The Vehicle Routing Problem with Drones: Extended Models and Connections

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The vehicle routing problem with drones (VRPD) is inspired by the increasing interest in commercial drone delivery by companies such as Amazon, Google, DHL, and Walmart. In our model, a fleet of m homogeneous trucks each carries k drones with a speed of α times that of the truck. Each drone may dispatch from the top of the truck and carry a package to a customer location. The drone then returns to the top of its truck to recharge or swap batteries (we assume instantaneously). The truck itself is allowed to move and deliver packages, but must be stationary at a delivery location or the depot when launching or retrieving drones. The goal is to minimize the completion time to deliver all packages and return all vehicles back to the central depot. In this article, we review and extend several worst-case results from an earlier paper and we make connections with another practical variant of the vehicle routing problem and with Amdahl's Law. We find that the VRPD model offers some important practical advantages. The drones allow the truck to *parallelize* tasks and they are able to take advantage of crow-fly distances. **NETWORKS, Vol. 70(1), 34–43 2017**

Keywords: vehicle routing; drones; close enough routing; bounds; worst-case analysis; package delivery; parallelization

1. INTRODUCTION

The importance of drones is increasing in commercial contexts. In particular, the delivery of packages via drone has been explored by a number of corporations including Amazon and Google. In our previous work [15], we introduced the vehicle routing problem with drones (VRPD) and we established some worst-case bounds under a number of assumptions (see Appendix 1 for connections to speed-up of computer programs). The VRPD allows for a fleet of m homogeneous trucks, each equipped with k drones, to either deliver packages directly, or bring the drones within range to deliver packages. This article seeks to expand on those results and to fit a much wider variety of circumstances. In addition, we establish bounds which relate the VRPD to two other well-established problems, which hint at future corresponding computational methods. In parallel with our somewhat theoretical approach, other researchers (e.g., see [1, 10]) propose computational approaches to address special cases of the VRPD.

This article is an example of numerous recent projects that illustrate the impact of new technologies on vehicle routing practice and models. Other examples include using radio frequency identification (RFID) to route meter readers more efficiently [7], strategically placing telemetry units at customer locations to reduce costs in inventory routing applications [13], and routing electric vehicles (to limit the emission of greenhouse gases) effectively, where recharging stations need to be visited on a regular basis [12].

Synchronicity plays a critical role in the VRPD model like a number of problems in the literature. Chao [4] describes the truck and trailer routing problem (TTRP). The ability of a truck to detach from its trailer at certain nodal locations to service customers bears some resemblance to the VRPD. However in TTRP, there is at most one trailer per truck and the trailer is unable to move while the truck is detached. The vehicle routing problem with time windows and multiple service (VRPTWMS) workers [6] allows for multiple workers to deploy from a single truck to deliver some set of homogeneous packages. However, unlike VRPD, it requires the truck to remain stationary at one of a finite set of parking locations while the workers are deployed.

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TABLE 1. Some of the problems studied

	P_t	P_{td}	$\sup\{Z(P_{\rm t})/Z(P_{\rm td})\}$
1	TSP	$\mathrm{VRPD}_{1,\alpha,k}$	$\alpha k+1$
2	TSP	$\mathrm{VRPD}_{m,\alpha,k}$	$m(\alpha k+1)$
3	VRP*	$\mathrm{VRPD}_{m,\alpha,k}$	$\alpha k+1$
4	$\mathrm{VRPD}_{m,\alpha,k}$	$\mathrm{VRPD}_{m,\beta,k}$	β/α

In Section 2, we more formally define VRPD, summarize key results from our previous paper [15], many of which serve as templates for further proofs. In Section 3, we extend some worst-case bounds to more generic distance/cost metrics. We also explicitly consider limited battery life and cost (in addition to time) objectives. Section 4 relates the VRPD to the min-max close-enough vehicle routing problem (CEVRP*) [14] and the min-max vehicle routing problem (VRP*). The asterisk notation indicates that we are minimizing the longest route in the problem. The bounds established cast the VRPD as an intermediate problem that connects the VRP* and the CEVRP*.

2. BACKGROUND

Our primary objective is to minimize the maximum duration route (i.e., minimize the completion time), since companies like Amazon want to make deliveries to customers as soon as possible. It should be clear that this min-max objective provides a good approximation to the time until the last delivery, especially when the last customer location is close to the depot.

The VRPD model assumes the following:

- m is the number of homogeneous trucks in the fleet.
- k is the number of drones on each truck.
- α is the ratio of drone speed to truck speed (without loss of generality, in this article, we assume drone speed is α and truck speed is 1).
- The recharge (or battery swap) of a drone's battery is instantaneous.
- We assume (until explicitly noted otherwise) that drones may only launch from or land on the truck, when the truck is located at a customer delivery location or the depot.
- A drone must land on the same truck from which it launched.

In [15], we proved a number of worst-case results comparing the optimal completion time using a fleet of trucks equipped with drones to the optimal completion time using a traditional fleet of only trucks. The results are summarized in Table 1. We refer readers who are interested in the proofs to [15]. Denote, by P_t, the routing problem with the fleet of trucks only, and, by P_{td}, the problem with the fleet of trucks and drones. $Z(P_{\rm t})$ and $Z(P_{\rm td})$ are optimal solution values, that is, the completion times, to P_t and P_{td}, respectively. We found tight upper bounds on the ratios $Z(P_{\rm t})/Z(P_{\rm td})$, which indicated the maximum benefit obtained from incorporating drones into the fleet.

In row 1 of Table 1, we compare the traveling salesman problem (TSP) to $\mathrm{VRPD}_{1,\alpha,k}$, that is, we have a fleet of only one truck carrying k drones. The worst-case ratio is $\alpha k+1$. The maximum benefit from using drones depends on the number of drones and the drone speed. If the truck carries two drones and the drones travel 50% times faster than the truck, the completion may be reduced by 75%, in the best case.

