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Problem 1. The following statements are true or false, you need only specify your final answer: "T" for true, or "F" for false:

$P(E\cap F)$ $\quad P$ $\quad P(E|F)$ $\quad \frac{E\cap F}{P(F)}$ $\quad PF>0<1$ ✓

- (a) $\emptyset \neq \{\emptyset\} \neq \{0\}$ **Answer:** (T)
- (b) For any events E and F with $P(F) > 0$, $P(E \cap F) \leq P(E \mid F)$ **Answer:** (T)
- (c) If the sets A, B are independent, then their complement sets A^c, B^c are also independent. **Answer:** (T)
- (d) If X follows binomial distribution $Binomial(n = 50, p = 0.4)$, then $50 - X$ also follows binomial distribution. **Answer:** (F) T $\quad X \sim (50, 0.4)$
- (e) If a set A is independent to all other sets, then A must be empty set $\emptyset$. **Answer:** (F) $\{\emptyset\}$

[4×5=20 marks]

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Problem 2.

If $P(A) = \frac{1}{5}, P(B) = \frac{7}{15}$, and $P(A \mid B) + P(B \mid A) = \frac{7}{10}$, find

(a) $P(A \cap B)$

(b) $P(A|A \cup B)$

[5×2=10 marks]

a) Let $P(A\cap B)=p$

$P(A|B)=\frac{P(A\cap B)}{P(B)} = \frac{p}{P(B)} = \cancel{5p}\ \frac{15}{7}p$

$P(B|A)=\frac{P(A\cap B)}{P(A)} = \frac{p}{P(A)} = 5p$

$\because P(A|B)+P(B|A)=\frac{7}{10}$

$\therefore \frac{15}{7}p+5p=\frac{7}{10}$

$\frac{50}{7}p=\frac{7}{10}$

$p=\frac{49}{500}$

$\therefore P(A\cap B)=\frac{49}{500}$

b) ~~$P(A|A\cup B)=\frac{P(A)\ P(A\cup}{P(A\cup B)}$~~

$P(A\cup B) = P(A)+P(B) - P(A\cap B)$

$= \frac{1}{5}+\frac{7}{15}-\frac{49}{500}$

$= \frac{853}{1500}$

$P(A|A\cup B)=\frac{\frac{1}{5}}{\frac{853}{1500}} = \frac{300}{853}$

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Problem 3. The average waiting time that people spend going through airport security for planes at an airport is 18 minutes with a standard deviation of 5.2 minutes.

(a) What is the probability that the average waiting time for 50 people is more than 20 minutes?

(b) The probability that 80 randomly chosen people have an average waiting time less than a certain average time is 0.1788. What is the average time?

[5×2=10 marks]

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a) Let $\bar{X}$ be num of minutes spend going through airport security, $\because n=50>30$ $\therefore$ Apply CLT

$\bar{X} \sim N(18, \frac{5.2^2}{50})$

$P(X>20) = P(Z > \frac{20-18}{5.2/\sqrt{50}})$

$= P(Z>2.72)$

$= 0.00326$

The probability that average waiting time for 50 people more then 20 min is 0.00326

b) ~~$X \sim N(18, \frac{5.2^2}{80})$, Let Y be average time~~ Let $\hat{A}$ be num of average waiting minutes

$\because n=80>30$ $\hat{A} \sim N(18, \frac{5.2^2}{80})$, Y be unknown certain waiting time

$\therefore$ Apply CLT

$P(\hat{A}<Y) = P(Z < \frac{Y-18}{5.2/\sqrt{50}}) = 0.1788$

$\frac{Y-18}{5.2/\sqrt{50}} = 0.82$

$Y-18 = 0.6030$

$Y = 18.6030$

So average time is 18.6030

$P(Z < \frac{Y-18}{5.2/\sqrt{50}}) = 0.1788$

$\frac{Y-18}{5.2/\sqrt{50}} = -0.92$

$Y = 17.4651$ minute

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Problem 4. The joint probability mass function of X and Y is given by

$$p(x,y) = \frac{x+y}{30}, \text{ for } x = 0,1,2,3 \text{ and } y = 0,1,2.$$

Find

(a) $P(X \le 2, Y = 1)$

(b) $P(X \le 2 \mid Y = 2)$

(c) $P(X \le 2, Y > 2)$

[5×3=15 marks]

a) $P(X \leq 2, Y=1) = P(X=0, Y=1) + P(X=1, Y=1) + P(X=2, Y=1)$

$= \frac{1}{30} + \frac{1+1}{30} + \frac{2+1}{30}$

$= \frac{1}{5}$

b) $P(X \leq 2 \mid Y=2) = \frac{\cancel{P(Y=2)}\ P(Y=2 \cap X \leq 2)}{P(Y=2)} = \cancel{\frac{P(Y=2)}{P(Y=2)}} \cancel{=1} = \frac{\frac{3}{10}}{\frac{7}{15}} = \frac{9}{14}$

