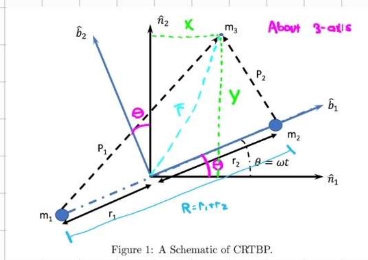
Preliminary Report

Team Member	PSU ID	Contributions	Percentages
Thomas Waltz	tiw5261	Part A, Review Work	34%
Ankit Gupta	apg5667	Part C, Review Work	33%
Kameron Metcalf	kmm8076	Part B, Review Work	33%

Part A:

a)

- $\bar{r}_1 = \frac{-m_2 R}{m_1 + m_2}\hat{b}_1 = -\mu \hat{b}_1$
- $\mu = \frac{m_2}{m_1 + m_2}$
- $\bar{r}_2 = \frac{m_1 R}{m_1 + m_2}\hat{b}_1 = (1-\mu)\hat{b}_1$
- $R = r_1 + r_2 = 1$
- $\bar{r}_3 = x\hat{b}_1 + y\hat{b}_2$
- $t = \frac{1}{\omega}T$



Kinetic Energy Of Spacecraft, and Lagrangian

Goal:

$$\boxed{\begin{aligned}\ddot{x} - 2\omega\dot{y} - \omega^2 x &= -\omega^2\frac{(1-\mu)(x+\mu)}{p_1^3} - \omega^2\frac{\mu(x-1+\mu)}{p_2^3}\\ \ddot{y} + 2\omega\dot{x} - \omega^2 y &= -\omega^2\frac{(1-\mu)y}{p_1^3} - \omega^2\frac{\mu y}{p_2^3}\end{aligned}}$$

Gravitational Potential Energy:

- $V = \frac{-G m_3 m_1}{p_1} - \frac{G m_3 m_2}{p_2}$
- $\omega^2 = G\frac{m_1 + m_2}{(r_1+r_2)^3}$

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}}\right) - \frac{\partial \mathcal{L}}{\partial x} = 0$$

- $\mathcal{L} = T - V = K.E. - P.E.$

$\bar{r}_3 = x\hat{b}_1 + y\hat{b}_2$

$\vec{\omega}_{B/N} = \dot{\theta}\hat{b}_3 = \dot{\theta}\hat{n}_3$

$\dot{\bar{r}}_3 = \dot{x}\hat{b}_1 + \dot{y}\hat{b}_2 + x\dot{\hat{b}}_1 + y\dot{\hat{b}}_2$

$\dot{\hat{b}}_1 = \vec{\omega}_{B/N} \times \hat{b}_1 = \begin{vmatrix} 0 & 0 & \dot{\theta} \\ 1 & 0 & 0 \end{vmatrix} = \cancel{0\hat{b}_1} + \dot{\theta}\hat{b}_2 + \cancel{0\hat{b}_3} = \dot{\theta}\hat{b}_2$

$\dot{\hat{b}}_2 = \vec{\omega}_{B/N} \times \hat{b}_2 = \begin{vmatrix} 0 & 0 & \dot{\theta} \\ 0 & 1 & 0 \end{vmatrix} = -\dot{\theta}\hat{b}_1 + \cancel{0\hat{b}_2} + \cancel{0\hat{b}_3} = -\dot{\theta}\hat{b}_1$

$\dot{\bar{r}}_3 = \dot{x}\hat{b}_1 + \dot{y}\hat{b}_2 + x(\dot{\theta}\hat{b}_2) + y(-\dot{\theta}\hat{b}_1)$

Note: $\dot{\theta} = \omega$

$\dot{\bar{r}}_3 = \hat{b}_1(\dot{x} - y\dot{\theta}) + \hat{b}_2(\dot{\theta}x + \dot{y}) \longrightarrow \hat{b}_1(\dot{x} - y\omega) + \hat{b}_2(\omega x + \dot{y})$

$T = \frac{1}{2}m_3(\dot{\bar{r}}_3 \cdot \dot{\bar{r}}_3) = \frac{1}{2}m_3\left(\hat{b}_1(\dot{x} - y\omega)^2 + \hat{b}_2(\omega x + \dot{y})^2\right)$