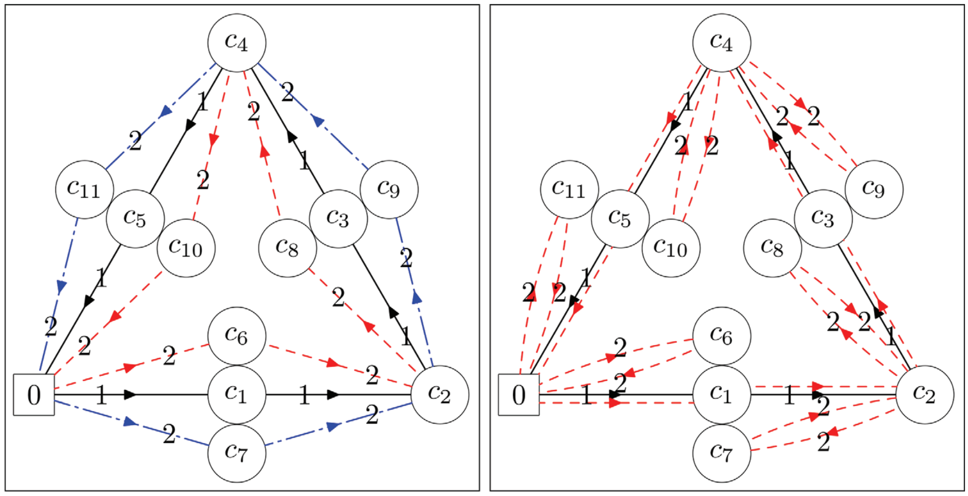
In row 2, we compare the TSP to $\mathrm{VRPD}_{m,\alpha,k}$, that is, we have a fleet of m trucks each carrying k drones. The maximum amount saved depends on the number of trucks, the number of drones, and the speed of the drones.

In row 3, we compare the VRP* with a fleet of m trucks to $\mathrm{VRPD}_{m,\alpha,k}$. Both the VRP* and $\mathrm{VRPD}_{m,\alpha,k}$ have m trucks in the fleet and the worst-case ratio is $\alpha k+1$, the same as the ratio when we compared the TSP to $\mathrm{VRPD}_{1,\alpha,k}$.

An interesting observation is that the speed of drones, α, and the number of drones per truck, k, play the same role in the worst-case bound. If we have more resources, do we invest in faster drones or in carrying more drones on a truck? In terms of the maximum benefit, doubling the drone speed and doubling the number of drones per truck can produce the same effect, but in a typical case, the problem is not straightforward. A larger number of drones has the advantage of serving more customers in parallel; greater drone speed has the advantage of serving more customers in serial. In our toy examples, we found that if there are times when not all drones are in service (service not fully parallelized), greater drone speed dominates. On the other hand, if drone range or capacity is severely limited, a larger number of drones may dominate. It would be interesting to explore the phenomenon in a simulation study given a computational procedure for the VRPD.

It is easy to design instances where a single fast drone is more beneficial than two slow drones. In a trivial case, we can have a single depot and a single package to be delivered to a location d units of distance from the depot. Assume the truck and drone are operating on the same metric. We have the choice of two drones with speed $\alpha_1 = 2$ or one drone with speed $\alpha_2 = 4$. In both cases, the optimal solution is a trivial out-and-back route, launching a single drone directly from the depot. However, in the first case, the optimal route duration is $(d+d)/\alpha_1 = d$, whereas the second case has an optimal route duration of $(d+d)/\alpha_2 = \frac{1}{2}d$.

In Figure 1, we show an example where two slow drones are more efficient than one fast drone. There are 11 customers. The distances between two nodes (customer or depot) are labeled on the arcs connecting them in Figure 1a. If there is no arc between the two nodes, the distance is the length of the (undirected) shortest path between them. For example, the distance between C_1 and C_6 is the sum of distances between C_1 & C_2 and C_2 & C_6, and thus equals $1+2=3$. In Figure 1a, we show the optimal solution with slower drones. The fleet has one truck with speed 1 and two drones with speed 2. The solid black line represents the truck path and the red



(a) Two slow drones of speed 2 (b) One faster drone of speed 4

FIG. 1. A larger number of slower drones is better for this network.

and blue lines represent the two drone paths, respectively. The drones are dispatched at the depot to serve customers C_6 and C_7, respectively, while the truck is dispatched to serve customer C_1, and then C_2. The three vehicles arrive at C_2 at the same time. Then the two drones are sent immediately to serve customers C_8 and C_9. The truck continues to serve C_3, and then C_4. The truck and drones resynchronize at C_4. The drones redeploy to C_{10} and C_{11}, while the truck delivers to C_5. All vehicles regather at the depot. The objective function value of the solution is 6.

In Figure 1b, we show the best solution over the same network with a single faster drone. The fleet has one truck carrying one drone with speed 4 whose path is in red. The drone is dispatched from the depot to serve customer C_6, while the truck is dispatched to serve customer C_1. The truck waits at C_1 for 0.25 time units to pick up the drone, which is immediately sent to serve customer C_7. The truck continues to serve C_2, where it waits for another 0.25 time units to pick up the drone, and so on. The pattern continues, where the truck will eventually serve C_3, C_4, and C_5, waiting at each of those stops for 0.25 time units for the drone to pick up its next package. It can be calculated that the objective function value of the solution is 7.5, which is worse than 6.

In this example, two slower drones are more efficient than one drone that is twice as fast. The limited carrying capacity of a single drone (i.e., one package) forces the single fast drone to resynchronize with the truck on six different occasions (namely C_1, C_2, C_3, C_4, C_5, and the depot), whereas two slower drones only require three resynchronization points at C_2, C_4, and finally back at the depot.

In row 4, we compare two VRPDs: $\text{VRPD}_{m,\alpha,k}$ and $\text{VRPD}_{m,\beta,k}$. The two problems have the same number of trucks each carrying the same number of drones, but the speeds of the drones are different. If we assume $\alpha < \beta$, then the worst-case ratio indicates the maximum savings if a new generation of faster drones is used.

