HW 11

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August 2023

1 FDM for Elliptic Problems in 2D

Consider the Poisson equation and given boundary conditions on a uniform grid with $h = \frac{1}{5}$. Using central difference approximations for the second derivatives, we have:

$$\begin{aligned}
u_{xx}(x_i, y_j) &\approx \frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{h^2}, \\
u_{yy}(x_i, y_j) &\approx \frac{u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{h^2}.
\end{aligned}$$

Substituting into the Poisson equation, for the interior grid points, we get:

$$25u_{i+1,j} - 50u_{i,j} + 25u_{i-1,j} + 25u_{i,j+1} - 50u_{i,j} + 25u_{i,j-1} = 1. \tag{1}$$

Now we write down the system of linear equations for the unknowns $u_{i,j}$ considering the boundary conditions:

$$\begin{aligned}
25u_{2,1} - 100u_{1,1} + 25u_{1,2} &= 26, \\
25u_{3,1} - 100u_{2,1} + 25u_{1,1} + 25u_{2,2} &= 1, \\
25u_{4,1} - 100u_{3,1} + 25u_{2,1} + 25u_{3,2} + 5 &= 1, \\
25u_{3,2} - 100u_{2,2} + 25u_{1,2} + 25u_{2,3} + 25u_{2,1} &= 1, \\
25u_{4,2} - 100u_{3,2} + 25u_{2,2} + 25u_{3,3} + 10 &= 1, \\
25u_{4,3} - 100u_{3,3} + 25u_{2,3} + 25u_{3,4} + 15 &= 1, \\
25u_{3,3} - 100u_{2,3} + 25u_{1,3} + 25u_{2,4} + 25u_{2,2} &= 1, \\
25u_{4,3} - 100u_{3,3} + 25u_{2,3} + 25u_{3,4} + 20 &= 1, \\
25u_{4,4} - 100u_{3,4} + 25u_{2,4} + 20 &= 1, \\
25u_{3,4} - 100u_{2,4} + 25u_{1,4} + 25u_{2,3} + 15 &= 1.
\end{aligned}$$

Where we have used the boundary conditions:

$$\begin{aligned}
u(0, y) = 0, &\quad u(1, y) = y, \quad 0 \le x \le 1, \\
u(x, 0) = 0, &\quad u(x, 1) = x, \quad 0 \le y \le 1.
\end{aligned}$$

2 Heat equation

2.1 a

Consider the given heat equation problem with initial and boundary conditions:

$$\begin{aligned} u_t(t,x) &= 4u_{xx}(t,x) + 1, \quad 0 < x < 1, \quad t > 0, \\ u(0,x) &= 0, \quad 0 < x < 1, \\ u(t,0) &= 0, \quad u(t,1) = 0, \quad t \geq 0. \end{aligned}$$

We discretize in space and time with increments $h = \frac{1}{N}$ and Δt. Using the forward Euler method and central differences, we obtain the approximation:

$$u_i^{n+1} = u_i^n + \Delta t \left(4\frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{h^2} + 1 \right).$$

The initial and boundary conditions are:

$$\begin{aligned} u_i^0 &= 0, \quad i = 1, \ldots, N-1, \\ u_0^n &= u_N^n = 0, \quad n \geq 0. \end{aligned}$$

2.2 b

Consider the given heat equation problem with initial and boundary conditions:

$$\begin{aligned} u_t(t,x) &= 4u_{xx}(t,x) + 1, \quad 0 < x < 1, \quad t > 0, \\ u(0,x) &= 0, \quad 0 < x < 1, \\ u(t,0) &= 0, \quad u(t,1) = 0, \quad t \geq 0. \end{aligned}$$

Using the backward Euler method, the equation for the unknowns u_i^{n+1} at each interior point $i = 1, \ldots, N-1$ is:

$$-u_{i+1}^{n+1} + (4\lambda + 2)u_i^{n+1} - u_{i-1}^{n+1} = u_i^n + \Delta t,$$

where $\lambda = \frac{4\Delta t}{h^2}$.

The boundary conditions are:

$$u_0^{n+1} = u_N^{n+1} = 0.$$

In matrix form, this system can be written as:

$$\mathbf{A}\mathbf{u}^{n+1} = \mathbf{u}^n + \Delta t \mathbf{1},$$

where

$$\mathbf{A} = \begin{bmatrix} 4\lambda + 2 & -1 & 0 & \cdots & 0 \\ -1 & 4\lambda + 2 & -1 & \cdots & 0 \\ 0 & -1 & 4\lambda + 2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & -1 & 4\lambda + 2 \end{bmatrix}.$$

3 Laplace Equation in 2D

Consider the elliptic boundary value problem:

$$\begin{aligned} 2u_{xx} + u_{yy} &= 0, & 0 < x < 1,\ 0 < y < 1, \\ u(x,0) &= 0, & u(x,1) = 0, 0 < x < 1, \\ u(0,y) &= \sin(\pi y), & u(1,y) = 0, 0 < y < 1. \end{aligned}$$

