AMA1131 Calculus: Solution to Assignment 2

1. $f(x) = e^x - \frac{1}{2} - \cos(2x) + 2\sin x$. Then f is continuous everywhere. $f'(x) = e^x + 2\sin(2x) + 2\cos(x) > 0$ on $\left(0, \dfrac{\pi}{4}\right)$, f is increasing on $\left(0, \dfrac{\pi}{4}\right)$ and thus one-to-one on $\left(0, \dfrac{\pi}{4}\right)$.

2. Differentiating both sides with respect to x gives
$$-\sin(x^2 + 2y)(2x + 2y') + 5e^y + 4xe^y y' = \frac{1}{1+y^2}y' + 6y'.$$
Hence,
$$y' = \frac{5e^y - 2x\sin(x^2+2y)}{2\sin(x^2+2y) - 4xe^y + 6 + \dfrac{1}{1+y^2}}.$$
At $(x, y) = (0, 0)$, we get $y'(0) = \dfrac{5}{7}$.

3. $f'(x) = -6x^2 - 6x + 12 = -6(x^2 + x - 2) = -6(x-1)(x+2)$. Set $f'(x) = 0$, we have $x = -2$ or $x = 1$, $x \in [-4, 2]$. Therefore, the stationary points are $x = -2$ and $x = 1$.Moreover $f(-2) = -27$ and $f(1) = 0$.

   $f'(x) > 0$ when $x \in (-2, 2)$ and $f'(x) < 0$ when $x \in [-4, -2) \cup (1, 2]$. Hence $(-2, 2)$ is increasing interval; $[-4, -2)$ and $(1, 2]$ are decreasing intervals.

   Hence, $f(-2) = -27$ is a local minimum value of $f(x)$ and $f(1) = 0$ is a local maximum value of $f(x)$. (Or use the second derivative test $f''(x) = -12x - 6$. So $f''(-2) = 18 > 0$ and $f''(1) = -18 < 0$.)

   Check the end points: $f(-4) = 25$ and $f(2) = -11$. Hence, $f(-4) = 25$ is the global maximum and $f(-2) = -27$ is the global minimum.

4. (a) $\int \left(\frac{2x^3-4x+7}{x^2} + 3\sin x \right) dx = x^2 - 4\ln|x| - \frac{7}{x} - 3cosx + C.$

(b) Let $u = x+8$, then $du = dx$.

$$\begin{aligned} & \int \frac{x}{\sqrt[3]{x+8}} dx \\ = & \int \frac{u-8}{\sqrt{3}u} du \\ = & \frac{3}{5}u^{\frac{5}{3}} - 12u^{\frac{2}{3}} + C \\ = & \frac{3}{5}(x+8)^{\frac{2}{3}}(x-12) + C. \end{aligned}$$

(c)

$$= \int \frac{1}{2\sqrt{1-u^2}} du \quad = \frac{1}{2}\arcsin(u)$$

$$= \frac{1}{2}\arcsin\left(e^{2x}\right) + C$$

(d)

$$= \int \frac{2\sin(2x)}{\cos(2x)+11} dx$$

$$= 2 \cdot \int -\frac{1}{2u} du \quad = 2\left(-\frac{1}{2}\ln|u|\right)$$

$$= -\ln\left|\cos(2x)+11\right| + C$$

(e) Let $x = u^2,\ (u > 0)$, then $dx = 2udu$ and $\sqrt{x} = u$.

$$\begin{aligned} & \int \frac{1}{\sqrt{x}(4-x)}dx \\ = \ & 2\int \frac{2u}{u(4-u^2)}du \\ = \ & \frac{1}{2}\ln|\frac{2+\sqrt{x}}{2-\sqrt{x}}| + C. \end{aligned}$$

(f)

$$\begin{aligned} & \int sin(8x)sin(4x)dx \\ = \ & \int \frac{cos4x - cos12x}{2}dx \\ = \ & \frac{sin4x}{8} - \frac{sin12x}{24} + C. \end{aligned}$$

