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Assignment 1

Question 1

1) a) $f(x) = \frac{\sqrt{9-x^2}}{x}$ $\quad g(x) = \arccos\left(\frac{x}{4}\right)$

Dom (f) ① $\sqrt{9-x^2} \geq 0 \Rightarrow x^2 \leq 9 \Rightarrow -3 \leq x \leq 3$

② $x \neq 0$

Dom (f) = $x \in [-3, 0) \cup (0, 3]$

Dom (g) arcos : $-1 \leq \frac{x}{4} \leq 1$

$-4 \leq x \leq 4$

Dom(g) = $x \in [-4, 4]$

Largest Domain of f is 3, of g is 4

b) To show f(x) 1 to 1 on (0,2)

$f(x_1) = f(x_2) \Rightarrow x_1 = x_2$

Assume $f(a) = f(b)$ $\quad x_1, x_2 \in (0,2)$

$\frac{\sqrt{9-a^2}}{a} = \frac{\sqrt{9-b^2}}{b}$

$\sqrt{9-a^2} \cdot b = \sqrt{9-b^2} \cdot a$

$(9-a^2) \cdot b^2 = (9-b^2) \cdot a^2$

$9b^2 - a^2b^2 = 9a^2 - b^2a^2$

$9b^2 - 9a^2 = 0$

$9(b^2 - a^2) = 0$

$9(b-a)(b+a) = 0 \Rightarrow (b-a)(b+a) = 0$

As $a, b \in (0,2)$, $b+a = 0$ can't hold

$\therefore b - a = 0$

$b = a \Rightarrow x_2 = x_1$

Thus f(x) is 1 to 1 on (0,2)

$y = f(x) = \frac{\sqrt{9-x^2}}{x}$

$y = \frac{\sqrt{9-x^2}}{x}$

$yx = \sqrt{9-x^2}$

$y^2x^2 = 9 - x^2$

$y^2x^2 + x^2 = 9$

$x^2(y^2+1) = 9$

$x^2 = \frac{9}{y^2+1}$

$x = \pm\sqrt{\frac{9}{y^2+1}} \Rightarrow$

$\because x \in (0,2)$

$y^2 > 0$, $y^2+1 > 0$

$x = \frac{3}{\sqrt{y^2+1}}$

$\therefore x = -\sqrt{\frac{9}{y^2+1}}$ rejected

$f^{-1}(y) = \frac{3}{\sqrt{y^2+1}}$

c) $f(x) = \frac{\sqrt{9-x^2}}{x}$ $g(x) = \arccos\left(\frac{x}{4}\right)$

$$fg = \frac{\sqrt{9-x^2}}{x} \cdot \arccos\left(\frac{x}{4}\right) = \frac{\sqrt{9-x^2} \cdot \arccos\left(\frac{x}{4}\right)}{x}$$

d) $$\frac{g}{f} = \frac{\arccos\left(\frac{x}{4}\right)}{\frac{\sqrt{9-x^2}}{x}} = \frac{\arccos\left(\frac{x}{4}\right) \cdot x}{\sqrt{9-x^2}}$$

e) $f \circ g$

$= f(g(x))$

$= f\left(\arccos\left(\frac{x}{4}\right)\right)$

$= \frac{\sqrt{9-\left(\arccos\left(\frac{x}{4}\right)\right)^2}}{\arccos\left(\frac{x}{4}\right)}$

$$f \circ g = \frac{\sqrt{9-\arccos\left(\frac{x}{4}\right)^2}}{\arccos\left(\frac{x}{4}\right)}$$

$g \circ f$

$= g(f(x))$

$= g\left(\frac{\sqrt{9-x^2}}{x}\right)$

$= \arccos\left(\frac{\frac{\sqrt{9-x^2}}{x}}{4}\right)$

$= \arccos\left(\frac{\sqrt{9-x^2}}{4x}\right)$

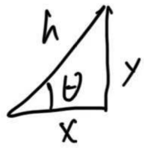
$$g \circ f = \arccos\left(\frac{\sqrt{9-x^2}}{4x}\right)$$

