AERSP 301 Aerospace Structures Extra Credit Project: Beam Bending

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This experiment was set up to compare experimental data with different analytical methods of finding deformations of a bent beam. Three methods were used to determine the deformations at the end of a cantilever beam at different loads, which included Euler-Bernoulli Beam Bending equations, Finite Element Analysis, and SolidWorks simulations. We applied 3 different loads at the endpoint of the beam. Comparing data revealed that the experimental data had more deformation compared to other analytical methods. The data also revealed that SolidWorks had the largest deformations compared to experimental data.

I. Nomenclature

σ = stress
M_x or u" = moment in the x direction
M_y or v" = moment in the y direction
I_{xx} = second moment of area with respect to the x-axis
I_{yy} = second moment of area with respect to the y-axis
I_{xy} = product of inertia
w_y = distributed load in the y direction
w_x = distributed load in the x direction
S_y = force in the y direction
S_x = force in the x direction
z = direction parallel to the length of the beam
u' = angle of beam in the x direction
v' = angle of beam in the y direction
u = displacement in the x direction
v = displacement in the y direction
C = constant of integration
F = force vector
K_s = stiffness matrix
q = displacement and angle vector
w = displacement for a beam
θ = angle
P = load

II. Introduction

This experiment was conducted with the purpose of testing the accuracy of beam-bending principles and simulation methods when compared to experimental data. The Aerospace Structures course teaches several different methods of predicting deformation of a beam that is subjected to an external force. The first method was Euler-Bernoulli Method (EBM). This method considers direct stress at a point and models a deflection relationship by considering load, load placement, and geometry of the beam. Euler-Bernoulli assumes that plane sections perpendicular to the mid-plane remain plane and perpendicular to the beam axis after deformation. It also assumes that the object under analysis is a long, slender beam. The second method covered was Finite Element Analysis (FEA). This method breaks a beam up into an N number of sections for analysis. This method considers energy methods which are strain energy and external work, which we use to model displacement and angle of a beam's Nth element. The third method in this experiment was the SolidWorks computer software. SolidWorks uses FEA methods to simulate the behavior of objects and assemblies, considering loads applied. With these three different methods of collecting deflection data, this experiment was set up to analyze the difference in data collected from all these methods and compare it to experimental data. This will also show the accuracy of these methods.

III. Objectives

The objectives of this experiment were:

- Determine the deflection of a beam under loading using an experimental set-up
- Determine the deflection of the same beam under corresponding loading using the Euler-Bernoulli Method
- Determine the deflection of the same beam under corresponding loading using Finite Element Analysis
- Determine the deflection of the same beam under corresponding loading using SolidWorks' Simulation Software
- Analyze the differences in data between these four methods of data collection
- Determine the accuracy of the data

IV. Experimental Set Up

This experiment used the following materials:

- A 3ft long weldable steel beam
- A clamp
- A meter length ruler
- A table
- Loads (water bottle, racket & hammer)
- String
- Weighing Scale
- Duct Tape