3. EXTENSIONS: COST ISSUES, OTHER METRICS, AND LIMITED BATTERY LIFE

In the previous paper, we ignored cost, assumed that the truck and the drone follow the same distance metric, and ignored the limited battery life of a drone. In this section, we begin to relax these simplifications and provide some initial results for others to build upon.

3.1. *Limited Battery and Maximum Savings*

The following theorem takes into account explicitly the limited battery life (in time units), U, of a drone, which we did not consider in detail in the previous paper. A lower bound on $Z(\text{VRPD}_{1,\alpha,k})$ is given by Theorem 1.

Theorem 1. *If the triangle inequality is valid, then*

$$Z(VRPD_{1,\alpha,k}) \geq Z(TSP) - nU\alpha, \tag{1}$$

where n is the number of customers served by drones in the optimal $VRPD_{1,\alpha,k}$ solution and U is the battery life of a drone.

Proof of Theorem 1. We construct a feasible TSP solution from the optimal $\text{VRPD}_{1,\alpha,k}$ solution. We insert the customers served by drones one by one onto the truck route whose duration was initially equal to $Z(\text{VRPD}_{1,\alpha,k})$. Denote the distance between customers i and j by L_{ij}. If a drone is launched at node i to service customer k and is then picked up at node j, the distance covered by the drone is $L_{ik} + L_{kj} \leq \alpha U$ (we assume the truck speed is 1 and the drone speed is α.). If $L_{ik} \leq L_{kj}$, we insert k just after node i on the truck route. If $L_{ik} > L_{kj}$, we insert k just after node j on the truck route. The increase in the distance of the truck route is no more than αU, if the triangle inequality is valid. After all n customers served by the drone are added, the increase in distance (and duration) of the truck route is no more than $n\alpha U$, that is, the duration of the feasible TSP solution is $Z^f(\text{TSP}) \leq Z(\text{VRPD}_{1,\alpha,k}) + n\alpha U$. Since $Z(\text{TSP}) \leq Z^f(\text{TSP})$, we have

$$Z(\text{VRPD}_{1,\alpha,k}) \geq Z(\text{TSP}) - nU\alpha$$

after rearranging the terms. ∎

The result in Theorem 1 is in a different style from those in Table 1. In Table 1, we consider the ratios of optimal objective function values, that is, the maximum relative benefit from using drones. But in Theorem 1, we consider the difference in objective function values, which indicates the maximum absolute benefit from using drones.

The maximum amount we can save by adding drones to trucks, that is, $nU\alpha$, is directly proportional to drone battery life and the number of drone deployments. In other words, long range drones and high utilization rates both could help reduce costs. If the operating range in distance ($U\alpha$) is small due to battery constraints and the number of drone deployments n is also small (perhaps due to practical constraints like a small number of available batteries or customer locations that are very spread out), this lower bound may be more restrictive.

The inequality $Z(\text{VRPD}_{1,\alpha,k}) \geq \frac{Z(\text{TSP})}{\alpha k+1}$ from Theorem 4 in [15] is still valid if the drones have limited battery life. Considering both theorems, we have $Z(\text{VRPD}_{1,\alpha,k}) \geq \max\left\{\frac{Z(\text{TSP})}{\alpha k+1}, Z(\text{TSP}) - nU\alpha\right\}$.

3.2. *Truck and Drones Utilizing Different Metrics*

In [15], the drones and the trucks follow the same distance metric. In practice, we expect the drones to more or less follow the crow-fly distance and the trucks to be restricted to the street network. Therefore, the worst-case ratios in [15] are conservative in practice. Of course, this dichotomy ignores the reality of high-rise buildings and other aerial obstructions.

We show what happens to the worst-case result if the drone and the truck follow different distance metrics in the following theorem. The distance matrices followed by a truck and a drone are denoted by Q_t and Q_d, respectively. The $(i,j)^{\text{th}}$ entry of Q_t (or Q_d), denoted by $Q_\text{t}(i,j)$ (or $Q_\text{d}(i,j)$), is the distance traveled by the truck (or drone) from node i to node j. We denote the duration of the optimal TSP solution by $Z(\text{TSP}, Q_\text{t})$, and we denote the optimal $\text{VRPD}_{m,\alpha,k}$ solution by $Z(\text{VRPD}_{m,\alpha,k}, Q_\text{t}, Q_\text{d})$. We also make the additional assumption that $Q_\text{d}(i,j) \leq Q_\text{t}(i,j),\ \forall i,j$. This implies drones will never travel further between two nodes than a truck.

Theorem 2.

$$\frac{Z(TSP, Q_\text{t})}{Z(VRPD_{m,\alpha,k}, Q_\text{t}, Q_\text{d})} \leq \frac{Z(TSP, Q_\text{t})}{Z(TSP, Q_\text{d})} m(\alpha k + 1).$$

Proof of Theorem 2. In our previous paper, we have shown that

$$\frac{Z(\text{TSP}, Q_\text{d})}{Z(\text{VRPD}_{m,\alpha,k}, Q_\text{d}, Q_\text{d})} \leq m(\alpha k + 1).$$

Divide by $Z(\text{TSP}, Q_\text{d})$ to get

$$\frac{1}{Z(\text{VRPD}_{m,\alpha,k}, Q_\text{d}, Q_\text{d})} \leq \frac{1}{Z(\text{TSP}, Q_\text{d})} m(\alpha k + 1).$$

Next, multiply both sides by $Z(\text{TSP}, Q_\text{t})$ to obtain

$$\frac{Z(\text{TSP}, Q_\text{t})}{Z(\text{VRPD}_{m,\alpha,k}, Q_\text{d}, Q_\text{d})} \leq \frac{Z(\text{TSP}, Q_\text{t})}{Z(\text{TSP}, Q_\text{d})} m(\alpha k + 1). \quad (2)$$

