**Guidelines and Rules:**

1) You can attempt this project in a group of three members. Please restrict your collaborations within your group. You are allowed to collaborate in problem formulation, approach and code development.
2) Each group will turn in one final report in a pdf format online at CANVAS. The report will contain problem formulation, solution methodology and discussion of results.
3) **Due Dates:**
   - Final Report and Matlab Code: Monday April 3rd, 2023.
4) You will get zero for discussion of results if your submitted Matlab code does not work.
5) Each report should clearly indicate the contribution of each group member in percentage. Only group members contributing 33% or more will get full earned points on the report.
6) Each group member should sign the honor statement indicating that **they have neither given nor received assistance on this project.**
7) The minimum penalty for the academic integrity violation will be ZERO in the Project. Most frequent examples of academic integrity violation includes but not limited to sharing your report or code with other group, writing the Matlab code for other group, coping the results figures and/or result discussion from other group students, etc.

The purpose of this assignment is to use the concept of rigid body dynamics and Lagrangian mechanics to model the motion of a quadcopter. A quadcopter is a flying vehicle with four equally spaced rotors, usually attached at the corner of a square frame (cf. Fig. 1). With six degrees of freedom (like conventional fixed wing airplane) and only four independent motors, quadcopter is severely under-actuated. To understand their motion, let us consider

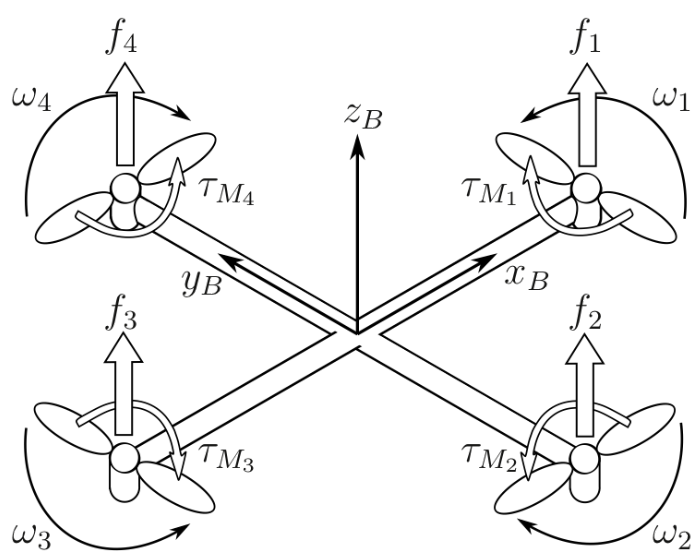


Fig. 1. A Schematic of a Quadcopter

two frames to analyze the motion of a quadcopter. The inertial frame is assumed to be attached to the ground with gravity pointing in the negative $z$-direction. The body frame is attached to the quadcopter with the rotor axis pointed in the positive body $z$-direction, denoted by $z_B$ and the arms pointing in the $x_B$ and $y_B$ direction as shown in Fig. 1. If we define the position of the quadcopter in the inertial frame as $\mathbf{r} = \{x, y, z\}^T$, then the Lagrangian formulation can be used to derive the following translational and rotational equations of motion:

$$\ddot{\mathbf{r}} = -g \left\{ \begin{array}{c} 0 \\ 0 \\ 1 \end{array} \right\} + \frac{T}{m} \left\{ \begin{array}{c} \cos\psi \sin\theta \cos\phi + \sin\psi \sin\phi \\ \sin\psi \sin\theta \cos\phi - \cos\psi \sin\phi \\ \cos\theta \cos\phi \end{array} \right\} \tag{1}$$

$$\left\{ \begin{array}{c} p \\ q \\ r \end{array} \right\} = A \left\{ \begin{array}{c} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{array} \right\}, \quad A = \begin{bmatrix} 1 & 0 & -\sin\theta \\ 0 & \cos\phi & \cos\theta \sin\phi \\ 0 & -\sin\phi & \cos\theta \cos\phi \end{bmatrix} \tag{2}$$

$$I\dot{\omega} = -\omega \times (I\omega) + \tau \tag{3}$$

where $(\psi, \theta, \phi)$ denote the $3-2-1$ Euler angle sequence and $\omega = (p, q, r)^T$ denotes the body angular rates of the quadcopter. Notice that the moment of inertia matrix, $I$, is a diagonal matrix due to the symmetric geometric shape of the quadcopter. $T$ is the combined forces of rotors in the direction of body $z$-axis, i.e., $z_B$:

$$T = \sum_{i=1}^{4} T_i = k \sum_{i=1}^{4} \Omega_i^2 \tag{4}$$

$\tau$ consists of roll, pitch and yaw moments. Notice that the roll moment is generated by decreasing the second rotor angular velocity and increasing the fourth rotor angular velocity. Similarly, the pitch moment is generated by decreasing the first rotor angular velocity and increasing the third rotor angular velocity. The yaw moment is generated by increasing the angular velocities of two opposite rotors and decreasing the angular velocities of the other two rotors.

$$\tau = \left\{ \begin{array}{c} L \\ M \\ N \end{array} \right\} = \left\{ \begin{array}{c} kl(-\Omega_2^2 + \Omega_4^2) \\ kl(-\Omega_1^2 + \Omega_3^2) \\ b(\Omega_1^2 - \Omega_2^2 + \Omega_3^2 - \Omega_4^2) \end{array} \right\} \tag{5}$$

where, $l$ is the distance between the rotor and center of mass of the quadcopter. You are required to complete following tasks to get more insights into quadcopter motion:

