hw 9

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1 Scalar ODE

1.1 a

$$f(t,x) = 2x^2 + x - 1$$
$$x_{n+1} = x_n + h * f(t_n, x_n)$$

Calculations:

For $n = 0 (t = 1.0) : x_0 = 1, f(t_0, x_0) = 2 * 1^2 + 1 - 1 = 2, x_1 = x_0 + h * f(t_0, x_0) = 1 + 0.1 * 2 = 1.2$
For $n = 1 (t = 1.1) : x_1 = 1.2, f(t_1, x_1) = 2 * 1.2^2 + 1.2 - 1 = 3.44, x_2 = x_1 + h * f(t_1, x_1) = 1.2 + 0.1 * 3.44 = 1.544$

Thus, by Euler's method, x(1.2) $\approx$ 1.544.

1.2 b

$$\text{Predict: } y_p = y_n + h * f(t_n, y_n)$$
$$\text{Correct: } y_{n+1} = y_n + h/2 * [f(t_n, y_n) + f(t_{n+1}, y_p)]$$

Calculations:

For $n = 0 (t = 1.0) : y_0 = 1, f(t_0, y_0) = 2, y_p = y_0 + h * f(t_0, y_0) = 1 + 0.1 * 2 = 1.2, f(t1, y_p) = 3.44, y_1 = y_0 + h/2 * [f(t_0, y_0)$
For $n = 1 (t = 1.1) : y_1 = 1.272, f(t_1, y_1) = 3.8144, y_p = y_1 + h * f(t_1, y_1) = 1.272 + 0.1 * 3.8144 = 1.65344, f(t_2, y_p) = 5.699,$

Thus, by Heun's method, x(1.2) $\approx$ 1.7257.

1.3 c

$$k1 = h * f(t_n, y_n)$$
$$k2 = h * f(t_n + h/2, y_n + k1/2)$$
$$k3 = h * f(t_n + h/2, y_n + k2/2)$$
$$k4 = h * f(t_n + h, y_n + k3)$$
$$y_{n+1} = y_n + 1/6 * (k1 + 2 * k2 + 2 * k3 + k4)$$

Calculations:

For $n = 0 (t = 1.0) : y_0 = 1, f(t_0, y_0) = 2, k1 = h * f(t_0, y_0) = 0.1 * 2 = 0.2, k2 = h * f(t_0 + h/2, y_0 + k1/2) = 0.1 * f(1.05, 1.1$
For $n = 1 (t = 1.1) : y_1 = 1.281, f(t_1, y_1) = 3.381, k1 = h * f(t_1, y_1) = 0.1 * 3.381 = 0.3381, k2 = h * f(t_1 + h/2, y_1 + k1/2) =$

Thus, by 4-th order Runge-Kutta method, x(1.2) $\approx$ 1.589.

1.4 d

Given: $x' = 2x^2 + x1, \quad x(1) = 1, \quad x(1.1) = 1.7257$
The 2-nd order Adams-Bashforth-Moulton (ABM) method is:
Predictor: $x_{n+1}^p = x_n + \dfrac{h}{2}[3f(t_n, x_n) - f(t_{n-1}, x_{n-1})],$
Corrector: $x_{n+1} = x_n + \dfrac{h}{2}[f(t_{n+1}, x_{n+1}^p) + f(t_n, x_n)], \quad$ where $f(t, x) = 2x^2 + x - 1.$
For $n = 1$ (i.e., $t = 1.1$) :
$x_0 = 1, \quad x_1 = 1.7257,$
$x_2^p = x_1 + \dfrac{h}{2}[3f(t_1, x_1) - f(t_0, x_0)] = 1.7257 + \dfrac{0.1}{2}[3(2 * 1.7257^2 + 1.7257 - 1) - (2 * 1^2 + 1 - 1)] = 2.3934,$
$x_2 = x_1 + \dfrac{h}{2}[f(t_2, x_2^p) + f(t_1, x_1)] = 1.7257 + \dfrac{0.1}{2}[(2 * 2.3934^2 + 2.3934 - 1) + (2 * 1.7257^2 + 1.7257 - 1)] = 2.3339.$
So, by ABM method, $x(1.2) \approx 2.3339.$
For $n = 2$ (i.e., $t = 1.2$) :
$x_1 = 1.7257, \quad x_2 = 2.3339,$
$x_3^p = x_2 + \dfrac{h}{2}[3f(t_2, x_2) - f(t_1, x_1)] = 2.3339 + \dfrac{0.1}{2}[3(2 * 2.3339^2 + 2.3339 - 1) - (2 * 1.7257^2 + 1.7257 - 1)] = 3.7204,$
$x_3 = x_2 + \dfrac{h}{2}[f(t_3, x_3^p) + f(t_2, x_2)] = 2.3339 + \dfrac{0.1}{2}[(2 * 3.7204^2 + 3.7204 - 1) + (2 * 2.3339^2 + 2.3339 - 1)] = 3.5307.$
So, by ABM method, $x(1.3) \approx 3.5307.$

