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Assignment 2

Question 1

Question 1

$f(x) = e^x - \frac{1}{2} - \cos(2x) + 2\sin(x)$

$f'(x) = \frac{d}{dx}(e^x - \frac{1}{2} - \cos(2x) + 2\sin(x))$

$= e^x - 0 + 2\sin 2x + 2\cos x$

$= e^x + 2\sin 2x + 2\cos x$

In interval $[0, \frac{\pi}{4}]$ Range $f'(x)$:

$e^0 + 2\sin 2\cdot 0 + 2\cos 0 \leq f'(x) \leq e^{\frac{\pi}{4}} + 2\sin\frac{\pi}{2} + 2\cos\frac{\pi}{4}$

$\Rightarrow \quad 3 \leq f'(x) \leq e^{\frac{\pi}{4}} + 2 + \sqrt{2}$

$\because 3 > 0, \ e^{\frac{\pi}{4}} + 2 + \sqrt{2} > 3$

$\therefore$ $f'(x)$ is strictly increasing in the interval $[0, \frac{\pi}{4}]$

$\therefore f(x) = e^x - \frac{1}{2} - \cos(2x) + 2\sin(x) \Rightarrow f(x)$ is one to one $\Leftrightarrow f^{-1}$ exist

Find $(f^{-1})'(-\frac{1}{2})$

$y = f(x) = e^x - \frac{1}{2} - \cos(2x) + 2\sin(x)$, $x = f^{-1}(y)$ $\quad (f^{-1})'(-\frac{1}{2}) = \frac{dx}{dy}\Big|_{y=-\frac{1}{2}}$

$\frac{dx}{dy} = \frac{1}{\frac{dx}{dy}} = \frac{1}{e^x + 2\sin 2x + 2\cos x}$

$e^x - \frac{1}{2} - \cos(2x) + 2\sin(x) = -\frac{1}{2}$

$e^x - \cos(2x) + 2\sin(x) = 0$

$x = 0$

$\therefore \frac{dx}{dy}\Big|_{y=-\frac{1}{2}} \overset{x=0}{=} \frac{1}{e^x + 2\sin 2x + 2\cos x} = \frac{1}{1+0+2\cdot 1} = \frac{1}{3}$

$\therefore (f^{-1})'(-\frac{1}{2}) = \frac{1}{3}$

Question 2

Question 2

Function $y = y(x)$

$y'(x) = \frac{d}{dx}\left[\cos(x^2+2y) + \frac{d}{dx}[5xe^y)\right]$

$= \frac{d}{dx}[\cos(x^2+2y)] + \frac{d}{dx}(5xe^y)$

$= -\sin(x^2+2y) \cdot (2x + 2\frac{dy}{dx})$

$+$

$5(e^y + xe^y\frac{dy}{dx})$

$= -\sin(x^2+2y) \cdot (2x+2y') + 5e^y + 5xe^y y'$

$y' = \frac{d}{dx}[\tan^{-1}(y) + 1 + 6y]$

$= \frac{d}{dx}(\tan^{-1}(y)] + \frac{d}{dx}(1) + \frac{d}{dx}(6y)$

$= \frac{y'}{1+y^2} + 0 + 6y'$

$= \frac{y'}{1+y^2} + 6y'$

$\therefore \frac{d}{dx}[\cos(x^2+2y) + 5xe^y = \tan^{-1}(y) + 1 + 6y]$

$\Rightarrow -\sin(x^2+2y) \cdot (2x+2y') + 5e^y + 5xe^y y' = \frac{y'}{1+y^2} + 6y'$

Find y' at $(x, y) = (0,0)$

$\therefore$ Substitute $(0,0) \Rightarrow -\sin(0+2\cdot 0) \cdot (0+2y') + 5e^0 + 5 \cdot 0 \cdot e^0 \cdot y' = \frac{y'}{1+0^2} + 6y'$

$\therefore \ 5 = y' + 6y'$

$7y' = 5$

$y' = \frac{5}{7}$

$\therefore$ Derictive y' at $(x,y) = (0,0)$ for function $y = y(x)$ is $\frac{5}{7}$