1) Let us consider the following value of parameters to simulate the quadcopter motion in MATLAB while using the in-built command “ODE45” to solve the quadcopter equations of motion:

$$g = 9.81 m/s^2, \; m = 0.450 kg, \; l = 0.225 m, \; k = 2.98 \times 10^{-7}, \; b = 1.14 \times 10^{-6} \tag{6}$$

$$I_{xx} = I_{yy} = 4.85 \times 10^{-3} kg - m^2, \; I_{zz} = 8.80 \times 10^{-3} kg - m^2 \tag{7}$$

The initial conditions corresponds to the quadcopter being at rest at the inertial frame origin with body frame aligned with the inertial frame, i.e.

$$\mathbf{r}(0) = \{0, 0, 0\}^T, \; \dot{\mathbf{r}}(0) = \{0, 0, 0\}^T, \; \psi(0) = \theta(0) = \phi(0) = 0, \; p(0) = q(0) = r(0) = 0$$

The control input, i.e., angular velocities of the four rotor are computed as follows:

a) For first one second, the quadcopter is ascended by increasing all of the rotor velocities from the hover thrust. Then, the ascend is stopped by decreasing the rotor velocities for the following one second.

$$\Omega_i = \Omega_{i_{hover}} + 70 \sin(2\pi t/4), \quad t < 1, \quad i = 1, 2, 3, 4 \tag{8}$$

$$\Omega_i = \Omega_{i_{hover}} - 77 \sin(2\pi t/4), \quad 1 \leq t \leq 2 \tag{9}$$

The $\Omega_{i_{hover}}$ is equal for all four rotors and corresponds to thrust produced being to the weight of the quadcopter, i.e.

$$T = \sum_{i=1}^{4} T_i = 4k\Omega_{i_{hover}}^2 = mg$$

The quadcopter should be hovering at an altitude after this thrust profile.

b) For the next one second, i.e., 2-3 second period, the quadcopter is put into a roll motion by increasing the velocity of the fourth rotor and decreasing the velocity of the second rotor. In the next one second, the roll rate is made zero by the increasing the velocity of the second rotor and decreasing the velocity of the fourth rotor.

$$\Omega_2^2 = \Omega_{2_{hover}}^2 - 70^2 \sin(2\pi(t-2)/4), \quad \Omega_4^2 = \Omega_{4_{hover}}^2 + 70^2 \sin(2\pi(t-2)/4), \quad 2 < t < 3 \tag{10}$$

$$\Omega_2^2 = \Omega_{2_{hover}}^2 + 70^2 \sin(2\pi(t-2)/4), \quad \Omega_4^2 = \Omega_{4_{hover}}^2 - 70^2 \sin(2\pi(t-2)/4), \quad 3 \le t < 4 \tag{11}$$

At the end of the maneuver, the quadcopter flying at constant roll angle. Comment on the quadcopter translational and rotational motion.

c) For the next one second, the quadcopter is put into a pitch motion by increasing the velocity of the third rotor and decreasing the velocity of the first rotor. In the next one second, the pitch rate is stopped by the opposite maneuver.

$$\Omega_1^2 = \Omega_{1_{hover}}^2 - 70^2 \sin(2\pi(t-4)/4), \quad \Omega_3^2 = \Omega_{3_{hover}}^2 + 70^2 \sin(2\pi(t-4)/4), \quad 4 \le t < 5 \tag{12}$$

$$\Omega_1^2 = \Omega_{1_{hover}}^2 + 70^2 \sin(2\pi(t-4)/4), \quad \Omega_3^2 = \Omega_{3_{hover}}^2 - 70^2 \sin(2\pi(t-4)/4), \quad 5 \le t \le 6 \tag{13}$$

Comment on what happened to quadcopter translational and rotational motion.

Plot quadcopter position in an inertial frame, velocity in body frame, Euler angles and angular velocity $(\omega)$ versus time for the whole duration of the simulation.

2) Let us consider the following auto pilot for the quadcopter:

$$T = (g + (\dot{z}_r - \dot{z}) + (z_r - z)) \frac{m}{\cos\phi\cos\theta} \tag{14}$$

$$L = I_{xx}\left((\dot{\phi}_r - \dot{\phi}) + (\phi_r - \phi)\right) \tag{15}$$

$$M = I_{yy}\left((\dot{\theta}_r - \dot{\theta}) + (\theta_r - \theta)\right) \tag{16}$$

$$N = I_{zz}\left((\dot{\psi}_r - \dot{\psi}) + (\psi_r - \psi)\right) \tag{17}$$

where subscript 'r' represents the reference signal for the variable. Show that rotor angular velocities can be computed as follows for the aforementioned auto pilot:

$$\Omega_1^2 = \frac{T}{4k} - \frac{M}{2kl} + \frac{N}{4b} \tag{18}$$

$$\Omega_2^2 = \frac{T}{4k} - \frac{L}{2kl} - \frac{N}{4b} \tag{19}$$

$$\Omega_3^2 = \frac{T}{4k} + \frac{M}{2kl} + \frac{N}{4b} \tag{20}$$

$$\Omega_4^2 = \frac{T}{4k} + \frac{L}{2kl} - \frac{N}{4b} \tag{21}$$

Simulate the motion of the quadcopter for 120 seconds with following initial conditions:

$$x = y = \dot{x} = \dot{y} = \dot{z} = p = q = r = 0, \; z = 1m, \; \theta = \phi = \psi = 10^\circ$$

The desired or reference position of the altitude is $z_r = 10m$. All other reference variables are zero. Once again, plot the quadcopter position in an inertial frame, velocity in body frame, Euler angles, angular velocity $(\omega)$ and rotor angular velocities versus time for the whole duration of the simulation. Comment on translational and rotational motion of the quadcopter. Discuss that what changes are required in the aforementioned auto-pilot (if any) to make the quadcopter hover at an altitude of 10m at a given $x$ and $y$ location.