Due: 1 pm on Wednesday, October 9
Submit by sending to: PKU.Quantum-Computing@outlook.com

Assignment 1

1 Product and entangled states

Determine which of the following states are entangled. If the state is not entangled, show how to write it as a tensor product; if it is entangled, prove this.

1) $\frac{2}{3}|00\rangle + \frac{2}{3}|01\rangle - \frac{1}{3}|11\rangle$
2) $\frac{1}{2}|00\rangle - \frac{i}{2}|01\rangle + \frac{i}{2}|10\rangle + \frac{1}{2}|11\rangle$
3) $\frac{1}{2}|00\rangle - \frac{1}{2}|01\rangle + \frac{1}{2}|10\rangle + \frac{1}{2}|11\rangle$

2 Unitary operations and measurements

Consider the state

$$|\psi\rangle = \frac{2}{3}|00\rangle + \frac{2}{3}|01\rangle - \frac{1}{3}|11\rangle.$$

1) Let $|\phi\rangle = (I \otimes H)|\psi\rangle$, where H denotes the Hadamard gate. Write $|\phi\rangle$ in the computational basis.
2) Suppose the first qubit of $|\phi\rangle$ is measured in the computational basis. What is the probability of obtaining 0, and in the event that this outcome occurs, what is the resulting state of the second qubit?
3) Suppose the second qubit of $|\phi\rangle$ is measured in the computational basis. What is the probability of obtaining 0, and in the event that this outcome occurs, what is the resulting state of the first qubit?
4) Suppose both qubits of $|\phi\rangle$ are measured in the computational basis. What are the probabilities of the four possible outcomes? Show that they are consistent with the marginal probabilities you computed in the previous two parts.

3 Distinguishing quantum states

Let $\theta \in [0, \pi/2]$ be a fixed, known angle. Suppose someone flips a fair coin and, depending on the outcome, either gives you the state

$$|0\rangle \quad \text{or} \quad \cos\theta|0\rangle + \sin\theta|1\rangle$$

(but does not tell you which). Among the measurements consisting of an orthonormal qubit basis for guessing which state you were given, which measurement gives you the highest success probability? Prove the optimality of your measurement.

4 More properties of the Bloch sphere

In our lecture, we introduced that the quantum state

$$\cos\frac{\theta}{2}|0\rangle + e^{i\phi}\sin\frac{\theta}{2}|1\rangle$$

can be presented by a point on the Bloch sphere below with coordinates

$$\hat{n} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta).$$

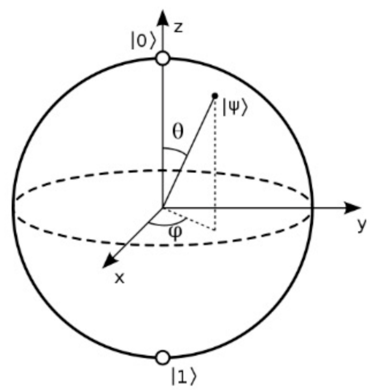

Figure 1: Bloch sphere.

1) Show that any two antipodal points on the Bloch sphere give two quantum states that are orthogonal. Note: For a point (x, y, z) on a sphere, its antipodal point is $(-x, -y, -z)$.

2) Given a point $\hat{n} = (n_x, n_y, n_z)$ on the Bloch sphere, show that the operator

$$\sigma_{\hat{n}} := n_x X + n_y Y + n_z Z$$

has eigenvectors which correspond to the points $\pm\hat{n}$ on the Bloch sphere. Here X, Y, Z are Pauli X, Y, Z operators, respectively.

3) Show that if the vectors $\hat{n}$ and $\hat{m}$ are orthogonal, then $\sigma_{\hat{n}}$ and $\sigma_{\hat{m}}$ anti-commute, i.e., $\sigma_{\hat{n}}\sigma_{\hat{m}} + \sigma_{\hat{m}}\sigma_{\hat{n}} = 0$.

5 Teleporting through a Hadamard gate

1) Write the state $(I \otimes H)|\beta_{00}\rangle$ in the computational basis.

2) Suppose Alice has a qubit in the state $|\psi\rangle$ and also, Alice and Bob share a copy of the state $(I \otimes H)|\beta_{00}\rangle$. If Alice measures her two qubits in the Bell basis, what are the probabilities of the four possible outcomes, and in each case, what is the post-measurement state for Bob?

3) Suppose Alice sends her measurement result to Bob. In each possible case, what operation should Bob perform in order to have the state $H|\psi\rangle$?