$P(Y=2) = P(X=0, Y=2) + P(X=1, Y=2) + P(X=2, Y=2) + P(X=3, Y=2)$

$= \frac{2}{30} + \frac{3}{30} + \frac{4}{30} + \frac{5}{30}$

$= \frac{7}{15}$

$P(Y=2 \cap X \leq 2) = P(X=0, Y=2) + P(X=1, Y=2) + P(X=2, Y=2)$

$= \frac{3}{10}$

c) $\because$ Range for $y = 0, 1, 2$

$\therefore$ $Y > 2$ is impossible

$\therefore$ $P(X \leq 2, Y > 2) = 0$

Problem 5. Let X and Y be continuous random variables having the joint density function

$$f(x,y) = \begin{cases} 4xy, & 0 < x < 1, \quad 0 < y < 1 \\ 0 & \text{elsewhere} \end{cases}$$

(a) Find $P(4X > Y)$

(b) Find $E(X)$

[5×2=10 marks]

a) $P(4X>Y) = P(X>\frac{1}{4}Y) = \int_{\frac{1}{4}}^{1}\int_0^1 4xy\, dy\, dx$

$\cancel{= 4\int_0^1 \left[x\frac{y^2}{2}\right]_0^4}$

$\cancel{= 4\int_0^1 8x\, dx}$

$\cancel{= 4\int_0^1 4x^2 \quad 4 \quad [4x^2]_0^1}$

$\cancel{= 16}$

$= \int_{\frac{1}{4}}^{1}\int_0^1 4xy\, dy\, dx$

$= 4\int_{\frac{1}{4}}^{1}\int_0^1 xy\, dy\, dx$

$= 4\int_{\frac{1}{4}}^{1}\left[x\frac{y^2}{2}\right]_0^1 dx$

$= 2\int_{\frac{1}{4}}^{1} x\, dx$

$= \cancel{2\int_0^1\left[\frac{x^2}{2}\right]_0^{\frac{1}{4}}} \quad 2\left[\frac{x^2}{2}\right]_{\frac{1}{4}}^{1}$

$= \frac{7}{8}$

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b) $\cancel{E[X] = \int_0^1 \int 4xy\, dx}$ $\quad f(x) = \int_0^1 4xy\, dy = 4x \cdot \left[\frac{y^2}{2}\right]_0^1 = 2x$

$E[X] = \sum_{-\infty}^{\infty} x \cdot f(x)$

$= \sum_{-\infty}^{\infty} x \cdot 2x$

$= 2x^2$

$E(x) = \int_0^1 x f(x)\, dx$

$= \int_0^1 x \int_0^1 f(x,y)\, dy\, dx$

$= \int_0^1 2x^2\, dx$

$= 2\left[\frac{x^3}{3}\right]_0^1 \quad = \frac{2}{3}$

$P(4X>Y) = \text{Part I} + \text{Part II}$

$= \int_0^{\frac{1}{4}}\int_0^{4x} 4xy\, dy\, dx + \int_{\frac{1}{4}}^{1}\int_0^1 4xy\, dy\, dx$

$= \int_0^{\frac{1}{4}} 32x^3\, dx + \int_{\frac{1}{4}}^{1} 2x\, dx$

$= \frac{1}{32} + \frac{15}{16}$

$= \frac{31}{32}$

Problem 6. Let X be a continuous random variable with density function:

$$f_X(x) = \begin{cases} x^2\left(\frac{1}{8}x + \frac{3}{16}\right) & 0 < x \le 2 \\ 0 & \text{otherwise} \end{cases}$$

If $Y = \frac{2}{X} + 5$, find Var(Y). [10 marks]

$\text{Var}Y = \cancel{\text{Var}\,E[Y^2]}\; E[Y^2] - [E[Y]]^2$

$= \sum y^2 \cdot f(y) - [\sum y \cdot f(y)]^2$

$f_X(x) = \begin{cases} x^2(\frac{1}{8}x + \frac{3}{16}) & 0<x\le 2 \\ 0 \end{cases} = \frac{1}{8}x^3 + \frac{3}{16}x \quad 0<x\le 2$

$\frac{d}{dx} f_X(x) = \frac{3}{8}x^2 + \frac{3}{16}$

$X = \int_0^2 x^2(\frac{1}{8}x + \frac{3}{16})\,dx$

$= \int_0^2 \frac{1}{8}x^3 dx + \int_0^2 \frac{3}{16}x\,dx$

$\cancel{= \frac{1}{8}(\frac{x^3}{3})^2_0 + \frac{3}{16}(\frac{x^2}{2}}$

$= \frac{1}{8}\cdot[\frac{x^4}{4}]_0^2 + \frac{3}{16}\cdot[\frac{x^2}{2}]_0^2$

$= \frac{1}{2} + \frac{3}{8}$

$= \frac{7}{8}$

$\cancel{E[Y]=}$

$Y = \frac{2}{X} + 5 = \frac{2}{\frac{7}{8}} + 5 = \frac{16}{7} + 5 = \frac{51}{7} \quad \therefore E[Y] = \frac{51}{7}$