(g)

$$\begin{aligned} & \int 32sin^2xcos^2xdx \\ = \ & 8\int sin^2 2xdx \\ = \ & 8\int \frac{1-cos4x}{2}dx \\ = \ & 4x - sin4x + C. \end{aligned}$$

(h) Let u = cos2x. Then $du = -2sin2xdx$.

$$\int e^{sin^2x}sin(2x)dx = -\frac{1}{2}\int e^{\frac{1-u}{2}}du.$$

Let $v = \frac{1-u}{2}$, then $dv = -\frac{1}{2}du$.

$$\int e^{\frac{1-u}{2}}du = -2\int e^v dv = -2e^v + C.$$

Therefore,

$$\int e^{sin^2x}sin(2x)dx = e^{sin^2x} + C.$$

(i)

$$\begin{aligned}
& \int \sqrt{x} \ln x dx \\
= & \frac{2}{3}\int \ln x dx^{3/2} \\
= & \frac{2}{3}x^{3/2}\ln x - \frac{2}{3}\int x^{1/2}dx \\
= & \frac{2}{3}x^{3/2}\ln x - \frac{4}{9}x^{3/2} + C.
\end{aligned}$$

(j) Use integration by part, we can obtain

$$\begin{aligned}
& \int xcos^5xdx \\
= & \int x(1-sin^2x)^2dsinx = \int xdsinx - 2\int xsin^2xdsinx + \int xsin^4xdsinx
\end{aligned}$$

$$\int xdsinx = xsinx - \int sinxdx = xsinx + cosx + C_1.$$

$$\begin{aligned}
& 2\int xsin^2xdsinx = \frac{2}{3}\int xdsin^3x \\
= & \frac{2}{3}xsin^3x - \frac{2}{3}\int sin^3xdx \\
= & \frac{2}{3}xsin^3x + \frac{2}{3}\int (1-cos^2x)dcosx \\
= & \frac{2}{3}xsin^3x + \frac{2}{3}cosx - \frac{2}{9}cos^3x + C_2.
\end{aligned}$$

$$\begin{aligned}
& \int xsin^4xdsinx = \frac{1}{5}\int xdsin^5x \\
= & \frac{1}{5}xsin^5x - \frac{1}{5}\int sin^5xdx = \frac{1}{5}xsin^5x + \frac{1}{5}\int sin^4xdcosx \\
= & \frac{1}{5}xsin^5x + \frac{1}{5}cosx - \frac{2}{15}cos^3x + \frac{1}{25}cos^5x + C_3
\end{aligned}$$

Therefore,

$$\begin{aligned}
& \int xcos^5xdx \\
= & xsinx - \frac{2}{3}xsin^3x + \frac{1}{5}xsin^5x + \frac{8}{15}cosx + \frac{4}{45}cos^3x + \frac{1}{25}cos^5x + C.
\end{aligned}$$

(k)

$$= \int \frac{1}{x-1} + \frac{4}{(x-2)^2} dx$$

$$= \ln|x-1| - \frac{4}{x-2} + C$$

(l)

$$\begin{aligned} \frac{1}{x(x^2+1)} &= \frac{A}{x} + \frac{Bx+C}{x^2+1} \\ &= \frac{(A+B)x^2+Cx+A}{x(x^2+1)}. \end{aligned}$$

Therefore,

$$\begin{aligned} A+B=0, \\ C=0, \\ A=2. \end{aligned}$$

Therefore,

$$A=2, B=-2, C=0.$$

Thus,

$$\begin{aligned} \int \frac{2}{x(x^2+1)} dx &= \int \frac{2}{x} dx + \frac{-2x}{x^2+1} dx \\ &= 2\ln|x| - \ln(x^2+1) + C \\ &= \ln \frac{x^2}{1+x^2} + C. \end{aligned}$$