定理7.6.1. 曲线（段）Γ： $\begin{cases} x=\varphi(t), \\ y=\psi(t), \end{cases}$ $(\alpha\leqslant t\leqslant\beta)$可求长的充分必要条件是$\varphi(t),\psi(t)\in BV[\alpha,\beta]$.

证明. 对Γ的任一分割Δ： $\left\{\overline{M_{i-1}M_i}\right\}_{i=1}^{n}$， 其中$M_i=(\varphi(t_i),\psi(t_i))$, $i=0,1,\cdots,n$; $\alpha=t_0<t_1<\cdots<t_n=\beta$.

由于$|M_{i-1}M_i|=\sqrt{|\varphi(t_i)-\varphi(t_{i-1})|^2+|\psi(t_i)-\psi(t_{i-1})|^2}$,

所以, $$\sum_{i=1}^{n}|\varphi(t_i)-\varphi(t_{i-1})|\leqslant\sum_{i=1}^{n}|M_{i-1}M_i|\leqslant\sum_{i=1}^{n}|\varphi(t_i)-\varphi(t_{i-1})|+\sum_{i=1}^{n}|\psi(t_i)-\psi(t_{i-1})|,$$

$$\sum_{i=1}^{n}|\psi(t_i)-\psi(t_{i-1})|\leqslant\sum_{i=1}^{n}|M_{i-1}M_i|\leqslant\sum_{i=1}^{n}|\varphi(t_i)-\varphi(t_{i-1})|+\sum_{i=1}^{n}|\psi(t_i)-\psi(t_{i-1})|.$$

所以,

$$\sup_{\forall\Delta}\sum_{i=1}^{n}|\varphi(t_i)-\varphi(t_{i-1})|\leqslant\sup_{\forall\Delta}\sum_{i=1}^{n}|M_{i-1}M_i|\leqslant\sup_{\forall\Delta}\sum_{i=1}^{n}|\varphi(t_i)-\varphi(t_{i-1})|+\sup_{\forall\Delta}\sum_{i=1}^{n}|\psi(t_i)-\psi(t_{i-1})|,$$

$$\sup_{\forall\Delta}\sum_{i=1}^{n}|\psi(t_i)-\psi(t_{i-1})|\leqslant\sup_{\forall\Delta}\sum_{i=1}^{n}|M_{i-1}M_i|\leqslant\sup_{\forall\Delta}\sum_{i=1}^{n}|\varphi(t_i)-\varphi(t_{i-1})|+\sup_{\forall\Delta}\sum_{i=1}^{n}|\psi(t_i)-\psi(t_{i-1})|.$$

即 $$\bigvee_{\alpha}^{\beta}\varphi(t)\leqslant\sup_{\forall\Delta}\sum_{i=1}^{n}|M_{i-1}M_i|\leqslant\bigvee_{\alpha}^{\beta}\varphi(t)+\bigvee_{\alpha}^{\beta}\psi(t);\qquad \bigvee_{\alpha}^{\beta}\psi(t)\leqslant\sup_{\forall\Delta}\sum_{i=1}^{n}|M_{i-1}M_i|\leqslant\bigvee_{\alpha}^{\beta}\varphi(t)+\bigvee_{\alpha}^{\beta}\psi(t).$$

故 $$\sup_{\forall\Delta}\sum_{i=1}^{n}|M_{i-1}M_i|<+\infty\ \Leftrightarrow\ \begin{cases}\bigvee_{\alpha}^{\beta}\varphi(t)<+\infty,\\ \bigvee_{\alpha}^{\beta}\psi(t)<+\infty.\end{cases}$$ □

记 $\lambda = \max\limits_{1\leqslant i\leqslant n} |M_{i-1}M_i|$. 由于$\varphi(t), \psi(t) \in C[\alpha, \beta]$, 类似命题7.2.18, 可以证明