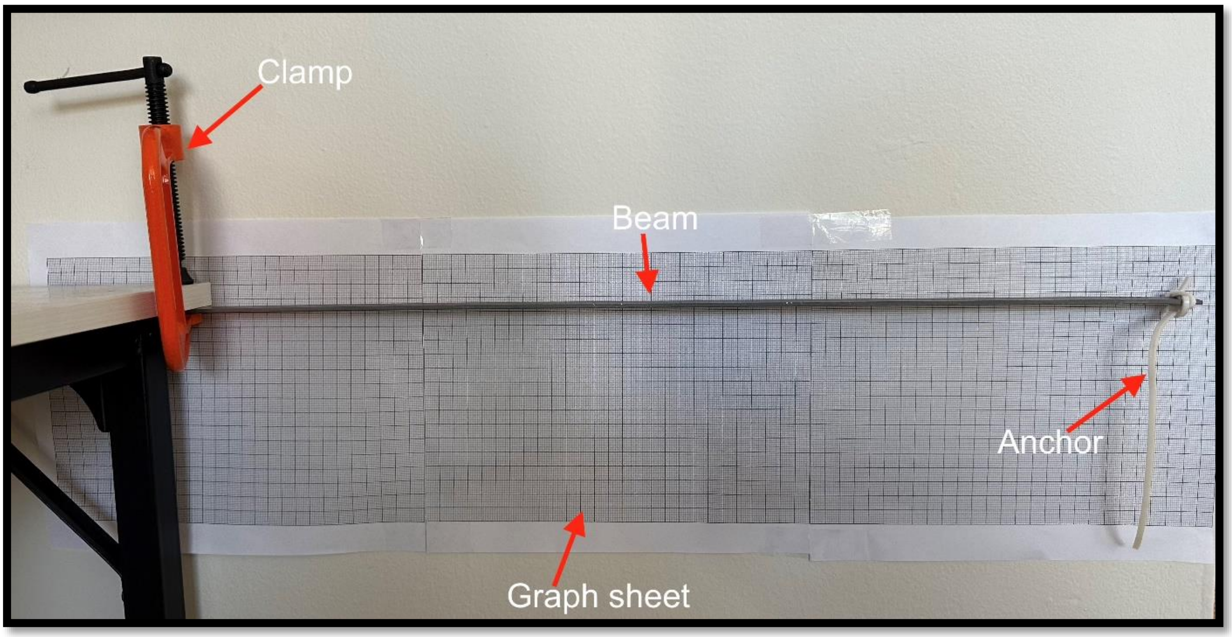

Fig.1 Experimental Setup

A 3ft long beam was used to do beam bending analysis. This beam has a thickness of 0.125 inches, width of 0.5 inches and is 3 ft long. For the experiment we clamped it 1 ft to the table so only 2 ft of beam was bending during the experiment. We add a string hook (Anchor) at the end of the beam to load the beam with different loads. We added a graph paper to the background of the beam to collect data. We loaded the beam with 3 different loads and each load five times to get standard deviation. This data was then inserted into Excel to plot a graph.

V. Modeling

Euler Bernoulli Method (EBM)

Euler-Bernoulli Method considers direct stress at a point and models a deflection relationship due to pure bending. Euler-Bernoulli also assumes that the plane section perpendicular to the mid-plane remains plane and perpendicular to the beam axis after deformation. It also assumes that the object under analysis is a long, slender beam. The direct stress of a beam in pure bending can be calculated using

$$\sigma = \frac{M_y I_{xx} - M_x I_{xy}}{I_{xx} I_{yy} - I_{xy}^2} x + \frac{M_x I_{yy} - M_y I_{xy}}{I_{xx} I_{yy} - I_{xy}^2} x \tag{1}$$

A relationship between distributed loads, point loads, and moments can also be made through

$$w_y = \frac{-\partial S_y}{\partial z} = \frac{-\partial^2 M_x}{\partial z^2} \tag{2}$$

and

$$w_x = \frac{-\partial S_x}{\partial z} = \frac{-\partial^2 M_y}{\partial z^2} \tag{3}$$

Thus, a relationship can be made that

$$\begin{pmatrix} u" \\ v" \end{pmatrix} = \frac{-1}{E(I_{xx} I_{yy} - I_{xy}^2)} \begin{bmatrix} -I_{xy} & I_{xx} \\ I_{yy} & -I_{xy} \end{bmatrix} \begin{pmatrix} M_x \\ M_y \end{pmatrix} \tag{4}$$

The equations for the secondary moment of areas for a symmetric, square cross-section are

$$I_{xx} = \frac{bh^3}{12} \tag{5}$$

$$I_{yy} = \frac{hb^3}{12} \tag{6}$$

$$I_{xy} = 0 \tag{7}$$

Fig.2 Cross Section of the Beam

As per figure 2, the beam has a rectangular cross-section with a thickness of 0.125 in and a length of 2 ft. Thus, the secondary moment of areas is calculated to be

$$I_{xx} = 0.5 * \frac{0.125^3}{12} = 0.000081\ in^4 = 3.371475 * 10^{-11}\ m^4$$

$$I_{yy} = \frac{0.125 * 0.5^3}{12} = 0.001302\ in^4 = \ 5.41933 * 10^{-10}\ m^4$$

$$M_x = P(l - z) \tag{8}$$

Combing the Equation (2), (8) and (3)

$$v" = (-M_x)/(E * I_{xx}) = (-P(l - z))/(E * I_{xx})$$

$$v' = \frac{P}{E * I_{xx}} * \left(\frac{z^2}{2} - (l * z)\right) + C_1$$

$$v(z) = \frac{P}{E*I_{xx}} * \left(\frac{z^3}{6} - \left(\frac{L*z^2}{2}\right)\right) + (C_1 * z) + C_2$$

But with the boundary conditions

$$v(0) = 0$$

$$v'(0) = 0$$

then

$$C_1 = 0$$

$$C_2 = 0$$

$$v(L) = \frac{P}{E * I_{xx}} * \left(\frac{L^3}{6} - \frac{L^3}{2}\right) = \frac{-P}{E * I_{xx}} * \frac{L^3}{3}\ \ m$$

$$L = 0.6096\ m$$

$$E = 190 * 10^9\ Pa$$

Table 1 – Euler Bernoulli Method	
Load (P)	Displacement (cm)
4.2 N	4.9 cm
5.9 N	6.9 cm
9.1 N	10.7 cm

Finite Elements Analysis (FEA)

