Modelling Process Notes

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1 Modelling Process

Factors

Anything that will have some effect on the your calculations e.g. drag, friction, mass, buoyancy, area. Note that separate things qualities will count separately e.g. a triangle's area and a square's.

Assumptions

A statement that makes the problem simpler - cancels factors. e.g. 'Total length of wood is not reduced when it is cut' or 'There are no significant currents'

Precise Problem Statement

Given (key factors and assumptions), Find (the value you're asked to find)

Formulating a Model

$$x \propto y, 1/z \implies x = \frac{ky}{z}$$

Modelling Forces

Use Newton's 2nd law, subtract negative forces and add positive ones. Use the ones from the list.

2 Differentiation

Implicit Differentiation

Series and Approximation

Numerical Differentiation

3 Integration

Integration Techniques

1. U substitution

2. Integration by Parts

Partial Fractions

4 Ordinary Differential Equations

Ordinary differential equations are equations containing one or more functions of one independent variable. You can recognise an ODE from a PDE (partial differential equation) because a PDE will contain ∂ (pronounced 'del') and ODEs have standard 'd'. $\frac{dy}{dx}$ means y is the dependent variable and x is the independent variable. $\frac{dx}{dt}$ x is dependent, t is independent.

Properties

Order

Highest derivative (also equal to number of values needed to find a particular solution) e.g.

$$\frac{dy}{dx} = 5x \text{ is 1st Order}$$
$$\frac{d^4y}{dx^4} = \frac{dy}{dx} + 2 \text{ is 4th Order}$$

Linear

Involves only derivatives of y and terms of y to the 1st power e.g. ONLY $\frac{dy}{dx}$, y etc.

$$\frac{d^4y}{dx^4} + \frac{dy}{dx} = 2 \text{ is linear}$$
$$\frac{dy}{dx} = 2y + 3 \text{ is linear}$$

Homogeneity

If all (non-zero) terms involve the dependent variable then the equation is homogeneous

$$\frac{dx}{dt} = x \text{ is homogeneous}$$
$$\frac{dy}{dx} = 2y + 3 \text{ is not homogeneous (3 doesn't involve x)}$$

Forming Differential Equations

In typical exam questions there are few points at which you will form a differential equation: modelling a set of forces in the typical modelling questions, using proportionality or previous knowledge. Typically the modelling questions will use Newton's 2nd law which states $\sum F = ma$ and then you can sum the forces and use it to find mass/acceleration (or their derivatives).

Solving Differential Equations

1. Direct Integration
2. Separation of Variables
3. Euler's Method
4. Integrating Factor

5 Probability

6 Vectors

7 Matrices

Vectors are just a special case of Matrices (one column/row) so a lot of their properties are shared with matrices. For instance, addition/subtraction are performed component wise and scalar multiplication works just like it does in vectors i.e

$$\begin{bmatrix} a & d \\ b & e \\ c & f \end{bmatrix} + \begin{bmatrix} g & j \\ h & k \\ i & l \end{bmatrix} = \begin{bmatrix} a+g & d+j \\ b+h & e+k \\ c+i & f+l \end{bmatrix}$$

$$3\begin{bmatrix} -1 & 0 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} -3 & 0 \\ 6 & 3 \end{bmatrix}$$

The only thing you will need to keep in mind for matrix addition/subtraction is that you can only add matrices of the same order. What's order?

Order of Matrices

By convention, the order of a matrix is expressed like $m \times n$. Where m is the number of rows and n the number of columns. This is opposite to the traditional 'x & y' way of thinking but you'll see why when you get to multiplication.

Transpose

The first of the new operations we will learn for matrices is something called 'transpose'. You can think of this like the rotation of a matrix. It is represented by a superscript capital T.

$$\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}^{\mathsf{T}} = \begin{bmatrix} 1 & 2 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 0 & -2 \\ 2 & 12 \\ 0 & -3 \end{bmatrix}^{\mathsf{T}} = \begin{bmatrix} -2 & 12 & 3 \\ 0 & 2 & 0 \end{bmatrix}$$

Matrix Multiplication

For two $m \times n$ matrices,

- Cover everything but the first column in the second matrix
- Take the dot product (just like vectors) of that and each of the m rows of the first matrix (where the first is the first row of the matrix output, the second the second row etc)
- Repeat for each of the n columns in the second matrix

For two $m \times n$ matrices, the first must have the same number of rows as columns in the second.

Example

$$\begin{bmatrix} 1 & 3 & 1 \\ 2 & 4 & 5 \\ 6 & 1 & 2 \end{bmatrix} \begin{bmatrix} 2 & 3 & 1 \\ 4 & 2 & 1 \\ 6 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 20 & 10 & 5 \\ 50 & 19 & 11 \\ 28 & 22 & 9 \end{bmatrix}$$

$$\begin{bmatrix} [1 & 3 & 1] \\ [2 & 4 & 5] \\ [6 & 1 & 2] \end{bmatrix} \begin{bmatrix} 2 & 3 & 1 \\ 4 & 2 & 1 \\ 6 & 1 & 1 \end{bmatrix}$$

$$[2, 4, 6] \cdot [1, 3, 1] = 2 \cdot 1 + 4 \cdot 3 + 6 \cdot 1 = 20$$

Matrix Multiplication as a Transformation

Taking two equations, that move two points x and y to two new points x_{new} and y_{new}.

$$2x - y = x_{new}$$
$$x + y = y_{new}$$

These can be represented as matrices, (the matrix containing the co-efficients is called a ‘transformation matrix’)

$$\begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x_{new} \\ y_{new} \end{bmatrix}$$