命题7.2.18. 设$f(x) \in C[a,b] \bigcap BV[a,b]$. 对$[a,b]$的任意分割$\Delta$： $a = x_0 < x_1 < \cdots < x_n = b$,

$$记\lambda_\Delta = \max_{1\leqslant i\leqslant n}(x_i - x_{i-1}). 则有 \bigvee_a^b f(x) = \lim_{\lambda_\Delta\to 0}\sigma(f,\Delta).$$

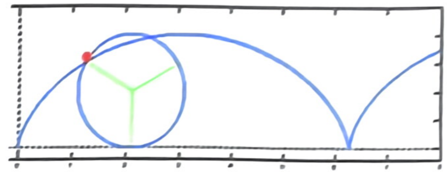
记 $\lambda = \max\limits_{1\leqslant i\leqslant n} |M_{i-1}M_i|$. 由于$\varphi(t),\psi(t)\in C[\alpha,\beta]$, 类似命题7.2.18, 可以证明

命题7.6.1. 对可求长的连续曲线（段）$\Gamma:\begin{cases} x=\varphi(t),\\ y=\psi(t),\end{cases} \quad (\alpha\leqslant t\leqslant\beta),\qquad |\Gamma| = \sup\limits_{\forall\Delta}\sum\limits_{i=1}^{n}|M_{i-1}M_i| = \lim\limits_{\lambda_\Delta\to 0}\sum\limits_{i=1}^{n}|M_{i-1}M_i|.$

命题7.6.2. 设$\varphi(t),\psi(t)\in C^1[\alpha,\beta]$, $\varphi'(t)^2+\psi'(t)^2\neq 0, t\in[\alpha,\beta]$. 则 $|\Gamma| = \int_\alpha^\beta \sqrt{\varphi'(t)^2+\psi'(t)^2}\,\mathrm{d}t.$

证明. $\forall\Delta$, $|M_{i-1}M_i| = \sqrt{|\varphi(t_i)-\varphi(t_{i-1})|^2+|\psi(t_i)-\psi(t_{i-1})|^2} = \sqrt{\varphi'(\xi_i)^2+\psi'(\eta_i)^2}\Delta t_i$, 其中, $\xi_i,\eta_i\in[t_{i-1},t_i]$, $i=1,\cdots,n$.

$$\text{所以, } |\Gamma| = \lim_{\lambda_\Delta\to 0}\sum_{i=1}^{n}|M_{i-1}M_i| = \lim_{\lambda_\Delta\to 0}\sum_{i=1}^{n}\sqrt{\varphi'(\xi_i)^2+\psi'(\eta_i)^2}\Delta t_i = \int_\alpha^\beta\sqrt{\varphi'(t)^2+\psi'(t)^2}\,\mathrm{d}t. \quad \square$$



例7.6.20. 求旋轮线$\begin{cases} x=a(t-\sin t),\\ y=a(1-\cos t)\end{cases}$ 的一拱$(0\leqslant t\leqslant 2\pi)$的弧长.

解. $L=\int_0^{2\pi}\sqrt{x'(t)^2+y'(t)^2}\,\mathrm{d}t = \int_0^{2\pi}\sqrt{a^2(1-\cos t)^2+a^2\sin^2 t}\,\mathrm{d}t = 2a\int_0^{2\pi}\sin\frac{t}{2}\,\mathrm{d}t = 8a.$ $\square$

例7.6.21. 求椭圆$\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$ 的周长（$a\neq b$）.

解. 椭圆参数式为$\begin{cases} x=a\cos\theta,\\ y=b\sin\theta,\end{cases}$ $(0\leqslant\theta\leqslant 2\pi)$. 所以椭圆周长 $L=4\int_0^{\frac{\pi}{2}}\sqrt{a^2\sin^2\theta+b^2\cos^2\theta}\,\mathrm{d}\theta = 4\int_0^{\frac{\pi}{2}}\sqrt{b^2+(a^2-b^2)\sin^2\theta}\,\mathrm{d}\theta.$

此为椭圆积分, 暂时不论...