Finite Element Analysis (FEA) breaks a beam up into an N number of sections for analysis. This theory considers energy methods, both strain energy and external work, to model displacement and angle of a beam's Nth element. This method allows for a more variety of models, which can include springs, rollers, and torsional springs. FEA uses an N number of elements to determine angle and displacement of the beam. It does this by assigning an element a displacement and angle at each end of an element of length *l*. The equation to find the angles and deformation is

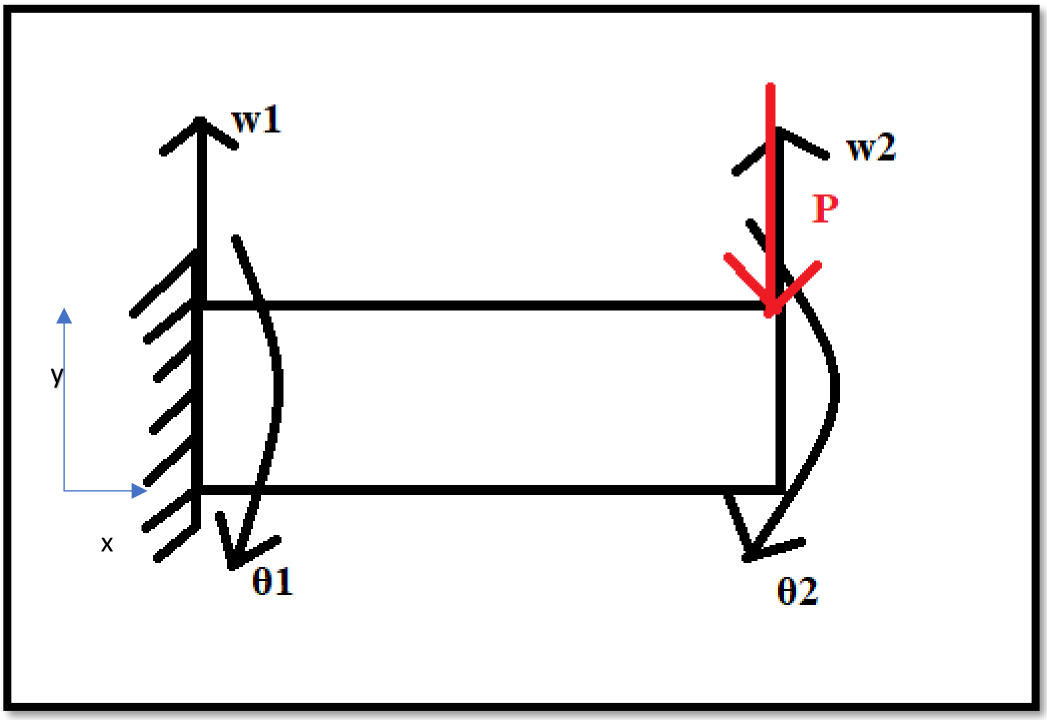

Fig.3 FEA Model for a Cantilever Beam

$$\vec{F} = [K_s]\vec{q}$$

Where F is the force vector, K_s is the stiffness vector and q is the displacement/angle vector. When under beam bending, with the force in the y direction, the stiffness matrix of each element is

$$K_s = \frac{2EI_{xx}}{l^3}\begin{bmatrix} 6 & -3l & -6 & -31 \\ -3l & 2l^2 & 3l & l^2 \\ -6 & 3l & 6 & 3l \\ -3l & l^2 & 3l & 2l^2 \end{bmatrix}$$

The experimental beam's dimensions matrix, using two elements, is

$$\begin{bmatrix} w_1 \\ \theta_1 \\ w_2 \\ \theta_2 \end{bmatrix}$$

With a corresponding force vector of

$$F = \begin{bmatrix} 0 \\ 0 \\ P \\ 0 \end{bmatrix}$$

Thus, using the boundary conditions

$$w_1 = \theta_1 = 0$$

Constants

$$l = 0.6096\ m$$

$$I_{xx} = 0.5 * \frac{0.125^3}{12} = 0.000081\ in^4 = 3.371475 * 10^{-11}\ m^4$$

$$E = 190 * 10^9\ Pa$$

$$Reduced\ K_s = \frac{2 * E * I_{xx}}{l^3} * \begin{bmatrix} 6 & 3l \\ 3l & 2l^2 \end{bmatrix}$$

$$(Reduced\ K_s)^{-1} = \frac{l^3}{2 * E * I_{xx}} * \begin{bmatrix} \frac{2}{3} & -\frac{1}{l} \\ -\frac{1}{l} & \frac{2}{l^2} \end{bmatrix}$$

$$\begin{bmatrix} w_2 \\ \theta_2 \end{bmatrix} = \frac{l^3}{2 * E * I_{xx}} * \begin{bmatrix} \frac{2}{3} & -\frac{1}{l} \\ -\frac{1}{l} & \frac{2}{l^2} \end{bmatrix} * \begin{bmatrix} P \\ 0 \end{bmatrix}$$