2 Solving ODE backward in time

Given the ODE: x'(t) = -tx^2 with initial condition $x(0) = 2$.
To solve at $t = -0.2$ using Taylor Series method of order 2:
$f(t, x) = -tx^2, \quad f'(t, x) = -x^2 - 2txf(t, x).$
$x(t + h) \approx x(t) + hf(t, x) + \frac{h^2}{2} f'(t, x),$
$x(-0.2) \approx x(0) - 0.2f(0, 2) - 0.02f'(0, 2),$
$\approx 2 - 0.2 * (-0 * 2^2) - 0.02 * (-2^2 - 2 * 0 * -0 * 2^2) = 2 + 0.08 = 2.08.$
To solve using Runge-Kutta method of order 2:
$k_1 = hf(t, x), \quad k_2 = hf(t + h, x + k_1), \quad x(t + h) \approx x(t) + \frac{1}{2}(k_1 + k_2),$
$k_1 = -0.2f(0, 2) = 0,$
$k_2 = -0.2f(-0.2, 2) = -0.2 * (-(-0.2) * 2^2) = 0.16,$
$x(-0.2) \approx x(0) + \frac{1}{2}(k_1 + k_2) = 2 + \frac{1}{2}(0 + 0.16) = 2.08.$
So, for both methods, $x(-0.2) \approx 2.08.$

3 A simple high order ODE

3.1 a

Let $\mathrm{y}_1(t) = x(t), \quad y_2(t) = x'(t) = y_1'(t), \quad y_3(t) = x''(t) = y_2'(t),$
then the system of first order equations is:
$y_1'(t) = y_2(t),$
$y_2'(t) = y_3(t),$
$y_3'(t) = -2y_3(t) + y_1(t)t,$
with the initial conditions: $y_1(0) = 1, \quad y_2(0) = 2, \quad y_3(0) = 3.$

3.2 b

$y_1(t+0.1) = y_1(t) + 0.1 * y_2(t)$,
$y_2(t+0.1) = y_2(t) + 0.1 * y_3(t)$,
$y_3(t+0.1) = y_3(t) + 0.1 * (-2y_3(t) + y_1(t)t)$.

3.3 c

$$x(1) = 3.8616$$

4 Higher Order ODEs with various methods

4.1 a

For Euler's method, $y_{n+1} = y_n + h * f(t_n, y_n)$,
where $f(t, y_1, y_2) = [y_2, ty_1]$.
Given $h = 0.1, y_1(0) = 1,$ and $y_2(0) = 1$:
$y_1(0.1) = y_1(0) + h * y_2(0) = 1 + 0.1 * 1 = 1.1$,
$y_2(0.1) = y_2(0) + h * 0 * 1 = 1 + 0 = 1$.
For the next step, $y_1(0.2) = y_1(0.1) + h * y_2(0.1) = 1.1 + 0.1 * 1 = 1.2$,
$y_2(0.2) = y_2(0.1) + h * 0.1 * y_1(0.1) = 1 + 0.01 * 1.1 = 1.011$.
So, $x(0.1) = y_1(0.1) = 1.1$ and $x(0.2) = y_1(0.2) = 1.2$

4.2 b

The 2nd order Runge-Kutta (Heun's method) is as follows:

$$\begin{aligned}
k_{1,1} &= h * y_2(t_n), \\
k_{1,2} &= h * t_n * y_1(t_n), \\
k_{2,1} &= h * (y_2(t_n) + k_{1,2}), \\
k_{2,2} &= h * (t_n + h) * (y_1(t_n) + k_{1,1}), \\
y_{1,n+1} &= y_{1,n} + 0.5 * (k_{1,1} + k_{2,1}), \\
y_{2,n+1} &= y_{2,n} + 0.5 * (k_{1,2} + k_{2,2}).
\end{aligned}$$

