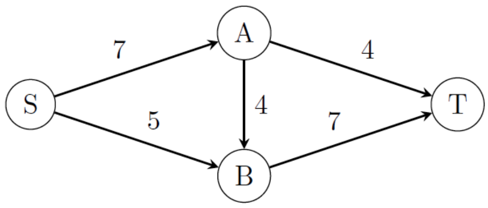
Assignment 8

1 Max Flow, Min Cut, and Duality

In this exercise, we will demonstrate that LP duality can be used to show the max-flow min-cut theorem.

Consider the following instance of max flow:



Let f_1 be the flow pushed on the path $\{S, A, T\}$, f_2 be the flow pushed on the path $\{S, A, B, T\}$, and f_3 be the flow pushed on the path $\{S, B, T\}$. The following is an LP for max flow in terms of the variables f_1, f_2, f_3:

$$
\begin{aligned}
\max\ f_1 + f_2 + f_3 & \\
f_1 + f_2 \leqslant 7 & \qquad \text{(Constraint for } (S, A)\text{)} \\
f_3 \leqslant 5 & \qquad \text{(Constraint for } (S, B)\text{)} \\
f_1 \leqslant 4 & \qquad \text{(Constraint for } (A, T)\text{)} \\
f_2 \leqslant 4 & \qquad \text{(Constraint for } (A, B)\text{)} \\
f_2 + f_3 \leqslant 7 & \qquad \text{(Constraint for } (B, T)\text{)} \\
f_1, f_2, f_3 \geqslant 0 &
\end{aligned}
$$

(a) Find the dual of this LP, where the variables in the dual are x_e for each edge e in the graph.

(b) Show that the dual of the LP for any max-flow problem is an LP for the corresponding min-cut problem.

2 Network Flow with Vertex Capacities

Let $G = (V, E)$ be a directed graph with a source vertex $s \in V$ and a sink vertex $t \in V$. Whereas the standard network flow problem involves capacities for edges, here we suppose instead that every vertex $v \in V$ has an integer capacity $c_v \geqslant 0$. A *vertex-capacitated* flow in G is a function $f : E \to [0, +\infty)$ such that

- (Capacity Constraint) For each vertex $v \in V$, we have

$$\sum_{e \text{ into } v} f(e) \leqslant c_v \text{ and } \sum_{e \text{ out of } v} f(e) \leqslant c_v$$

- (Conservation Constraint) For each vertex $v \in V/\{s, t\}$, we have

$$\sum_{e \text{ into } v} f(e) = \sum_{e \text{ out of } v} f(e)$$

As usual, the value of a flow is defined as $\sum_{e \text{ out of } s} f(e)$. Give an efficient algorithm to find a maximum vertex-capacitated flow in G from s to t and analyze its running time.

3 Restoring the Balance!

We are given a network $G = (V, E)$ whose edges have integer capacities $c(e)$, and a maximum flow f from source s to sink t. Explicitly, f is given to us in the representations of integer flows along every edge e, $(f(e))$.

However, we find out that one of the capacity values of G was wrong: for edge (u, v), we used $c(u, v)$ whereas it should have been $c(u, v) - 1$. This is unfortunate because the flow f uses that particular edge at full capacity (i.e., $f(u, v) = c(u, v)$). We could rerun Ford-Fulkerson (or its improved versions) from scratch, but there shall be a faster way.

Design an algorithm to fix the max-flow for this network in $O(|V| + |E|)$ time. **Please give a 3-part solution.**

4 Optimal Edge Removal to Reduce Max Flow

Given a flow network $G = (V, E)$ with source s, sink t, and unit-capacity edges ($c(e) = 1$ for all $e \in E$), and an integer k, find a set $F \subseteq E$, $|F| \leq k$, to minimize the max s-to-t flow in $G' = (V, E - F)$.

Task: Give an algorithm to solve this problem. **Please give a 3-part solution.**