And with w_3 being the deformation at the end of the beam, then the deformation is

$$w_2 = \frac{l^3 * 2 * P}{2 * E * I_{xx} * 3}\ \text{m}$$

Table 2 – FEA	
Load (P)	Displacement (cm)
4.2 N	4.95 cm
5.9 N	6.95 cm
9.1 N	10.73 cm

MATLAB was used to calculate displacement for each load. We could use a code such that it would run FEA through a loop that calculated displacements for an N number of elements, but for now N was 1. We can also assume that the displacement per number of elements converged right away. Thus, one element was used to run the FEA numbers, out of personal preference.

SolidWorks

SolidWorks uses FEA methods to simulate the behavior of objects and assemblies, considering loads applied. The beam under analysis was recreated in SolidWorks using exact dimensions. A simulation was set up, under the Simulations tab, to recreate the boundary conditions and the forces. The beam used in the experiment is an alloy and the information about this type of material it unknown. However, the beam is a steel variant called weldable steel. In addition, SolidWorks does not have an exact material type for this steel. Thus, Cast Alloy Steel was used for the SolidWorks simulation. The material properties of this steel are very similar to weldable steel. In addition, Young's Modulus is the only material property used in both Euler-Bernoulli and in FEA. It can be assumed that using this in the simulation will not deem large differences in results. A fixed boundary condition was applied to one end of the beam using the Fixtures tab. A point load was applied to the opposite end using the External Loads tab. The simulation was then run. Displacement of the entire beam can be visualized under the Displacement tab, but only the displacement at the loaded end was recorded. In the simulation, this displacement is visualized as the deepest red color.

