定理8.2.8. 设 $f(x)$ 于 $[a,+\infty)$ 单调, $g(x)\in C(\mathbb{R}), g(x)\not\equiv 0,\ g(x)=g(x+T)\ (T>0)$ 则

$$\int_a^{+\infty} f(x)\,\mathrm{d}x \text{ 收敛} \Leftrightarrow \int_a^{+\infty} f(x)|g(x)|\,\mathrm{d}x \text{ 收敛}.$$

证明. "$\Leftarrow$" 首先证明: $\boxed{f(x)\text{单降}}$

$$\int_{n_0T}^{nT} f(x)\,\mathrm{d}x = \sum_{k=n_0}^{n-1}\int_{kT}^{(k+1)T} f(x)\,\mathrm{d}x \leqslant \sum_{k=n_0}^{n-1} f(kT)T$$

$$= \frac{1}{m}\sum_{k=n_0}^{n-1} f(kT)\ mT \ = \frac{1}{m}\sum_{k=n_0}^{n-1} f(kT)\int_{(k-1)T}^{kT}|g(x)|\,\mathrm{d}x$$

$$\leqslant \frac{1}{m}\sum_{k=n_0}^{n-1}\int_{(k-1)T}^{kT} f(x)|g(x)|\,\mathrm{d}x$$

不妨设 $f(x)\geqslant 0$,

$|g(x)|\leqslant M,\ x\in\mathbb{R}$.

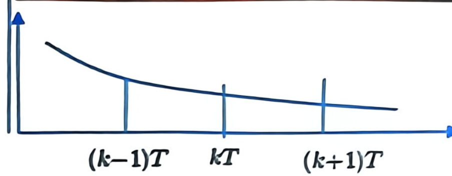
$m=\dfrac{1}{T}\displaystyle\int_0^T|g(x)|\,\mathrm{d}x,\ \ (m>0).$

已知 $\displaystyle\int_a^{+\infty} f(x)|g(x)|\,\mathrm{d}x$ 收敛,

往证 $\displaystyle\int_a^{+\infty} f(x)\,\mathrm{d}x$ 收敛.

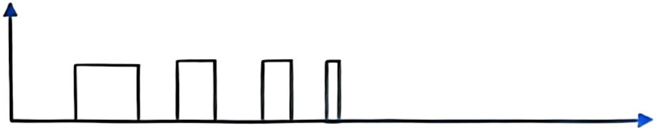
设 $a\leqslant n_0T,\ \ 1<n_0\in\mathbb{N}$.

$\boxed{\text{往证}\displaystyle\int_{n_0T}^{nT} f(x)\,\mathrm{d}x,\ (n>n_0)\text{ 有界.}}$



例8.2.17. 设$\int_a^{+\infty} f(x)\,\mathrm{d}x$绝对收敛，$\int_a^{+\infty} g(x)\,\mathrm{d}x$绝对收敛，

则$\int_a^{+\infty} f(x)g(x)\,\mathrm{d}x$的敛散性不可确定.



证明. 已知 $\lim\limits_{n\to\infty}\sum\limits_{k=1}^{n}\frac{1}{k}=+\infty$，$\lim\limits_{n\to\infty}\sum\limits_{k=1}^{n}\frac{1}{k^2}\ \exists \Rightarrow \sum\limits_{k=1}^{\infty}\frac{1}{k}$ 发散，$\lim\limits_{n\to\infty}\sum\limits_{k=1}^{\infty}\frac{1}{k^2}$ 收敛.

令 $f(x)=\begin{cases} n, & x\in[n,n+\frac{1}{n^3}) \\ 0, & x\in[n+\frac{1}{n^3},n+1) \end{cases}$，$n=1,2,\cdots$.

$$\text{则}\int_1^{+\infty} f(x)\,\mathrm{d}x=\sum_{n=1}^{\infty}\int_n^{n+1} f(x)\,\mathrm{d}x=\sum_{n=1}^{\infty}\frac{1}{n^2}\ \text{收敛.}$$

，$f^2(x)=\begin{cases} n^2, & x\in[n,n+\frac{1}{n^3}) \\ 0, & x\in[n+\frac{1}{n^3},n+1) \end{cases}$，$n=1,2,\cdots$.

$$\Rightarrow \int_1^{+\infty} f^2(x)\,\mathrm{d}x=\sum_{n=1}^{\infty}\int_n^{n+1} f^2(x)\,\mathrm{d}x=\sum_{n=1}^{\infty}\frac{1}{n}\ \text{发散.}\quad\square$$