For the initial condition at t = 0, we have

$$\begin{aligned}
y_1(0) &= x(0) = 1, \\
y_2(0) &= x'(0) = 1.
\end{aligned}$$

For t = 0.1, we get

$$\begin{aligned}
k_{1,1} &= h * y_2(0) = 0.1 * 1 = 0.1, \\
k_{1,2} &= h * 0 * y_1(0) = 0, \\
k_{2,1} &= h * (y_2(0) + k_{1,2}) = 0.1 * 1 = 0.1, \\
k_{2,2} &= h * (0.1 * (y_1(0) + k_{1,1})) = 0.1 * 0.11 = 0.011, \\
y_1(0.1) &= y_1(0) + 0.5 * (k_{1,1} + k_{2,1}) = 1 + 0.5 * (0.1 + 0.1) = 1.1, \\
y_2(0.1) &= y_2(0) + 0.5 * (k_{1,2} + k_{2,2}) = 1 + 0.5 * (0 + 0.011) = 1.0055.
\end{aligned}$$

For t = 0.2, we repeat the process:

$$\begin{aligned}
k_{1,1} &= h * y_2(0.1) = 0.1 * 1.0055 = 0.10055, \\
k_{1,2} &= h * 0.1 * y_1(0.1) = 0.1 * 0.11 = 0.011, \\
k_{2,1} &= h * (y_2(0.1) + k_{1,2}) = 0.1 * 1.01655 = 0.101655, \\
k_{2,2} &= h * (0.2 * (y_1(0.1) + k_{1,1})) = 0.1 * 0.22 = 0.022, \\
y_1(0.2) &= y_1(0.1) + 0.5 * (k_{1,1} + k_{2,1}) = 1.1 + 0.5 * (0.10055 + 0.101655) = 1.2011025, \\
y_2(0.2) &= y_2(0.1) + 0.5 * (k_{1,2} + k_{2,2}) = 1.0055 + 0.5 * (0.011 + 0.022) = 1.0195.
\end{aligned}$$

Therefore, the approximation at x(0.2) using Heun's method is 1.2011025.

4.3 c

The 4th order Runge-Kutta method is as follows:

$$\begin{aligned}
k_{1,1} &= h * y_2(t_n), \\
k_{1,2} &= h * t_n * y_1(t_n), \\
k_{2,1} &= h * (y_2(t_n) + 0.5 * k_{1,2}), \\
k_{2,2} &= h * (t_n + 0.5 * h) * (y_1(t_n) + 0.5 * k_{1,1}), \\
k_{3,1} &= h * (y_2(t_n) + 0.5 * k_{2,2}), \\
k_{3,2} &= h * (t_n + 0.5 * h) * (y_1(t_n) + 0.5 * k_{2,1}), \\
k_{4,1} &= h * (y_2(t_n) + k_{3,2}), \\
k_{4,2} &= h * (t_n + h) * (y_1(t_n) + k_{3,1}), \\
y_{1,n+1} &= y_{1,n} + (1/6) * (k_{1,1} + 2 * k_{2,1} + 2 * k_{3,1} + k_{4,1}), \\
y_{2,n+1} &= y_{2,n} + (1/6) * (k_{1,2} + 2 * k_{2,2} + 2 * k_{3,2} + k_{4,2}).
\end{aligned}$$

For the initial condition at t = 0, we have

$$\begin{aligned}
y_1(0) &= x(0) = 1, \\
y_2(0) &= x'(0) = 1.
\end{aligned}$$

$$\text{At } t = 0, y_1(0) = 1, y_2(0) = 1:$$

$$\begin{aligned}
k_{1,1} &= h * y_2(0) = 0.1 * 1 = 0.1, \\
k_{1,2} &= h * 0 * y_1(0) = 0, \\
k_{2,1} &= h * (y_2(0) + 0.5 * k_{1,2}) = 0.1, \\
k_{2,2} &= h * 0.05 * (y_1(0) + 0.5 * k_{1,1}) = 0, \\
k_{3,1} &= h * (y_2(0) + 0.5 * k_{2,2}) = 0.1, \\
k_{3,2} &= h * 0.05 * (y_1(0) + 0.5 * k_{2,1}) = 0, \\
k_{4,1} &= h * (y_2(0) + k_{3,2}) = 0.1, \\
k_{4,2} &= h * 0.1 * (y_1(0) + k_{3,1}) = 0.01.
\end{aligned}$$

$$\begin{aligned}
y_{1,1} &= y_{1,0} + \frac{1}{6} * (k_{1,1} + 2 * k_{2,1} + 2 * k_{3,1} + k_{4,1}) \\
&= 1 + \frac{1}{6} * (0.1 + 2 * 0.1 + 2 * 0.1 + 0.1) = 1.1333333, \\
y_{2,1} &= y_{2,0} + \frac{1}{6} * (k_{1,2} + 2 * k_{2,2} + 2 * k_{3,2} + k_{4,2}) \\
&= 1 + \frac{1}{6} * (0 + 2 * 0 + 2 * 0 + 0.01) = 1.0016667.
\end{aligned}$$

