Computer Project 2: Thermal Protection System for Hypersonic Vehicle

Out: 11/14/2022, Due: See Canvas

The computer project is assigned to a team of about 5 students. The team needs to submit one report (electronically in one single PDF file, scanned or typed) detailing (1) problem definition, (2) analytical solution, (3) numerical method, (4) results, (5) discussion, (6) conclusions, and (7) the commented code. If MATLAB is used, the code should be exported in PDF and attached to the report file. The example report is the same as CP1. The figures have to be generated using a computer language, with the axis labels, titles, and legends added. After the report submission, everyone will be asked to complete an online form that rates the contributions of all the team members, including themselves. The outcome of the online form will potentially result in differences in scores for different members of the same team; so that (1) those who do not contribute enough may end up with low scores even if the project is done well; (2) those who made sufficient contributions may end up with high scores even if the project is not done well.

A company Merheus is developing a new hypersonic civil aircraft that can fly in the atmosphere. However, atmospheric flight at hypersonic speed will result in strong aerodynamic heating on the surface of the vehicle, which could cause the structure to *melt*. The chief engineer is asking for your help to design a cooling system for the vehicle to mitigate the aerodynamic heating on the vehicle structure.

After some simplification and modeling, the unsteady heat transfer in the vehicle structure is governed by a 1D parabolic PDE (all quantities nondimensionalized),

$$\begin{aligned} u_t &= \kappa(u)u_{yy} \\ u(y,0) &= 0 \\ u_y(0,t) &= -Q_1(u(0,t)) \\ u_y(H,t) &= -Q_2 \end{aligned} \tag{1}$$

where u is the temperature increase in the structure, κ is a coefficient characterizing the structural properties, Q_1 is the aerodynamic heating on the surface of the vehicle, $y = 0$, and Q_2 is the heat flux that is absorbed by the cooling system at $y = H$. Note that due to the nonlinearity of aerodynamic heating, Q_1 is a function of vehicle surface temperature, $u(0,t)$; in addition, κ is also a function of temperature because the heat capacity of the structural material may change as temperature increases. During the hypersonic flight, the aerodynamic heating will cause the temperature to increase throughout the structure thickness H.

The goal of this project is to choose the value of Q_2 such that the maximum temperature increase of the structure is less than a prescribed value $u_{\max}$ during the entire mission profile $0 < t \leq T$.

1 Analytical Solutions

In this section, assume κ and Q_1 are constants.

(1) Show that the solution to Eq. (1) can be split into two parts,

$$u(y,t) = [w(y) + g(t)] + v(y,t) \tag{2}$$

where $w(y) + g(t)$ form the long-term solution,

$$w(y) = \frac{Q_1 - Q_2}{2H} y^2 - Q_1 y \tag{3}$$

$$g(t) = \frac{(Q_1 - Q_2)\kappa}{H} t \tag{4}$$

and $v(y,t)$ is a transient solution satisfying the following PDE,

$$\begin{aligned} v_t &= \kappa v_{yy} \\ v(y,0) &= -w(y) \\ v_y(0,t) &= 0 \\ v_y(H,t) &= 0 \end{aligned} \tag{5}$$

Hint: You only need to verify that the assumed form satisfies Eq. (1); but if you are able to derive the form of $w(y)$ and $g(t)$, you could earn some bonus. Physically, when $Q_1 > Q_2$, thermal energy will accumulate in the structure and result in a non-steady increase in temperature (thus the $g(t)$ term).

(2) Show that the complete solution to Eq. (1) is,

$$u(y,t) = w(y) + g(t) + A_0 + \sum_{n=1}^{N} A_n \cos(p_n y) \exp(-\kappa p_n^2 t) \tag{6}$$

where $p_n = \dfrac{n\pi}{H}$, and

$$A_0 = \frac{H}{6}(2Q_1 + Q_2) \tag{7}$$

$$A_n = \frac{2H}{n^2\pi^2}[(-1)^n Q_2 - Q_1] \tag{8}$$

You may cite solutions that are already in the notes.

2 Numerical Discretization

In this section, we will take a discretization approach that is slightly different from the lecture. In the lecture, space and time are discretized simultaneously. In the project, space is discretized first, resulting in a first-order ODE. Subsequently, this ODE is discretized in time using the Crank-Nicolson scheme. Eventually, both approaches result in the same discretization. There are two takeaways: (1) The Crank-Nicolson scheme is a general implicit algorithm for first-order ODE's, and (2) Eq. (1) could be solved using many other ODE algorithms, such as implicit Euler and Runge-Kutta methods.