Question 2

2) $\sin(2\theta) = 2\sin(\theta)\cos(\theta)$

Let $\theta = \arctan(\frac{3}{5})$, hypotenuse $= h$

Graph:



$$h = \sqrt{3^2+5^2} = \sqrt{34}$$

$$\sin(\theta) = \frac{y}{h} = \frac{3}{\sqrt{34}}$$

$$\cos(\theta) = \frac{x}{h} = \frac{5}{\sqrt{34}}$$

$$\sin(2\theta) = 2\sin(\theta)\cos(\theta)$$

$$= 2 \times \frac{3}{\sqrt{34}} \times \frac{5}{\sqrt{34}}$$

$$= \frac{15}{17}$$

Thus, $\underline{\sin(2\arctan(\frac{3}{5}) = \frac{15}{17}}$

Question 3

3) $f(x) = \frac{3x-2}{x+1}$

To prove f is 1 to 1

$\cdot x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2)$

$f(x_1) = f(x_2) \Rightarrow x_1 = x_2$

Let $f(x_1) = f(x_2)$, $x_1, x_2 \in Dom(f)$

Assume $f(a) = f(b)$

$$\frac{3a-2}{a+1} = \frac{3b-2}{b+1}$$

$$(3a-2)(b+1) = (3b-2)(a+1)$$

$$3ab+3a-2b-2 = 3ab+3b-2a-2$$

$$3a-2b = 3b-2a$$

$$5a = 5b$$

$$a = b$$

$$x_1 = x_2$$

Thus, $f(x)$ is 1 to 1

Find $f^{-1}(y)$

$$y = f(x) = \frac{3x-2}{x+1}$$

$$y = \frac{3x-2}{x+1}$$

$$y(x+1) = 3x-2$$

$$yx+y = 3x-2$$

$$yx-3x = -2-y$$

$$x(y-3) = -2-y$$

$$x = \frac{-y-2}{y-3}$$

Thus $f^{-1}(y) = \frac{-y-2}{y-3}$

Question 4

4) $f(x) = \sin(x^2) \rightarrow$ prove $\lim_{h \to 0} = \frac{f(x+h) - f(x)}{h} = 2x\cos(x^2)$

Substitute: $\Rightarrow \lim_{h \to 0} = \frac{\sin((x+h)^2) - \sin x^2}{h}$

$$= \frac{\sin(x^2 + 2xh + h^2) - \sin x^2}{h}$$

$\because \sin A - \sin B = 2\cos\frac{A+B}{2} \sin\frac{A-B}{2}$

$\therefore \lim_{h \to 0} = \frac{2\cos(x^2 + xh + \frac{h^2}{2}) \sin(xh + \frac{h^2}{2})}{h}$

As $h \to 0$ $x^2 + xh + \frac{h^2}{2} \to x^2$

$\sin(xh + \frac{h^2}{2}) \to xh + \frac{h^2}{2} \to$ Sandwich Principle

As $\lim_{\theta \to 0} \frac{\sin(\theta)}{\theta} = 1$

$\therefore \lim_{h \to 0} = \frac{2\cos(x^2) \cdot (xh + \frac{h^2}{2})}{h}$

$\therefore$ As $_{\theta \to 0}$ $\sin(\theta) = \theta$

Take $\theta = xh + \frac{h^2}{2}$

$\therefore \sin(xh + \frac{h^2}{2}) = xh + \frac{h^2}{2}$

$$= \frac{2\cos(x^2) \cdot h(x + \frac{h}{2})}{h}$$

$$= 2\cos(x^2) \cdot (x + \frac{h}{2})$$

As $h \to 0$ $x + \frac{h}{2} \to x$

$\therefore \lim_{h \to 0} \frac{f(x+h) - f(h)}{h} = 2x\cos(x^2)$ as $f(x) = \sin(x^2)$