Question 3

Question 3

$f'(x) = \frac{d}{dx}(-2x^3 - 3x^2 + 12x - 7) = -6x^2 - 6x + 12$ in $[-4, 2]$

$f'(x) = 0 \Rightarrow -6x^2 - 6x + 12 = 0$

$(x+2)(x-1) = 0$

$x = -2 \quad x = 1$

$\therefore$ Stationary points are $x = -2$ and $x = 1$

At $f(-4) = -2(-4)^3 - 3(-4)^2 + 12(-4) - 7 = 25$

$f(-2) = -2(-2)^3 - 3(-2)^2 + 12(-2) - 7 = -27$

$f(1) = -2(1)^3 - 3(1)^2 + 12(1) - 7 = 0$

$f(2) = -2(2)^3 - 3(2)^2 + 12(2) - 7 = -11$

$\therefore$ Local maxima: stationary point $f(1) = 0$ $(1, 0)$ $(+ \Rightarrow -)$

Local minima: stationary point $f(-2) = -27$ $(-2, 27)$ $(- \Rightarrow +)$

Global maximum: In interval $[-4, 2]$ is $f(-4) = 25$ $(-4, 25)$

Global minimum: In interval $[-4, 2]$ is $f(-2) = -27$ $(-2, 27)$

$\because f'(x) = 0 \Rightarrow x = -2 \quad x = 1, \quad [-4, 2]$

$\therefore$ 3 intervals ① $[-4, -2)$ ② $(-2, 1)$ ③ $(1, 2]$

① : $f'(-3) = -6(-3+2)(-3-1) = -24 < 0 \Rightarrow$ decreasing

② : $f'(0) = -6(0+2)(0-1) = 12 > 0 \Rightarrow$ increasing

③ : $f'(2) = -6(2+2)(2-1) = -24 < 0 \Rightarrow$ decreasing

Relative	$[-4,-2)$	-2	$(-2,1)$	1	$(1,2]$
$f'(x)$	$-$	0	$+$	0	$-$
$f(x)$	$\searrow$	$\cup$ rel.min	$\nearrow$	$\cap$ rel.max	$\searrow$

$\therefore$ Increasing intervals: $(-2, 1)$

Decreasing intervals: $[-4, -2)$ and $(1, 2]$

Question 4

Question 4

a) $\int \left(\frac{2x^3-4x+7}{x^2}\right)dx$

$= \int \left(2x - \frac{4x}{x} + \frac{7}{x^2}\right)dx$

$\int 2x\,dx = 2\cdot\frac{x^2}{2} = x^2$

$\int -\frac{4}{x}\,dx = -4\ln|x|$

$\int \frac{7}{x^2} = 7\int x^{-2}dx = 7\cdot\left(-\frac{1}{x}\right) = -\frac{7}{x}$

$\therefore \int \left(\frac{2x^3-4x+7}{x^2}\right)dx$

$= x^2 - 4\ln|x| - \frac{7}{x} + C$

b) $\int \frac{x}{\sqrt[3]{x+8}}\,dx$

Let $u = x+8 \Rightarrow \frac{du}{dx} = 1$

$\Rightarrow du = dx$

$\Rightarrow x = u-8$

$\therefore \int \frac{x}{\sqrt[3]{x+8}}\,dx$

$= \int \frac{u-8}{\sqrt[3]{u}}\,du$

$= \int \frac{u}{\sqrt[3]{u}}du - \int \frac{8}{\sqrt[3]{u}}$

$= \int u^{\frac{2}{3}}du - \int 8u^{-\frac{1}{3}}du$

$= \int \left(u^{\frac{2}{3}} - 8u^{-\frac{1}{3}}\right)du$

$\int u^{\frac{2}{3}}du = \frac{u^{\frac{2}{3}+1}}{\frac{2}{3}+1} = \frac{3}{5}u^{\frac{5}{3}}$

$\int -8u^{-\frac{1}{3}}du = -8\cdot\frac{u^{-\frac{1}{3}+1}}{-\frac{1}{3}+1} = -12u^{\frac{2}{3}}$

As $u = x+8$

$\therefore \int \frac{x}{\sqrt[3]{x+8}}\,dx$

$= \frac{3}{5}(x+8)^{\frac{5}{3}} - 12(x+8)^{\frac{2}{3}} + C$

c) $\int \frac{e^{2x}}{\sqrt{1-e^{4x}}}\, dx$

Let $u = e^{2x}$, $\frac{du}{dx} = 2e^{2x} \Rightarrow du = 2u\, dx$

$\Rightarrow dx = \frac{du}{2u}$

$\int \frac{e^{2x}}{\sqrt{1-e^{4x}}}\, dx$

$= \int \frac{u}{\sqrt{1-u^2}} \cdot \frac{du}{2u}$

$= \frac{1}{2} \int \frac{1}{\sqrt{1-u^2}}\, du$

$= \frac{1}{2} \cdot (\arcsin(u))$

As $u = e^{2x}$

$\therefore \int \frac{e^{2x}}{\sqrt{1-e^{4x}}}\, dx = \frac{\arcsin(e^{2x})}{2} + C$

