

# Modelling Process Notes

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## 1 Modelling Process

### Factors

Anything that will have some effect on the your calculations e.g. drag, friction, mass, buoyancy, area. Note that separate things qualities will count separately e.g. a triangle's area and a square's.

### Assumptions

A statement that makes the problem simpler - cancels factors. e.g. 'Total length of wood is not reduced when it is cut' or 'There are no significant currents'

### Precise Problem Statement

Given (key factors and assumptions), Find (the value you're asked to find)

### Formulating a Model

$$x \propto y, 1/z \implies x = \frac{ky}{z}$$

### Modelling Forces

Use Newton's 2nd law, subtract negative forces and add positive ones. Use the ones from the list.

## 2 Differentiation

Implicit Differentiation

Series and Approximation

Numerical Differentiation

## 3 Integration

Integration Techniques

1. U substitution
2. Integration by Parts

Partial Fractions

## 4 Ordinary Differential Equations

Ordinary differential equations are equations containing one or more functions of one independent variable. You can recognise an ODE from a PDE (partial differential equation) because a PDE will contain  $\partial$  (pronounced 'del') and ODEs have standard 'd'.  $\frac{dy}{dx}$  means  $y$  is the dependent variable and  $x$  is the independent variable.  $\frac{dx}{dt}$   $x$  is dependent,  $t$  is independent.

**Properties**

**Order**

Highest derivative (also equal to number of values needed to find a particular solution) e.g.

$$\frac{dy}{dx} = 5x \text{ is 1st Order}$$
$$\frac{d^4y}{dx^4} = \frac{dy}{dx} + 2 \text{ is 4th Order}$$

## Linear

Involves only derivatives of  $y$  and terms of  $y$  to the 1st power e.g. ONLY  $\frac{dy}{dx}$ ,  $y$  etc.

$$\frac{d^4y}{dx^4} + \frac{dy}{dx} = 2 \text{ is linear}$$
$$\frac{dy}{dx} = 2y + 3 \text{ is linear}$$

## Homogeneity

If all (non-zero) terms involve the dependent variable then the equation is homogeneous

$$\frac{dx}{dt} = x \text{ is homogeneous}$$
$$\frac{dy}{dx} = 2y + 3 \text{ is not homogeneous (3 doesn't involve } x)$$

## Forming Differential Equations

In typical exam questions there are few points at which you will form a differential equation: modelling a set of forces in the typical modelling questions, using proportionality or previous knowledge. Typically the modelling questions will use Newton's 2nd law which states  $\sum F = ma$  and then you can sum the forces and use it to find mass/acceleration (or their derivatives).

## Solving Differential Equations

1. Direct Integration
2. Separation of Variables
3. Euler's Method
4. Integrating Factor
5. Exponential Substitution

## Euler's Method

Euler's method is a numerical method for solving D.E.s that uses a 2 term Taylor Series and is simply stated by the equation.

$$f(x+h) \approx f(x) + hf'(x)$$

These questions are easiest to complete if you use a table for the values of  $x$ ,  $f(x)$ ,  $f'(x)$  and then calculate  $f(x+h)$  e.g.

$x$	$f(x)$	$f'(x)$	$f(x) + hf'(x)$
1	$1^2$	2	4
4	...	...	...

## Exponential Substitution

Exponential Substitution allows you to solve a 2nd order O.D.E using something called a 'characteristic equation'. You substitute in a trial solution  $e^{\lambda t}$  and then complete the substitution (differentiate) and then divide through by your trial solution leaving you with the characteristic equation which you can then solve for  $\lambda$ . You then substitute your values into the equation:

$$x = C_1 e^{\lambda t} + C_2 e^{\lambda t}$$

This is a general solution, to calculate the exact solution you need two points (for a 2nd order equation) to calculate values of  $C_1$  and  $C_2$ . e.g

$$\begin{aligned}\frac{d^2x}{dt^2} - \frac{dx}{dt} - 6x &= 0 \\ \lambda^2 - \lambda - 6 &= 0 \\ \implies \lambda &= 3, -2 \\ x &= c_1 e^{3t} + c_2 e^{-2t} \text{ gen. soln.} \\ \text{Given } x(0) &= 0, x'(0) = 5 \\ 0 &= C_1 + C_2 \implies C_1 = -C_2 \\ 5 &= 3C_1 + -2C_2 \implies C_2 = -1 \\ &\implies C_1 = 1 \\ x &= e^{3t} - e^{-2t} \text{ exact soln.}\end{aligned}$$