$$
\begin{aligned}
k_{1,1} &= h * y_2(0.1) = 0.1 * 1.0016667 = 0.10016667,\\
k_{1,2} &= h * 0.1 * y_1(0.1) = 0.1 * 0.1133333 = 0.01133333,\\
k_{2,1} &= h * (y_2(0.1) + 0.5 * k_{1,2}) = 0.1 * 1.00708333 = 0.100708333,\\
k_{2,2} &= h * 0.15 * (y_1(0.1) + 0.5 * k_{1,1}) = 0.1 * 0.168125 = 0.0168125,\\
k_{3,1} &= h * (y_2(0.1) + 0.5 * k_{2,2}) = 0.1 * 1.0090625 = 0.10090625,\\
k_{3,2} &= h * 0.15 * (y_1(0.1) + 0.5 * k_{2,1}) = 0.1 * 0.17204167 = 0.017204167,\\
k_{4,1} &= h * (y_2(0.1) + k_{3,2}) = 0.1 * 1.01927083 = 0.101927083,\\
k_{4,2} &= h * 0.2 * (y_1(0.1) + k_{3,1}) = 0.1 * 0.23423333 = 0.023423333.
\end{aligned}
$$

$$
\begin{aligned}
y_{1,2} &= y_{1,1} + \frac{1}{6} * (k_{1,1} + 2 * k_{2,1} + 2 * k_{3,1} + k_{4,1})\\
&= 1.1333333 + \frac{1}{6} * (0.10016667 + 2 * 0.100708333 + 2 * 0.10090625 + 0.101927083) = 1.26949028,\\
y_{2,2} &= y_{2,1} + \frac{1}{6} * (k_{1,2} + 2 * k_{2,2} + 2 * k_{3,2} + k_{4,2})\\
&= 1.0016667 + \frac{1}{6} * (0.01133333 + 2 * 0.0168125 + 2 * 0.017204167 + 0.023423333) = 1.021823153.
\end{aligned}
$$

4.4 d

The first order system is:

$$
\begin{aligned}
y_1' &= y_2,\\
y_2' &= ty_1,\\
\text{with } y_1(0) = 1, &\quad y_2(0) = 1.
\end{aligned}
$$

The 2nd order Adams-Bashforth-Moulton (ABM) method:

Prediction:

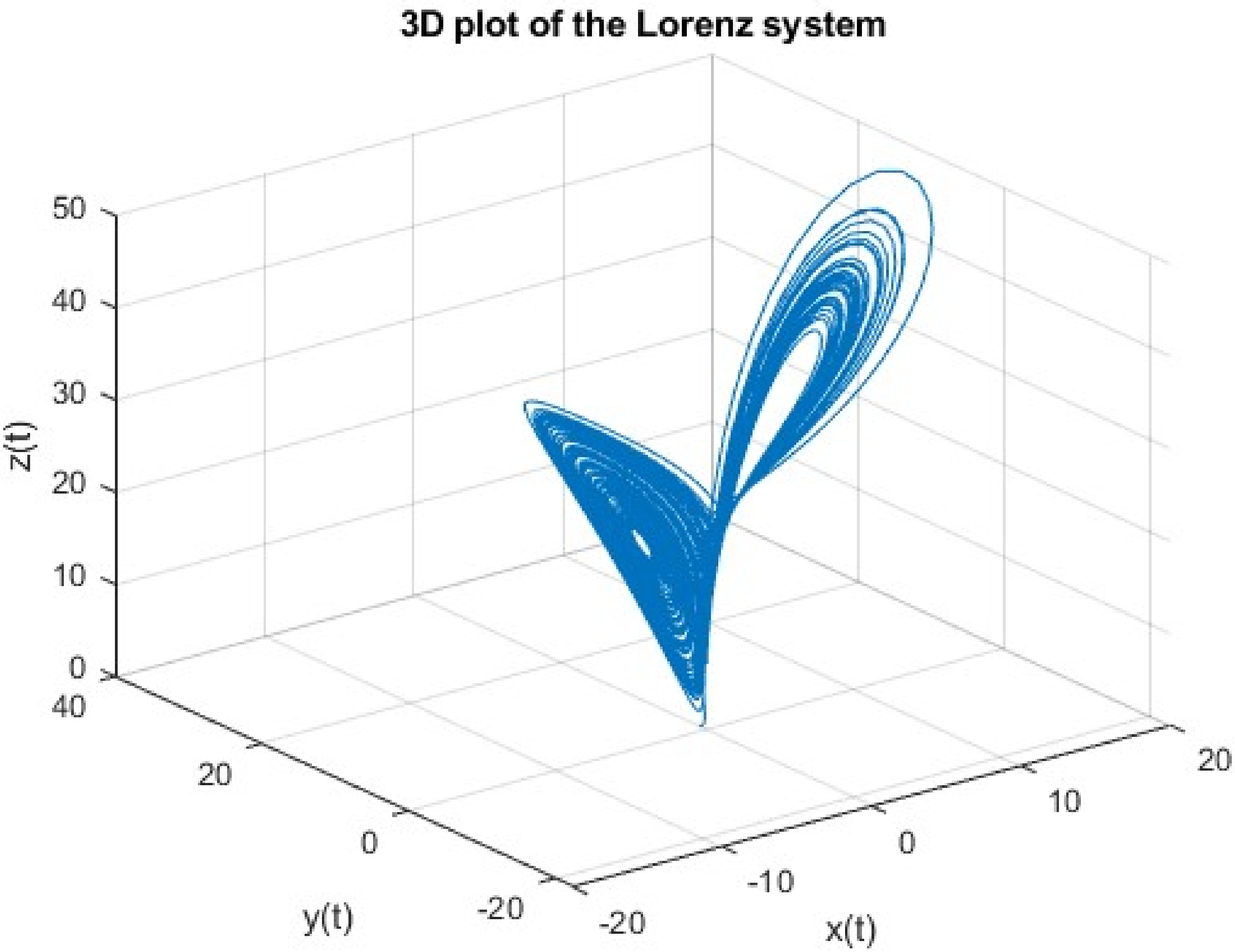
$$
\begin{aligned}
y_{1,pred}(0.1) &= y_1(0) + h \cdot y_2(0) = 1 + 0.1 \cdot 1 = 1.1,\\
y_{2,pred}(0.1) &= y_2(0) + h \cdot ty_1(0) = 1 + 0 \cdot 1 = 1.
\end{aligned}
$$

