1 Governing equations

In this assignment, we will use basic numerical methods to solve the linear shallow water equations (LSWE) in 1DH:

$$\begin{cases} \dfrac{\partial \eta}{\partial t} + \dfrac{\partial hU}{\partial x} = 0 \\ \\ \dfrac{\partial U}{\partial t} = -g\dfrac{\partial \eta}{\partial x} \end{cases}, \tag{1}$$

where η is the free surface elevation, U is the horizontal flow velocity, h is the still water depth, and g is the gravitational acceleration.

2 Discretization

We can employ the Lax-Friedrichs method to discretize (1) as

$$\begin{cases} \eta_i^{(n+1)} = \dfrac{\eta_{i+1}^{(n)} + \eta_{i-1}^{(n)}}{2} - \dfrac{\Delta t}{2\Delta x}\left[U_{i+1}^{(n)} h_{i+1} - U_{i-1}^{(n)} h_{i-1}\right] \\ \\ U_i^{(n+1)} = \dfrac{U_{i+1}^{(n)} + U_{i-1}^{(n)}}{2} - \dfrac{\Delta t}{2\Delta x} g\left[\eta_{i+1}^{(n)} - \eta_{i-1}^{(n)}\right] \end{cases}, \tag{2}$$

where (n) denotes variables at the current time step, $(n+1)$ denotes variables at the new time step, i denotes the i-th discretized node in space, Δx is the step size in space, and Δt is the step size in time

The relation between Δx and Δt is required by the CFL (Courant, Friedrichs, and Lewy) condition:

$$\Delta t = C_{\text{CFL}} \frac{\Delta x}{\sqrt{gh_0}}, \tag{3}$$

where the constant C_{CFL} is often referred to as the *Courant number* or the CFL number, and h_0 is the *maximum water depth* in the problem.

The Lax-Friedrichs method can be proved to be stable for $0 \leq C_{\text{CFL}} \leq 1$ (see for example the textbook by LeVeque, 1992). In practice, $C_{\text{CFL}} = 0.9$ is found to be an optimal choice, which we will use in this assignment.

3 Boundary conditions

In this assignment, let us consider a numerical wave flume spanning from $x = -12$ m to $x = 24$ m. The water depth is constant, $h = h_0 = 0.3$ m. The two ends of the wave flume are solid walls. At the left end, $x = -12$ m for example, the wall boundary condition means

$$\begin{cases} U_{i=1} = 0 \\ \\ \dfrac{\partial U}{\partial t} = 0 = -g\dfrac{\partial \eta}{\partial x} \;\; \Rightarrow \;\; (\dfrac{\partial \eta}{\partial x})_{i=1} = 0 \end{cases}. \tag{4}$$

This also means the ghost cells to the left of $x = -12$ have the values

$$\begin{cases} \eta_{i=0} = \eta_{i=2} \\ \\ U_{i=0} = 0 \end{cases}. \tag{5}$$

4 Initial conditions

We need to specify the initial conditions – $\eta(x, 0)$ and $U(x, 0)$. In constant water depth, we know that any wave of translation (平移波) moving at the speed $\sqrt{gh}$ is a solution to the 1DH LSWE (1). This can be verified by checking that a function of the form $f(x - \sqrt{gh} \cdot t)$ is a solution to (1) in constant water depth.

In a addition, based on the linear wave theory, which we reviewed on the first day of class, the horizontal flow velocity for linear shallow water waves can be determined as

$$U(x, t) = \frac{\eta(x, t)}{h} \sqrt{gh}. \tag{6}$$

As the initial conditions for this assignment, we will use a wave of translation of the form

$$\eta(x, t) = H\mathrm{sech}^2\Big(K(x - Ct)\Big), \quad K = \frac{1}{h}\sqrt{\frac{3H}{4h}}, \tag{7}$$

where H is the wave height, C denotes the wave speed ($C = \sqrt{gh}$ for LSWE), and sech(x) is the hyperbolic secant function. K can be seen as the effective wave number for this wave, and an effective wavelength L can be defined as

$$L = \frac{2\pi}{K}. \tag{8}$$

An effective wave period T can also be defined:

$$T = \frac{L}{C}. \tag{9}$$

A wave of the form (7), whose flow velocity can be calculated from (6), is called the *solitary wave* (孤立波). It is often used as a benchmark wave in many long-wave studies. In this assignment, we will use $H = 0.04$ m. In a water depth of $h = 0.3$ m, this means that the effective wavelength is $L = 5.961$ m, and the effective wave period is $T = 3.475$ s.