Distribute and Simplify:

$T = \frac{1}{2}m_3\left(\dot{x}^2 + y^2\omega^2 + \omega^2 x^2 - 2\dot{x}y\omega + \dot{y}^2 + 2x\dot{y}\omega\right)$

Figure 1: Finding Velocity Vector, and Kinetic Energy

Gravity as a force: (has direction) Note P.E. = mgh

$$\vec{F}_G = \frac{-G m_3 m_1}{\rho_1} \cdot \hat{\rho}_1 - \frac{G m_3 m_2}{\rho_2} \hat{\rho}_2$$ (need in terms of b-frame)

$$\bar{\rho}_1 = (x+\mu)^2 \hat{b}_1 + y^2 \hat{b}_2 \xrightarrow[\text{Vector}]{\text{Unit}} \hat{\rho}_1 = \frac{\vec{\rho}_1}{|\vec{\rho}_1|} = \frac{1}{\rho_1}\left((x+\mu)^2 \hat{b}_1 + y^2 \hat{b}_2\right)$$

$$\bar{\rho}_2 = (x-(1-\mu))^2 \hat{b}_1 + y^2 \hat{b}_2 \to (x-1+\mu)^2 \hat{b}_1 + y^2 \hat{b}_2 \xrightarrow[\text{vector}]{\text{Unit}} \hat{\rho}_2 = \frac{\vec{\rho}_2}{|\vec{\rho}_2|} = \frac{1}{\rho_2}\left((x-1+\mu)^2 \hat{b}_1 + y^2 \hat{b}_2\right)$$

$$\vec{F}_G = \frac{-G m_3 m_1}{\rho_1} \cdot \frac{1}{\rho_1}\left((x+\mu)^2 \hat{b}_1 + y^2 \hat{b}_2\right)^{-3/2} - \frac{G m_3 m_2}{\rho_2} \cdot \frac{1}{\rho_2}\left((x-1+\mu)^2 \hat{b}_1 + y^2 \hat{b}_2\right)^{-3/2}$$

$$\bar{F}_G = \frac{-G m_3 m_1}{\rho_1^2}\left((x+\mu)^2 \hat{b}_1 + y^2 \hat{b}_2\right)^{-3/2} - \frac{G m_3 m_2}{\rho_2^2}\left((x-1+\mu)^2 \hat{b}_2 + y^2 \hat{b}_2\right)^{-3/2}$$

$$\mathcal{L} = \frac{1}{2} m_3 \left(\dot{x}^2 + y^2 \omega^2 + \omega^2 x^2 - 2\dot{x} y \omega + \dot{y}^2 + 2 x \dot{y} \omega \right) + \frac{G m_3 m_1}{\rho_1^2}\left((x+\mu)^2 \hat{b}_1 + y^2 \hat{b}_2\right)^{-3/2} + \frac{G m_3 m_2}{\rho_2^2} \cdot \left((x-1+\mu)^2 \hat{b}_2 + y^2 \hat{b}_2\right)^{-3/2}$$

Figure 2: Vectorizing Force of Gravity for Potential Energy. Defining P1 and P2 in terms of the b-frame. Solving for Lagrangian.

X - Degree of freedom:

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}}\right) - \frac{\partial \mathcal{L}}{\partial x} = 0$$

$$\omega^2 = \frac{G(m_1 + m_2)}{(r_1 + r_2)^3}$$

$$\frac{\partial \mathcal{L}}{\partial \dot{x}} = \cancel{\tfrac{2}{2} m_3}\, \dot{x} - \cancel{\tfrac{2}{2} m_3}\, y\omega \rightarrow \dot{x} - y\omega$$

$$G = \frac{\omega^2 \cancel{(r_1 + r_2)}^3}{m_1 + m_2}$$

$$G = \frac{\omega^2}{m_1 + m_2}$$

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}}\right) = \ddot{x} - \dot{y}\omega$$

