Mathematical Modelling Notes

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1 Modelling Process

Factors

Anything that will have some effect on the your calculations e.g. drag, friction, mass, buoyancy, area. Note that separate things qualities will count separately e.g. a triangle's area and a square's.

Assumptions

A statement that makes the problem simpler - cancels factors. e.g. 'Total length of wood is not reduced when it is cut' or 'There are no significant currents'

Precise Problem Statement

Given (key factors, assumptions and values needed to solve), Find (the value you're asked to find)

Formulating a Model

$$x \propto y, 1/z \implies x = \frac{ky}{z}$$

Modelling Forces

Use Newton's 2nd law, subtract negative forces and add positive ones. These are some common models:

Force	Model
Gravity	$f_g = mg$
Spring	$f_s = -kx$
Friction	$f_f = \mu f_n$
Buoyancy	$f_b = \rho V g$
Drag Force	$f_d = -cv$
Drag Force	$f_d = cv^2$

Dimensions

By performing dimensional analysis, we can produce a model or determine the dimension of a property. Some common dimensions include,

Length	L
Mass	M
Time	T
Temperature	θ
Area	L^2
Density	$\frac{M}{L^3}$
Force	$\frac{ML}{T^2}$
Energy	$\frac{ML^2}{T^2}$

2 Probability

Some General Rules

$$\begin{aligned} Pr(A|B) &= \text{Probability of A happening given B has already happened} \\ &= \frac{Pr(A\&B)}{Pr(B)} \\ Pr(A\&B) &= Pr(A) \times Pr(B) \text{ If they are independent} \end{aligned}$$

Continuous Random Variables

Continuous Random Variables that are infinite i.e height as there are an infinite amount of possible heights with infinite accuracy. When graphed the area under a graph (probability density function) will always sum to one i.e

$$\int_{\infty}^{\infty} f(x)dx = 1$$

PDFs and CDFs

The probability density function (PDF) is the density of a continuous random variable, it provides a value for a 'probability' that the random variable would equal a given sample. The integral of a PDF is the cumulative distribution function (CDF) gives the probability for less than or greater than the given sample.

Frequency/Probablity Trees

A probability tree is a tree in which you label edges with the probability of an event and label vertices with each event. A frequency tree is very similar but you multiply those probabilities by some fixed amount i.e 1000 people etc. These will carry through so you multiply the number in the parent vertex by the amount on that edge.

Bayes Theorem

$$Pr(A|B) = \frac{Pr(A)}{Pr(A)Pr(B|A) + Pr(\ A)Pr(B|\ A)}$$

3 Differentiation

Series and Approximation

These infinite series allow you to approximate any function for as many or as few terms as you like. It can be important for these questions to remember that 'linear' means 2 terms, 'quadratic' means 3 terms and 'cubic' means 4.

Maclaurin Series

The Maclaurin Series is an approximation that is accurate around $x = 0$, it is a special case of the Taylor Series.

$$f(x) = f(0)f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(x)}{3!}x^3 \dots$$

Taylor Series

The Taylor Series is an approximation that is accurate around $x = a$.

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 + \frac{f'''(0)}{3!}(x-a)^3 \dots$$

Binomial Series

You will likely have to manipulate/factor your expression to get into the binomial form.

$$(1+x)^n = 1 + \frac{n}{1!}x + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 \dots$$

Implicit Differentiation

Implicit Differentiation is simply differentiating term by term for implicit equations. That is, equations that have both x and y on either side of the equation. e.g.

$$\begin{gathered}
\text{Finding } \frac{dy}{dx} \text{ for } x^2 - 2x + 2y^3 = 1 \\
2x\frac{dx}{dx} - 2\frac{dx}{dx} + 6y^2\frac{dy}{dx} = 0 \\
\implies \frac{dy}{dx} = \frac{1-x}{3y^2}
\end{gathered}$$