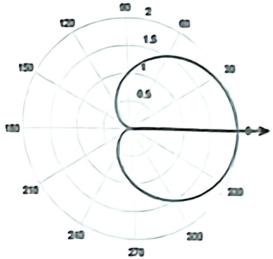
设平面曲线为极坐标形式 $r=r(\theta)$, $\theta_1\leqslant\theta\leqslant\theta_2$, $r'(\theta)\in C[\theta_1,\theta_2]$, 则曲线可求长.

事实上, $\begin{cases} x=x(\theta)=r(\theta)\cos\theta, \\ y=y(\theta)=r(\theta)\sin\theta. \end{cases}$

x,y作为参数θ的函数都是导数有界的, 从而是有界变差函数, 从而据前述可求长定理, 曲线可求长.

并且, $\begin{cases} \mathrm{d}x=[r'(\theta)\cos\theta-r(\theta)\sin\theta]\,\mathrm{d}\theta \\ \mathrm{d}y=[r'(\theta)\sin\theta+r(\theta)\cos\theta]\,\mathrm{d}\theta \end{cases}$, 故弧长微元为 $\mathrm{d}s=\sqrt{(\mathrm{d}x)^2+(\mathrm{d}y)^2}=\sqrt{r'(\theta)^2+r^2(\theta)}\,\mathrm{d}\theta$.

曲线弧长为 $L=\displaystyle\int_{\theta_1}^{\theta_2}\sqrt{r'(\theta)^2+r^2(\theta)}\,\mathrm{d}\theta$.



例7.6.25. 求心形线$r=a(1+\cos\theta)$的周长.

解. $L=2\displaystyle\int_0^{\pi}\sqrt{r^2+(r')^2}\,\mathrm{d}\theta=2a\int_0^{\pi}\sqrt{2(1+\cos\theta)}\,\mathrm{d}\theta=4a\int_0^{\pi}\left|\cos\frac{\theta}{2}\right|\mathrm{d}\theta=8a.$

例7.6.26. 求双纽线$r^2=2a^2\cos2\theta$的全长.

解. $r^2=2a^2\cos2\theta\Rightarrow 2rr'=-4a^2\sin2\theta\Rightarrow(r')^2=\dfrac{4a^4\sin^2 2\theta}{r^2}=\dfrac{2a^2\sin^2 2\theta}{\cos2\theta}$,

$r^2+r'(\theta)^2=2a^2\cos2\theta+\dfrac{2a^2\sin^2 2\theta}{\cos2\theta}=\dfrac{2a^2}{\cos2\theta}$, $\quad L=4\displaystyle\int_0^{\frac{\pi}{4}}\sqrt{r^2+r'(\theta)^2}\,\mathrm{d}\theta=4\sqrt2 a\int_0^{\frac{\pi}{4}}\frac{\mathrm{d}\theta}{\sqrt{\cos2\theta}}$,

$\displaystyle\int_0^{\frac{\pi}{4}}\frac{\mathrm{d}\theta}{\sqrt{\cos2\theta}}=\int_0^{\frac{\pi}{4}}\frac{\mathrm{d}\theta}{\sqrt{\cos^2\theta-\sin^2\theta}}=\int_0^{\frac{\pi}{4}}\frac{\mathrm{d}\theta}{\cos\theta\sqrt{1-\tan^2\theta}}=\int_0^{\frac{\pi}{4}}\frac{\sqrt{\sec^2\theta}}{\sqrt{1-\tan^2\theta}}\mathrm{d}\theta$ 故$L=4\sqrt2 a\displaystyle\int_0^1\frac{\mathrm{d}t}{\sqrt{1-t^4}}$. 此为椭圆积分...

