例10.2.1. 设 $f_n(x) \in C[a,b], \forall n \in \mathbb{N}, \mathbb{1}\{f_n(x)\} + f(a,b) -$ 致收敛,则 $\{f_n(x)\} + f[a,b] -$ 致收敛. (闭区间上连续函数列的一致收敛不可能抛开端点.)

证明. 往证: $f_n(a), f_n(b)$ 都收敛.

设
$$\{f_n(x)\}$$
于 (a,b) 一致收敛于 $f(x)$,则 $\forall \varepsilon>0$ $\exists N$ $s.t.$ $|f_n(x)-f_m(x)|<\varepsilon$, $\forall n,m>N, \ \forall x\in (a,b)$.

在上式中任意固定
$$n$$
和 m , 分别令 $x \to a + 0$ 和 $x \to b - 0$, 则由 $f_n(x)$ 在 $[a,b]$ 的连续性得

$$|f_n(a) - f_m(a)| \leqslant \varepsilon, \quad |f_n(b) - f_m(b)| \leqslant \varepsilon, \quad \forall n, m > N.$$

即
$$\{f_n(a)\}$$
, $\{f_n(b)\}$ 是两个 $Cauchy$ 列, 所以存在 $A,B \in \mathbb{R}$ 使得 $\lim_{n \to \infty} f_n(a) = A$, $\lim_{n \to \infty} f_n(b) = B$.

$$\exists N \text{ of } |f(a)| \quad A = c \quad \forall n > N \quad \exists N \text{ of } |f(b)| \quad P = c \quad \forall n > N.$$

i.e.
$$\exists N_1$$
 s.t. $|f_n(a) - A| < \varepsilon$, $\forall n > N_1$; $\exists N_2$ s.t. $|f_n(b) - B| < \varepsilon$, $\forall n > N_2$.

令
$$\tilde{f}(x) = \left\{ \begin{array}{ll} A, & x = a \\ f(x), & x \in (a,b) \end{array}, \; \text{则} \forall \varepsilon > 0, \; \mathbb{R}N_0 = \max\{N,N_1,N_2\}$$
时, $B, \quad x = b$

$$|f(x)| < \varepsilon, \quad \forall n > N_0, \quad \forall x \in [a,b].$$
所以, $f_n(x) \rightrightarrows \tilde{f}(x), \quad x \in [a,b].$

$$|f_n(x)- ilde{f}(x)|N_0,\;\;orall x\in [a,b]$$
.所以, $f_n(x)
ightrightarrows ilde{f}(x),\;\;x\in [a,b]$.

例10.2.4. 证明函数列
$$f_n(x) = n^2(e^{\frac{1}{nx}} - 1)\sin\frac{1}{nx} \quad (n \in \mathbb{N}) + f_n(x) = 0$$
 一致收敛性. 解:

(1)对任意固定的 $x \in [a, +\infty)$,
$$\lim_{n \to \infty} f_n(x) = \lim_{n \to \infty} n^2(e^{\frac{1}{nx}} - 1)\sin\frac{1}{nx} \quad \frac{\frac{1}{nx} = t}{t \to 0+0} \lim_{t \to 0+0} \frac{1}{x^2t^2}(e^t - 1)\sin t = \frac{1}{x^2},$$
i.e. $f_n(x) \to \frac{1}{x^2} \quad (n \to \infty), \quad x \in [a, +\infty).$

$$i.e. \ f_n(x) \to \frac{1}{x^2} \ (n \to \infty), \ x \in [a, +\infty).$$

$$(2) \left| f_n(x) - \frac{1}{x^2} \right| = \left| n^2 \left(e^{\frac{1}{nx}} - 1 \right) \sin \frac{1}{nx} - \frac{1}{x^2} \right| \qquad f(t) = e^t, \ f(t) = f(0) + f'(0)t + \frac{f''(\xi)}{2}t^2, \ \xi \in (0, t). \ \Rightarrow e^t - 1 = t + \frac{e^{\xi}}{2}t^2.$$

$$\Rightarrow \lim_{t \to \infty} \frac{1}{nx} - 1 - \frac{1}{nx} + \frac{e^{\xi}}{nx} \left(\frac{1}{nx} \right)^2 \qquad 0 < \xi < \frac{1}{nx}.$$

$$\sin \frac{1}{nx} = \frac{1}{nx} + \frac{\sin(\eta + \frac{3\pi}{2})}{nx} \left(\frac{1}{nx} \right)^3, \quad 0 \le \eta \le \frac{1}{nx}.$$

