$$X_1 \sim B(n_1, p_1), X_2 \sim B(n_2, p_2)$$

$$H_0 : p_1 = p_2 = p \text{ vs. } H_1 : p_1 \neq p_2$$

The test statistic

$$T = \frac{\hat{p}_1 - \hat{p}_2}{\sqrt{\hat{p}(1-\hat{p})(\frac{1}{n_1} + \frac{1}{n_2})}}$$

Under the alternative,

$$\hat{p}_1 - \hat{p}_2 \sim N(p_1 - p_2, \frac{p_1 q_1}{n_1} + \frac{p_2 q_2}{n_2})$$

If $p_1 > p_2$, the power

$$\begin{aligned}
1 - \beta &= P\left(\frac{\hat{p}_1 - \hat{p}_2}{\sqrt{\hat{p}(1-\hat{p})(\frac{1}{n_1} + \frac{1}{n_2})}} > Z_{1-\alpha/2} | p_1, p_2\right) \\
&= P\left(\frac{\hat{p}_1 - \hat{p}_2 - (p_1 - p_2) + (p_1 - p_2)}{\sqrt{\hat{p}(1-\hat{p})(\frac{1}{n_1} + \frac{1}{n_2})}} > Z_{1-\alpha/2}\right) \\
&= P\left(\frac{\frac{\hat{p}_1 - \hat{p}_2 - (p_1 - p_2)}{\sqrt{\frac{p_1 q_1}{n_1} + \frac{p_2 q_2}{n_2}}} + \frac{(p_1 - p_2)}{\sqrt{\frac{p_1 q_1}{n_1} + \frac{p_2 q_2}{n_2}}}}{\sqrt{\frac{\hat{p}(1-\hat{p})(\frac{1}{n_1} + \frac{1}{n_2})}{\frac{p_1 q_1}{n_1} + \frac{p_2 q_2}{n_2}}}} > Z_{1-\alpha/2}\right) \\
&\approx P\left(\frac{Z + \frac{(p_1 - p_2)}{\sqrt{\frac{p_1 q_1}{n_1} + \frac{p_2 q_2}{n_2}}}}{\sqrt{\frac{\bar{p}(1-\bar{p})(\frac{1}{n_1} + \frac{1}{n_2})}{\frac{p_1 q_1}{n_1} + \frac{p_2 q_2}{n_2}}}} > Z_{1-\alpha/2}\right), Z \sim N(0,1), \bar{p} = \frac{p_1 + k p_2}{1 + k} \\
&= P\left(Z > Z_{1-\alpha/2}\sqrt{\frac{\bar{p}(1-\bar{p})(\frac{1}{n_1} + \frac{1}{n_2})}{\frac{p_1 q_1}{n_1} + \frac{p_2 q_2}{n_2}}} - \frac{p_1 - p_2}{\sqrt{\frac{p_1 q_1}{n_1} + \frac{p_2 q_2}{n_2}}}\right)
\end{aligned}$$

then we have,

$$Z_{1-\alpha/2}\sqrt{\frac{\bar{p}(1-\bar{p})(\frac{1}{n_1} + \frac{1}{n_2})}{\frac{p_1 q_1}{n_1} + \frac{p_2 q_2}{n_2}}} - \frac{p_1 - p_2}{\sqrt{\frac{p_1 q_1}{n_1} + \frac{p_2 q_2}{n_2}}} = Z_\beta = -Z_{1-\beta} \tag{1}$$

If $p_1 < p_2$, then

$$
\begin{aligned}
1-\beta &= P\left(\frac{\hat{p}_1-\hat{p}_2}{\sqrt{\hat{p}(1-\hat{p})(\frac{1}{n_1}+\frac{1}{n_2})}} < Z_{\alpha/2}|p_1,p_2\right) \\
&= P\left(\frac{\hat{p}_1-\hat{p}_2-(p_1-p_2)+(p_1-p_2)}{\sqrt{\hat{p}(1-\hat{p})(\frac{1}{n_1}+\frac{1}{n_2})}} < Z_{\alpha/2}\right) \\
&= P\left(\frac{\frac{\hat{p}_1-\hat{p}_2-(p_1-p_2)}{\sqrt{\frac{p_1q_1}{n_1}+\frac{p_2q_2}{n_2}}}+\frac{(p_1-p_2)}{\sqrt{\frac{p_1q_1}{n_1}+\frac{p_2q_2}{n_2}}}}{\sqrt{\frac{\hat{p}(1-\hat{p})(\frac{1}{n_1}+\frac{1}{n_2})}{\frac{p_1q_1}{n_1}+\frac{p_2q_2}{n_2}}}} < Z_{\alpha/2}\right) \\
&\approx P\left(\frac{Z+\frac{(p_1-p_2)}{\sqrt{\frac{p_1q_1}{n_1}+\frac{p_2q_2}{n_2}}}}{\sqrt{\frac{\bar{p}(1-\bar{p})(\frac{1}{n_1}+\frac{1}{n_2})}{\frac{p_1q_1}{n_1}+\frac{p_2q_2}{n_2}}}} < Z_{\alpha/2}\right) \\
&= P\left(Z < Z_{\alpha/2}\sqrt{\frac{\bar{p}(1-\bar{p})(\frac{1}{n_1}+\frac{1}{n_2})}{\frac{p_1q_1}{n_1}+\frac{p_2q_2}{n_2}}}-\frac{p_1-p_2}{\sqrt{\frac{p_1q_1}{n_1}+\frac{p_2q_2}{n_2}}}\right) \\
&= P\left(Z < -Z_{1-\alpha/2}\sqrt{\frac{\bar{p}(1-\bar{p})(\frac{1}{n_1}+\frac{1}{n_2})}{\frac{p_1q_1}{n_1}+\frac{p_2q_2}{n_2}}}-\frac{p_1-p_2}{\sqrt{\frac{p_1q_1}{n_1}+\frac{p_2q_2}{n_2}}}\right)
\end{aligned}
$$

we have,

$$
-Z_{1-\alpha/2}\sqrt{\frac{\bar{p}(1-\bar{p})(\frac{1}{n_1}+\frac{1}{n_2})}{\frac{p_1q_1}{n_1}+\frac{p_2q_2}{n_2}}}-\frac{p_1-p_2}{\sqrt{\frac{p_1q_1}{n_1}+\frac{p_2q_2}{n_2}}} = Z_{1-\beta} \tag{2}
$$

$$
(1)+(2) \Rightarrow Z_{1-\alpha/2}\sqrt{\frac{\bar{p}(1-\bar{p})(\frac{1}{n_1}+\frac{1}{n_2})}{\frac{p_1q_1}{n_1}+\frac{p_2q_2}{n_2}}}-\frac{|p_1-p_2|}{\sqrt{\frac{p_1q_1}{n_1}+\frac{p_2q_2}{n_2}}} = -Z_{1-\beta}
$$

and

$$
\begin{cases}
n_1 = \left[Z_{1-\alpha/2}\sqrt{\bar{p}(1-\bar{p})(1+1/k)}+Z_{1-\beta}\sqrt{p_1q_1+p_2q_2/k}\right]^2/\Delta^2 \\
n_2 = kn_1
\end{cases}
$$