20240228作业

1. 求极限$\lim\limits_{x\to 0}\dfrac{\displaystyle\int_0^x (\sin t)^\alpha \,\mathrm{d}t}{x^{1+\alpha}}$ $(\alpha > 0)$.

2. 设函数$f(x) \in R[a,b]$并且对$\forall x \in [a,b]$有$f(x) > 0$. 证明$\displaystyle\int_a^b f(x)\,\mathrm{d}x > 0$.

3. 证明下列极限成立

   (1) $\displaystyle\lim_{n\to+\infty}\int_{-1}^1 (1-x^2)^n \,\mathrm{d}x = 0$;

   (2) 设函数$f(x) \in C[-1,1]$, $\displaystyle\lim_{n\to+\infty}\frac{\displaystyle\int_{-1}^1 f(x)(1-x^2)^n\,\mathrm{d}x}{\displaystyle\int_{-1}^1 (1-x^2)^n\,\mathrm{d}x} = f(0)$.

4. 设$f(x) \in C[a,b]$ 满足：对任意的$\varphi(x) \in C[a,b]$, 只要$\displaystyle\int_a^b \varphi(x)\,\mathrm{d}x = 0$, 就有$\displaystyle\int_a^b f(x)\varphi(x)\,\mathrm{d}x = 0$. 证明: $f(x) = const.$, $x \in [a,b]$.

5. 设$f(x) \in R[0,1]$, $0 < m \leqslant f(x) \leqslant M$, 求证: $\displaystyle\int_0^1 f(x)\,\mathrm{d}x\int_0^1 \frac{\mathrm{d}x}{f(x)} \leqslant \frac{(m+M)^2}{4mM}$.

6. 【连续性Jensen不等式】

   设$f(x), p(x) \in R[a.b]$, $m \leqslant f(x) \leqslant M$, $p(x) \geqslant 0$, $x \in [a,b]$, $\displaystyle\int_a^b p(x)\,\mathrm{d}x > 0$; $\varphi(x) \in D[m,M]$且在$[m,M]$凸. 则 $\varphi\left(\frac{\displaystyle\int_a^b p(x)f(x)\,\mathrm{d}x}{\displaystyle\int_a^b p(x)\,\mathrm{d}x}\right) \leqslant \frac{\displaystyle\int_a^b p(x)\varphi(f(x))\,\mathrm{d}x}{\displaystyle\int_a^b p(x)\,\mathrm{d}x}$.