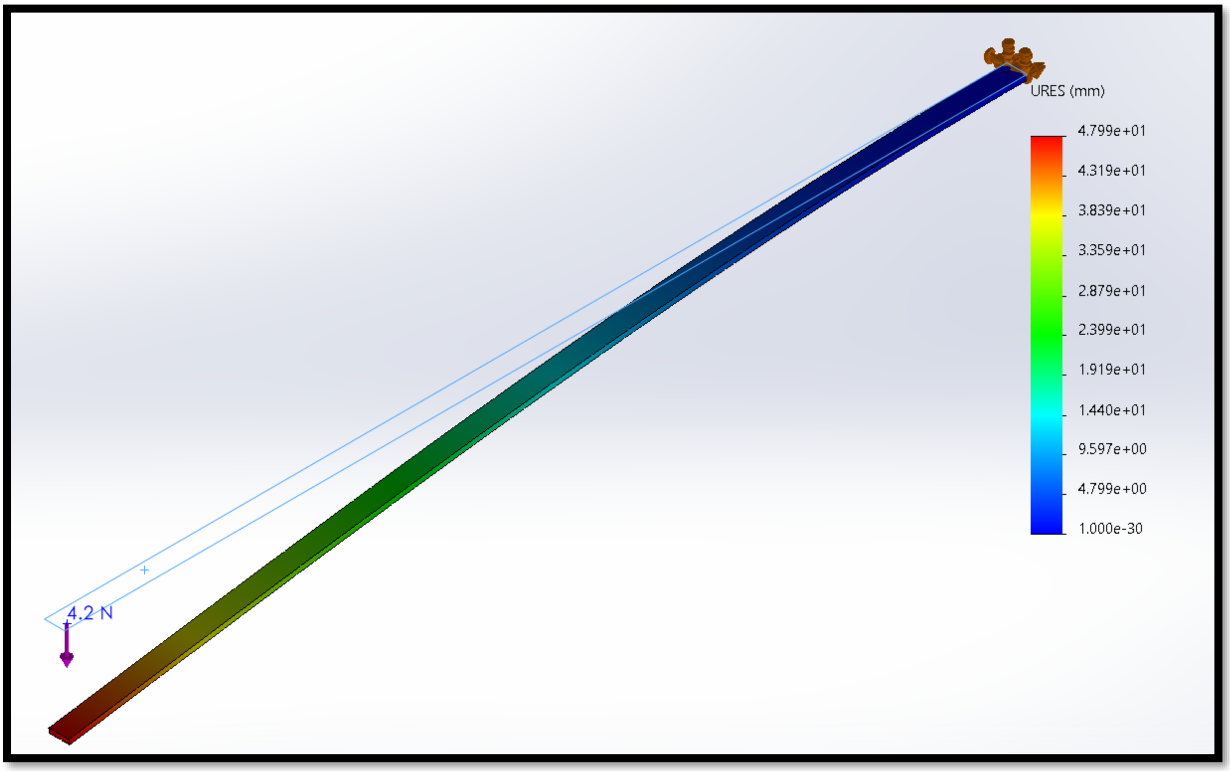

Fig. 5 SolidWorks Simulation for 4.2 N load

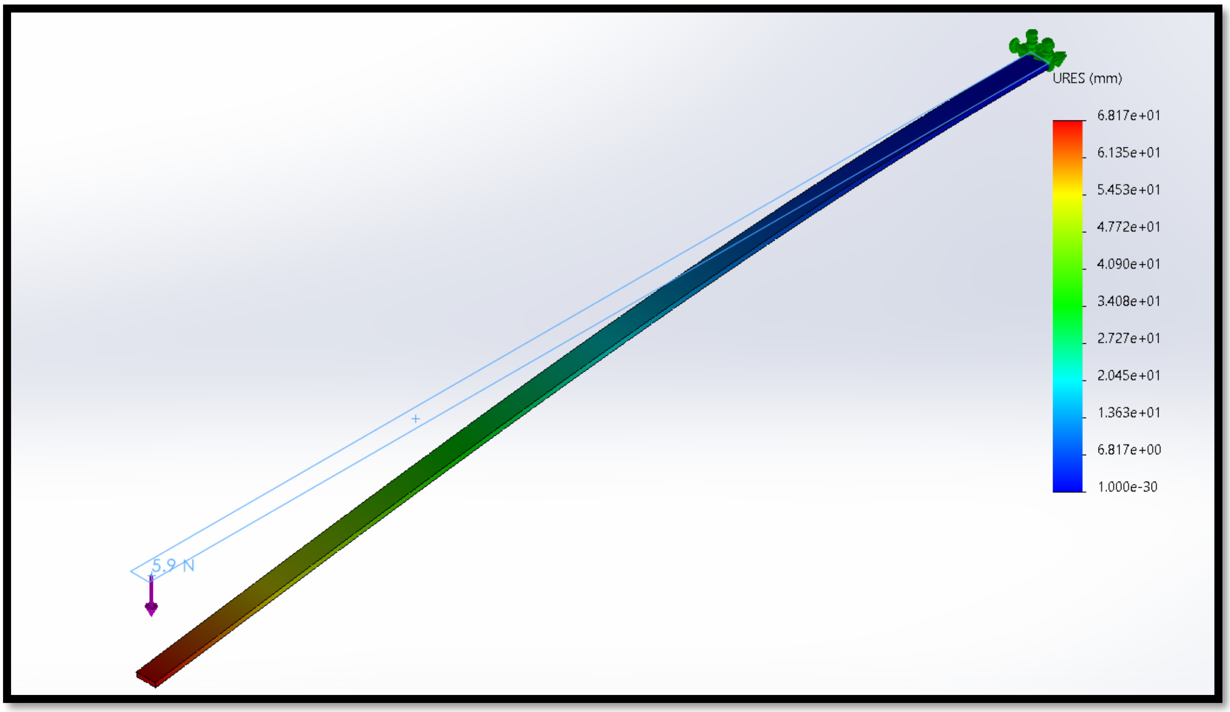

Fig. 4 SolidWorks Simulation for 5.9N load