$\xlongequal{t=\tan\theta}\displaystyle\int_0^1\frac{\sqrt{1+t^2}}{\sqrt{1-t^2}}\frac{\mathrm{d}t}{1+t^2}=\int_0^1\frac{\mathrm{d}t}{\sqrt{1-t^4}}$ (瑕积分)

$|\Gamma|=\displaystyle\int_{(A)}^{(B)}\mathrm{d}s$

$\mathrm{d}s=\sqrt{(\mathrm{d}x)^2+(\mathrm{d}y)^2}$

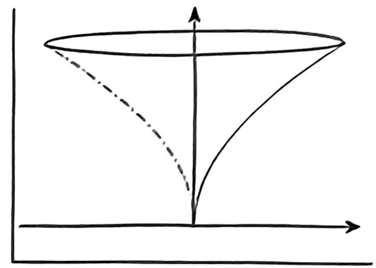
$\mathrm{d}s=\sqrt{\varphi'(t)^2+\psi'(t)^2}\,\mathrm{d}t$

$\tau=(\mathrm{d}x,\mathrm{d}y)$

$\mathrm{d}s=\sqrt{1+f'(x)^2}\,\mathrm{d}x$

$\mathrm{d}s=\sqrt{1+g'(y)^2}\,\mathrm{d}y$

例7.6.28. 求曲线段$x=y^2$，$0\leqslant y\leqslant 1$绕y轴旋转一周而成的旋转面面积.



解. $x=y^2$, $\dfrac{\mathrm{d}x}{\mathrm{d}y}=2y.$ $\qquad S=2\pi\displaystyle\int_{y=0}^{y=1}x\sqrt{1+\left(\frac{\mathrm{d}x}{\mathrm{d}y}\right)^2}\,\mathrm{d}y$

$$=2\pi\int_{y=0}^{y=1}y^2\sqrt{1+4y^2}\,\mathrm{d}y=\frac{2\pi}{3}\int_0^1\sqrt{1+4y^2}\,\mathrm{d}y^3$$

$$=\frac{2\pi}{3}y^3\sqrt{1+4y^2}\Big|_0^1-\frac{2\pi}{3}\int_0^1 y^3\,\frac{4y}{\sqrt{1+4y^2}}\,\mathrm{d}y=\frac{2\sqrt{5}\pi}{3}-\frac{2\pi}{3}\int_0^1 y^3\,\frac{4y}{\sqrt{1+4y^2}}\,\mathrm{d}y$$

$$=\frac{2\sqrt{5}\pi}{3}-\frac{2\pi}{3}\int_0^1 y^2\,\frac{1+4y^2-1}{\sqrt{1+4y^2}}\,\mathrm{d}y=\frac{2\sqrt{5}\pi}{3}-\frac{2\pi}{3}\int_0^1 y^2\left[\sqrt{1+4y^2}-\frac{1}{\sqrt{1+4y^2}}\right]\mathrm{d}y$$

$$=\frac{2\sqrt{5}\pi}{3}-\frac{1}{3}S+\frac{2\pi}{3}\int_0^1\frac{y^2}{\sqrt{1+4y^2}}\,\mathrm{d}y.$$

$$\int_0^1\frac{y^2}{\sqrt{1+4y^2}}\,\mathrm{d}y=\left[\frac{1}{8}y\sqrt{1+4y^2}-\frac{1}{16}\ln\left(2y+\sqrt{1+4y^2}\right)\right]\Bigg|_0^1$$

故, $\dfrac{4}{3}S=\dfrac{2\sqrt{5}\pi}{3}+\dfrac{\sqrt{5}\pi}{12}-\dfrac{\pi}{24}\ln(2+\sqrt{5})=\dfrac{3\sqrt{5}\pi}{4}-\dfrac{\pi}{24}\ln(2+\sqrt{5})$, $\qquad S=\dfrac{9\sqrt{5}\pi}{16}-\dfrac{\pi}{32}\ln(2+\sqrt{5}).$