### Integrating Factor

$$\text{if } h(x) = \int p(x)dx$$

$$\frac{dy}{dx} + p(x)y = r(x) \text{ has solution } y = e^{-h(x)} \int e^{h(x)}r(x)dx$$

## 5 Probability

### Some General Rules

$Pr(A|B)$  = Probability of A happening given B has already happened

$$= \frac{Pr(A \& B)}{Pr(B)}$$

$Pr(A \& B) = Pr(A) \times Pr(B)$  If they are independent

### Continuous Random Variables

Continuous Random Variables that are infinite i.e height as there are an infinite amount of possible heights with infinite accuracy. When graphed the area under a graph (probability density function) will always sum to one i.e

$$\int_{-\infty}^{\infty} f(x)dx = 1$$

### Bayes Theorem

$$Pr(A|B) = \frac{Pr(A)}{Pr(A)Pr(B|A) + Pr(\neg A)Pr(B|\neg A)}$$

## 6 Vectors

## 7 Matrices

Vectors are just a special case of Matrices (one column/row) so a lot of their properties are shared with matrices. For instance, addition/subtraction are performed component wise and scalar multiplication works just like it does in vectors i.e

$$\begin{bmatrix} a & d \\ b & e \\ c & f \end{bmatrix} + \begin{bmatrix} g & j \\ h & k \\ i & l \end{bmatrix} = \begin{bmatrix} a+g & d+j \\ b+h & e+k \\ c+i & f+l \end{bmatrix}$$
$$3 \begin{bmatrix} -1 & 0 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} -3 & 0 \\ 6 & 3 \end{bmatrix}$$

The only thing you will need to keep in mind for matrix addition/subtraction is that you can only add matrices of the same order. What's order?

### Order of Matrices

By convention, the order of a matrix is expressed like  $m \times n$ . Where m is the number of rows and n the number of columns. This is opposite to the traditional ' $x$  &  $y$ ' way of thinking but you'll see why when you get to multiplication.

### Transpose

The first of the new operations we will learn for matrices is something called 'transpose'. You can think of this like the rotation of a matrix. It is represented by a superscript capital T.

$$\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}^T = \begin{bmatrix} 1 & 2 & 1 \end{bmatrix}$$
$$\begin{bmatrix} 0 & -2 \\ 2 & 12 \\ 0 & -3 \end{bmatrix}^T = \begin{bmatrix} -2 & 12 & 3 \\ 0 & 2 & 0 \end{bmatrix}$$

## Matrix Multiplication

For two  $m \times n$  matrices,

- Cover everything but the first column in the second matrix
- Take the dot product (just like vectors) of that and each of the  $m$  rows of the first matrix (where the first is the first row of the matrix output, the second the second row etc)
- Repeat for each of the  $n$  columns in the second matrix

For two  $m \times n$  matrices, the first must have the same number of rows as columns in the second.

### Example

$$\begin{bmatrix} 1 & 3 & 1 \\ 2 & 4 & 5 \\ 6 & 1 & 2 \end{bmatrix} \begin{bmatrix} 2 & 3 & 1 \\ 4 & 2 & 1 \\ 6 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 20 & 10 & 5 \\ 50 & 19 & 11 \\ 28 & 22 & 9 \end{bmatrix}$$
$$\begin{bmatrix} [1 & 3 & 1] \\ [2 & 4 & 5] \\ [6 & 1 & 2] \end{bmatrix} \begin{bmatrix} 2 & 3 & 1 \\ 4 & 2 & 1 \\ 6 & 1 & 1 \end{bmatrix}$$
$$[2, 4, 6] \cdot [1, 3, 1] = 2 \cdot 1 + 4 \cdot 3 + 6 \cdot 1 = 20$$

## Matrix Multiplication as a Transformation

Taking two equations, that move two points  $x$  and  $y$  to two new points  $x_{new}$  and  $y_{new}$ .

$$2x - y = x_{new}$$

$$x + y = y_{new}$$

These can be represented as matrices, (the matrix containing the co-efficients is called a ‘transformation matrix’)

$$\begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x_{new} \\ y_{new} \end{bmatrix}$$