$$\frac{\partial \mathcal{L}}{\partial x} = \cancel{\tfrac{2}{2} m_3}\, x\omega^2 + \cancel{\tfrac{2}{2} m_3}\, \dot{y}\omega - \frac{G\cancel{m_3} m_1 (x+\mu)}{\rho_1^2} - \frac{G\cancel{m_3} m_2 (x-1+\mu)}{\rho_2^2} \rightarrow x\omega^2 + \dot{y}\omega - \frac{G m_1 (x-\mu)}{\rho_1^2} - \frac{G m_2 (x-1+\mu)}{\rho_2^2}$$

$$\frac{\partial \mathcal{L}}{\partial x} = x\omega^2 + \dot{y}\omega - \frac{\omega^2 m_1 (x-\mu)}{\rho_1^2 (m_1 + m_2)} - \frac{\omega^2 m_2 (x-1+\mu)}{\rho_2^2 (m_1 + m_2)}$$

$$\frac{\partial \mathcal{L}}{\partial x} = x\omega^2 + \dot{y}\omega - \frac{\omega^2 (1-\mu)(x+\mu)}{\rho_1^3} - \frac{\omega^2 \mu (x-1+\mu)}{\rho_2^3}$$

$$\ddot{x} - \dot{y}\omega - x\omega^2 - \dot{y}\omega + \frac{\omega^2 (1-\mu)(x+\mu)}{\rho_1^3} + \frac{\omega^2 \mu (x-1+\mu)}{\rho_2^3} = 0$$

$$\boxed{\ddot{x} - 2\omega\dot{y} - x\omega^2 = -\frac{\omega^2 (1-\mu)(x+\mu)}{\rho_1^3} - \frac{\omega^2 \mu (x-1+\mu)}{\rho_2^3}}$$

Y - Degree of freedom:

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{y}}\right) - \frac{\partial \mathcal{L}}{\partial y} = 0$$

$$\frac{\partial \mathcal{L}}{\partial \dot{y}} = \cancel{\tfrac{2}{2} m_3}\, \dot{y} + \cancel{\tfrac{2}{2} m_3}\, x\omega$$

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{y}}\right) = \ddot{y} + \dot{x}\omega$$

$$\frac{\partial \mathcal{L}}{\partial y} = \cancel{\tfrac{2}{2} m_3}\, y\omega^2 - \cancel{\tfrac{2}{2} m_3}\, \dot{x}\omega - \frac{G m_1 \cancel{m_3} y}{\rho_1^3} - \frac{G m_2 \cancel{m_3} y}{\rho_2^3} \rightarrow y\omega^2 - \dot{x}\omega - \frac{\omega^2 m_1 y}{\rho_1^3 (m_1 + m_2)} - \frac{\omega^2 m_2 y}{\rho_2^3 (m_1 + m_2)}$$

$$\ddot{y} + \dot{x}\omega - y\omega^2 + \dot{x}\omega + \frac{\omega^2 m_1 y}{\rho_1^3 (m_1 + m_2)} + \frac{\omega^2 m_2 y}{\rho_2^3 (m_1 + m_2)} = 0$$

$$\ddot{y} + 2\dot{x}\omega - \omega^2 y = -\frac{\omega^2 m_1 y}{\rho_1^3 (m_1 + m_2)} + \frac{\omega^2 m_2 y}{\rho_2^3 (m_1 + m_2)} \rightarrow \boxed{\ddot{y} + 2\dot{x}\omega - \omega^2 y = \frac{-\omega^2 (1-\mu) y}{\rho_1^3} - \frac{\omega^2 \mu y}{\rho_2^3}}$$

Figure 3: Using Lagrangian method to formulate two separate equations of motion around each degree of freedom.

Goal:

$$x'' - 2y' = \frac{\partial U}{\partial x} = U_x$$
$$y'' + 2x' = \frac{\partial U}{\partial y} = U_y$$