Assignment (due online on Sunday, 2020/3/22)

Please type up your work as a report. You should do as much as you can to make the report "self-contained" (自我完整獨立的). A reader should be able to get all the necessary information from your report in order to understand your work. This means that just like this document, you should discuss the governing equations, numerical methods, numerical domain size, boundary conditions, and initial conditions in your report.

1. Plot the initial conditions, i.e., $\eta(x, 0)$ and $U(x, 0)$, to be used in the simulations. The results would look similar to Figure 1.

2. Write a program to solve the 1DH LSWE using these initial conditions, for $-12 \leq x \leq 24$ (m) and $0 \leq t \leq 6.95$ (s). You can try using the step size $\Delta x = 0.06$ m as a start.

3. Make sure your code runs correctly. To gain confidence in your results, you may want to compare your numerical results for η against the analytical solution at $t = 6.95$ s, i.e., (7).

4. Run two additional simulations with different step sizes Δx. Then plot the analytical solution and all your numerical results in one plot at $t = 6.95$ s. The results would look similar to Figure 2.

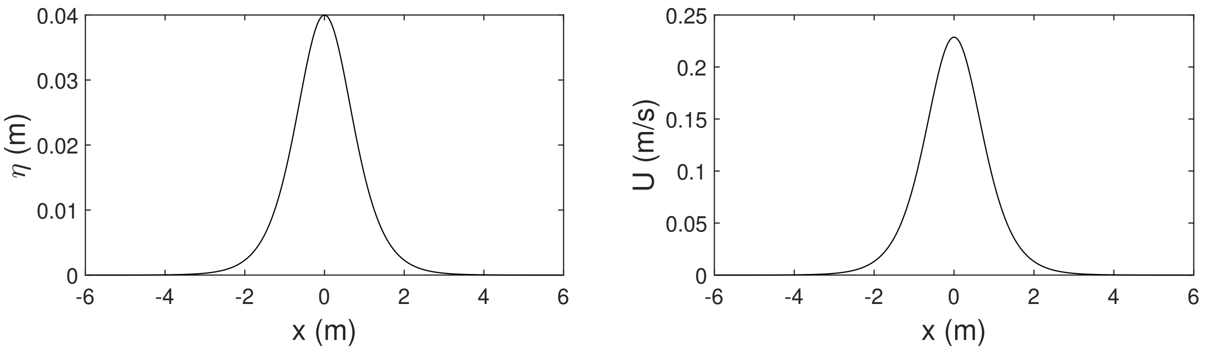

Figure 1: The initial conditions to use in the simulations, as given by (7) and (6) at $t = 0$ s.

5. Show that the convergence rate of the Lax-Friedrichs numerical scheme is first-order; i.e., $\Delta x \propto L^2$-norm, by plotting the three different step sizes against the corresponding L^2-norms.

   For each data set at $t = 6.95$, the L^2-norm is calculated as

$$L^2\text{-norm} = \sqrt{\frac{1}{N}\sum_{i=1}^{N}(\eta_{i,\text{num}} - \eta_{i,\text{true}})^2}, \tag{10}$$

   where N is the number of discretized nodes in space, $\eta_{i,\text{true}}$ is the analytical solution based on (7) at the i-th node, $\eta_{i,\text{num}}$ is the model-predicted value at the i-th node.

   Calculate the three different L^2-norms for the three different step sizes you use. Then plot Δx vs. L^2-norm. The results would look similar to Figure 3. All points should more or less fall on a straight line, because we expect $\Delta x \propto L^2$-norm.

Tips

1. If you are using a MATLAB-based language, keep in mind that operations with and without a period sign (such as '.*' vs. '*', and './' vs. '/') mean different things in vector operations.

2. If you want to plot all your results at exactly $t = 6.95$ s, you may need to adjust the Δt for the last time step to reach exactly $t = 6.95$ s.

3. Alternatively, you may just plot your results at slightly different times, such as $t = 6.9250$ s, $t = 6.9408$, $t = 6.9486$ s, etc. When calculating the L^2-norms, be sure to use the actual time to determine the analytical solution – for example, if the numerical results end at $t = 6.9250$ s, then calculate the analytical solution also at $t = 6.9250$ s.