Question 5

Q5

a) $\lim_{x \to 4} \frac{x^2+5x+4}{x^2+3x-4}$

$= \frac{4^2+5\times4+4}{4^2+3\times4-4}$

$= \frac{40}{24}$

$= \frac{5}{3}$

$\therefore \lim_{x \to 4} \frac{x^2+5x+4}{x^2+3x-4} = \frac{5}{3}$

b) $\lim_{x \to 0} \frac{x}{\sqrt{1-x}-1}$

$= \frac{x}{\sqrt{1-x}-1} \cdot \frac{\sqrt{1-x}+1}{\sqrt{1-x}+1}$

$= \frac{x(\sqrt{1-x}+1)}{(\sqrt{1-x}-1)(\sqrt{1-x}+1)}$

$= \frac{x(\sqrt{1-x}+1)}{-x}$

$= -(\sqrt{1-x}+1)$

As $x \to 0$

$\sqrt{1-x} = 1$

$-(\sqrt{1-x}+1)$

$= -2$

$\therefore \lim_{x \to 0} \frac{x}{\sqrt{1-x}-1} = -2$

c) $\lim_{t \to 1} \left(\frac{1}{(t-1)\sqrt{t}} - \frac{1}{t-1} \right)$

$\frac{1}{(t-1)\sqrt{t}} - \frac{1}{t-1}$

$= \frac{1}{(t-1)\sqrt{t}} - \frac{\sqrt{t}}{(t-1)(\sqrt{t})}$

$= \frac{1-\sqrt{t}}{(t-1)\sqrt{t}}$

$\lim_{t \to 1} \frac{1-\sqrt{t}}{(t-1)\sqrt{t}}$ $\quad 1-\sqrt{t} = (1-\sqrt{t}) \times \frac{(1+\sqrt{t})}{(1+\sqrt{t})} = \frac{(1-t)}{1+\sqrt{t}}$

$= \frac{(1-t)}{(t-1)\sqrt{t}(1+\sqrt{t})}$

$= \frac{-(t-1)}{(t-1)\sqrt{t}(1+\sqrt{t})}$

$= \frac{-1}{\sqrt{t}(1+\sqrt{t})}$

$t \to 1 \therefore \frac{-1}{\sqrt{t}(1+\sqrt{t})} = -\frac{1}{2}$

$\therefore \lim_{t \to 1} \left(\frac{1}{(t-1)\sqrt{t}} - \frac{1}{t-1} \right) = -\frac{1}{2}$

d) $\lim_{x \to 1} \frac{x^4-1}{\sin(x-1)}$

$x^4-1 = (x^2-1)(x^2+1) = (x+1)(x-1)(x^2+1)$

$\Rightarrow \lim_{x \to 1} \frac{(x+1)(x-1)(x^2+1)}{\sin(x-1)}$

$= \lim_{x \to 1} (x+1)(x^2+1) \times \frac{x-1}{\sin(x-1)}$

$\because \lim_{x \to 0} \frac{\sin ax}{bx} = \frac{a}{b}, \quad \therefore \lim_{x \to 0} \frac{bx}{\sin ax} = \frac{b}{a}$

Let $x = x-1$ $\quad \therefore \frac{x-1}{\sin(x-1)} = \frac{1x}{\sin 1x}$

$\therefore b = a = 1 \quad \therefore \frac{1}{1} = 1$

$\because x+1 = 2 \quad x^2+1 = 2 \quad \frac{x-1}{\sin(x-1)} = 1$

$\therefore 2 \times 2 \times 1 = 4$ $\quad$ Thus $\lim_{x \to 1} \frac{x^4-1}{\sin(x-1)} = 4$

Question 6

6)

a) $f(0)$ $x=0 \Rightarrow f(x)=3x$

$3\times 0=0$

$\therefore f(0)=0$

b) $x \to 0^-$

$x \leq 0$ $f(x)=3x=3\cdot 0=0$

$x \to 0^+$ $0<x\leq 1$

$\Rightarrow f(x)=\frac{2x^3-2}{x^2-1}=\frac{2(0)^3-2}{0^2-1}=2$

$\because \lim_{x\to 0^-} f(x)=0 \neq \lim_{x\to 0^+} f(x)=2$

$\therefore \lim_{x\to 0} f(x)$ does not exist

c) $\lim_{x\to 1} f(x)$ exist $\Rightarrow \lim_{x\to 1^+} f(x)=\lim_{x\to 1^-} f(x)$

$\lim_{x\to 1^-} f(x)=\frac{2x^3-2}{x^2-1}=\frac{2(x^3-1)}{(x+1)(x-1)}=\frac{2(x-1)(x^2+x+1)}{(x+1)(x-1)}=\frac{2(x^2+x+1)}{x+1}$

$\because x\to 1^-$ $\therefore \lim_{x\to 1^-} f(x)=\frac{2(x^2+x+1)}{x+1}=\frac{2(1+1+1)}{2}=3$

$\lim_{x\to 1^+} f(x)=\frac{x^2+2ax+b}{x-1}$ $\because x\to 1^+$ $x-1=0 \Rightarrow \frac{x^2+2ax+b}{0} \Rightarrow$ undefined

$\therefore$ $x-1$ will need to be factored out to calculcte

$\because \lim_{x\to 1^-} f(x)=\lim_{x\to 1^+} f(x) \Rightarrow \lim_{x\to 1} f(x)$ exist $\Rightarrow \lim_{x\to 1^+} f(x)=\frac{x^2+2ax+b}{x-1}=3$, $x-1$ factored out

