^2 = \frac{N^2(N+1)^2}{4}$$

$$\sum_1^\infty \frac{(-1)^{n+1}}{n} = 1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots = \ln 2 \qquad \text{[see expansion of } \ln(1+x)\text{]}$$

$$\sum_1^\infty \frac{(-1)^{n+1}}{2n-1} = 1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\cdots = \frac{\pi}{4} \qquad \text{[see expansion of } \tan^{-1}x\text{]}$$

$$\sum_1^\infty \frac{1}{n^2} = 1+\frac{1}{4}+\frac{1}{9}+\frac{1}{16}+\cdots = \frac{\pi^2}{6}$$

$$\sum_1^N n(n+1)(n+2) = 1.2.3+2.3.4+\cdots+N(N+1)(N+2) = \frac{N(N+1)(N+2)(N+3)}{4}$$

This last result is a special case of the more general formula,

$$\sum_1^N n(n+1)(n+2)\ldots(n+r) = \frac{N(N+1)(N+2)\ldots(N+r)(N+r+1)}{r+2}.$$

Plane wave expansion

$$\exp(ikz) = \exp(ikr\cos\theta) = \sum_{l=0}^{\infty}(2l+1)i^l j_l(kr)P_l(\cos\theta),$$

where $P_l(\cos\theta)$ are Legendre polynomials (see section 11) and $j_l(kr)$ are spherical Bessel functions, defined by

$$j_l(\rho) = \sqrt{\frac{\pi}{2\rho}}J_{l+1/2}(\rho), \quad \text{with } J_l(x) \text{ the Bessel function of order } l \text{ (see section 11).}$$

2. Vector Algebra

If $\boldsymbol{i}$, $\boldsymbol{j}$, $\boldsymbol{k}$ are orthonormal vectors and $\boldsymbol{A} = A_x\boldsymbol{i} + A_y\boldsymbol{j} + A_z\boldsymbol{k}$ then $|\boldsymbol{A}|^2 = A_x^2 + A_y^2 + A_z^2$. [Orthonormal vectors $\equiv$ orthogonal unit vectors.]

Scalar product

$$\boldsymbol{A}\cdot\boldsymbol{B} = |\boldsymbol{A}|\,|\boldsymbol{B}|\cos\theta \qquad \text{where } \theta \text{ is the angle between the vectors}$$

$$= A_xB_x + A_yB_y + A_zB_z = [\,A_xA_yA_z\,]\begin{bmatrix} B_x \\ B_y \\ B_z \end{bmatrix}$$

Scalar multiplication is commutative: $\boldsymbol{A}\cdot\boldsymbol{B} = \boldsymbol{B}\cdot\boldsymbol{A}$.

Equation of a line

A point $\boldsymbol{r} \equiv (x, y, z)$ lies on a line passing through a point $\boldsymbol{a}$ and parallel to vector $\boldsymbol{b}$ if

$$\boldsymbol{r} = \boldsymbol{a} + \lambda\boldsymbol{b}$$

with λ a real number.

Equation of a plane

A point $\boldsymbol{r} \equiv (x, y, z)$ is on a plane if either

(a) $\boldsymbol{r} \cdot \widehat{\boldsymbol{d}} = |\boldsymbol{d}|$, where $\boldsymbol{d}$ is the normal from the origin to the plane, or

(b) $\dfrac{x}{X} + \dfrac{y}{Y} + \dfrac{z}{Z} = 1$ where X, Y, Z are the intercepts on the axes.

Vector product

$\boldsymbol{A} \times \boldsymbol{B} = \boldsymbol{n}\, |\boldsymbol{A}|\, |\boldsymbol{B}| \sin\theta$, where θ is the angle between the vectors and $\boldsymbol{n}$ is a unit vector normal to the plane containing $\boldsymbol{A}$ and $\boldsymbol{B}$ in the direction for which $\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{n}$ form a right-handed set of axes.

$\boldsymbol{A} \times \boldsymbol{B}$ in determinant form

$$\begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$

$\boldsymbol{A} \times \boldsymbol{B}$ in matrix form

$$\begin{bmatrix} 0 & -A_z & A_y \\ A_z & 0 & -A_x \\ -A_y & A_x & 0 \end{bmatrix} \begin{bmatrix} B_x \\ B_y \\ B_z \end{bmatrix}$$

Vector multiplication is not commutative: $\boldsymbol{A} \times \boldsymbol{B} = -\boldsymbol{B} \times \boldsymbol{A}$.