解:

$$= n^2 \left| \frac{e^{\xi}}{2} \left(\frac{1}{nx} \right)^3 + \frac{\sin(\eta + \frac{3\pi}{2})}{6} \left(\frac{1}{nx} \right)^4 + \frac{e^{\xi} \sin(\eta + \frac{3\pi}{2})}{12} \left(\frac{1}{nx} \right)^5 \right| \le \frac{e^{\xi}}{2} \frac{1}{nx^3} + \frac{1}{6} \frac{1}{n^2 x^4} + \frac{e^{\xi}}{12} \frac{1}{n^3 x^5}$$

$$= n^2 \left| \frac{e^5}{2} \left(\frac{1}{nx} \right)^3 + \frac{\sin(\eta + \frac{\pi}{2})}{6} \left(\frac{1}{nx} \right)^4 + \frac{e^5 \sin(\eta + \frac{\pi}{2})}{12} \left(\frac{1}{nx} \right)^5 \right| \le \frac{e^5}{2} \frac{1}{nx^3} + \frac{1}{6} \frac{1}{n^2 x^4} + \frac{e^5}{12} \frac{1}{n^3 x^5}$$

$$e^{\frac{1}{na}} \frac{1}{n^2} \frac$$

例10.2.4. 证明函数列 $f_n(x) = n^2(e^{\frac{1}{nx}} - 1)\sin{\frac{1}{nx}}$ $(n \in \mathbb{N})$ 于 $[a, +\infty)$ (a > 0)一致收敛性.

注10.2.4. 这里使用Lagrange余项型Taylor公式是关键之所在!

注意到当 $t \to 0$ 时有 $e^t - 1 \sim t + o(t)$, $\sin t \sim t + o(t)$, 从而当 $x \in [a, +\infty)$ 且 $n \to \infty$ 时有

因此我们有
$$\sup_{x \in [a, +\infty)} \left| n^2 (e^{\frac{1}{n\omega}} - 1) \sin \frac{1}{nx} - \frac{1}{x^2} \right| \le n^2 o\left(\frac{1}{n^2 a^2}\right) \to 0, (n \to \infty).$$

注意到当
$$t \to 0$$
时有 $e^t - 1 \sim t + o(t)$, $\sin t \sim t + o(t)$, 从而当 $x \in [a, +\infty)$ 且 $n \to \infty$ 时有
$$\left| n^2(e^{\frac{1}{n\omega}} - 1)\sin\frac{1}{nx} - \frac{1}{x^2} \right| = n^2 \left| (e^{\frac{1}{n\omega}} - 1)\sin\frac{1}{nx} - \frac{1}{n^2x^2} \right|$$

 $= n^2 \Big| \Big(\frac{1}{nx} + o\Big(\frac{1}{nx} \Big) \Big) \Big(\frac{1}{nx} + o\Big(\frac{1}{nx} \Big) \Big) - \frac{1}{n^2 x^2} \Big\| = n^2 o\Big(\frac{1}{n^2 x^2} \Big).$

例10.2.4. 证明函数列 $f_n(x) = n^2(e^{\frac{1}{nx}} - 1)\sin{\frac{1}{nx}} \quad (n \in \mathbb{N}) + f[a, +\infty) \quad (a > 0)$ 一致收敛性. 注10.2.4. 这里使用Lagrange余项型Taylor公式是关键之所在!

注意到当
$$t \to 0$$
时有 $e^t - 1 \sim t + o(t)$, $\sin t \sim t + o(t)$, 从而当 $x \in [a, +\infty)$ 且 $n \to \infty$ 时有
$$\left| n^2(a^{\frac{1}{12}} - 1) \sin \frac{1}{1} - \frac{1}{1} \right| = n^2(a^{\frac{1}{12}} - 1) \sin \frac{1}{1} - \frac{1}{1} \right|$$

$$\left| n^2 (e^{\frac{1}{n\omega}} - 1) \sin \frac{1}{nx} - \frac{1}{x^2} \right| = n^2 \left| (e^{\frac{1}{n\omega}} - 1) \sin \frac{1}{nx} - \frac{1}{n^2 x^2} \right|$$

$$\begin{aligned} \left| n^2 (e^{\frac{1}{nw}} - 1) \sin \frac{1}{nx} - \frac{1}{x^2} \right| &= n^2 \left| (e^{\frac{1}{nw}} - 1) \sin \frac{1}{nx} - \frac{1}{n^2 x^2} \right| \\ &= n^2 \left| \left(\frac{1}{nx} + o(\frac{1}{nx}) \right) \left(\frac{1}{nx} + o(\frac{1}{nx}) \right) - \frac{1}{n^2 x^2} \right| = n^2 o\left(\frac{1}{n^2 x^2} \right). \end{aligned}$$

$$= n^2 \left| \left(\frac{1}{nx} + o(\frac{1}{nx}) \right) \left(\frac{1}{nx} + o(\frac{1}{nx}) \right) - \frac{1}{n^2 x^2} \right| = n^2 o\left(\frac{1}{n^2 x^2} \right).$$

$$\text{(i)} = \sup_{x \to \infty} \left| n^2 \left(e^{\frac{1}{nx}} - 1 \right) \sin \frac{1}{n} - \frac{1}{n^2} \right| \le n^2 o\left(\frac{1}{n^2 x^2} \right) \to 0, (n \to \infty).$$

$$= n^{2} \left| \left(\frac{1}{nx} + o\left(\frac{1}{nx} \right) \right) \left(\frac{1}{nx} + o\left(\frac{1}{nx} \right) \right) - \frac{1}{n^{2}x^{2}} \right| = n^{2} o\left(\frac{1}{n^{2}x^{2}} \right).$$

