AI 中的数学 第五次作业

2300012929 尹锦润

教材 3.3

令 $x=r\cos\theta, y=r\sin\theta, \theta\in[0,2\pi]$，则有

$$
\begin{aligned}
\iint_{x^2+y^2\leqslant R^2} c(R-\sqrt{x^2+y^2})\mathrm{d}x\mathrm{d}y &= \iint_{r\leqslant R,\theta\in[0,2\pi]} c(R-r)r\cos^2\theta\mathrm{d}\theta\mathrm{d}r \\
&= \int_{r\leqslant R} c(R-r)r\mathrm{d}r \int_0^{2\pi} \cos^2\theta\mathrm{d}\theta \\
&= \int_{r\leqslant R} \pi c(R-r)r\mathrm{d}r \\
&= \frac{c\pi}{6R^3} = 1
\end{aligned}
$$

因此 $c=\dfrac{3R^3}{\pi}$。

对于 (X, Y) 落入 $x^2+y^2\leqslant r^2$ 的概率，我们有：

$$
\begin{aligned}
\iint_{x^2+y^2\leqslant t^2} c(R-\sqrt{x^2+y^2})\mathrm{d}x\mathrm{d}y &= \iint_{r\leqslant t,\theta\in[0,2\pi]} c(R-r)r\mathrm{d}\theta\mathrm{d}r \\
&= \int_{r\leqslant t} c(R-r)r\mathrm{d}r \int_0^{2\pi} 1\mathrm{d}\theta \\
&= \int_{r\leqslant t} 2\pi c(R-r)r\mathrm{d}r \\
&= \frac{3t^2}{R^2} - \frac{2t^3}{R^3}
\end{aligned}
$$

代入原题目变量，答案为 $\dfrac{3r^2}{R^2}-\dfrac{2r^3}{R^3}$

教材 3.4

也就是求 D 的面积 S，有：

$$
S = \iint_{(x,y)\in D} 1\mathrm{dxdy} = 2\iint_{(u,v)\in E} 1\mathrm{dxdy} = \pi ab
$$

其中 $(u,v)=(x+y,x-y)$，$|J|=\dfrac{1}{2}$，$E:\dfrac{u^2}{(\sqrt{2}a)^2}+\dfrac{v^2}{(\sqrt{2}b)^2}\leqslant 1$，$E$ 的面积根据椭圆公式可以得到为 $\dfrac{\pi ab}{2}$。

因此有 (X,Y) 的联合密度为 $\begin{cases}\frac{1}{\pi ab}, & (X,Y)\in D \\ 0, & otherwise\end{cases}$

教材 3.8

$$\begin{aligned}
p_{X+Y}(z) &= \int_0^z p_X(x)p_Y(z-x)\mathrm{d}x \\
&= \int_0^z \frac{1}{2^{(m+n)/2}\Gamma(m/2)\Gamma(n/2)}x^{m/2-1}e^{-x/2}(z-x)^{n/2-1}e^{-(z-x)/2}\mathrm{d}x \\
&= \frac{1}{2^{(m+n)/2}\Gamma(m/2)\Gamma(n/2)}e^{-z/2}\int_0^z x^{m/2-1}(z-x)^{n/2-1}\mathrm{d}x \\
&= \frac{1}{2^{(m+n)/2}\Gamma(m/2)\Gamma(n/2)}e^{-z/2}z^{(n+m-1)}\frac{\Gamma(m/2)\Gamma(n/2)}{\Gamma((m+n)/2)} \\
&= \frac{1}{2^{(m+n)/2}\Gamma(m/2)\Gamma(n/2)}e^{-z/2}z^{(n+m-1)}\frac{\Gamma(m/2)\Gamma(n/2)}{\Gamma((m+n)/2)} \\
&= \frac{1}{2^{(m+n)/2}\Gamma((m+n)/2)}e^{-z/2}z^{(n+m-1)}
\end{aligned}$$