Scalar triple product

$$\boldsymbol{A} \times \boldsymbol{B} \cdot \boldsymbol{C} = \boldsymbol{A} \cdot \boldsymbol{B} \times \boldsymbol{C} = \begin{vmatrix} A_x & A_y & A_z \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix} = -\boldsymbol{A} \times \boldsymbol{C} \cdot \boldsymbol{B}, \quad \text{etc.}$$

Vector triple product

$$\boldsymbol{A} \times (\boldsymbol{B} \times \boldsymbol{C}) = (\boldsymbol{A} \cdot \boldsymbol{C})\boldsymbol{B} - (\boldsymbol{A} \cdot \boldsymbol{B})\boldsymbol{C}, \qquad (\boldsymbol{A} \times \boldsymbol{B}) \times \boldsymbol{C} = (\boldsymbol{A} \cdot \boldsymbol{C})\boldsymbol{B} - (\boldsymbol{B} \cdot \boldsymbol{C})\boldsymbol{A}$$

Non-orthogonal basis

$$\begin{aligned} \boldsymbol{A} &= A_1\boldsymbol{e}_1 + A_2\boldsymbol{e}_2 + A_3\boldsymbol{e}_3 \\ A_1 &= \boldsymbol{\epsilon}' \cdot \boldsymbol{A} \quad \text{where} \quad \boldsymbol{\epsilon}' = \frac{\boldsymbol{e}_2 \times \boldsymbol{e}_3}{\boldsymbol{e}_1 \cdot (\boldsymbol{e}_2 \times \boldsymbol{e}_3)} \end{aligned}$$

Similarly for A_2 and A_3.

Summation convention

$$\begin{aligned} \boldsymbol{a} &= a_i\boldsymbol{e}_i && \text{implies summation over } i = 1\ldots3 \\ \boldsymbol{a} \cdot \boldsymbol{b} &= a_i b_i \\ (\boldsymbol{a} \times \boldsymbol{b})_i &= \varepsilon_{ijk} a_j b_k && \text{where } \varepsilon_{123} = 1; \quad \varepsilon_{ijk} = -\varepsilon_{ikj} \\ \varepsilon_{ijk}\varepsilon_{klm} &= \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl} \end{aligned}$$

3. Matrix Algebra

Unit matrices

The unit matrix I of order n is a square matrix with all diagonal elements equal to one and all off-diagonal elements zero, i.e., $(I)_{ij} = \delta_{ij}$. If A is a square matrix of order n, then $AI = IA = A$. Also $I = I^{-1}$.
I is sometimes written as I_n if the order needs to be stated explicitly.

Products

If A is a $(n \times l)$ matrix and B is a $(l \times m)$ then the product AB is defined by

$$(AB)_{ij} = \sum_{k=1}^{l} A_{ik} B_{kj}$$

In general $AB \neq BA$.

Transpose matrices

If A is a matrix, then transpose matrix A^T is such that $(A^T)_{ij} = (A)_{ji}$.

Inverse matrices

If A is a square matrix with non-zero determinant, then its inverse A^{-1} is such that $AA^{-1} = A^{-1}A = I$.

$$(A^{-1})_{ij} = \frac{\text{transpose of cofactor of } A_{ij}}{|A|}$$

where the cofactor of A_{ij} is $(-1)^{i+j}$ times the determinant of the matrix A with the j-th row and i-th column deleted.

Determinants

If A is a square matrix then the determinant of A, $|A|$ ($\equiv \det A$) is defined by

$$|A| = \sum_{i,j,k,\ldots} \epsilon_{ijk\ldots} A_{1i} A_{2j} A_{3k} \ldots$$

where the number of the suffixes is equal to the order of the matrix.

2×2 matrices

If $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ then,

$$|A| = ad - bc \qquad A^T = \begin{pmatrix} a & c \\ b & d \end{pmatrix} \qquad A^{-1} = \frac{1}{|A|} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

Product rules

$$(AB \ldots N)^T = N^T \ldots B^T A^T$$

$$(AB \ldots N)^{-1} = N^{-1} \ldots B^{-1} A^{-1} \qquad \text{(if individual inverses exist)}$$

$$|AB \ldots N| = |A|\,|B| \ldots |N| \qquad \text{(if individual matrices are square)}$$

Orthogonal matrices

An orthogonal matrix Q is a square matrix whose columns q_i form a set of orthonormal vectors. For any orthogonal matrix Q,

$$Q^{-1} = Q^T, \quad |Q| = \pm 1, \quad Q^T \text{ is also orthogonal.}$$

Solving sets of linear simultaneous equations

If A is square then $A\boldsymbol{x} = \boldsymbol{b}$ has a unique solution $\boldsymbol{x} = A^{-1}\boldsymbol{b}$ if A^{-1} exists, i.e., if $|A| \neq 0$.