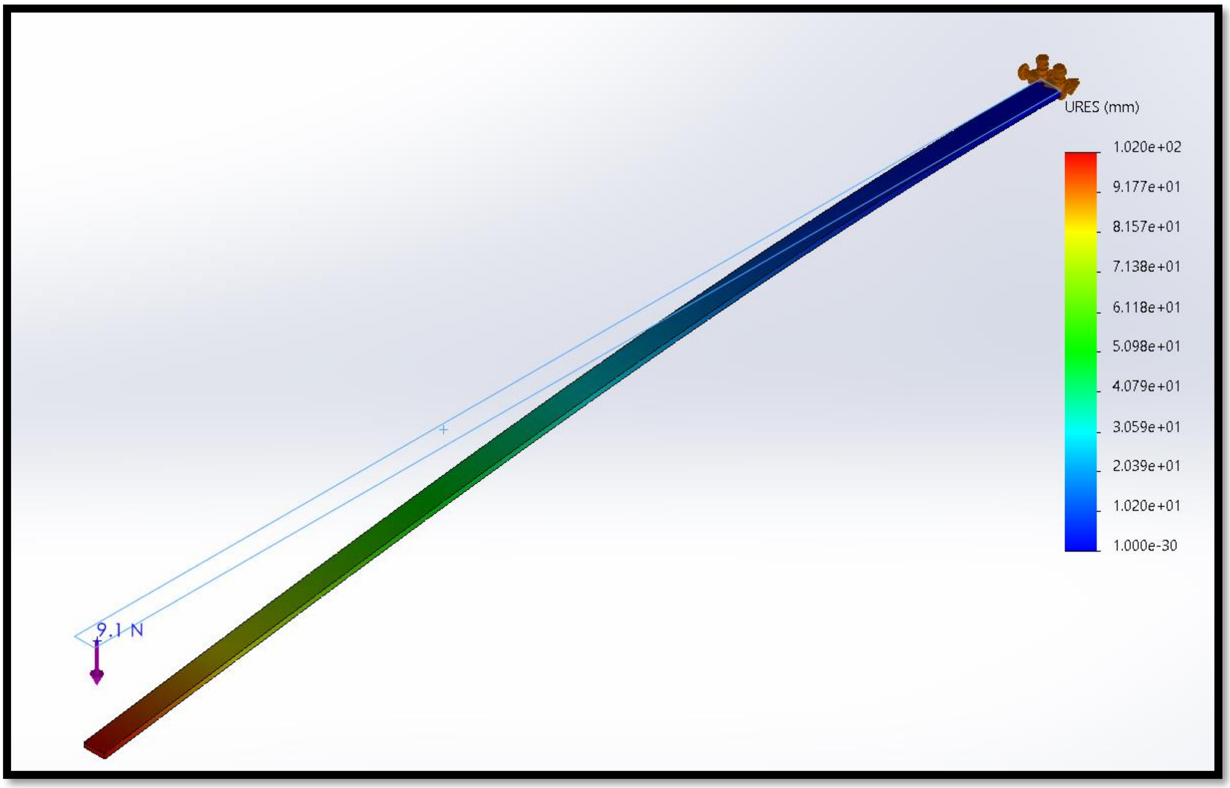

Fig. 6 SolidWorks Simulation for 9.1 N load

Table 3 – SolidWorks	
Load (P)	Displacement (cm)
4.2 N	4.799 cm
5.9 N	6.817cm
9.1 N	10.2 cm