d) $\int \frac{\sin(2x)}{5+\cos^2(x)}\, dx$

$= \int \frac{2\sin x \cos x}{5+\cos^2(x)}\, dx$

Let $u = 5+\cos^2(x)$

$\frac{du}{dx} = -2\cos(x)\sin(x)$

$\Rightarrow du = dx \cdot (-2\cos(x)\sin(x))$

$\Rightarrow dx = \frac{du}{-2\sin x \cos x}$

$\therefore \int \frac{2\sin x\cos x}{5+\cos^2(x)}\, dx$

$= \int \frac{2\sin x \cos x}{u} \cdot \frac{du}{-2\sin x\cos x}$

$= \int -\frac{1}{u} \cdot du$

$\because \int \frac{1}{x} dx = \ln|x|$

$\therefore \int -\frac{1}{u} \cdot du$

$= -\ln|u|$

As $u = 5+\cos^2(x)$

$\int \frac{\sin(2x)}{5+\cos^2 x}\, dx$

$= -\ln|5+\cos^2 x| + C$

e) $\int \frac{1}{\sqrt{x}(4-x)} dx$

$= -\int \frac{1}{\sqrt{x}(x-4)} dx$

Let $u = \sqrt{x} \Rightarrow \frac{du}{dx} = \frac{1}{2\sqrt{x}} \Rightarrow du = \frac{dx}{2\sqrt{x}}$

$-\int \frac{1}{\sqrt{x}(x-4)} dx$ $\Rightarrow dx = du \cdot 2\sqrt{x}$

$= -\int \frac{1}{\sqrt{x}(u^2-4)} \cdot 2\sqrt{x}\, du$

$= 2\int \frac{1}{u^2-4} du$

$= -2\int \frac{1}{(u-2)(u+2)} du$

$= -2\int \left(\frac{1}{4(u-2)} - \frac{1}{4(u+2)}\right) du$

$= -\frac{1}{2}\int \frac{1}{u-2} + \frac{1}{2}\int \frac{1}{u+2}$

$\because \int \frac{1}{x} = \ln|x|$

$\therefore \int \frac{1}{u+2} = \ln|u+2|$

$\int \frac{1}{u-2} = \ln|u-2|$

$\therefore -\frac{1}{2}\int \frac{1}{u-2} + \frac{1}{2}\int \frac{1}{u+2}$

$= \frac{\ln|u+2|}{2} - \frac{\ln|u-2|}{2}$

As $u = \sqrt{x}$

$\therefore \int \frac{1}{(4-x)\sqrt{x}} dx = \frac{\ln|\sqrt{x}+2| - \ln|\sqrt{x}-2|}{2} + C$

f) $\int \sin(8x)\sin(4x)\, dx$

$\sin(8x)\sin(4x)$

$= \frac{1}{2}[\cos(4x-8x) - \cos(4x+8x)]$

$= \frac{1}{2}[\cos(-4x) - \cos(12x)]$

$= \frac{1}{2}[\cos(4x) - \cos(12x)]$

$\therefore \int \sin(4x)\sin(8x)dx$

$= \frac{1}{2}\int [\cos(4x) - \cos(12x)]$

$= \frac{1}{2}\int \cos(4x) - \frac{1}{2}\int \cos(12x)$

$\frac{1}{2}\int \cos(4x) = \frac{\frac{1}{4}\sin(4x)}{2}$

$\frac{1}{2}\int \cos(12x) = \frac{\frac{1}{12}\sin(12x)}{2}$

$\therefore \int \sin(4x)\sin(8x)\, dx$

$= \frac{\sin(4x)}{8} - \frac{\sin(12x)}{24} + C$

g) $\int 32\sin^2 x\cos^2 x\,dx$

$= 32\int \sin^2 x\cos^2 x\,dx$

$= 32\int \left(\frac{1-\cos 2x}{2}\right)\cdot\left(\frac{1+\cos 2x}{2}\right)dx$