Since $Q_\text{d}(i,j) \leq Q_\text{t}(i,j)$, it follows that

$$\begin{aligned} Z(\text{VRPD}_{m,\alpha,k}, Q_\text{d}, Q_\text{d}) &\leq Z(\text{VRPD}_{m,\alpha,k}, Q_\text{t}, Q_\text{d}) \\ &\leq Z(\text{VRPD}_{m,\alpha,k}, Q_\text{t}, Q_\text{t}) \quad (3) \end{aligned}$$

because in the worst case, when a vehicle utilizes the Q_d metric, it is possible to use the same set of routes, but inject artificial waiting periods to simulate the Q_t metric. Theorem 2 follows directly from inequalities (2) and (3) above. ∎

This is similar to our bound from the previous paper:

$$\frac{Z(\text{TSP}, Q_\text{t})}{Z(\text{VRPD}_{m,\alpha,k}, Q_\text{t}, Q_\text{t})} \leq m(\alpha k + 1).$$

In Theorem 2, we have an additional factor $B = \frac{Z(\text{TSP}, Q_\text{t})}{Z(\text{TSP}, Q_\text{d})}$ which compensates for the different metrics. If drones travel as the crow flies, we know that $B \geq 1$.

Theorem 3.

$$\frac{Z(VRP^*, Q_\text{t})}{Z(VRPD_{m,\alpha,k}, Q_\text{t}, Q_\text{d})} \leq \frac{Z(VRP^*, Q_\text{t})}{Z(VRP^*, Q_\text{d})} (\alpha k + 1).$$

Proof. We know from Theorem 6 of the previous paper that

$$\frac{Z(\text{VRP}^*, Q_\text{d})}{Z(\text{VRPD}_{m,\alpha,k}, Q_\text{d}, Q_\text{d})} \leq \alpha k + 1.$$

If we divide both sides by $Z(\text{VRP}^*, Q_\text{d})$, we obtain

$$\frac{1}{Z(\text{VRPD}_{m,\alpha,k}, Q_\text{d}, Q_\text{d})} \leq \frac{1}{Z(\text{VRP}^*, Q_\text{d})} (\alpha k + 1).$$

Next, multiply both sides by $Z(\text{VRP}^*, Q_\text{t})$ and we get

$$\frac{Z(\text{VRP}^*, Q_\text{t})}{Z(\text{VRPD}_{m,\alpha,k}, Q_\text{d}, Q_\text{d})} \leq \frac{Z(\text{VRP}^*, Q_\text{t})}{Z(\text{VRP}^*, Q_\text{d})} (\alpha k + 1).$$

As with the previous theorem, we note inequality (3). Theorem 3 follows directly. ∎

The implication of the above theorem is that with the VRPD model, it is not only possible to take advantage of parallelization (with a speed-up factor of up to $\alpha k + 1$ relative to VRP*), but the use of the crow-fly metric allows for an additional speed-up (up to a factor of $\frac{Z(\text{VRP}^*, Q_\text{t})}{Z(\text{VRP}^*, Q_\text{d})}$). In Appendix 3, we display a simple geometric example where the VRPD speed-up ratio actually exceeds $\alpha k + 1$ due to the ability to use the crow-fly metric.

3.3. Economic Savings

While minimizing the completion time is the primary objective, a company will want to consider costs, as well. In the next theorem, we combine the completion time and the variable costs of using the truck and drone to form a new objective function, denoted by Y. Therefore, the new objective function for a TSP solution is given by $Y(\text{TSP}) = Z(\text{TSP}) + \theta X(\text{TSP})$, where $X(\text{TSP})$ denotes the variable cost of truck usage and θ allows us to attach weights to the two components of the objective function. When $\theta = 0$, we are minimizing the completion time. When θ is very large, we are minimizing the sum of the variable costs. The new objective function value of the optimal $\text{VRPD}_{1,\alpha,k}$ solution is calculated by $Y(\text{VRPD}_{1,\alpha,k}) = Z(\text{VRPD}_{1,\alpha,k}) + \theta X(\text{VRPD}_{1,\alpha,k})$, where $X(\text{VRPD}_{1,\alpha,k}) = X_\text{t} + X_\text{d}$, the sum of truck and drone usage costs. We assume the cost per unit time of the drone is a times the cost per unit time of the truck. We expect a to be much less than 1 because we assume that drones will fly autonomously once they leave the truck. The drone usage cost is incurred only when the drone is airborne. We ignore the fixed costs for now.

Theorem 4. *If the triangle inequality is valid, then*

$$Y(VRPD_{1,\alpha,k}) \geq Y(TSP) - \left[\frac{\alpha}{a} + \left(\frac{\alpha}{a} - 1\right)\theta\right] X_\text{d},$$

where X_d is the variable cost of k drones in the optimal $VRPD_{1,\alpha,k}$ solution.

The coefficient $\left[\frac{\alpha}{a} + \left(\frac{\alpha}{a} - 1\right)\theta\right]$ is positive if $\alpha > a$. The potential savings from using a drone is large if α, θ, and X_d are large while a is small. We also point out the similar structure of the inequalities in Theorems 1 and 4.

Proof of Theorem 4. As noted earlier, we assume the truck speed is 1 and the drone speed is α. We further assume that the truck usage cost is 1 per unit time and the drone usage cost is a per unit time, so that $X(\text{TSP}) = 1 \times Z(\text{TSP})$. If not, we can modify the parameter θ to normalize the usage costs of the vehicles. Note also that $Y(\text{TSP}) = Z(\text{TSP}) + \theta X(\text{TSP}) = (1 + \theta)Z(\text{TSP})$. Therefore, a TSP solution that minimizes duration also minimizes the total cost Y.