$\therefore x^2+2ax+b=(x-1)(x-y)$, after $x-1$ factored $\Rightarrow \lim_{x\to 1^+} f(x)=x-y=3$, as $x\to 1^+$ $y=-2$

$\therefore x^2+2ax+b=(x-1)(x+2)=x^2+x-2$

From above $2ax=x$ $b=-2$

$a=\frac{1}{2}$

Thus, to make $\lim_{x\to 1} f(x)$ exist, $a=\frac{1}{2}$, $b=-2$

Question 7

7) a) $\log_2(x-2) = 3 - \log_2(x-1)$

$\log_2(x-2) + \log_2(x-1) = 3$

$\log_2[(x-2)+(x-1)] = 3$

$(x-2)(x-1) = 2^3$

$x^2 - 3x + 2 = 8$

$x^2 - 3x - 6 = 0$

$a = 1 \quad b = -3 \quad c = -6$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-(-3) \pm \sqrt{(-3)^2 - 4 \times 1 \times (-6)}}{2}$$

$$= \frac{3 \pm \sqrt{33}}{2}$$

$\because x - 2 > 0$

$\therefore x > 2$

$\because \frac{3-\sqrt{33}}{2} < 2$ (rejected)

$\therefore x = \frac{3+\sqrt{33}}{2}$

b) $\log_{16}(2x+3) + \log_{16}(x+5) = \log_4(x)$

$\log_{16}[(2x+3)+(x+5)] = \log_4(x)$

$\frac{1}{2}\log_4[(2x+3)+(x+5)] = \log_4(x)$

$\log_4[(2x+3)+(x+5)] = 2\log_4(x)$

$\log_4[(2x+3)+(x+5)] = \log_4(x^2)$

$(2x+3)(x+5) = x^2$

$2x^2 + 13x + 15 = x^2$

$x^2 + 13x + 15 = 0$

$a = 1 \quad b = 13 \quad c = 15$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2}$$

$$= \frac{-13 \pm \sqrt{(13)^2 - 1 \times 15 \times 4}}{2}$$

$$= \frac{-13 \pm \sqrt{109}}{2}$$

$\because \log_4(x)$

$\therefore x > 0$

$\because \frac{-13+\sqrt{109}}{2} < 0$ (rejected), $\frac{-13-\sqrt{109}}{2} < 0$ (rejected)

$\therefore$ No solution

Question 8

8) Define $f(x) = 3x - (1 + \tan^{-1}(x) + \sin(x)) \in (0, 1)$

As $x = 0$

$f(0) = 3(0) - (1 + \tan^{-1}(0) + \sin(0)) = 0 - (1+0+0) = -1$

As $x = 1$

$f(1) = 3(1) - (1 + \tan^{-1}(1) + \sin(1)) = 3 - (1 + \frac{\pi}{4} + 0.8415) \approx 0.3731$

$\therefore$ $f(0) = -1$ (Negative) $\quad f(1) = 0.3731$ (Positive)

$\because$ Since $f(x) = 3x - (1 + \tan^{-1}(x) + \sin(x))$ is a polynomial and thus is continuous on $\mathbb{R}$ and hence on $(0,1)$, $f(0) = -1 < 0$ and $f(1) = 0.3731 > 0$, By IVT, $f(x) = 3x - (1 + \tan^{-1}(x) + \sin(x))$,

There exists at least one solution to the equation $3x = 1 + \tan^{-1}(x) + \sin(x)$ $c \in (0,1)$ such that $f(c) = 0$

Question 9

9)

a) $\lim_{x\to 2} \tan(x^2-x+1)\frac{\sin(x-2)}{2(x-2)}$

$x=2 \quad x^2-x+1=4-2+1=3$

$\Rightarrow \lim_{x\to 2} \tan(3)\frac{\sin(x-2)}{2(x-2)}$

$\because \lim_{u\to 0}\frac{\sin u}{u}=1$

$\therefore u=x-2 \quad x\to 2 \quad u\to 0$

$\frac{\sin(x-2)}{2(x-2)}=\frac{\sin u}{2u}$

$\therefore \lim_{u\to 0}\frac{\sin u}{u}=\frac{1}{2}$

$\Rightarrow \lim_{x\to 2}\tan(x^2-x+1)\cdot\frac{\sin(x-2)}{2(x-2)}$

$=\tan(3)\cdot\frac{1}{2}$

$=\frac{\tan(3)}{2}$

$=0.02620$

Thus $\lim_{x\to 2}\tan(x^2-x+1)\frac{\sin(x-2)}{2(x-2)}=\frac{\tan(3)}{2}$
$=0.02620$