4. Just for fun, you can try changing the Courant number to $C_{\text{CFL}} = 1$ and $C_{\text{CFL}} = 0.1$. You'll notice that there is minimal numerical dissipation only in the special case with $C_{\text{CFL}} = 1$; in cases with $C_{\text{CFL}} < 1$, numerical dissipation of the Lax-Friedrichs scheme is highly significant. As a result, the Lax-Friedrichs method is pretty much useless in practice without adjustments, because it is rarely possible to use $C_{\text{CFL}} = 1$ in complex problems.

5. Since we use the wall boundary condition, the wave should be reflected at the two ends. You can shorten your numerical domain or increase the simulation time to see if wave reflection happens.

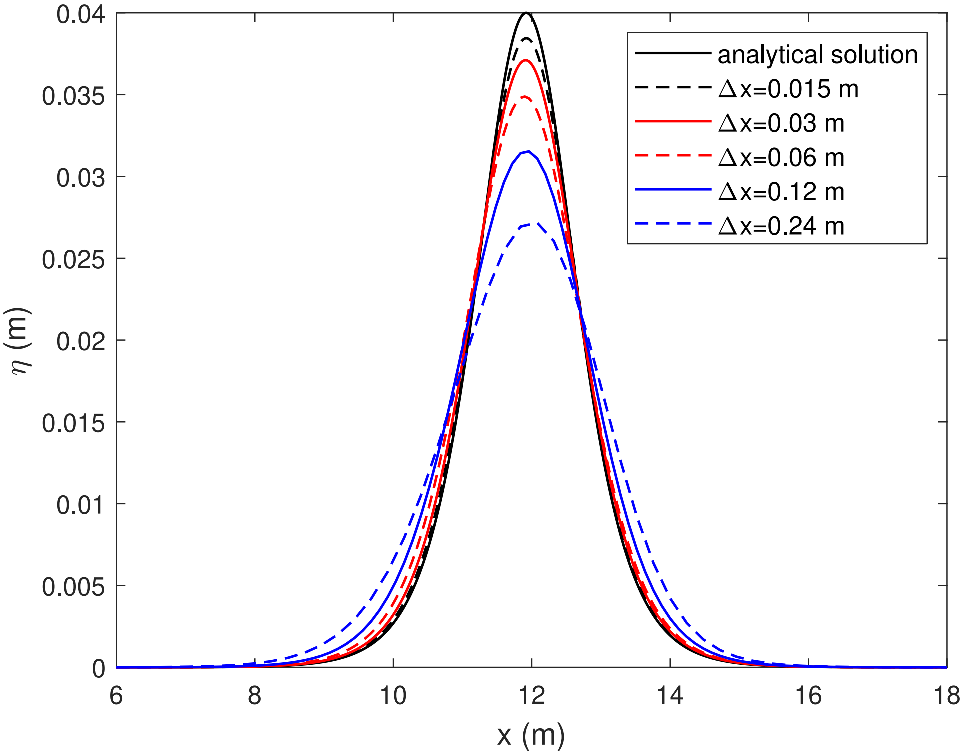

Figure 2: Comparison of the analytical solution against numerical results using different step sizes, after the wave has propagated two wavelengths.

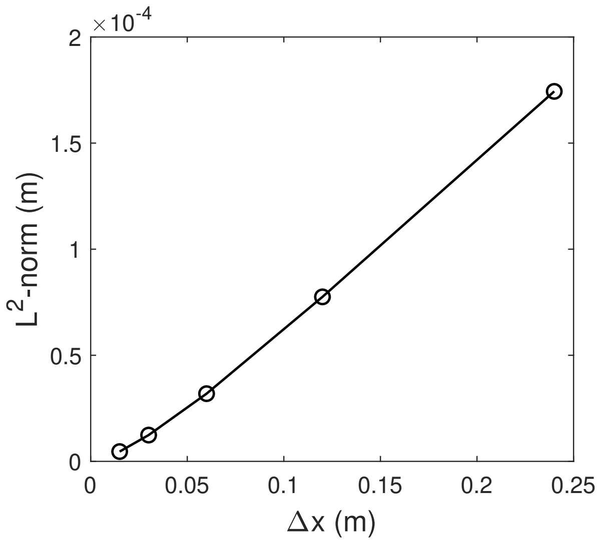

Figure 3: Step size plotted against the L^2-norm. The linear trend demonstrates the first-order convergence rate of the Lax-Friedrichs numerical scheme.