If A is square then $A\boldsymbol{x} = 0$ has a non-trivial solution if and only if $|A| = 0$.

An over-constrained set of equations $A\boldsymbol{x} = \boldsymbol{b}$ is one in which A has m rows and n columns, where m (the number of equations) is greater than n (the number of variables). The best solution x (in the sense that it minimizes the error $|A\boldsymbol{x} - \boldsymbol{b}|$) is the solution of the n equations $A^T A\boldsymbol{x} = A^T\boldsymbol{b}$. If the columns of A are orthonormal vectors then $\boldsymbol{x} = A^T\boldsymbol{b}$.

Hermitian matrices

The Hermitian conjugate of A is $A^\dagger = (A^*)^T$, where A^* is a matrix each of whose components is the complex conjugate of the corresponding components of A. If $A = A^\dagger$ then A is called a Hermitian matrix.

Eigenvalues and eigenvectors

The n eigenvalues λ_i and eigenvectors $\boldsymbol{u}_i$ of an $n \times n$ matrix A are the solutions of the equation $A\boldsymbol{u} = \lambda\boldsymbol{u}$. The eigenvalues are the zeros of the polynomial of degree n, $P_n(\lambda) = |A - \lambda I|$. If A is Hermitian then the eigenvalues λ_i are real and the eigenvectors u_i are mutually orthogonal. $|A - \lambda I| = 0$ is called the characteristic equation of the matrix A.

$$\operatorname{Tr} A = \sum_i \lambda_i, \quad \text{also } |A| = \prod_i \lambda_i.$$

If S is a symmetric matrix, Λ is the diagonal matrix whose diagonal elements are the eigenvalues of S, and U is the matrix whose columns are the normalized eigenvectors of A, then

$$U^T SU = \Lambda \qquad \text{and} \qquad S = U\Lambda U^T.$$

If x is an approximation to an eigenvector of A then $\boldsymbol{x}^T A\boldsymbol{x}/(\boldsymbol{x}^T\boldsymbol{x})$ (Rayleigh's quotient) is an approximation to the corresponding eigenvalue.

Commutators

$$\begin{aligned}
[A, B] &\equiv AB - BA \\
[A, B] &= -[B, A] \\
[A, B]^\dagger &= [B^\dagger, A^\dagger] \\
[A + B, C] &= [A, C] + [B, C] \\
[AB, C] &= A[B, C] + [A, C]B \\
[A, [B, C]] &+ [B, [C, A]] + [C, [A, B]] = 0
\end{aligned}$$

Hermitian algebra

$$\boldsymbol{b}^\dagger = (b_1^*, b_2^*, \ldots)$$

	Matrix form	Operator form	Bra-ket form
Hermiticity	$\boldsymbol{b}^* \cdot A \cdot \boldsymbol{c} = (A \cdot \boldsymbol{b})^* \cdot \boldsymbol{c}$	$\int \psi^* O\phi = \int (O\psi)^*\phi$	$\langle\psi\|O\|\phi\rangle$
Eigenvalues, λ real	$A\boldsymbol{u}_i = \lambda_{(i)}\boldsymbol{u}_i$	$O\psi_i = \lambda_{(i)}\psi_i$	$O\,\|i\rangle = \lambda_i\,\|i\rangle$
Orthogonality	$\boldsymbol{u}_i \cdot \boldsymbol{u}_j = 0$	$\int \psi_i^*\psi_j = 0$	$\langle i\|j\rangle = 0 \qquad (i \neq j)$
Completeness	$\boldsymbol{b} = \sum_i \boldsymbol{u}_i(\boldsymbol{u}_i \cdot \boldsymbol{b})$	$\phi = \sum_i \psi_i \left(\int \psi_i^*\phi\right)$	$\phi = \sum_i \|i\rangle\,\langle i\|\phi\rangle$
Rayleigh–Ritz			
Lowest eigenvalue	$\lambda_0 \leq \dfrac{\boldsymbol{b}^* \cdot A \cdot \boldsymbol{b}}{\boldsymbol{b}^* \cdot \boldsymbol{b}}$	$\lambda_0 \leq \dfrac{\int \psi^* O\psi}{\int \psi^*\psi}$	$\dfrac{\langle\psi\|O\|\psi\rangle}{\langle\psi\|\psi\rangle}$