Numerical Differentiation

Numerical Differentiation essentially comes down to manipulation of the definition of a derivative. Mostly using these 4 equations:

$$f'(x) \approx \frac{f(x+h) - f(x)}{h} \text{ (first forward difference)}$$

$$f'(x) \approx \frac{f(x) - f(x-h)}{h} \text{ (first backward difference)}$$

$$f'(x) \approx \frac{f(x+h) - f(x-h)}{h} \text{ (first central difference)}$$

$$f''(x) \approx \frac{f(x+h) - 2f(x) + f(x-h)}{h^2} \text{ (second central difference)}$$

4 Integration

Integration Techniques

1. u substitution $dx = du \ldots$
2. Integration by Parts $\int u \frac{dv}{dx} dx = uv - \int v \frac{du}{dx} dx$

Partial Fractions

Partial Fractions is a method to split up rational functions into easy to integrate functions. It comes in 3 different forms:

Simple Form

$$\frac{p(x)}{(x-a)(x-b)(x-c)\ldots} = \frac{A}{x-a} + \frac{B}{x-b} + \frac{C}{x-c} + \ldots$$

Repeated Factors

$$\frac{p(x)}{(x-a)(x-b)^3\ldots} = \frac{A}{x-a} + \frac{B_1}{x-b} + \frac{B_2}{(x-b)^2} + \frac{B_3}{(x-b)^3} \ldots$$

Irreducible quadratic factors

$$\frac{p(x)}{(x-a)(x^2+bx+c)} = \frac{A}{x-a} + \frac{Bx+c}{x^2+bx+c} + \ldots$$

The cover-up rule

When the denominator of the split up fraction is of the form $x - a$ you can 'cover-up' that factor in the denominator and then plugin the x value from $x - a = 0$ and solve. This will give you the 'A' value.

5 Ordinary Differential Equations

Ordinary differential equations are equations containing one or more functions of one independent variable. You can recognise an ODE from a PDE (partial differential equation) because a PDE will contain ∂ (pronounced 'del') and ODEs have standard 'd'. $\frac{dy}{dx}$ means y is the dependent variable and x is the independent variable. $\frac{dx}{dt}$ x is dependent, t is independent.

Properties

Order

Highest derivative (also equal to number of values needed to find a particular solution) e.g.

$$\frac{dy}{dx} = 5x \text{ is 1st Order}$$
$$\frac{d^4y}{dx^4} = \frac{dy}{dx} + 2 \text{ is 4th Order}$$

Linear

Involves only derivatives of y and terms of y to the 1st power e.g. ONLY $\frac{dy}{dx}$, y etc.

$$\frac{d^4y}{dx^4} + \frac{dy}{dx} = 2 \text{ is linear}$$
$$\frac{dy}{dx} = 2y + 3 \text{ is linear}$$

Homogeneity

If all (non-zero) terms involve the dependent variable then the equation is homogeneous

$$\frac{dx}{dt} = x \text{ is homogeneous}$$
$$\frac{dy}{dx} = 2y + 3 \text{ is not homogeneous (3 doesn't involve x)}$$

Forming Differential Equations

In typical exam questions there are few points at which you will form a differential equation: modelling a set of forces in the typical modelling questions, using proportionality or previous knowledge. Typically the modelling questions will use Newton's 2nd law which states $\sum F = ma$ and then you can sum the forces and use it to find mass/acceleration (or their derivatives).

Solving Differential Equations

1. Direct Integration
2. Separation of Variables
3. Euler's Method
4. Integrating Factor
5. Exponential Substitution

Euler's Method

Euler's method is a numerical method for solving D.E.s that uses a 2 term Taylor Series and is simply stated by the equation.

$$f(x+h) \approx f(x) + hf'(x)$$

These questions are easiest to complete if you use a table for the values of x, $f(x)$, $f'(x)$ and then calculate $f(x+h)$ e.g.

x	$f(x)$	$f'(x)$	$f(x) + hf'(x)$
1	1^2	2	4
4	...	...	...