VI. Results

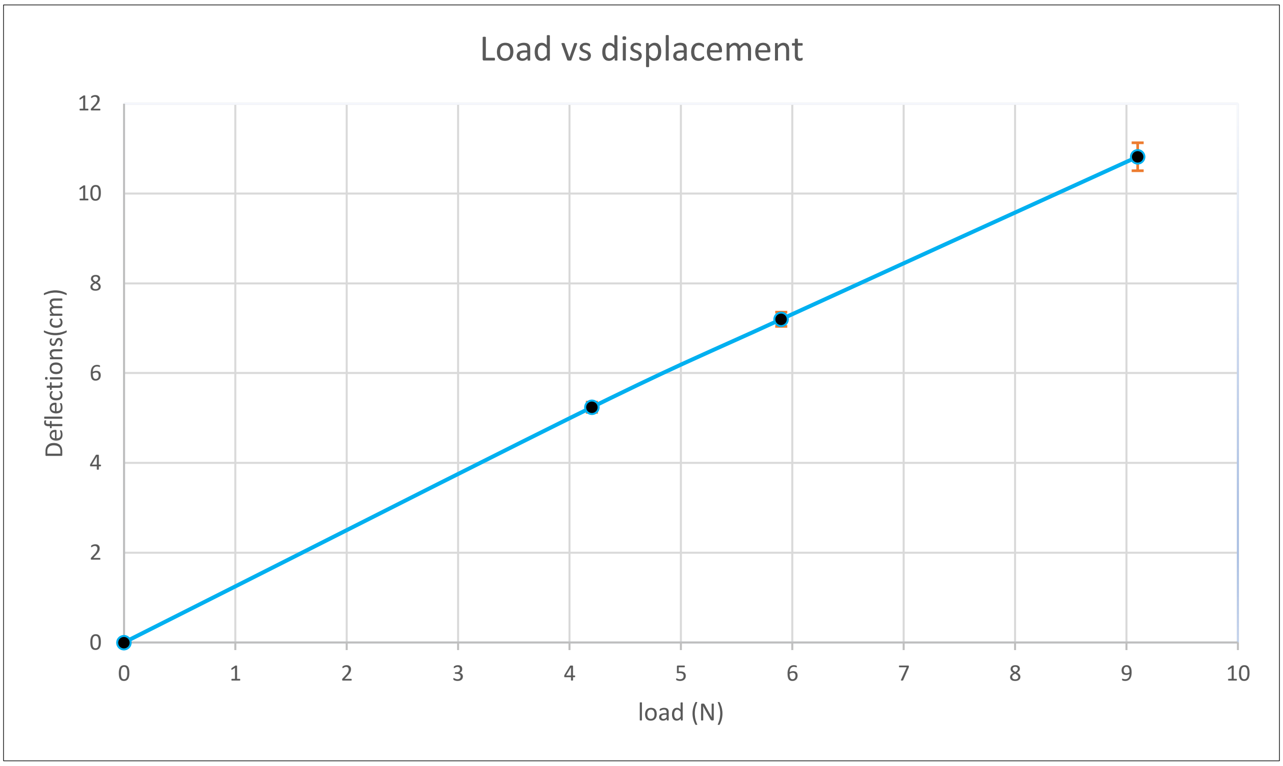

Fig.7 Force vs Displacement for Experimental Data

Table 4 Experimental Data and Analytical Results

P, Loads (N)	Experimental (cm)	EBM (cm)	FEA (cm)	SolidWorks (cm)
4.2 N	5.24 cm	4.9 cm	4.95 cm	4.799 cm
5.9 N	7.2 cm	6.9 cm	6.95 cm	6.817 cm
9.1 N	10.82 cm	10.7 cm	10.73 cm	10.2 cm

Table 5 Percent Error Compared to Experiment

P, Loads (N)	Experimental	EBM	FEA	SolidWorks
4.2 N	0%	6.5%	5.53%	8.4%
5.9 N	0%	4.2%	3.47%	5.32%
9.1 N	0%	1.11%	0.832%	5.73%