证毕。

教材 3.10

令 $x=r\cos\theta$，$y=r\sin\theta, r\in[0,+\infty), \theta\in[0,\frac{\pi}{2})$，有 $|J|=r$，进而

$$\begin{aligned}
\iint 4xy\sqrt{x^2+y^2}\exp\{-(x^2+y^2)\}\mathrm{dxdy} &= \iint 4r^4\sin\theta\cos\theta\exp\{-r^2\}\mathrm{d}r\mathrm{d}\theta \\
&= \int_0^\infty 2r^4e^{-r^2}\mathrm{dr} \\
&= \int_0^\infty t\sqrt{t}e^{-t}\mathrm{d}t \\
&= \Gamma\left(\frac{5}{2}\right) \\
&= \frac{3}{4}\sqrt{\pi}
\end{aligned}$$

教材 3.11

$$E(X) = 2\int_0^1 x^2 \mathrm{d}x = \frac{2}{3}$$

$$var(X) = 2\int_0^1 (x-\frac{2}{3})^2 x \mathrm{d}x = \frac{1}{18}$$

$$E(Y) = 2\int_0^1 y(1-y)\mathrm{d}y = \frac{1}{3}$$

$$var(Y) = 2\int_0^1 (y-\frac{1}{3})^2(1-y)\mathrm{d}y = \frac{1}{18} E(XY) = 2\int_0^1 x \int_0^x y \mathrm{d}x\mathrm{d}y = \frac{1}{36}$$

$$cov(X,Y) = E(XY) - E(X)E(Y) = \frac{1}{36}$$

$$\rho(X,Y) = \frac{cov(X,Y)}{\sqrt{var(X)var(Y)}} = \frac{1}{2}$$

教材 3.12

N 为偶数时候， $E(XY)=0, E(X)E(Y)=0$，因此 $\rho(X,Y)=0$。

N 为奇数时候，$E(X)=E(Y)=0, var(X)=0$,

$$
\begin{aligned}
var(Y) &= \int_{-\infty}^{+\infty} x^{2n} \frac{1}{\sqrt{2\pi}} \exp\{-\frac{x^2}{2}\} dx \\
&= \int_{-\infty}^{+\infty} \frac{2^{n-1}}{\sqrt{2\pi}} t^{n-1} \frac{\sqrt{2}}{\sqrt{t}} \exp\{-t\} dt \\
&= \frac{2^{n-1}}{\sqrt{\pi}} \Gamma(n-\frac{1}{2}+1) = \frac{2^{n-1}}{\sqrt{\pi}} \Gamma(n+\frac{1}{2}) \quad \frac{2n-1}{2} \cdots \\
&= \frac{2^{n-1}}{\sqrt{\pi}} (n-\frac{1}{2}) \cdots (\frac{1}{2}) \cdot \sqrt{\pi} \\
&= \frac{2^{n-1}}{1} \frac{1}{2^n} \cdot (2n-1)!! \\
&= \frac{(2n-1)!!}{2}
\end{aligned}
$$

$$
\begin{aligned}
E(XY) &= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} x^{n+1} \exp\{-\frac{x^2}{2}\} dx \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} 2^{\frac{n}{2}} (\sqrt{t})^n \exp\{-t\} dt \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} 2^{\frac{n}{2}} t^{\frac{n}{2}} \exp\{-t\} dt \\
&= \frac{2^{\frac{n}{2}}}{\sqrt{2}\sqrt{\pi}} \Gamma(\frac{n}{2}+1) \\
&= \frac{2^{\frac{n}{2}-\frac{1}{2}}}{\sqrt{\pi}} (\frac{n}{2})(\frac{n-2}{2}) \cdots (\frac{1}{2}) \cdot \sqrt{\pi} \\
&= \frac{2^{\frac{n}{2}-\frac{1}{2}} \, n!!}{2^{\frac{n+1}{2}}} \\
&= \frac{n!!}{2}
\end{aligned}
$$

因此$\rho(X,Y) = \frac{E(XY)-E(X)E(Y)}{\sqrt{var(Y)var(X)}} = \frac{n!!}{\sqrt{2}(2n-1)!!}$