Need magnitude of P_1 and P_2

$$\|P_1\| = \sqrt{(x+\mu)^2 + y^2} \qquad \|P_2\| = \sqrt{(x-1+\mu)^2 + y^2}$$

$$U = \frac{1}{2}(x^2+y^2) + \frac{1-\mu}{P_1} + \frac{\mu}{P_2} \rightarrow U = \frac{1}{2}(x^2+y^2) + \frac{1-\mu}{\sqrt{(x+\mu)^2+y^2}} + \frac{\mu}{\sqrt{(x-1+\mu)^2+y^2}}$$

$$\frac{dU}{dx} = \frac{\cancel{2}}{\cancel{2}}x + \frac{1-\mu}{((x+\mu)^2+y^2)^{3/2}} + \frac{\mu}{((x-1+\mu)^2+y^2)^{3/2}}$$

$$U_x = x + \frac{(1-\mu)(x+\mu)}{P_1^3} + \frac{\mu(x-1+\mu)}{P_2^3}$$

$$x'' - 2y' + x = -\frac{(1-\mu)(x+\mu)}{P_1^3} - \frac{\mu(x-1+\mu)}{P_2^3}$$

$$x'' - 2y' = x - \frac{(1-\mu)(x+\mu)}{P_1^3} - \frac{\mu(x-1+\mu)}{P_2^3}$$

$$\boxed{x'' - 2y = U_x}$$

$$U = \frac{1}{2}(x^2+y^2) + \frac{1-\mu}{\left(\sqrt{(x+\mu)^2+y^2}\right)} + \frac{\mu}{\left(\sqrt{(x-1+\mu)^2+y^2}\right)}$$

$$\frac{dU}{dx} = \frac{\cancel{2}}{\cancel{2}}y + \frac{(1-\mu)}{((x+\mu)^2+y^2)^{3/2}} + \frac{\mu}{((x-1+\mu)^2+y^2)^{3/2}}$$

$$U_y = y + \frac{(1-\mu)y}{P_1^3} + \frac{\mu y}{P_2^3}$$

$$y'' + 2x' - y = -\frac{(1-\mu)y}{P_1^3} - \frac{\mu y}{P_2^3}$$

$$y'' + 2x' = y - \frac{(1-\mu)y}{P_1^3} - \frac{\mu y}{P_2^3}$$

$$\boxed{y'' + 2x' = U_y}$$

Figure 4: Solving for Ux and Uy, then rewriting Equations of Motion.

Part B:

