Assignment (due online on Sunday, 2020/4/5)

Please type up your work as a report. You should do as much as you can to make the report "self-contained" (自我完整獨立的). A reader should be able to get all the necessary information from your report in order to understand your work. To prepare this report, you may use your report for Assignment#2 as a base – simply modify it and add new content to it.

1. Solve the same wave propagation problem as in Assignment#2. Instead of using the Lax-Friedrichs method, now use the third-order Strong Stability-Preserving Runge-Kutta (SSP-RK) scheme in time, and the fourth-order central difference scheme in space (details are given in the lecture handout for 2020/3/23). With $C_{\mathrm{CFL}} = 0.9$, run a total of three simulations using a total of three different step sizes Δx. Then plot the analytical solution and all your numerical results in one plot at $t = 6.95$ s. The results would look similar to Figure 1.

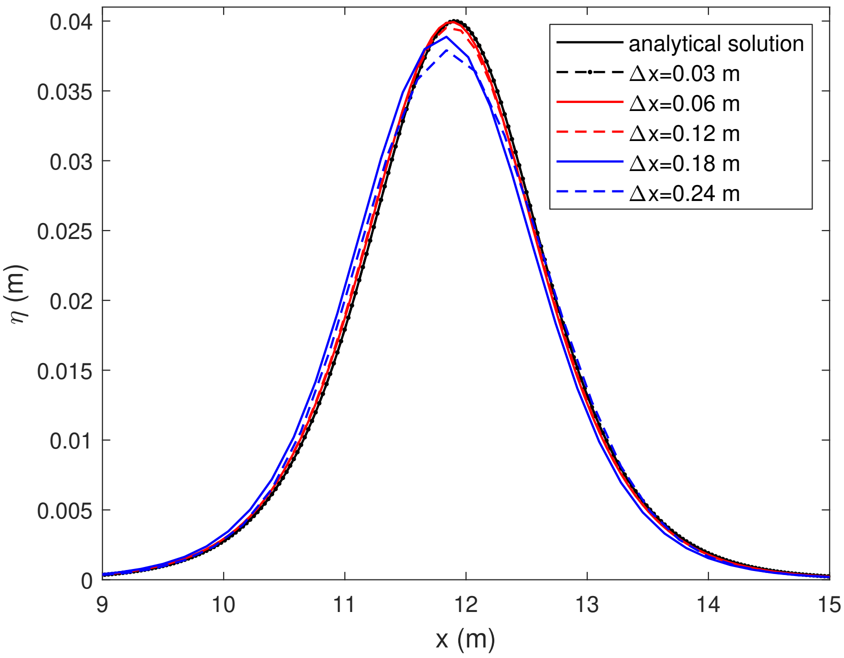

Figure 1: Comparison of the analytical solution against numerical results using different step sizes, after the wave has propagated two wavelengths.

2. Show that the overall convergence rate of this numerical scheme is third-order; i.e., $\Delta x^3 \propto L^2$-norm, by plotting the three different step sizes against the corresponding L^2-norms. The results would look similar to Figure 2. All points should more or less fall on a straight line, because we expect $\Delta x^3 \propto L^2$-norm.

3. Pick a Δx that you believe to be adequate to yield accurate results, increase the simulation time to $t = 13.9$ s, and change the water depth for $x > 12$ m to $h_2 = 0.1$ m (so that the water depth for $-12 \leq x \leq 12$ m is $h_1 = 0.3$ m). Calculate the ratio of the transmitted wave height to the incident wave height (透射波高除以入射波高), $H_{\mathrm{T}}/H_{\mathrm{I}}$, and the ratio of the reflected wave height to the incident wave height (反射波高除以入射波高), $H_{\mathrm{R}}/H_{\mathrm{I}}$. Compare the numerical results with the analytical solutions.

   For example, the wave height ratios determined from my numerical simulation are $H_{\mathrm{T}}/H_{\mathrm{I}} = 1.262$ and $H_{\mathrm{R}}/H_{\mathrm{I}} = 0.268$, while the analytical solutions are

$$\frac{H_{\mathrm{T}}}{H_{\mathrm{I}}} = \frac{2C_1}{C_1 + C_2} \simeq 1.268, \quad \frac{H_{\mathrm{R}}}{H_{\mathrm{I}}} = \frac{C_1 - C_2}{C_1 + C_2} \simeq 0.268. \tag{1}$$

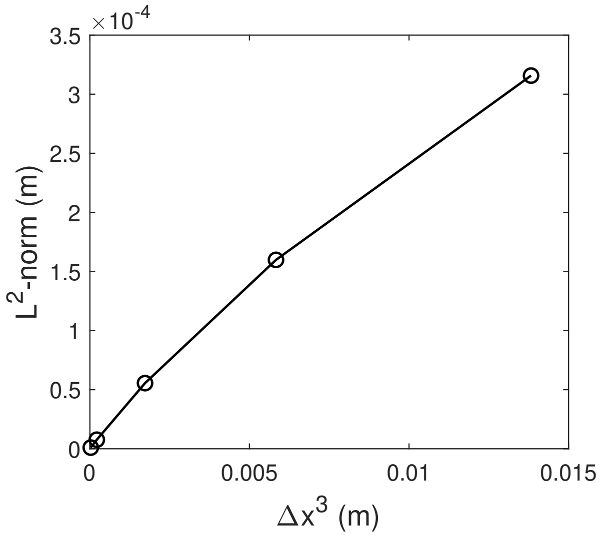

Figure 2: Step size cubed plotted against the L^2-norm. The linear trend in terms of Δx^3 vs. L^2-norm demonstrates the third-order convergence rate of this numerical scheme.

Tips

1. Try to allow users to specify the time intervals for data output. For example, if I specify that I want the numerical results at $t = 0.1 : 0.1 : 7.0$ (s), then your program should only save data at these precise times. The intermediate results need not be saved.

2. To be able to reach these user-specified times, you may need to modify the time step size Δt inside your loop in time. The CFL condition requires

$$\Delta t \leq C_{\text{CFL}} \frac{\Delta x}{\sqrt{gh}}, \tag{2}$$

so you can freely *decrease* Δt when necessary.

3. The goal is to minimize storage of your numerical model. If you save all your results, everything, everywhere, at all times, the size of your data and memory usage become unnecessarily large. Our calculations are only 1DH right now, so storage really isn't a big problem. However, when we solve 2DH problems, code efficiency becomes important.

4. Numerical methods *hate* discontinuities. The sudden change of the water depth from $h_1 = 0.3$ m to $h_2 = 0.1$ m is such a discontinuity. Therefore, you will see numerical oscillations caused by this jump. A common solution is to "smooth out" the transition. For example, you can use the hyperbolic tangent function, $\tanh(x)$, to obtain a "smooth jump". You can try defining the water depth $h(x)$ as

$$h(x) = 0.3 - 0.2 \frac{1 + \tanh\Big(a(x - 12)\Big)}{2}, \tag{3}$$

where the constant a controls the smoothness of this jump (try $a = 8$ as a start).

5. Can you overlap the free surface η predicted by the numerical model with that predicted by the analytical solution? Note: you will need to shift the time and space of the analytical solution (or, shift the numerical results instead), because in the analytical solution, the jump is located at $x = 0$ m, and the incident wave reaches $x = 0$ m at $t = 0$ s.