Correction:

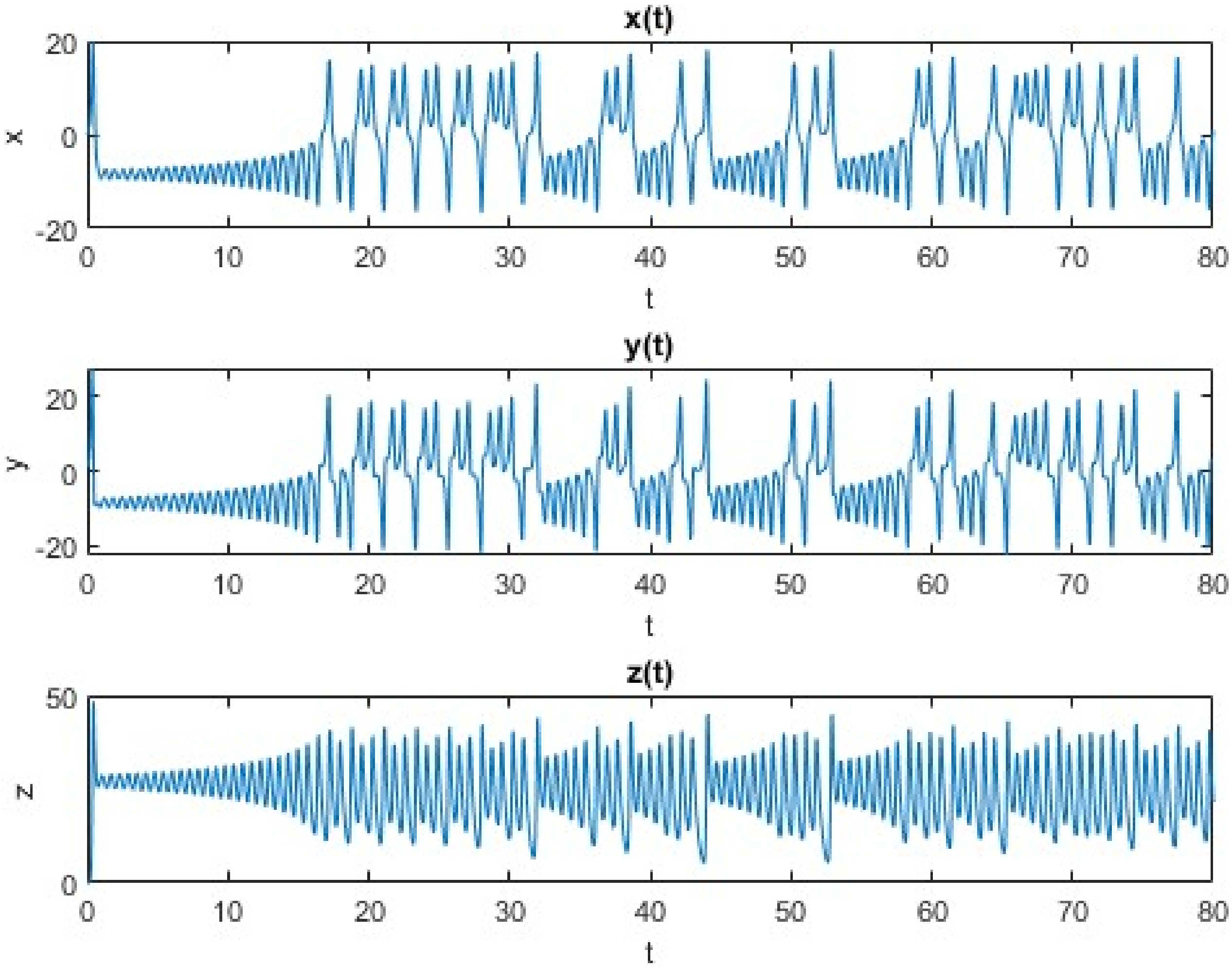
$$
\begin{aligned}
y_{1,corr}(0.1) &= y_1(0) + \frac{h}{2} \cdot (3y_2(0) - y_2(-0.1)) = 1 + 0.1 \cdot (3 \cdot 1 - 0) = 1.3,\\
y_{2,corr}(0.1) &= y_2(0) + \frac{h}{2} \cdot (3ty_1(0) - ty_1(-0.1)) = 1 + 0 \cdot (3 \cdot 1 - 0) = 1.
\end{aligned}
$$