In this section, κ and Q_1 are still assumed constants. However, you are welcomed to develop the formulation for temperature-dependent κ and Q_1, which would only require some minor modifications..

Suppose Eq. (1) is solved numerically on the domain $y \in [0, H]$ and $t \in [0, T]$ with step sizes of Δy and Δt in the y- and t-directions, respectively.

(1) Spatially, discretize the y direction into a grid of $(N_y + 1)$ points, where $N_y = \frac{H}{\Delta y}$. At the ith grid point, the solution is denoted by $u_i(t) = u(i\Delta y, t)$, $i = 0, 1, \cdots, N_y$. Show that,

$$\dot{\mathbf{u}} = \mathbf{A}\mathbf{u} + \tilde{\mathbf{f}} \tag{9}$$

where

$$\mathbf{u} = [u_1, u_2, \cdots, u_{N_y-1}]^T \tag{10}$$

$$\tilde{\mathbf{f}} = \frac{2\kappa}{3\Delta y}[Q_1, 0, \cdots, 0, -Q_2]^T \tag{11}$$

$$\mathbf{A} = \frac{\kappa}{\Delta y^2}\begin{bmatrix} -2/3 & 2/3 & & & \\ 1 & -2 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & -2 & 1 \\ & & & 2/3 & -2/3 \end{bmatrix} \tag{12}$$

Hint: $u_0(t)$ and $u_{N_y}(t)$ are determined by the boundary conditions using two special finite difference formulas,

$$u_y(0, t) = \frac{-3u_0(t) + 4u_1(t) - u_2(t)}{2\Delta y} = -Q_1 \tag{13}$$

$$u_y(H, t) = \frac{u_{N_y-2}(t) - 4u_{N_y-1}(t) + 3u_{N_y}(t)}{2\Delta y} = -Q_2 \tag{14}$$

which leads to

$$u_0(t) = \frac{2\Delta y Q_1}{3} + \frac{4}{3}u_1(t) - \frac{1}{3}u_2(t) \tag{15}$$

$$u_{N_y}(t) = -\frac{2\Delta y Q_2}{3} + \frac{4}{3}u_{N_y-1}(t) - \frac{1}{3}u_{N_y-2}(t) \tag{16}$$

(2) Temporally, discretize Eq. (9) using Crank-Nicolson scheme with a time step size of Δt. At the jth time step, the solution is denoted by $\mathbf{u}_j = \mathbf{u}(j\Delta t)$, $j = 0, 1, \cdots, N_t$, where $N_t = \frac{T}{\Delta t}$. Show that

$$\mathbf{T}\mathbf{u}_{n+1} = \mathbf{S}\mathbf{u}_n + \mathbf{f} \tag{17}$$

where the initial condition is $\mathbf{u}_0 = \mathbf{0}$ as given by Eq. (1), $\mathbf{f} = \Delta t\tilde{\mathbf{f}}$, and

$$\mathbf{T} = \mathbf{I} - \frac{\Delta t}{2}\mathbf{A}, \quad \mathbf{S} = \mathbf{I} + \frac{\Delta t}{2}\mathbf{A} \tag{18}$$

(3) Bonus (10%): Leverage the tool of eigenvalue decomposition to show that Eq. (17) is absolutely stable. *Hint: Start with the case with* $\mathbf{f} = 0$.

3 Computer Program

In this section, κ and Q_1 are still assumed constants. Even if you try to develop a code for the general temperature-dependent case, you are suggested to implement the constant case.

The computer program should have the following components/functions:

(1) An initialization procedure that generates the vectors and matrices involved in calculation (e.g. $\mathbf{f}$, $\mathbf{S}$ and $\mathbf{T}$) and the empty arrays for solution storage.

(2) A linear solver for the system in the form of $\mathbf{Tx} = \mathbf{d}$ (e.g. Eq. (17)) using the LU decomposition (i.e. Thomas algorithm when $\mathbf{T}$ is tridiagonal).

   (a) A procedure to decompose $\mathbf{T}$ into the $\mathbf{L}$ and $\mathbf{U}$ matrices. Since $\mathbf{T}$ is tridiagonal, $\mathbf{L}$ and $\mathbf{U}$ can be represented using three 1-D arrays $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, as specified in Eqs. (19-20). Note, this decomposition only needs to be done *once* for the same $\mathbf{T}$.