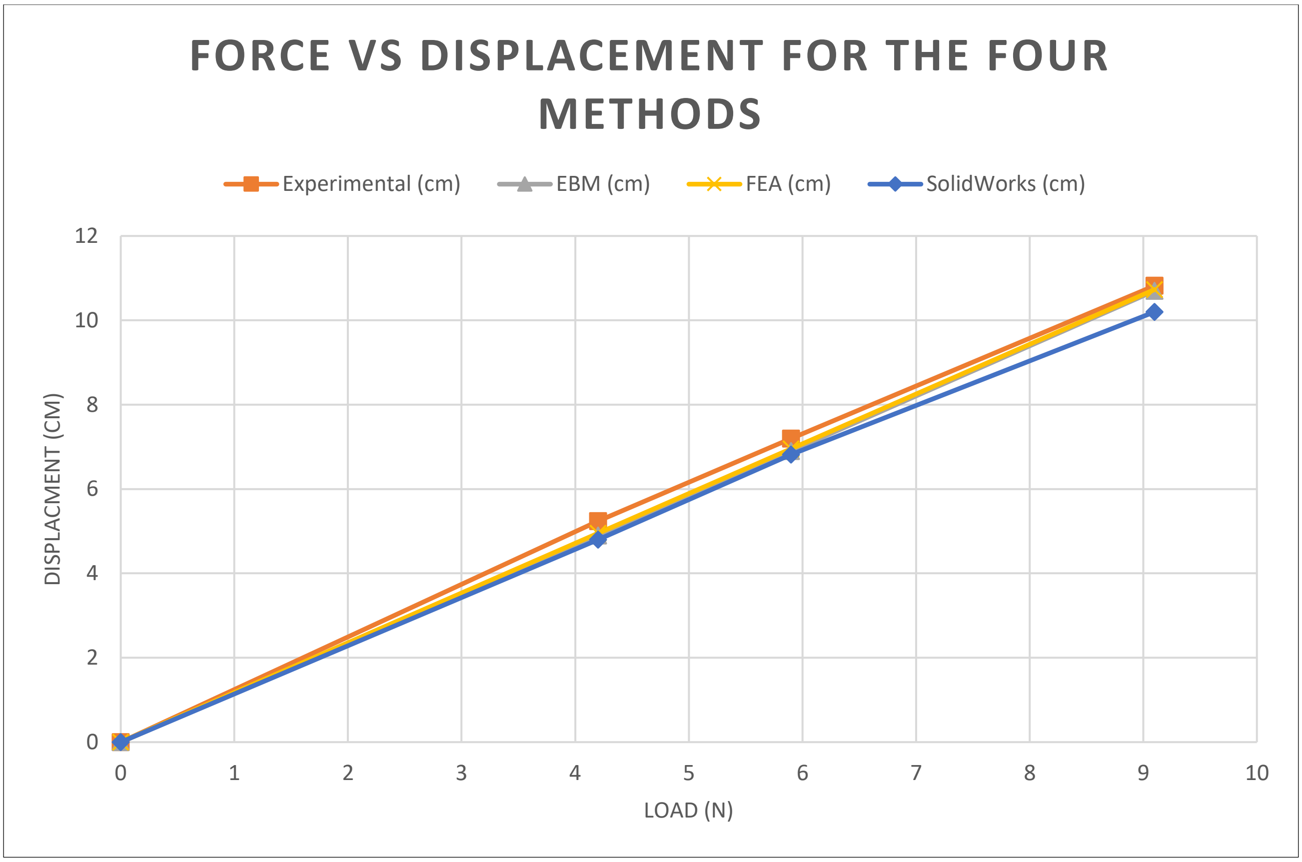

Fig. 8 Force vs Displacement for the Four Methods

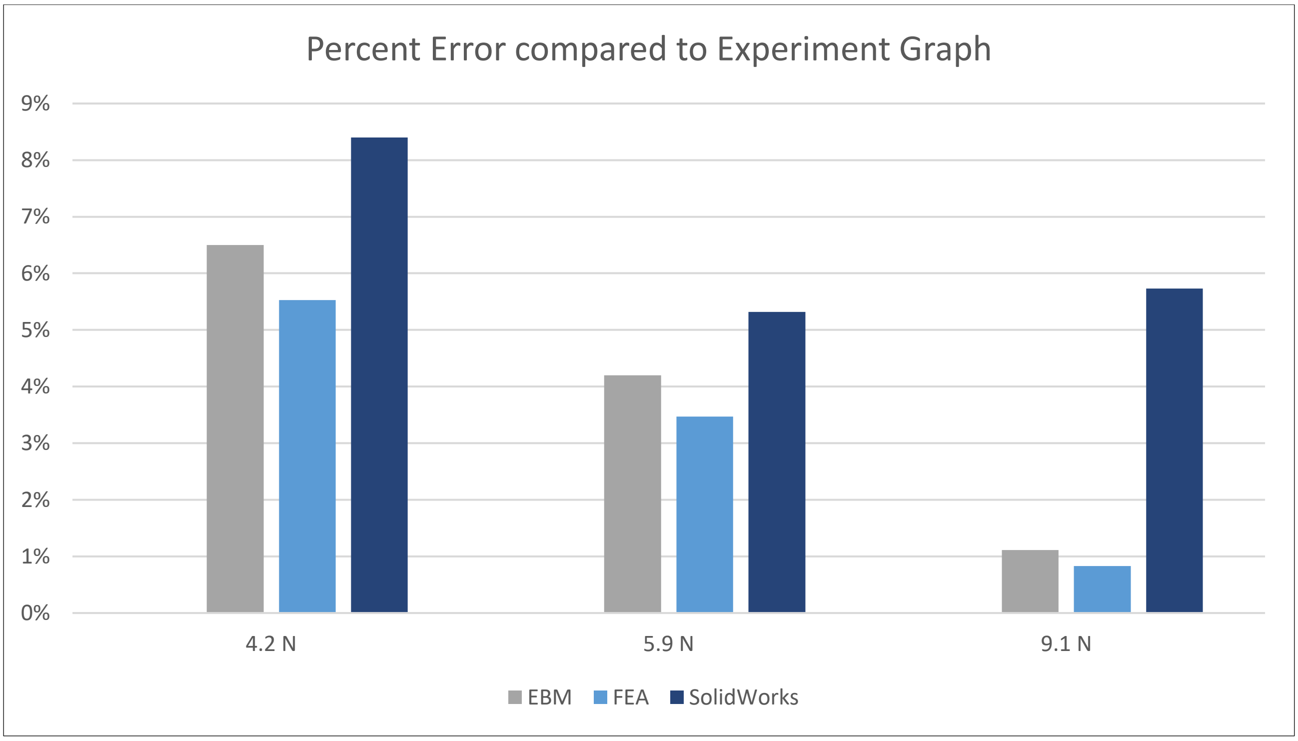

Fig.9 Percent Error compared to Experiment graph

VII. Conclusion

As shown in Table 4 Euler-Bernoulli Method (EBM) and FEA both produced calculations that were very close to the experimental data. SolidWorks also produced results that were similar to the Experimental data, however, they were off by a great percentage as shown in Table 5. From this it can be stated that the SolidWorks simulation is less reliable than the theoretical models. For FEA, if we have a greater number of elements then FEA would matchup with EBM, but we only used one element. Despite slight differences, all models produced linear relationships between force applied and deformation which is expected since this experiment was in the elastic region for the material. In conclusion, the models produced deformations which were less than the real-world results since they do rely on ideal conditions. Plus, the Young Modulus for material wasn't accurately known which also might have led to these errors. This is why it is so important to understand assumptions made and apply factors of safety to adjust for these slight differences in theoretical to real world.

VIII. Appendix

Experimental Data:

Load (N)	4.2 N	5.9 N	9.1 N
Displacement(cm)	5.2	7	11
Displacement(cm)	5.1	7.2	10.9
Displacement(cm)	5.3	7.1	11.1
Displacement(cm)	5.2	7.3	10.8
Displacement(cm)	5.4	7.4	10.3
Mean displacement(cm)	5.24	7.2	10.82
Standard Deviation	0.114018	0.158114	0.311448