5 The Lorenz system; A study in chaoes

5.1 a

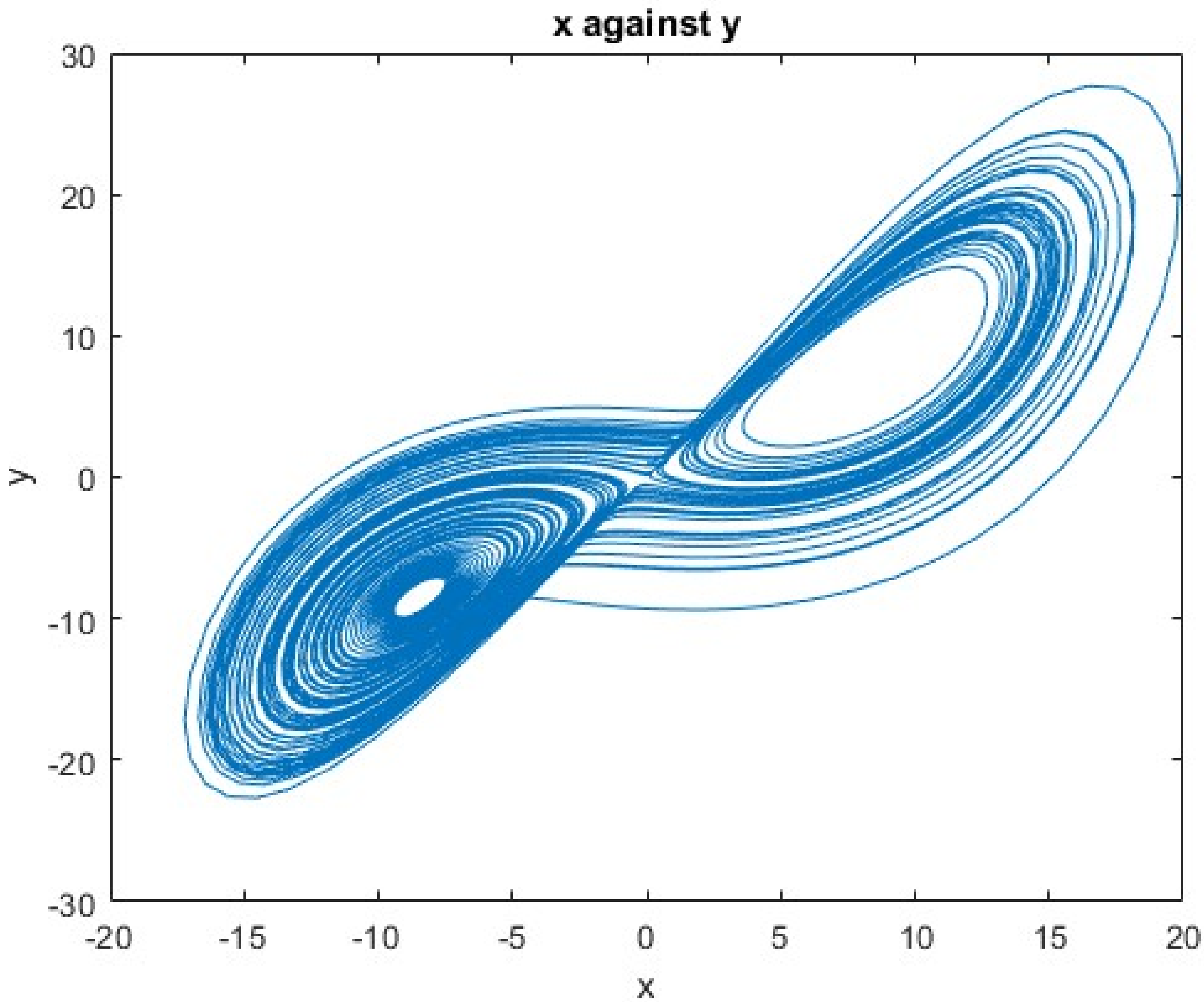


5.2 b

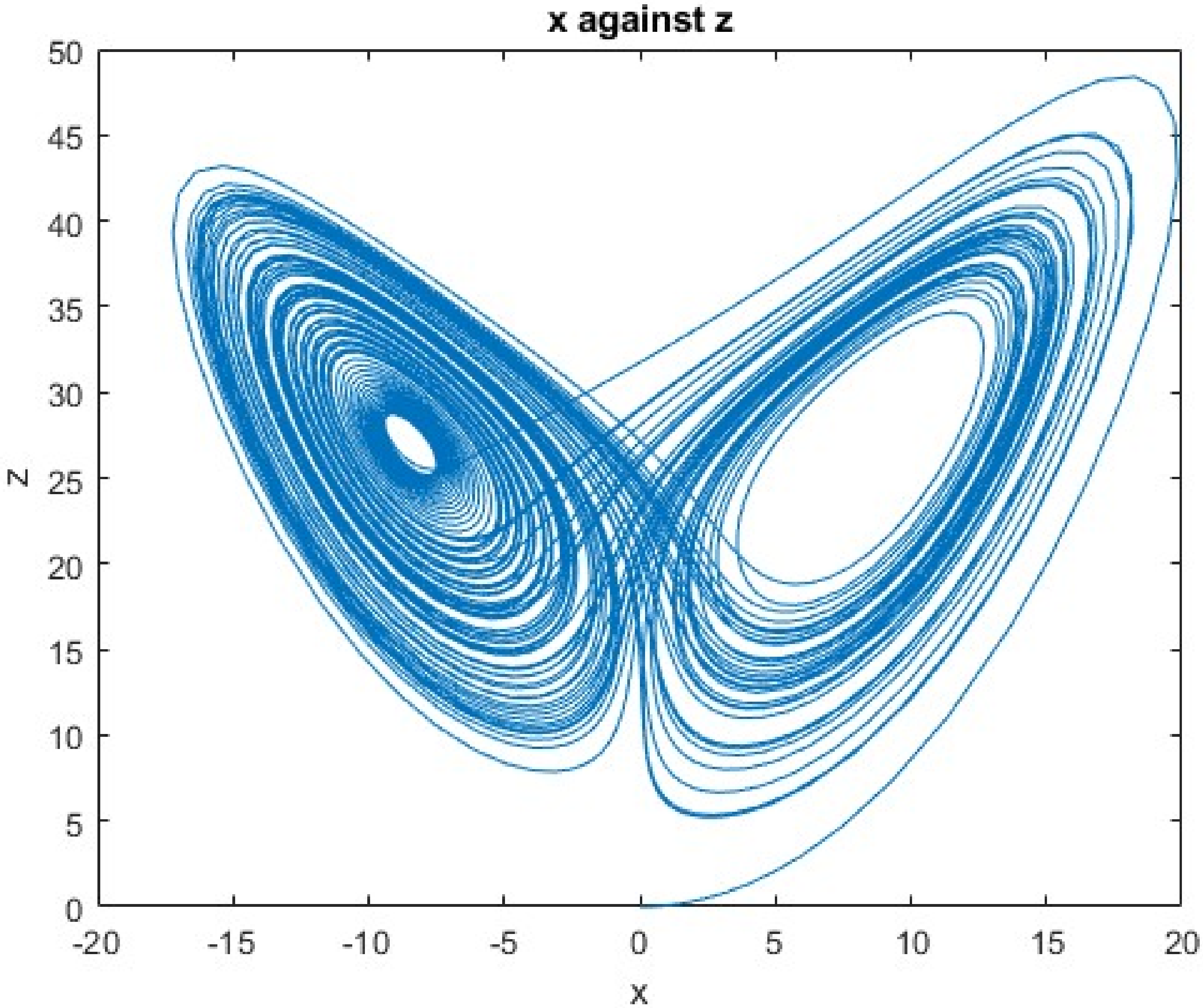


x(t)
20
0
-20
x
0
10
20
30
40
50
60
70
80
t
y(t)
20
0
-20
y
0
10
20
30
40
50
60
70
80
t
z(t)
50
0
z
0
10
20
30
40
50
60
70
80
t