We want to find a lower bound on $Y(\text{VRPD}_{1,\alpha,k})$ in terms of $Y(\text{TSP})$. This is similar to what we did in Theorem 1. From Table 1,

$$Z(\text{TSP}) \leq (\alpha k + 1) Z(\text{VRPD}_{1,\alpha,k}). \tag{4}$$

Since the truck usage cost is 1 per unit time, we have $X_\text{t} = Z(\text{VRPD}_{1,\alpha,k})$, where X_t is the truck usage cost in the optimal $\text{VRPD}_{1,\alpha,k}$ solution. Then,

$$Z(\text{TSP}) \leq (\alpha k + 1) X_\text{t} = X_\text{t} + \alpha k X_\text{t}. \tag{5}$$

Using the same construction process described in the proof of Theorem 1, we can show that an upper bound on the truck usage cost is given by

$$X(\text{TSP}) \leq X_\text{t} + \frac{X_\text{d}}{a}\alpha. \tag{6}$$

We construct a feasible TSP solution from the optimal $\text{VRPD}_{1,\alpha,k}$ solution. We insert the drone customers one by one onto the truck route whose variable cost was initially equal to X_t. The additional cost due to the drone customers is $\frac{X_\text{d}}{a}\alpha$. The factor $\frac{X_\text{d}}{a}$ gives the sum of usage time of the k drones. Multiplying it by the drone speed α gives the maximum total distance covered by the k drones. Since the truck has unit speed and unit usage cost, the term $\frac{X_\text{d}}{a}\alpha$ also gives the additional usage cost when we convert the optimal $\text{VRPD}_{1,\alpha,k}$ to a feasible TSP solution.

The left-hand sides of inequalities (5) and (6) are equal, that is, $Z(\text{TSP}) = X(\text{TSP})$ because we assume that truck usage cost is 1 per unit time. The two inequalities give two upper bounds on $Z(\text{TSP})$. The tighter upper bound is $X_\text{t} + \frac{X_\text{d}}{a}\alpha$ given by (6), because $\frac{X_\text{d}}{ak} \leq \frac{X_\text{t}}{1}$, as the average usage time per drone is never greater than the usage time of the truck.

Now,

$$\begin{aligned}
Y(\text{TSP}) &= (1 + \theta) Z(\text{TSP}) \\
&\leq (1 + \theta)\left(X_\text{t} + \frac{\alpha}{a} X_\text{d}\right) \\
&= X_\text{t} + \theta(X_\text{t} + X_\text{d}) + \left[\frac{\alpha}{a} + \left(\frac{\alpha}{a} - 1\right)\theta\right] X_\text{d} \\
&= Y(\text{VRPD}_{1,\alpha,k}) + \left[\frac{\alpha}{a} + \left(\frac{\alpha}{a} - 1\right)\theta\right] X_\text{d},
\end{aligned}$$

which yields the desired result. ∎

4. EXTENSION TO CEVRP*

Suppose there exists the following node locations along a street network $P = \{P_1, P_2, \ldots, P_{|P|}\}$, each requiring a package to be delivered to them from depot D. In the traditional TSP, one may insist that a truck stop at all of these locations, then finally return to D. The min-max close-enough traveling salesman problem (CETSP*) has the same objective as the ordinary TSP (i.e., minimize the time required to visit all node locations and return to the depot). However, in the CETSP*, we assume we need not necessarily visit location P_i itself. We only need to come within distance $R_i \geq 0$ of P_i [8]. Coming within distance R_i of a node P_i is "close enough" for some important applications. For example, utility companies use automated meter reading with RFID to read meters from a distance for billing purposes. Military applications involve surveillance from a distance.

In the min-max vehicle routing problem (VRP*), we wish for at least one truck out of a set of m homogeneous trucks to visit each customer location $P_i \in P$, then return to the depot. The objective is to minimize the time until all sites are visited and all trucks have returned to the depot. Analogously, we define the min-max close-enough vehicle routing problem (CEVRP*) to be the problem of minimizing the time required for at least one in a set of m trucks to come within some distance R_i of each customer location $P_i \in P$ before returning to the depot.

In this section, we will show that there exists a strong relationship between VRP*, VRPD, and CEVRP*. In future work, we hope to show how this relationship enlightens computational heuristics for finding solutions to the VRPD. Moreover, if we have reliable VRP* and CEVRP* solvers, this relationship will indicate whether our computational solutions are near-optimal.

4.1. *VRPD: An Intermediate Problem*

Let us define two problems. Firstly, let VRPD_{ur} be the unrestricted VRPD. This problem has the same characteristics as the VRPD, except that drone launch and retrieval locations are not restricted to nodes. This is more consistent with CEVRP*, which does not mandate a covering point of P_i to be a nodal point.

Secondly, let $\text{CEVRP}^*_{\text{nodes}}$ represent a problem similar to CEVRP*. However, $\text{CEVRP}^*_{\text{nodes}}$ is stricter. $\text{CEVRP}^*_{\text{nodes}}$ requires that for each customer P_i, there exists a *nodal* location on some truck route within distance R_i of customer P_i. This is consistent with the VRPD model where launch and retrieval points occur only at node locations.

Theorem 5.

$$Z(CEVRP^*) \le Z(VRPD_{ur}) \le Z(VRP^*);$$
$$Z(CEVRP^*_{nodes}) \le Z(VRPD) \le Z(VRP^*).$$

These claims are constructed from four inequalities which are proved formally in Appendix 2. In less formal terms, we note that the truck routes from the optimal VRP* solution act as feasible VRPD and VRPD_{ur} routes (that simply do not utilize any drone delivery capabilities). Thus VRPD and VRPD_{ur} can always do at least as well as VRP*. However, VRPD and VRPD_{ur} may be able to do better by making some drone deliveries.

Similarly, the truck routes from the optimal VRPD (or VRPD_{ur}) are feasible solutions to the $\text{CEVRP}^*_{\text{nodes}}$ (or CEVRP*) problems. Thus, the optimal solution to $\text{CEVRP}^*_{\text{nodes}}$ (or CEVRP*) is no worse than the optimal solution to VRPD (or VRPD_{ur}).

4.2. *VRPD in the Limit*

In this section, we will consider the limit cases of drone speed, namely when α approaches 0 and when α approaches ∞.