Exponential Substitution

Exponential Substitution allows you to solve a 2nd order O.D.E using something called a 'characteristic equation'. You substitute in a trial solution $e^{\lambda t}$ and then complete the substitution (differentiate) and then divide through by your trial solution leaving you with the characteristic equation which you can then solve for λ (note: if both solutions are negative, it is called 'stable'). You then substitute your values into the equation:

$$x = C_1 e^{\lambda t} + C_2 e^{\lambda t}$$

This is a general solution, to calculate the exact solution you need two points (for a 2nd order equation) to calculate values of C_1 and C_2. e.g

$$\frac{d^2x}{dt^2} - \frac{dx}{dt} - 6x = 0$$
$$\lambda^2 - \lambda - 6 = 0$$
$$\implies \lambda = 3, -2$$
$$x = c_1 e^{3t} + c_2 e^{-2t} \text{ gen. soln.}$$
$$\text{Given x(0) = 0, x'(0)=5}$$
$$0 = C_1 + C_2 \implies C_1 = -C_2$$
$$5 = 3C_1 + -2C_2 \implies C_2 = -1$$
$$\implies C_1 = 1$$
$$x = e^{3t} - e^{-2t} \text{ exact soln.}$$

Integrating Factor

$$\text{if } h(x) = \int p(x)dx$$

$$\frac{dy}{dx} + p(x)y = r(x) \text{ has solution } y = e^{-h(x)} \int e^{h(x)} r(x)dx$$

6 Vectors

Vectors are a special case of matrices that only involve one column (for column vectors) or one row (row vectors). It is usually described as a directed length or arrow i.e it has both direction and magnitude! Vectors can be written in a number of ways:

$$a\hat{\imath} + b\hat{\jmath} + c\hat{k}$$

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

$$\begin{bmatrix} a & b & c \end{bmatrix}$$

The magnitude (length) of a vector is given by

$$\sqrt{a^2 + b^2 + c^2}$$

The dot product of two vectors is described by the formula

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} \cdot \begin{bmatrix} d \\ e \\ f \end{bmatrix} = a \times d + b \times e + c \times f$$

Note that this formula works for vectors of all sizes. The dot product can also be defined geometrically as $|a||b|\cos\theta$ where a and b are the two input vectors.

The cross product of two vectors is given by

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} \times \begin{bmatrix} p \\ q \\ r \end{bmatrix} = \begin{bmatrix} br - cq \\ -(ar - cp) \\ aq - bp \end{bmatrix}$$

The cross product returns a vector that is normal to both of the input vectors. It has a magnitude $|a||b|\sin\theta$ where a and b are the two input vectors.

The projection of u onto v (the amount u in the direction of v) is given by

$$(\frac{u \cdot v}{v \cdot v})v$$

7 Matrices

Vectors are just a special case of Matrices (one column/row) so a lot of their properties are shared with matrices. For instance, addition/subtraction are performed component wise and scalar multiplication works just like it does in vectors i.e

$$\begin{bmatrix} a & d \\ b & e \\ c & f \end{bmatrix} + \begin{bmatrix} g & j \\ h & k \\ i & l \end{bmatrix} = \begin{bmatrix} a+g & d+j \\ b+h & e+k \\ c+i & f+l \end{bmatrix}$$

$$3\begin{bmatrix} -1 & 0 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} -3 & 0 \\ 6 & 3 \end{bmatrix}$$

The only thing you will need to keep in mind for matrix addition/subtraction is that you can only add matrices of the same order. What's order?

Order of Matrices

By convention, the order of a matrix is expressed like $m \times n$. Where m is the number of rows and n the number of columns. This is opposite to the traditional 'x & y' way of thinking but you'll see why when you get to multiplication.