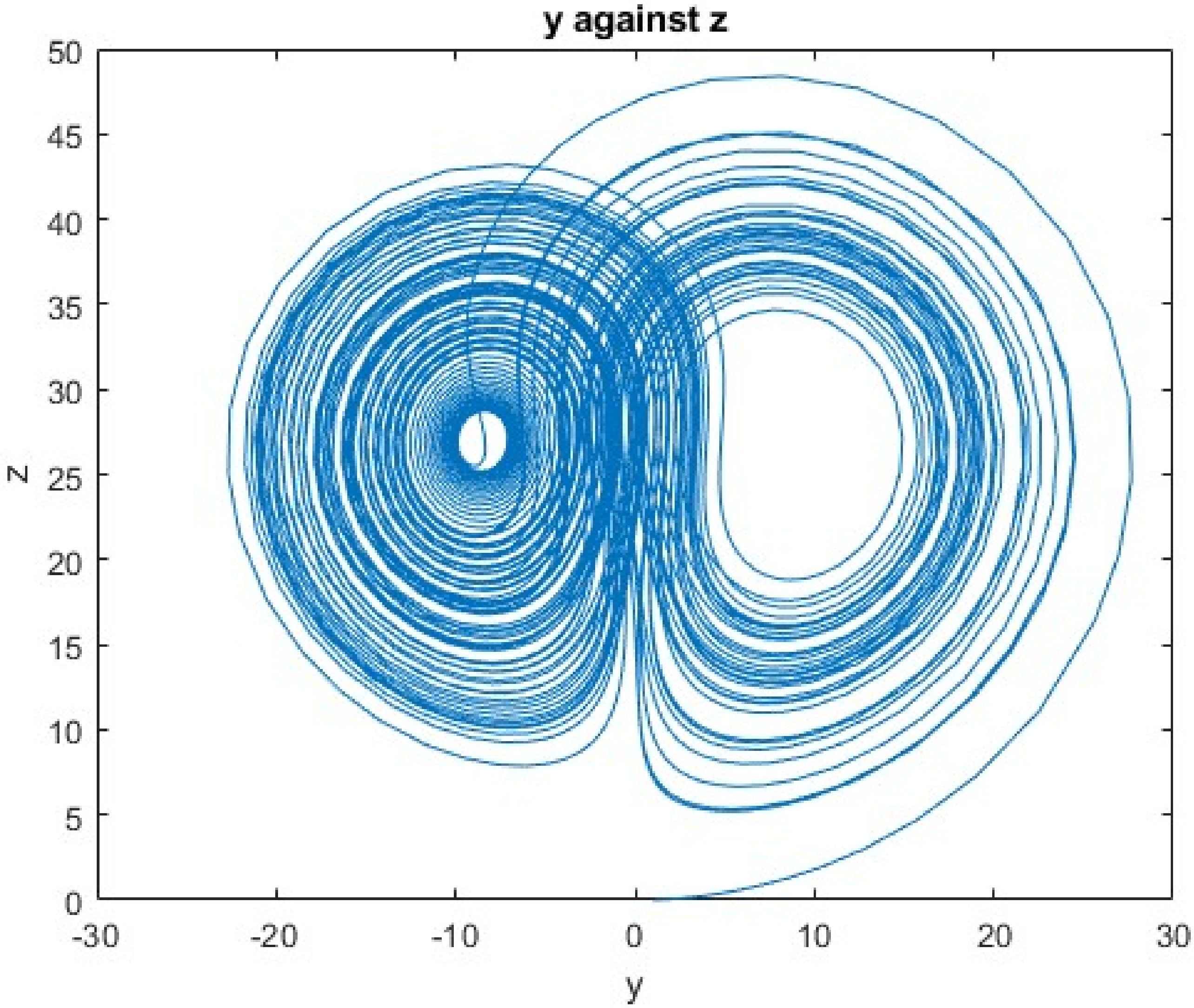
5.3 c



5.4 d



5.5 e



6 Solving the Airy Equation