Theorem 6.

$$\lim_{\alpha\to\infty} Z(VRPD, \alpha) = Z(CEVRP^*_{nodes}).$$

Proof. We establish in Appendix 2 that every $\text{CEVRP}^*_{\text{nodes}}$ solution can be converted into a VRPD solution. This is done by starting with the truck route of the $\text{CEVRP}^*_{\text{nodes}}$ solution. However, the truck waits at the drone release point until the drone delivers its package and returns to the truck. This trivial feasible solution to VRPD is called VRPD_f. Let W_j be the sum of all such wait times on the jth truck's VRPD route. Let $W = \max_j(W_j)$. By this construction, it is clear that

$$Z(\text{CEVRP}^*_{\text{nodes}}) + W \ge Z(\text{VRPD}_f) \ge Z(\text{VRPD}).$$

The upper bound distance on any drone flight, again, is $2R = V$. Thus $2R/\alpha$ is the maximum duration that a truck would wait for any drone to deliver a package. Let M be the maximum number of customers on any route. So $0 \le W \le 2MR/\alpha$. Given a finite number of customers,

$$\lim_{\alpha\to\infty} W = 0.$$

Furthermore, as $\alpha \to \infty$

$$Z(\text{CEVRP}^*_{\text{nodes}}) = Z(\text{VRPD}_f) \ge Z(\text{VRPD}).$$

However, we established in a previous theorem that $Z(\text{CEVRP}^*_{\text{nodes}}) \le Z(\text{VRPD})$. Therefore, as $\alpha \to \infty$

$$Z(\text{CEVRP}^*_{\text{nodes}}) = Z(\text{VRPD}_f) = Z(\text{VRPD}). \qquad \blacksquare$$

Theorem 7 (Fast Drone Theorem)**.**

$$\lim_{\alpha\to\infty} Z(VRPD_{ur}, \alpha) = Z(CEVRP^*).$$

Proof. The proof is identical in structure to Theorem 6 with one minor exception. Namely, we now may designate any point within distance R of a customer as a launch/retrieval point, rather than being restricted to nodal locations. We then force trucks to wait at these launch points until the drone(s) returns. The total required waiting time of all trucks (again) converges to 0 as $\alpha \to \infty$. ■

The two theorems above show that as drone speed goes to ∞, the VRPD is an equivalent problem to CEVRP*. However, it also hints that perhaps a CEVRP* solution would be a good approximation to the VRPD solution whenever the ratio of drone speed to truck speed is high. In environments with highly congested roadways, but a relatively unobstructed sky, or perhaps when utilizing very high speed drones, it may be worth starting with a CEVRP* solution, and adapting it into a VRPD solution.

Theorem 8 (Slow Drone Theorem)**.**

$$\lim_{\alpha\to 0} Z(VRPD, \alpha) = Z(VRP^*)$$

and

$$\lim_{\alpha\to 0} Z(VRPD_{ur}, \alpha) = Z(VRP^*).$$

Proof. If our optimal VRPD or VRPD_{ur} solution has no drone launches, then the solution is the same as the optimal VRP* solution. In this case, the above equality holds.

Now suppose our VRPD or VRPD_{ur} solution has some drone flight of non-zero length (out and back). Let L be the longest of such routes. Then as $\alpha \to 0$, the time required for such a route is $L/\alpha \to \infty$. This implies that as drone speed goes to 0, any VRPD or VRPD$_{ur}$ solution containing a drone launch (e.g., VRPD$_f$) is such that $\lim_{\alpha\to 0} Z(\text{VRPD}_f, \alpha) = \infty$. Supposing our truck speed is non-zero, a VRP* solution would require a finite amount of time. This proves that as $\alpha \to 0$, VRPD and VRPD$_{ur}$ solutions should not contain drone launches to remain optimal. ∎

In Section 4.1, we showed VRPD's objective value was bounded below by the objective value of CEVRP* and bounded above by the objective value of VRP*. Now in Section 4.2, we have shown that VRPD and CEVRP* are equivalent problems for an arbitrarily fast drone; VRPD and VRP* are equivalent problems for an arbitrarily slow drone.

Other than the bounds on optimal objective values, we do not yet know the relationship between VRPD optimal solutions and optimal solutions to CEVRP* and VRP* for intermediate values of α (i.e., $0 < \alpha < \infty$). Furthermore, we do not know how this relationship is affected by our choice of α, the underlying street network, or the drone network.

5. CONCLUSIONS AND FUTURE WORK

This article extends and strengthens previous results in [15]. VRPD is one model that attempts to complement the carrying capacity and range of a truck with the ability of a drone to help "parallelize" delivery and take advantage of crow-fly distances. This article has shown the theoretical maximum benefit of this model under ideal circumstances.

A connection between the CEVRP*, VRPD, and VRP* has been made in the form of objective value bounds and asymptotic results. We believe that a number of computational heuristics and benchmark instances could now be developed to find VRPD solutions as close to optimal as possible for practical values of α. Using solution methods for CEVRP* (such as in [3, 5, 11]), modifying them to maintain VRPD feasibility, and then applying some local optimization procedures is one new idea for obtaining computational solutions to VRPD. An alternative idea is starting with a VRP* solution and inserting drone deliveries in a smart way. In addition, one may compare computed objective values for VRPD with CEVRP* and VRP*, assessing the tightness of these theoretical bounds in practice for varying values of α.

Expanding the model to include limitations on drone package weight (while still allowing trucks to carry heavier packages) could add to the practical worth of this model. The study of other variations, such as allowing a drone to launch on one truck and land on another truck or allowing a drone to carry more than one package at a time, may also be considered.

APPENDIX 1: ANALOGS TO AMDAHL'S LAW

Many of the bounds established in our previous paper [15] show great similarities to Amdahl's Law. Amdahl's Law, in its original form, establishes the maximum speed-up potential for computer programs via parallelization [2]. The simultaneous use of multiple vehicles to deliver packages is an example of parallelization. Furthermore, the worst-case VRPD bounds from [15] match Amdahl's Law.