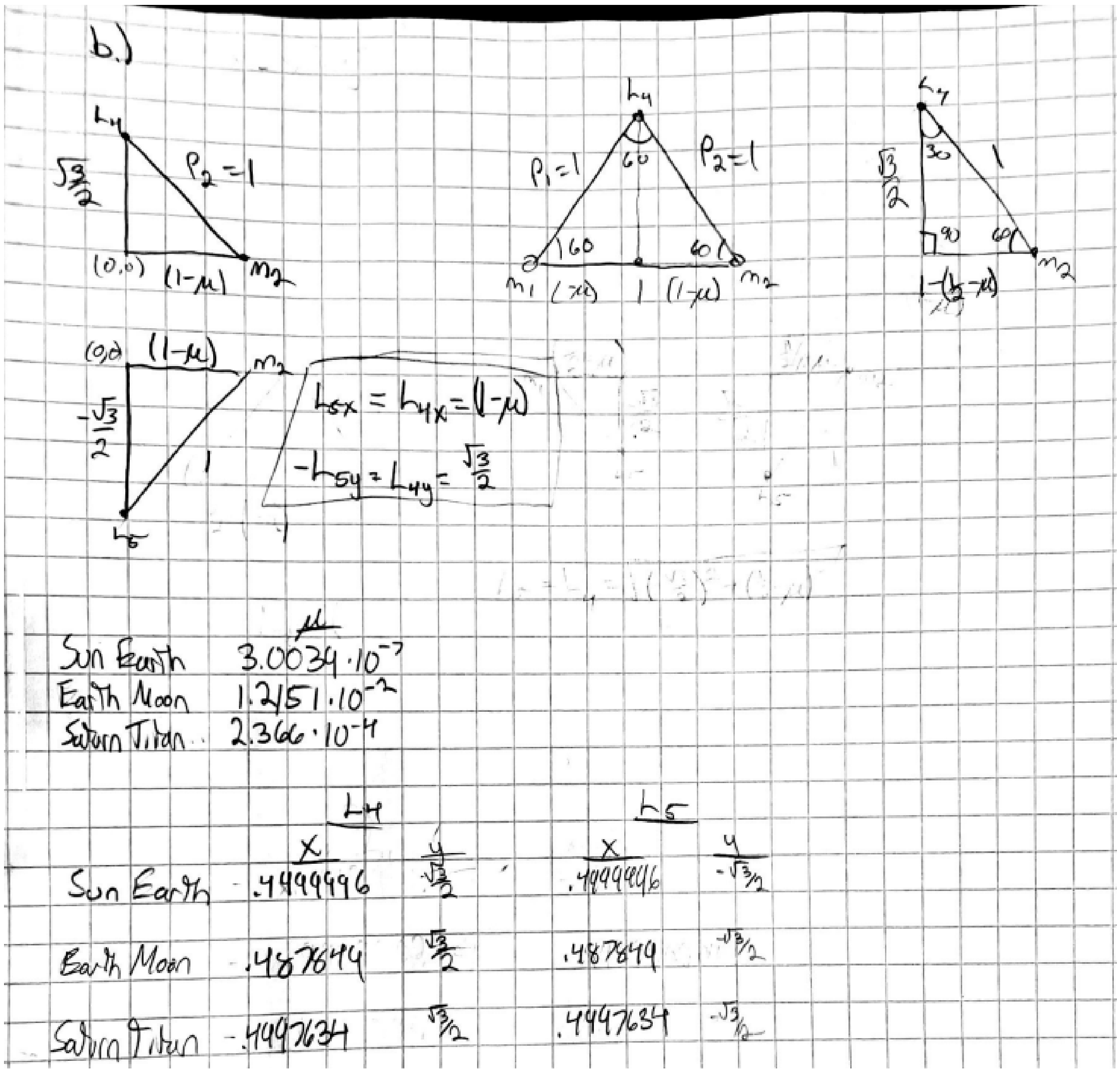


Figure 5: Using Lagrangian Points to find values of L4 and L5.

Part C:

$$x'' - 2y' = \frac{\partial U}{\partial x} = U_x \quad \Big| \quad y'' + 2x' = \frac{\partial U}{\partial y} = U_y$$

Linearize

$$x: \frac{\partial F}{\partial x}\Big|_0 \Delta x + \frac{\partial F}{\partial y}\Big|_0 \Delta y + \frac{\partial F}{\partial x'}\Big|_0 \Delta x' + \frac{\partial F}{\partial y'}\Big|_0 \Delta y' + \frac{\partial F}{\partial x''} \Delta x'' + \frac{\partial F}{\partial y''}\Big|_0 \Delta y''$$

$$f(x, y, x', y') = -U_{xx}\Delta x - U_{xy}\Delta y + 0 - 2\Delta y' + \Delta x'' + 0 = 0$$

$$\Delta x'' = U_{xx}\Delta x + U_{xy}\Delta y + 2\Delta y'$$

$$y = \frac{\partial F}{\partial x}\Big|_0 \Delta x + \frac{\partial F}{\partial y}\Big|_0 \Delta y + \frac{\partial F}{\partial x'} \Delta x' + \frac{\partial F}{\partial y'}\Big|_0 \Delta y' + \frac{\partial F}{\partial x''}\Big|_0 \Delta x'' + \frac{dF}{\partial y''}\Big|_0 \Delta y''$$

$$f(x, y, x', y') = -U_{xy}\Delta x - U_{yy}\Delta y + 2\Delta x' + \Delta y'' = 0$$

$$\Delta y'' = U_{xy}\Delta x + U_{yy}\Delta y - 2\Delta x'$$

$$A = \begin{array}{c} \begin{array}{cccc} \Delta x & \Delta y & \Delta x' & \Delta y' \end{array} \\ \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ U_{xx} & U_{xy} & 0 & 2 \\ U_{xy} & U_{yy} & -2 & 0 \end{bmatrix} \end{array} \begin{array}{c} \Delta x' \\ \Delta y' \\ \Delta x'' \\ \Delta y'' \end{array}$$

Figure 6: Steps of linearizing equations of motion.