##(1) $= \sup_{x \in \mathbb{R}^{2}} \left| \frac{1}{n^{2}x^{2}} - 1 \right| \sin \frac{1}{n^{2}} - \frac{1}{n^{2}} \left| \frac{1}{n^{2}x^{2}} - 1 \right| = n^{2} o\left(\frac{1}{n^{2}x^{2}} \right).$

 $\sup_{x\in[a,+\infty)}\left|n^2(e^{\frac{1}{nx}}-1)\sin\frac{1}{nx}-\frac{1}{x^2}\right|\leq n^2o\left(\frac{1}{n^2a^2}\right)\to 0, (n\to\infty).$

因此我们有
$$\sup_{x \in [a, +\infty)} \left| n^2(e^{\frac{1}{n\omega}} - 1) \sin \frac{1}{nx} - \frac{1}{x^2} \right| \le n^2 o\left(\frac{1}{n^2a^2}\right) \to 0, (n \to \infty).$$
 ?

 $= n^2 \left| \frac{e^{\xi}}{2} \left(\frac{1}{nx} \right)^3 + \frac{\sin(\eta + \frac{3\pi}{2})}{6} \left(\frac{1}{nx} \right)^4 + \frac{e^{\xi} \sin(\eta + \frac{3\pi}{2})}{12} \left(\frac{1}{nx} \right)^5 \right|$

所以, $\left| f_n(x) - \frac{1}{x^2} \right| = \left| n^2 \left[\frac{1}{nx} + \frac{e^{\xi}}{2} (\frac{1}{nx})^2 \right] \left[\frac{1}{nx} + \frac{\sin(\eta + \frac{3\pi}{2})}{6} (\frac{1}{nx})^3 \right] - \frac{1}{x^2} \right|$

 $\leq \frac{e^{\xi}}{2} \frac{1}{n^{3}} + \frac{1}{6} \frac{1}{n^{2}n^{4}} + \frac{e^{\xi}}{12} \frac{1}{n^{3}n^{5}} \leq \frac{e^{\frac{1}{na}}}{2} \frac{\frac{1}{na}}{12} + \frac{1}{6} \frac{1}{n^{2}} + \frac{e^{\frac{1}{na}}}{12} \frac{1}{n^{2}} \to 0 \quad (n \to \infty).$

更有甚者,在 $x_n'=a+\frac{1}{n^2}$ 时, $g_n(x_n')\to +\infty$ $(n\to\infty)$, 即 $\frac{w_n(x)}{\frac{1}{(na)^2}}$ 于 $[a,+\infty)$ 并不有界,所以不可能有 $w_n(x)\leq o((\frac{1}{na})^2)$, x>a. 问题出在哪里呢?事实上,

命题10.2.1. 若 $g(t) = o(t), (t \to 0)$ 且 $\alpha(x,n) \Rightarrow 0, n \to +\infty, x \in I$. 则 $o(\alpha(x,n)) = \alpha(x,n) \ o(1), n \to +\infty$.

本来,在只有一个变元的情况下,o(t)=t o(1), $t\to 0$,是完全正确的,但在含有两个及以上的变元的时候,等式(*)是不准确的.

比如,对任意 $x>a,\,w_n(x)=rac{1}{(nx)^2}rac{1}{(x-a)n}=o((rac{1}{nx})^2),\,\,\,(n o\infty),\,\,\,\,\,$ 但是 $g_n(x)=rac{w_n(x)}{rac{1}{(x-a)^2}}=(rac{a}{x})^2rac{1}{(x-a)n}
eq 0,\,\,\,x>a.$

(*)

 $\left| o((\frac{1}{nx})^2) \right| \leq o((\frac{1}{na})^2), \quad x > a, \ n \to \infty. \quad \left| \ o((\frac{1}{nx})^2) \right| = (\frac{1}{nx})^2 o(1), \quad x > a, \ n \to \infty.$

事实上, 当 $x_n = a + \frac{1}{n}$ 时, $g_n(x_n) = (\frac{a}{a+1})^2 \to 1$, $(n \to \infty) \Rightarrow g_n(x) \not = 0$, (x > a).

 $o((\frac{1}{n\pi})^2) = (\frac{1}{n\pi})^2 \alpha(x,n), \quad n \to +\infty, \ x \in I, \ \sharp \ \forall \lim_{n \to +\infty} \alpha(x,n) = 0, \ \forall x \in I.$

由此可见, 在使用 Taylor公式或等价无穷小量讨论一致收敛问题时, 要特别注意一致性.

如果(在某些情况下)一定要使用Peano余项型Taylor公式,则需要如下的结论.

如果 $\alpha(x,n) \not \Rightarrow 0, \ n \to +\infty, \ x \in I$, 则 $\alpha(x,n) \neq o(1), \quad n \to +\infty$.

它的右端o(1)是在 $n \to \infty$ 过程中关于x的一致无穷小量,而左端的无穷小量是否关于x一致是未知的.