Discretizing with $h = 0.25$, the finite difference approximation at the interior points is:

$$2\frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{h^2} + \frac{u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{h^2} = 0.$$

The system of linear equations for the unknowns is:

$$\begin{aligned} 8u_{1,1} - 2u_{2,1} - u_{1,2} &= \sin(\pi \cdot 0.25), \\ 8u_{2,1} - 2u_{1,1} - 2u_{3,1} - u_{2,2} &= 0, \\ 8u_{3,1} - 2u_{2,1} - u_{3,2} &= 0, \\ 8u_{1,2} - 2u_{2,2} - u_{1,3} - u_{1,1} &= 0, \\ 8u_{2,2} - 2u_{1,2} - 2u_{3,2} - u_{2,3} - u_{2,1} &= 0, \\ 8u_{3,2} - 2u_{2,2} - u_{3,3} - u_{3,1} &= 0, \\ 8u_{1,3} - 2u_{2,3} - u_{1,2} &= \sin(\pi \cdot 0.75), \\ 8u_{2,3} - 2u_{1,3} - 2u_{3,3} - u_{2,2} &= 0, \\ 8u_{3,3} - 2u_{2,3} - u_{3,2} &= 0. \end{aligned}$$

4 Explicit and implicit methods for heat equation

4.1 a

Consider the one-dimensional heat equation:

$$\begin{aligned} u_t &= 2u_{xx}, & 0 < x < 4,\ t > 0, \\ u(t,0) &= 0, & u(t,4) = 0, t > 0 \text{ (boundary conditions)}, \\ u(0,x) &= x(4-x), & 0 < x < 4 \text{ (initial condition)}. \end{aligned} \tag{2}$$

Let's discretize the domain using grid sizes Δx and Δt, and let u_i^k be the approximation to $u(t_k, x_i)$ at time $t_k = k\Delta t$ and location $x_i = i\Delta x$.

The forward-Euler method approximates the time derivative by the forward difference:

$$u_t \approx \frac{u_i^{k+1} - u_i^k}{\Delta t}. \tag{3}$$

And the second spatial derivative is approximated by the central difference:

$$u_{xx} \approx \frac{u_{i+1}^k - 2u_i^k + u_{i-1}^k}{(\Delta x)^2}. \tag{4}$$

Inserting these into the heat equation gives the forward-Euler update scheme:

$$\frac{u_i^{k+1} - u_i^k}{\Delta t} = 2\frac{u_{i+1}^k - 2u_i^k + u_{i-1}^k}{(\Delta x)^2}. \tag{5}$$

Rearranging, we have:

$$u_i^{k+1} = u_i^k + 2\frac{\Delta t}{(\Delta x)^2}(u_{i+1}^k - 2u_i^k + u_{i-1}^k). \tag{6}$$

The boundary conditions are implemented by setting:

$$u_0^k = u_N^k = 0 \quad \forall k, \tag{7}$$

and the initial condition by setting:

$$u_i^0 = i\Delta x(4 - i\Delta x) \quad \forall i. \tag{8}$$

4.2 b

Given the heat equation:

$$u_t = 2u_{xx}, \tag{9}$$

we use the forward-Euler method to discretize it:

$$u_i^{k+1} = u_i^k + 2\frac{\Delta t}{(\Delta x)^2}\left(u_{i+1}^k - 2u_i^k + u_{i-1}^k\right). \tag{10}$$

To analyze the stability, we assume a solution of the form:

$$u_i^k = \xi^k e^{ijm\Delta x}, \tag{11}$$

where ξ is the growth factor, and j is the imaginary unit.

Inserting this into the numerical scheme, we get:

$$\xi^{k+1}e^{ijm\Delta x} = \xi^k e^{ijm\Delta x} + 2\frac{\Delta t}{(\Delta x)^2}\left(\xi^k e^{ij(m+1)\Delta x} - 2\xi^k e^{ijm\Delta x} + \xi^k e^{ij(m-1)\Delta x}\right). \tag{12}$$

Dividing by $\xi^k e^{ijm\Delta x}$, we have:

$$\xi = 1 + 2\frac{\Delta t}{(\Delta x)^2}\left(e^{ij\Delta x} - 2 + e^{-ij\Delta x}\right). \tag{13}$$

Using Euler's formula:

$$\xi = 1 - 4\frac{\Delta t}{(\Delta x)^2}(1 - \cos(\Delta x)). \tag{14}$$

The method is stable if and only if $|\xi| \leq 1$. This leads to the stability condition (CFL condition):

$$\Delta t \leq \frac{(\Delta x)^2}{2}. \tag{15}$$

Hence, the forward-Euler method is conditionally stable for the given heat equation, and this CFL condition ensures stability.