Note: Code Submitted in Gradescope

Table 1: Earth and Moon

L1	-2.93210331736619 + 0.00000000000000i 2.93210331736619 + 0.00000000000000i 2.77555756156289e-17 + 2.33441574278267i 2.77555756156289e-17 - 2.33441574278267i	Unstable
L2	2.15875415806544 + 0.00000000000000i -2.15875415806544 + 0.00000000000000i 0.00000000000000 + 1.86269258455063i 0.00000000000000 - 1.86269258455063i	Unstable
L3	-1.43765208071578e-16 + 1.01042841824630i -1.43765208071578e-16 - 1.01042841824630i -0.177948963939713 + 0.00000000000000i 0.177948963939713 + 0.00000000000000i	Unstable
L4	2.07299455379228e-15 + 0.954499117832690i 2.07299455379228e-15 - 0.954499117832690i -1.63085690785270e-15 + 0.298213738879709i -1.63085690785270e-15 - 0.298213738879709i	Stable
L5	-2.07299455379228e-15 + 0.954499117832690i -2.07299455379228e-15 - 0.954499117832690i 1.63085690785270e-15 + 0.298213738879709i 1.63085690785270e-15 - 0.298213738879709i	Stable

Table 2: Sun and Earth

L1	-2.51927176915253 + 0.00000000000000i 2.51927176915253 + 0.00000000000000i -1.11022302462516e-16 + 2.07828726022754i -1.11022302462516e-16 - 2.07828726022754i	Unstable
L2	-2.50766802961189 + 0.00000000000000i 2.50766802961189 + 0.00000000000000i 1.66533453693773e-16 + 2.07121742796525i 1.66533453693773e-16 - 2.07121742796525i	Unstable
L3	-1.51788304147971e-18 + 0.999970636745927i -1.51788304147971e-18 - 0.999970636745927i 3.68628738645072e-18 + 0.00938545387897627i 3.68628738645072e-18 - 0.00938545387897627i	Stable
L4	9.43689570931383e-16 + 0.999998986181485i 9.43689570931383e-16 - 0.999998986181485i’ -8.55153397784201e-16 + 0.00142395084250021i -8.55153397784201e-16 - 0.00142395084250021i	Stable
L5	-9.43689570931383e-16 + 0.999998986181485i -9.43689570931383e-16 - 0.999998986181485i 8.55153397784201e-16 + 0.00142395084250021i 8.55153397784201e-16 - 0.00142395084250021i	Stable

Table 3: Saturn and Titan

L1	2.61513098346219 + 0.00000000000000i -2.61513098346219 + 0.00000000000000i 0.00000000000000 + 2.13698484537958i 0.00000000000000 - 2.13698484537958i	Unstable
L2	-2.47314812523973 + 0.00000000000000i 2.47314812523973 + 0.00000000000000i -4.99600361081320e-16 + 2.05023249232707i -4.99600361081320e-16 - 2.05023249232707i	Unstable
L3	2.08166817117217e-17 + 1.00020269334695i 2.08166817117217e-17 - 1.00020269334695i -0.0246621944870533 + 0.00000000000000i 0.0246621944870533 + 0.00000000000000i	Unstable
L4	1.38777878078145e-16 + 0.999200065215411i 1.38777878078145e-16 - 0.999200065215411i -3.98921177937727e-16 + 0.0399903697596923i -3.98921177937727e-16 - 0.0399903697596923i	Stable
L5	-1.38777878078145e-16 + 0.999200065215411i -1.38777878078145e-16 - 0.999200065215411i 3.98921177937727e-16 + 0.0399903697596923i 3.98921177937727e-16 - 0.0399903697596923i	Stable

Part D and E:

Future Approach:

• Uxx = Uxx(mu,x,y): used to calculate the second derivative U versus x. The function takes x and y as input, where x is the x-position of the Lagrange point and y is the y-position of the Lagrange point.

• Uyy = Uyy(mu,x,y): Used to calculate the second derivative of U with respect to y.

• Uxy = Uxy(mu,x,y): calculates the first derivative of U with respect to x, then for y.

The function ev takes x and y as input, where x is the x location of the Lagrange point and y is the y location of the Lagrange point and mu for different planets.

•Matrix A uses the equations found above and then the eig function produces the eigenvalues.

Procedure:

• Matlab function ode45 will be used to integrate.

• The x and y positions obtained from the integration will then be plotted.

• These x and y positions will then be converted from the body frame to the inertial frame using a loop.

• These new points will be then plotted using matlab functions.

• Simulating orbits in Earth and Moon system by adding the initial conditions found in files 'EML2304P1.mat'.

• Ode45 was reused, but with a different initial condition for Earth's sun system.