Transpose

The first of the new operations we will learn for matrices is something called 'transpose'. You can think of this like the rotation of a matrix. It is represented by a superscript capital T.

$$\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}^\intercal = \begin{bmatrix} 1 & 2 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 0 & -2 \\ 2 & 12 \\ 0 & -3 \end{bmatrix}^\intercal = \begin{bmatrix} -2 & 12 & 3 \\ 0 & 2 & 0 \end{bmatrix}$$

Note: $\mathbf{AB} \neq \mathbf{BA}$ but $(\mathbf{AB})^\intercal = \mathbf{B}^\intercal\mathbf{A}^\intercal$

Matrix Multiplication

For two $m \times n$ matrices,

- Cover everything but the first column in the second matrix
- Take the dot product (just like vectors) of that and each of the m rows of the first matrix (where the first is the first row of the matrix output, the second the second row etc)
- Repeat for each of the n columns in the second matrix

For two $m \times n$ matrices, the first must have the same number of rows as columns in the second.

Example

$$\begin{bmatrix} 1 & 3 & 1 \\ 2 & 4 & 5 \\ 6 & 1 & 2 \end{bmatrix} \begin{bmatrix} 2 & 3 & 1 \\ 4 & 2 & 1 \\ 6 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 20 & 10 & 5 \\ 50 & 19 & 11 \\ 28 & 22 & 9 \end{bmatrix}$$

$$\begin{bmatrix} [1 & 3 & 1] \\ [2 & 4 & 5] \\ [6 & 1 & 2] \end{bmatrix} \begin{bmatrix} 2 & 3 & 1 \\ 4 & 2 & 1 \\ 6 & 1 & 1 \end{bmatrix}$$

$$[2, 4, 6] \cdot [1, 3, 1] = 2 \cdot 1 + 4 \cdot 3 + 6 \cdot 1 = 20$$

Matrix Multiplication as a Transformation

Taking two equations, that move two points x and y to two new points x_{new} and y_{new}.

$$\begin{aligned} 2x - y &= x_{new} \\ x + y &= y_{new} \end{aligned}$$

These can be represented as matrices, (the matrix containing the co-efficients is called a 'transformation matrix')

$$\begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x_{new} \\ y_{new} \end{bmatrix}$$

Transformation Matrices

A transformation matrix is a square matrix that when you multiply a matrix of points by it will perform a transformation i.e scale it or rotate it etc. All of these examples are for a 'unit square' which for a length k is the matrix:

$$\begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix}$$

Stretch of k in the x direction

$$\begin{bmatrix} k & 0 \\ 0 & 1 \end{bmatrix}$$

Stretch of k in the y direction

$$\begin{bmatrix} 1 & 0 \\ 0 & k \end{bmatrix}$$

Shear of k in the x direction

$$\begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$$

Shear of k in the y direction

$$\begin{bmatrix} 1 & 0 \\ k & 1 \end{bmatrix}$$

Rotation about the origin by θ anti-clockwise

$$\begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$$

Combinations

To combine transformations we can multiply transformation matrices. The order that the transformations will happen will be the order you multiply them in. i.e **AB** means transformation **A** will be applied before **B**.

Determinants

Determinants are a property of matrices that can help you to understand a matrix transform. In 2D, it will tell you the ratio of the areas of the matrix and the transformed matrix. In 3D, it will tell you the ratio of the volumes. In other words, 'determinants give scale'. Additionally note, A negative determinant means the area or volume will be inverted.

Calculating Determinants

$$\det \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = a_{11} \times a_{22} - a_{12} \times a_{21}$$

$$\det \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = a_{11} \times \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \times \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \times \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

The Identity Matrix

The identity matrix $\mathbf{I}$ is a matrix that when multiplied with any other matrix will return that matrix. It is analogous to the scalar 'one'. An identity matrix will contain a single line of 1s through the leading diagonal and 0s on either side i.e for a 2x2 matrix:

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Inverses

An inverse given by $\mathbf{A}^{-1}$ reverses the transformation performed by $\mathbf{A}$ hence $\mathbf{A}^{-1}$ (if it exists) is the inverse transformation of $\mathbf{A}$. When you multiply any inverse $\mathbf{A}^{-1}$ by $\mathbf{A}$ it should return the identity matrix $\mathbf{I}$.

Calculating Inverses

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}^{-1} = \frac{1}{\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}} \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}$$