$E[Y^2] = (\frac{2}{X} + 5)^2$

$= \frac{4}{X^2} + \frac{20}{X} + 25$

$= \frac{4}{(\frac{7}{8})^2} + \frac{20}{(\frac{7}{8})} + 25$

$= 4\cdot\frac{64}{49} + \frac{160}{7} + 25$

$= \frac{256}{49} + \frac{1120}{49} + \frac{1225}{49}$

$= \frac{2601}{49}$

$\cancel{\text{Var}Y}$ $\text{Var}(Y) = E[Y^2] - [E[Y]]^2$

$= \frac{2601}{49} - (\frac{51}{7})^2$

$= 0$

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$\text{Var}(Y) = \text{Var}(\frac{2}{X} + 5) = 4\text{Var}(\frac{1}{X})$

$E[\frac{1}{X}] = \int_0^2 x(\frac{1}{8}x + \frac{3}{16})dx = \frac{17}{24}$

$E[\frac{1}{X^2}] = \int_0^2 (\frac{1}{8}x + \frac{3}{16})dx = \frac{5}{8}$

Thus $\text{Var}(\frac{1}{X}) = E[\frac{1}{X^2}] - E[\frac{1}{X}]^2$

$= \frac{71}{576}$

$\text{Var}(Y) = 4\text{Var}(\frac{1}{X}) = \frac{71}{144}$

≈ 0.493

Problem 7. Let $X \sim Normal(-4, 16)$. (μ, σ^2)

(a) Find $P(X < 1)$.

(b) Find $P(-6 < X < -2)$.

(c) Find $P(X > -2 \mid X > -4)$.

[5×3=15 marks]

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a) $P(X<1) = P(Z < \frac{1-(-4)}{4})$

$= P(Z < 1.2)$

$= \cancel{0.1151}\ 0.8859$

$P(Z < \frac{5}{4})$

$= P(Z < 1.25)$

$= 0.8944$

b) $P(-6 < X < -2)$

$= \cancel{P(\quad < Z < \frac{-2-(-4)}{4}}\ P(\frac{-6-(-4)}{4} < Z < \frac{-2-(-4)}{4})$

$= P(-\frac{1}{2} < Z < \frac{1}{2})$

$= \cancel{P(Z>\frac{1}{2}) - P(Z>\frac{1}{2})}\ 1 - 2P(Z > \frac{1}{2})$

$= 0.383$

c) $P(X>-2 \mid X>-4) = \frac{P(X>-2 \cap X>-4)}{P(X>-4)} = \frac{P(X>-2)}{P(X>-4)}$

$P(X>-2) = P(Z > \frac{-2-(-4)}{4}) = P(Z > \frac{1}{2}) = 0.3085$

$P(X>-4) = P(Z > \frac{-4-(-4)}{4}) = P(Z>0) = 0.5$

$\therefore P(X>-2 \mid X>-4) = \frac{\overset{0.3085}{\cancel{0.3065}}}{0.5} = 0.617$

Problem 8. If X is a generic **positive** random variable, compare the values of $E(X)$ and $\int_0^{+\infty} P(X > u)du$. Specifically, determine in which case they are equal. [10 marks]

- $E[X] = \sum_1^\infty x \cdot f(x)$

$\int_0^\infty P(X>u)\,du$ As $x>0$ $f_X(\alpha) \geqslant 0$ $\therefore E[X] = \sum_1^\infty x \cdot f(x) > 0$

$p(x>u)=1, x=1$ $\therefore \int_0^\infty P(X>u)du$ also >0

when ~~x=1~~ $E[X] = \int_0^\infty P(X>u)du$ ~~$\therefore$ when $x>1$, $x \cdot f(x)$ will be~~

~~p(x>u), x<1~~ $\because \sum_{i=1}^{M} P(x)\,(\alpha_i) = 1$, $F_X(\alpha) = \sum_{a_i \leq a} p_X(\alpha_i)$, $F_X(\alpha) = \int_{-\infty}^{a} f_X(u)\,du$

~~when $x<1$ $E[X] < \int_0^\infty P(X>u)du$~~ $\therefore$ when $x=1$ $E[X] = \int_0^\infty P(X>u)du$ $\boxed{P(X>u)=1}$

~~$x>1$ $E[X] > \int_0^\infty P(X>u)du$~~ as $\int_0^\infty f_X(u)\,du = 1$ at that point

p

- when $p(x>u)<1$, $x<1$ $E[X] < \int_0^\infty p(x>u)du$

~~p(x>u)~~ $x>1$ $E[X] > \int_0^\infty p(x>u)du$

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$P(X \geqslant u) = \int_u^\infty f_X(t)dt$

$\int_0^\infty \int_u^\infty f(x)\,dt\,du = EX$

$\int_0^{+\infty} P(X>u)\,du = \int_0^\infty \int_u^\infty f(x)\,dt\,du$

$= \int_0^\infty \int_0^t f(x)\,du\,dt$

$= \int_0^\infty f(x)(t)\left(\int_0^t 1\,du\right)dt$

$= \int_0^\infty t f_X(t)\,dt = EX$

X is positive so integral range only $[0, +\infty)$