   (b) A procedure to carry out the backward substitution using the three 1-D arrays $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ to solve $\mathbf{Tx} = \mathbf{d}$.

(3) A procedure to solve the PDE in time using a *for*-loop. At each step, the linear solver is used to generate the solution of the next step. In the skeleton code, this procedure has already been implemented in the function "heatSolver". <u>If you use the implementation provided, please comment on each line of code.</u>

(4) A procedure to compute the analytical solutions at any given time.

$$\mathbf{T} = \begin{bmatrix} \tilde{a}_1 & \tilde{c}_1 & & \\ \tilde{b}_1 & \tilde{a}_2 & \ddots & \\ & \ddots & \ddots & \tilde{c}_{N-1} \\ & & \tilde{b}_{N-1} & \tilde{a}_N \end{bmatrix} = \mathbf{LU} = \begin{bmatrix} 1 & & & \\ b_1 & 1 & & \\ & \ddots & \ddots & \\ & & b_{N-1} & 1 \end{bmatrix} \begin{bmatrix} a_1 & c_1 & & \\ & a_2 & \ddots & \\ & & \ddots & c_{N-1} \\ & & & a_N \end{bmatrix} \tag{19}$$

$$\mathbf{a} = [a_1, a_2, \cdots, a_N]^T, \ \mathbf{b} = [b_1, b_2, \cdots, b_{N-1}]^T, \ \mathbf{c} = [c_1, c_2, \cdots, c_{N-1}]^T \tag{20}$$

Suggestion for the implementation of the Thomas algorithm: First verify the code by applying it to a simple linear system, such as

$$\begin{bmatrix} 2 & -1 & & \\ 1 & 2 & -1 & \\ & 1 & 2 & -1 \\ & & 1 & 2 \end{bmatrix} \mathbf{x} = \begin{bmatrix} 0 \\ 6 \\ 2 \\ -5 \end{bmatrix}, \quad \text{where } \mathbf{x} = \begin{bmatrix} 1 \\ 2 \\ -1 \\ -2 \end{bmatrix} \tag{21}$$

Hint: If you find the linear solver in Step (2) challenging to implement, you may use the standard linear solver in MATLAB to solve $\mathbf{Tx} = \mathbf{d}$ *and proceed to tackle the rest of the project; and come back later.*

4 Numerical Solutions and Discussion

Using the code you developed, perform the following analysis:

(1) Assuming the following quantities,

$$Q_1 = \text{constant} = 10, \quad Q_2 = 5, \quad \kappa = \text{constant} = 1.0, \quad H = 1, \quad T = 5.0, \quad \Delta y = 0.1 \tag{22}$$

Solve the PDE using three time step sizes: (1) $\Delta t = 0.01$, (2) $\Delta t = 0.05$, (3) $\Delta t = 0.25$.

(2) For $\Delta t = 0.01$, plot and compare the numerical and analytical solutions for temperature profiles at $t = 0.05, 0.1, 0.2, 0.3, 0.6, 1.2, 2.0$. Comment on any trend you find interesting (either the physics or the numerical solutions).

(3) For $\Delta t = 0.01, 0.05, 0.25$, plot and compare the numerical and analytical solutions for the time history of temperature at the vehicle surface ($y = 0$). Comment on any trend you find interesting and the accuracy of the Crank-Nicolson scheme.

(4) Design the cooling system by determining the amount of cooling heat flux Q_2, such that the maximum structural temperature is less than $u_{\max} = 5$ over $0 < t \leq T$. *Hint: While a straightforward approach is to solve the PDE for multiple values of Q_2, there is a simple analytical solution for this question.*

(5) Bonus (10%): Now consider the nonlinear case where Q_1 and κ are functions of temperature. Assume,

$$u_y(0, t) = -Q_1 + 0.1u(0, t) \tag{23}$$

$$\kappa(u) = 1.0 - 0.1u \tag{24}$$

Try to modify the numerical discretization, solve for the new temperature distribution in the structure, and design the new cooling system. What are the effects of the nonlinearities? Now will the required cooling heat flux Q_2 increase or decrease? *Note: Here lies the true power of numerical solution, since the analytical solution is no longer available.*