Let us begin by defining the following:

- S is the speed-up factor of the whole computer program.
- p is the percentage of the program that is parallelizable.
- s is the speed-up factor among the portion of the program that is parallelizable.

Amdahl's Law states that:

$$S = \frac{1}{1 - p + p/s}.$$

Suppose one has K processors, instead of one processor. So $s = K$. Then, if the program is fully parallelizable, we get that $p = 1$, so $S = \frac{1}{1/K} = K$. So the maximum speedup factor is K.

If a delivery route is fully parallelizable, then $p = 1$. By fully parallelizable, we mean that the sum of distances traveled by vehicles is not more than the TSP distance to deliver all packages (i.e., no extra work is required), and all vehicles are in constant operation. If instead of one truck of speed 1, we have a truck of speed 1 and K drones of speed α, the speed-up of the fully parallelizable portions is $1 + \alpha K$. So, $s = 1 + \alpha K$. By Amdahl's Law, we would get $S = 1 + \alpha K$, which matches our results from [15].

If we have m trucks of speed 1 and mk drones of speed α, full parallelization, that is, $p = 1$, then $s = m(\alpha k + 1)$ and Amdahl's Law would yield $S = m(\alpha k + 1)$.

Amdahl's Law, like the results from our previous paper, exist as bounds. There would, in practice, exist certain overhead costs in the release of the drones. Analogously, spawning threads for a parallel program is mandatory overhead. Also, full parallelization generally only occurs when data (or a street network) has a particular structure. In parallel programs, there often exist data dependencies: For example, certain results must be written to memory, before other computations can begin. This is analogous to the need for drones to resynchronize with a truck. Finally, each individual processor may be limited in its ability to do work based on what information is currently available in memory. Similarly, drones are unable to do functional work without a truck placing them close enough to a customer location.

APPENDIX 2: PROOFS RELATING TO SECTION 4.1

Theorem 9. *It is always possible to convert a CEVRP* solution to an equivalent CEVRP$^*_{nodes}$ solution.*

Proof. Begin with our CEVRP* solution. For each customer location P_i for which there exists a vehicle route that visits a node that is within distance R_i, we will say that P_i is "covered." For each non-covered customer location P_j, find some node z such that the distance between P_j and z is less than R_j. Insert into some vehicle's route an extra "out and back" to node z. The existence of such a node z is guaranteed, as setting $z = P_j$ implies that the distance between P_j and z is 0, so P_j becomes covered with this altered route. ∎

As an example, suppose some truck called T_i has a route visiting the following customer locations in order: $3, 8, 10, 12, 14, 11, 3$. Now suppose that customer location P_j is not covered in the original CEVRP* solution. We then revise T_i to be the path $3, 8, 10, 12, P_j, 12, 14, 11, 3$. Every edge and customer location that was previously visited by T_i is still visited. However, now P_j is covered by vehicle T_i, as it certainly passes within distance R_j. We act similarly for all uncovered customer locations, creating a valid $\text{CEVRP}^*_{\text{nodes}}$ solution.

This particular construction is not necessarily efficient, but it is a simple way to show that we can always create a $\text{CEVRP}^*_{\text{nodes}}$ solution from a CEVRP* solution. In practice, we would look for a cheaper insertion method to create feasibility.

Theorem 10. *It is always possible to convert from a $CEVRP^*_{nodes}$ solution to a VRPD solution.*

Proof. Assume a drone has a range of V distance units. For each customer location P_i, set $R_i = R = V/2$. The $\text{CEVRP}^*_{\text{nodes}}$ solution requires that for every customer location, there exists a node within distance R of the customer location, which is traversed by a truck. For every customer location P_i, designate one node that is visited by a truck and within distance R as a "cover node." A simple feasible VRPD solution is the same as the $\text{CEVRP}^*_{\text{nodes}}$ solution, except that the "cover nodes" become release nodes for drones and the truck remains idle until the drone returns from its release location. Since the distance between P_i and its cover node is less than $R = V/2$, we do not exceed the range of the drone. ∎

Let VRPD_{ur} be the unrestricted VRPD problem. This relaxation of the VRPD allows for the launch and release points of a drone to occur anywhere along a truck's route (i.e., launch and release may occur along edges and at nodes on a truck's route).

Theorem 11. *It is always possible to convert from a CEVRP* solution to a $VRPD_{ur}$ solution.*

Proof. Every CEVRP* solution contains at least one point on a truck route within distance R of each customer location $P_i \in P$. For each customer location P_i, select one such corresponding point. These shall be the launch points. For the VRPD_{ur} solution, the truck routes shall remain the same as in the CEVRP* solution. However, at each launch point, the truck releases a drone and waits for the drone to return. Since each launch point is within distance R of the corresponding customer location, the drone never exceeds its $2R = V$ range. ∎

Theorem 12. *Every VRP* solution is also a VRPD and a $VRPD_{ur}$ solution. Therefore* $Z(VRPD) \leq Z(VRP^*)$ *and* $Z(VRPD_{ur}) \leq Z(VRP^*)$

Proof. Every customer location P_i is visited by a truck in the VRP* solution, and all the trucks return to the depot. VRPD and VRPD_{ur} require each location to be visited by a truck or a drone, where the drone routes have some restrictions. Since every location is visited by a truck in a VRP* solution, we need not even launch any drones. All conditions of a VRPD or a VRPD_{ur} solution are trivially satisfied. ∎

Theorem 13. $Z(CEVRP^*_{nodes}) \leq Z(VRPD)$ *and* $Z(CEVRP^*) \leq Z(VRPD_{ur})$

Proof. Every VRPD solution is easily convertible to a $\text{CEVRP}^*_{\text{nodes}}$ solution, as the distance traveled by the drone in the VRPD solution must have a total distance of less than $2R$. This implies that the drone's outbound and/or return flight was less than distance R. Thus, at least one of the launch or release nodes was within distance R of each customer location. Therefore, each customer location has a truck pass within distance R of it. Hence, the VRPD truck route also is a valid solution to $\text{CEVRP}^*_{\text{nodes}}$. Equivalently, every VRPD_{ur} solution is a CEVRP* solution. By similar reasoning, every VRPD_{ur} solution has a launch or release point within distance R of each customer location. In addition, we note that any time the truck spends waiting for a drone may be eliminated for CEVRP* and $\text{CEVRP}^*_{\text{nodes}}$ solutions. ∎