教材 3.13

$$
\begin{aligned}
E(X_1X_2) &= \int_0^1 2x \int_5^{+\infty} e^{-y+5} \mathrm{d}x\mathrm{d}y \\
&= \frac{2}{3} \times 6 \\
&= 4
\end{aligned}
$$

教材 3.14

$cov(X,Y) = \rho(X,Y) * \sqrt{var(X)var(Y)} = 12$，因此

$var(X+Y) = var(X) + var(Y) + 2cov(X,Y) = 85, var(X-Y) = var(X) + var(Y) - 2cov(X,Y) = 37$。

教材 3.15

$$
\begin{aligned}
p(x,y) &= \frac{1}{2\pi ab}\exp\{-\frac{u^2+v^2}{2}\} \\
E(D) &= \iint \frac{1}{2\pi ab}\exp\{-\frac{u^2+v^2}{2}\}\mathrm{d}x\mathrm{d}y \\
&= \iint \frac{1}{2\pi}\exp\{-\frac{u^2+v^2}{2}\}\mathrm{d}u\mathrm{d}v \\
&= \iint \frac{r}{2\pi}\exp\{-\frac{r^2}{2}\}\mathrm{d}u\mathrm{d}v \\
&= 1 - e^{-k^2/2}
\end{aligned}
$$

教材 3.16

$$
p(x) = e^{-x}\int_0^{+\infty} e^{-y}\mathrm{d}y \int_0^{+\infty} e^{-z}\mathrm{d}z = e^{-x}
$$

类似可得 $p(y) = e^{-y}, p(z) = e^{-z}$，进而 $p(x,y,z) = p(x)p(y)p(z) \Rightarrow X,Y,Z$ 相互独立。

教材 3.17

$X,Y,Z \sim N(0,1)$，那么有 $X^2+Y^2+Z^2 \sim \Gamma(3/2,1/2)$，即对于 $t \geqslant 0$

$$
p_{X^2+Y^2+Z^2}(t) = \frac{1}{2^{3/2}\Gamma(3/2)}t^{1/2}e^{-t/2}
$$

那么对于 $s \geqslant 0$ 有，

$$
p_\xi(s) = p_{X^2+Y^2+Z^2}(s^2)(s^2)' = \frac{1}{2^{1/2}\Gamma(3/2)}s^2e^{-\frac{s^2}{2}} = \frac{1}{2^{-1/2}\sqrt{\pi}}s^2e^{-\frac{s^2}{2}}
$$

于是对于 ξ 的概率分布有：

$$
p_\xi(s) = \begin{cases} \frac{1}{2^{-1/2}\sqrt{\pi}}s^2e^{-\frac{s^2}{2}} & s \geqslant 0 \\ 0 & s < 0 \end{cases}
$$

教材 3.18

对于 $z > 0$:

$$F_\xi(z) = 1 - \prod_{i=1}^{n}(1 - F_{X_i}(z)) = 1 - \exp^n\left\{-\left(\frac{z}{\eta}\right)^m\right\} = 1 - \exp\left\{-\left(\frac{zn^{1/m}}{\eta}\right)^m\right\}$$

对于 $z \leqslant 0$ 有 $F_\xi(z) = 0$，因此 ξ 也满足威布尔分布，$\eta' = \frac{\eta}{n^{1/m}}$。

教材 3.19

$E(X+Y+Z) = E(X) + E(Y) + E(Z) = 1$

$var(X+Y+Z) = var(X) + var(Y) + var(Z) + cov(X,Y) + cov(Y,Z) + cov(X,Z) = 3$

教材 3.20

对于 $|J| = \left|\frac{\partial(U,V)}{\partial(X,Y)}\right| = 2$，而 X, Y 的联合密度为

$$p_{X,Y}(x,y) = \frac{1}{2\pi}\exp\{-\frac{x^2+y^2}{2}\}$$

于是 U, V 的联合密度为:

$$p_{U,V}(u,v) = \frac{1}{4\pi}\exp\{-\frac{u^2+v^2}{4}\}$$