$= 32\int \frac{1-\cos^2(2x)}{4}$

$= 32\int \frac{\sin^2(2x)}{4}$

$= \int 8\sin^2(2x)$

$= \int 8\cdot\frac{1-\cos(4x)}{2}$

$= 4\int (1-\cos(4x))$

$\int 1 = x$

$\int -\cos(4x) = -\frac{1}{4}\sin(4x)$

$\therefore 4\int (1-\cos(4x))$

$= 4\left(x-\frac{1}{4}\sin(4x)\right)$

$= 4x-\sin(4x)+C$

$\therefore \int 32\sin^2 x\cos^2 x\,dx$

$= 4x-\sin(4x)+C$

h) $\int e^{\sin^2 x}\sin(2x)\,dx$

$= \int e^{\sin^2 x}\cdot 2\cos x\sin x\,dx$

Let $u=\sin^2(x) \Rightarrow \frac{du}{dx}=2\cos x\sin x$

$\Rightarrow du = dx\cdot 2\cos x\sin x \Rightarrow dx=\frac{du}{2\cos x\sin x}$

$\therefore = \int e^u\cdot 2\cos x\sin x\cdot\frac{du}{2\cos x\sin x}$

$= \int e^u\,du$

$= e^u + C$

As $u=\sin^2(x)$

$\therefore \int e^{\sin^2 x}\sin(2x)\,dx$

$= e^{\sin^2 x}+C$

i) $\int \sqrt{x} \ln x \, dx$

$\because \int u \, dv = uv - \int v \, du$

$\therefore$ Let $u = \ln(x) \Rightarrow \frac{du}{dx} = \frac{1}{x}$:

$dv = \sqrt{x} dx = x^{\frac{1}{2}} dx \Rightarrow v = \frac{2}{3} x^{\frac{3}{2}}$

$\therefore \; uv - \int v \, du$

$= \ln x \cdot \frac{2}{3} x^{\frac{3}{2}} - \int \frac{2}{3} x^{\frac{3}{2}} \cdot \frac{dx}{x}$

$= \frac{2x^{\frac{3}{2}} \ln(x)}{3} - \int \frac{2}{3} x^{\frac{3}{2}} \cdot \frac{dx}{x}$