Lemma 1. $Z(CETSP^*) \leq Z(TSP)$

Proof. From the above theorems in Appendix 2,

$$Z(\text{CEVRP}^*) \leq Z(\text{VRPD}_{\text{ur}}) \leq Z(\text{VRP}^*).$$

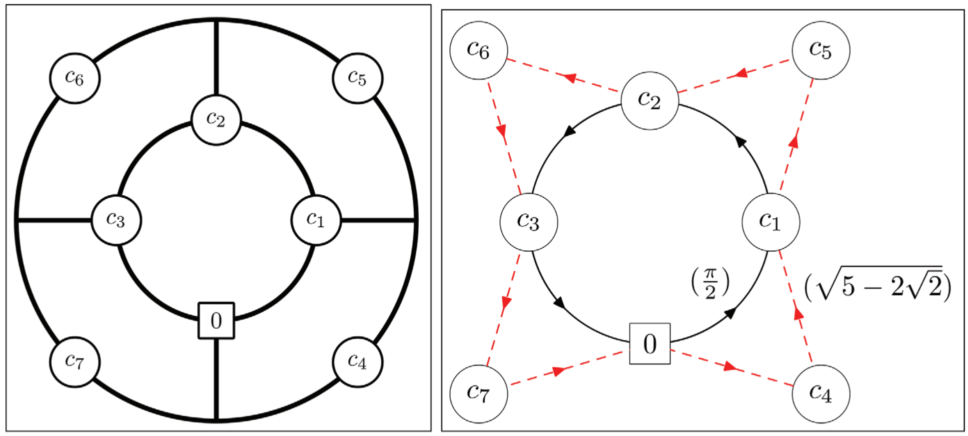
By letting $m = 1$, we have

$$Z(\text{CETSP}^*) \leq Z(\text{VRPD}_{\text{ur},1,\alpha,k}) \leq Z(\text{TSP}^*).$$

Now by removing the center term, we have the result by transitivity. ∎

APPENDIX 3: EXAMPLE OF SPEED-UP RATIO EXCEEDING $\alpha K + 1$

The next example illustrates, not surprisingly, that when trucks and drones follow different distance metrics, we can do better than the bound in Theorem 4 of [15], which states that the maximum speed-up ratio of VRPD with $m = 1$ relative to the optimal TSP is $\alpha k + 1$. In Figure 2, there are seven customers located on two concentric circles with radii 1 and 2. There are four edges of length 1 that link the two circles (see Fig. 2a). The seven customers are served by one truck carrying one drone. The speed of the truck is one and the speed of the drone is $\alpha = \frac{4}{\pi}\sqrt{5 - 2\sqrt{2}} \approx 1.876$. The polar coordinates of the depot (node 0) and the seven customers and their distances from the depot are given in Table 2.



(a) Problem (b) Optimum $VRPD_{1,\alpha,k}$ solution

FIG. 2. Truck and drone follow different distance metrics. [Color figure can be viewed at wileyonlinelibrary.com]

TABLE 2. Positions of the depot and the customers in Figure 2a

	Polar coordinates		Distance from the depot	
Node	Radial	Angular	Edge restriction	Crow-fly
0	1	$-\pi/2$	0	0
c_1	1	0	$\pi/2$	$\sqrt{2}$
c_2	1	$\pi/2$	π	2
c_3	1	π	$\pi/2$	$\sqrt{2}$
c_4	2	$-\pi/4$	$1+\pi/2$	$5-2\sqrt{2}$
c_5	2	$\pi/4$	$1+\pi$	$5+2\sqrt{2}$
c_6	2	$3\pi/4$	$1+\pi$	$5+2\sqrt{2}$
c_7	2	$-3\pi/4$	$1+\pi/2$	$5-2\sqrt{2}$

We assume the drone can fly as the crow flies, but the truck is restricted to the edges shown in Figure 2a. For example, the distance between customer c_1 and the depot is $Q_t(1,0) = \frac{\pi}{2} \approx 1.571$ for the truck, but $Q_d(1,0) = \sqrt{2} \approx 1.414$ for the drone using the coordinates given in Table 2. The crow-fly distance between points (r_1, θ_1) and (r_2, θ_2) is calculated by $\sqrt{r_1^2 + r_2^2 - 2r_1r_2\cos(\theta_1 - \theta_2)}$. We denote the optimal objective function value by $Z(\mathrm{VRPD}_{1,\alpha,1}, Q_t, Q_d)$. From a geometric point of view, we serve the four customers on the outer circle by the drone and the three customers on the inner circle by the truck. The optimal solution is shown in Figure 2b. The drone is dispatched from the depot to serve customer c_4 while the truck travels to serve c_1. The two vehicles arrive at c_1 at the same time when the drone is immediately launched to serve c_5, and so on. Eventually, the two vehicles arrive at the depot simultaneously with $Z(\mathrm{VRPD}_{1,\alpha,1}, Q_t, Q_d) = 2\pi \approx 6.283$.

If we serve all the customers using only the truck, $Z(\mathrm{TSP}, Q_t) = 2 + \frac{11}{2}\pi \approx 19.279$. The path followed by the truck is $0 \to c_1 \to c_5 \to c_4 \to c_7 \to c_6 \to c_2 \to c_3 \to 0$ (optimal solution obtained using Gurobi solver [9]). In this example, we have $\frac{Z(\mathrm{TSP}, Q_t)}{Z(\mathrm{VRPD}_{1,\alpha,1}, Q_t, Q_d)} = \frac{2+\frac{11}{2}\pi}{2\pi} \approx 3.068 > \alpha k + 1 \approx 2.876$.

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