THE HONG KONG POLYTECHNIC UNIVERSITY

Department of Applied Mathematics

Subject Code:	AMA1131	**Subject Title:**	Calculus
Session:	Semester 1, 2023/2024		
Date:	December 19, 2023	**Time:**	12:30 – 14:30
Time Allowed:	2 hours		

This question paper has 6 pages. Page 4-6 contains formula tables.

Instructions: This paper has **11** questions.
Attempt ALL questions in this paper.
Full mark is 100.

Subject Examiner: Dr. Yuan Yancheng, Dr. Leung Chun Sing

DO NOT TURN OVER THE PAGE UNTIL YOU ARE TOLD TO DO SO.

1. Find the domain and range of the function $f(x) = \sqrt{1-x^2}$. (4 marks)

2. Let $f(x) = 3x^2$, $0 \le x \le 5$, $g(x) = \ln x$, $2 \le x \le 10$. Find the domain of $f \circ g$. (5 marks)

3. Show that $\cos 5x = 5\cos x - 20\cos^3 x + 16\cos^5 x$. (5 marks)

4. Evaluate the following limits (if they exist) : (15 marks)

   (a) $\displaystyle\lim_{x\to 9} \frac{\sqrt{x}-3}{x-9}$.

   (b) $\displaystyle\lim_{x\to 0} \frac{\tan 3x}{\sin 5x}$.

   (c) $\displaystyle\lim_{x\to 0^+} x^3 \tan^2 x \cos\left(\frac{1}{\sqrt{x}}\right)$.

   (d) $\displaystyle\lim_{x\to 0^+} (1+\sin 7x)^{\cot x}$.

   (e) $\displaystyle\lim_{x\to\infty} \left(\sqrt{x^2+x} - x^2\right)$

5. Find $f'(x)$ for each of the following $f(x)$.

   (a) $f(x) = \dfrac{\ln x + 3e^{2x}}{x^5}$

   (b) $f(x) = (\cos 5x)^{\tan x}$

   (10 marks)

6. Given that $ye^{x-1} + \sin(3x^2 y) = \frac{\pi}{3}$. Find y' at $(1, \pi/3)$. (5 marks)

7. Let $f(x) = 2x^2 - \ln x$. Find the intervals on which the function $f(x)$ is increasing or decreasing, also find the relative extrema. (10 marks)

8. Define the function $f(x)$ by:

$$f(x) = \begin{cases} ax+1 & x \le 0, \\ \sin^2(3x) + b, & x > 0 \end{cases}$$

   If $f'(0)$ exists, find values of a and b. (You are required to express the one-sided derivatives by the first principle of differentiation.) (10 marks)

9. Evaluate the integrals. (24 marks)

(a) $\displaystyle\int_0^1 \left(3x - \frac{x^3}{4}\right) dx$

(b) $\displaystyle\int \ln x dx$ (Hint: Use integration by parts formula: $\int udv = uv - \int vdu$.)

(c) $\displaystyle\int \frac{2x}{x^2 + 4x + 13} dx$

(d) $\displaystyle\int x(x^2 + 3)^7 dx$

(e) $\displaystyle\int \frac{x^2 + 1}{(x-1)(x-2)(x+3)} dx$

(f) $\displaystyle\int x \cos^5 x dx$

(Hint: Evaluate $\int \cos^5 x dx$ first.)

10. Let $I_n = \displaystyle\int_0^1 (1 + x^3)^n dx$ for integer n. Show that $(3n + 1)I_n = 2^n + 3nI_{n-1}$. (6 marks)

11. Given a curve C: $y = \sqrt{x}$, $0 \leq x \leq 9$ and a line L: $y = \dfrac{x}{2}$, $0 \leq x \leq 9$.

(a) Show that intersection points of C and L are $x = 0, 4$. (1 marks)

(b) Evaluate $\displaystyle\int_0^4 (\sqrt{x} - \frac{x}{2})dx + \int_4^9 (\frac{x}{2} - \sqrt{x})dx$. (4 marks)

(c) Use part (b) or otherwise, find the area of the region enclosed by C and L for $0 \leq x \leq 9$. (1 marks)

	y	y'
(1)	constant	0
(2)	x^p , where $p \in \mathbb{R}$	px^{p-1}
(3)	e^x, where $e = 2.71828...$	e^x
(4)	$a^x, \quad a > 0$	$a^x \ln a$
(5)	$\ln x$	$\frac{1}{x}$
(6)	$\sin x$	$\cos x$
(7)	$\cos x$	$-\sin x$
(8)	$\tan x$	$\sec^2 x$
(9)	$\tan^{-1} x$	$\frac{1}{1+x^2}$
(10)	$\sin^{-1} x$	$\frac{1}{\sqrt{1-x^2}}$

	$f(x)$	$\int f(x)dx$
(0)	constant k	$kx + C$
(1)	x^p , where $p \in \mathbb{R} \setminus \{-1\}$,	$\frac{x^{p+1}}{p+1} + C$
(2)	$\frac{1}{x}$	$\ln\lvert x\rvert + C$
(3)	e^x, where $e = 2.71828...$	$e^x + C$
(4)	$a^x, \quad a > 0$	$\frac{a^x}{\ln a} + C$
(5)	$\sin x$	$-\cos x + C$
(6)	$\cos x$	$\sin x + C$
(7)	$\frac{1}{x^2+a^2}$	$\frac{1}{a}\tan^{-1}\frac{x}{a} + C$

$$\begin{aligned}
\sin(A \pm B) &= \sin A \cos B \pm \cos A \sin B \\
\cos(A \pm B) &= \cos A \cos B \mp \sin A \sin B \\
\sin A \cos B &= \frac{1}{2}[\sin(A+B) + \sin(A-B)] \\
\cos A \cos B &= \frac{1}{2}[\cos(A+B) + \cos(A-B)] \\
\sin A \sin B &= \frac{1}{2}[\cos(A-B) - \cos(A+B)] \\
\cos^2 A &= \frac{1+\cos 2A}{2} \\
\sin^2 A &= \frac{1-\cos 2A}{2} \\
\sin 2A &= 2\sin A \cos A
\end{aligned}$$

Consider a curve of the function $y = f(x)$ in the interval $[a, b]$, the length of curve is given by

$$\int_a^b \sqrt{1 + (\frac{dy}{dx})^2} dx.$$

Suppose that $f(x) \geq 0$ on $[a, b]$. The volume of revolution about the x-axis of the region R which is bounded by the curve $y = f(x)$, x-axis, the vertical lines $x = a$ and $x = b$ is given by

$$V = \pi \int_a^b (f(x))^2 dx.$$

Suppose that $f(x) \geq 0$ on $[a, b]$ and $a \geq 0$. Let R be the region bounded by the curve $y = f(x)$, x-axis, the vertical lines $x = a$ and $x = b$. Then the volume of revolution of the region R about the y-axis is

$$V = 2\pi \int_a^b x f(x) dx.$$