$= \frac{2x^{\frac{3}{2}} \ln(x)}{3} - \int \frac{2\sqrt{x}}{3} \, dx$

$- \int \frac{2\sqrt{x}}{3} \, dx$

$= -\frac{2}{3} \int \sqrt{x} \, dx$

$= -\frac{2}{3} \cdot \frac{2x^{\frac{3}{2}}}{3}$

$= -\frac{4x^{\frac{3}{2}}}{9}$

$\therefore \int \sqrt{x} \ln x \, dx$

$= \frac{2x^{\frac{3}{2}} \ln(x)}{3} - \frac{4x^{\frac{3}{2}}}{9} + C$

j) $\int x\cos^5 x\,dx$

$\because \int u\,dv = uv - \int v\,du$

$\therefore$ Let $u = x \Rightarrow \frac{du}{dx} = 1 \Rightarrow du = dx$

$dv = \cos^5(x)\,dx$

$\int \cos^5(x)\,dx$

Let $t = \sin x$, $dt = \cos(x)dt$

$\int \cos^5(x)\,dx$

$= \int (1-t^2)^2\,dt$

$= \int (1-2t^2+t^4)\,dt$

$= t - \frac{2}{3}t^3 + \frac{1}{5}t^5$

As $t = \sin x$

$= \sin x - \frac{2}{3}\sin^3 x + \frac{1}{5}\sin^5 x$

$\therefore uv - \int v\,du$

$= x\cdot(\sin x - \frac{2}{3}\sin^3 x + \frac{1}{5}\sin^5 x)$

$-$

$\int (\sin x - \frac{2}{3}\sin^3 x + \frac{1}{5}\sin^5 x)\,dx$

$\int (\sin x - \frac{2}{3}\sin^3 x + \frac{1}{5}\sin^5 x)\,dx$

$\int \sin x = -\cos x$

$\int -\frac{2}{3}\sin^3 x\,dx$

$= -\frac{2}{3}\int \sin^3 x$

$\because \sin^3 x = \sin x\cdot(1-\cos^2 x)$

$\therefore$ Let $a = \cos(x)$ $\frac{da}{dx} = -\sin x$

$\Rightarrow da = -\sin x\,dx \Rightarrow dx = -\frac{da}{\sin x}$

$\therefore -\frac{2}{3}\int \sin^3 x\,dx$

$= -\frac{2}{3}\int \sin x\cdot(1-a^2)\cdot\frac{da}{-\sin x}$

$= \frac{2}{3}\int (1-a^2)\,dv$

As $\int 1-x^2 = x - \frac{1}{3}x^3$

$\therefore \frac{2}{3}\int (1-a^2)\,dv$

$= \frac{2}{3}(\cos x - \frac{\cos^3 x}{3})$

$\int \frac{1}{5}\sin^5 x\,dx$

$= \frac{1}{5}\int \sin^5 x\,dx$

$= \frac{1}{5}\int \sin x(1-\cos^2(x))^2\,dx$

Let $b = \cos x \Rightarrow \frac{db}{dx} = -\sin x$

$\Rightarrow db = -\sin x\,dx$

$\Rightarrow dx = \frac{db}{-\sin x}$

$\therefore \frac{1}{5}\int \sin^5 x\,dx$

$= -\frac{1}{5}(1-b^2)^2\,db$

$= -\frac{1}{5}(b - \frac{2}{3}b^3 + \frac{b^5}{5})$

$= -\frac{1}{5}(\cos(x) - \frac{2}{3}\cos^3 x + \frac{1}{5}\cos^5 x)$

$-\int v\,du = -(-\cos x + \frac{2}{3}\cos x - \frac{2}{9}\cos^3 x - \frac{1}{5}\cos x + \frac{2}{15}\cos^3 x - \frac{1}{25}\cos^5 x)$

$= \frac{8}{15}\cos x + \frac{4}{45}\cos^3 x + \frac{1}{25}\cos^5 x$

$\therefore \int x\cos^5 x\,dx$

$= x\sin x - \frac{2}{3}x\sin^3(x) + \frac{1}{5}x\sin^5 x + \frac{8}{15}\cos x + \frac{4}{45}\cos^3 x + \frac{1}{25}\cos^5 x + C$

k) $\int \frac{x^2}{(x-1)(x-2)^2}\,dx = \frac{A}{(x-1)} + \frac{B}{(x-2)^2} + \frac{C}{x-2}$

$\Rightarrow A = 1 \quad B = 4 \quad C = 0$

$\therefore = \int \left(\frac{1}{x-1} + \frac{4}{(x-2)^2}\right) dx$

$= \int \frac{1}{x-1}dx + \int \frac{4}{(x-2)^2}\,dx$

Let $u = x-1$ $\quad \frac{du}{dx} = 1;\ du = dx$

$\because \int \frac{1}{u}\,du$

$\therefore \ln|u|$ As $u = x-1$ $\int \frac{1}{x-1} = \ln|x-1|$

$\int \frac{4}{(x-2)^2}\,dx$

$= 4\int \frac{1}{(x-2)^2}\,dx$

Let $u = x-2$ $\quad \frac{du}{dx} = 1 \Rightarrow du = dx$

$= 4\int \frac{1}{u^2}\,du = 4\cdot\left(-\frac{1}{u}\right)$

As $u = x-2$

$\therefore 4\int \frac{1}{(x-2)^2} = -\frac{4}{x-2}$

$\therefore \int \frac{x^2}{(x-1)(x-2)^2}\,dx$

$= \ln|x-1| - \frac{4}{x-2} + C$

l) $\int \frac{2}{x(x^2+1)}\,dx$

$= 2\int \frac{1}{x(x^2+1)}\,dx$

$\Rightarrow \frac{A}{x} + \frac{B}{x^2+1} \Rightarrow A = 1 \quad B = -x$

$\therefore = 2\int \frac{1}{x}dx - 2\int \frac{x}{(x^2+1)}\,dx$

$\int \frac{1}{x} = \ln|x|$

$\int \frac{x}{(x^2+1)} \Rightarrow$ Let $u = x^2+1$

$\Rightarrow \frac{du}{dx} = 2x \quad \Rightarrow du = 2x\,dx$

$\Rightarrow dx = \frac{du}{2x}$

$\therefore \int \frac{x}{(x^2+1)}\,dx$

$= \int \frac{x}{u}\cdot\frac{du}{2x}$

$= \frac{1}{2}\int \frac{1}{u}\,du$

$= \frac{1}{2}\ln|u|$

$= \frac{1}{2}\ln|x^2+1|$

$\therefore 2\int \frac{1}{x}dx - 2\int \frac{x}{(x^2+1)}dx$

$= 2\ln|x| - \ln|x^2+1| + C$

$\therefore \int \frac{2}{x(x^2+1)}\,dx$

$= 2\ln|x| - \ln|x^2+1| + C$