b) $\lim_{x\to\infty}\frac{x\cos x}{(x^2+1)\tan^{-1}x}$

$\because x\to\infty \therefore \tan^{-1}x\to\frac{\pi}{2}$

$\because x\to\infty, \therefore x^2+1\approx x^2$

$\lim_{x\to\infty}\frac{x\cos x}{(x^2+1)\tan^{-1}x}\approx\lim_{x\to\infty}\frac{x\cos x}{x^2\cdot\frac{\pi}{2}}=\lim\frac{2\cos x}{\pi x}$

$\because x\to\infty \therefore -1\le\cos x\le 1$

$\therefore -\frac{2}{\pi x}\le\frac{2\cos x}{\pi x}\le\frac{2}{\pi x}$

$\because x\to\infty, -\frac{2}{\pi x}\to 0 \quad \frac{2}{\pi x}\to 0$

$\therefore \lim_{x\to\infty}\frac{2\cos x}{\pi x}=0$

Thus $\lim_{x\to\infty}\frac{x\cos x}{(x^2+1)\tan^{-1}x}=0$

9)

c) $\lim_{x\to 0^+}\left(\sqrt{\frac{1}{x}} - \sqrt{\frac{1}{x}-\frac{1}{1130}}\right)$

$\Rightarrow \sqrt{\frac{1}{x}} - \sqrt{\frac{1}{x}-\frac{1}{1130}}\Big)$

$\times$

$\frac{\sqrt{\frac{1}{x}} + \sqrt{\frac{1}{x}-\frac{1}{1130}}}{\sqrt{\frac{1}{x}} + \sqrt{\frac{1}{x}-\frac{1}{1130}}}$

$= \frac{\frac{1}{x} - \frac{1}{x} + \frac{1}{1130}}{\sqrt{\frac{1}{x}} + \sqrt{\frac{1}{x}-\frac{1}{1130}}}$

$= \frac{\frac{1}{1130}}{\sqrt{\frac{1}{x}} + \sqrt{\frac{1}{x}-\frac{1}{1130}}}$

$= \frac{1}{1130\left(\sqrt{\frac{1}{x}} + \sqrt{\frac{1}{x}-\frac{1}{1130}}\right)}$

$\because x \to 0^+,\ \frac{1}{x} = \frac{1}{0^+}$

$\therefore \sqrt{\frac{1}{x}} = \sqrt{\frac{1}{0^+}} = \sqrt{\infty}$

$\sqrt{\frac{1}{x}-\frac{1}{1130}} = \sqrt{\frac{1}{0^+}-\frac{1}{1130}} = \sqrt{\infty}$

$\therefore \lim_{x\to 0^+} = \frac{1}{1130(\infty+\infty)}$

$= \frac{1}{\infty}$

$= 0$

Thus, $\lim_{x\to 0^+}\left(\sqrt{\frac{1}{x}} - \sqrt{\frac{1}{x}-\frac{1}{1130}}\right)$

$= 0$

d) $\lim_{x\to -\infty}\left(\sqrt{4x^2-x+1}+2x\right)$

$= \left(\sqrt{4x^2-x+1}+2x\right)\cdot\frac{\sqrt{4x^2-x+1}-2x}{\sqrt{4x^2-x+1}-2x}$

$= \frac{4x^2-x+1-4x^2}{\sqrt{4x^2-x+1}-2x}$

$= \frac{-x+1}{\sqrt{4x^2-x+1}-2x}$

$\Rightarrow$ Both divide by x

$\div x = \lim_{x\to -\infty}\frac{-\frac{x}{x}+\frac{1}{x}}{\frac{\sqrt{4x^2-x+1}-2x}{x}}$

$= \frac{-1+\frac{1}{x}}{\frac{\sqrt{4x^2-x+1}}{-\sqrt{x^2}}-2}$

$\because x \to -\infty$

$\therefore x < 0$

$\therefore \sqrt{x^2} = -x$

$= \frac{-1+\frac{1}{x}}{-\sqrt{4-\frac{1}{x}+\frac{1}{x^2}}-2}$

As $x \to -\infty$

$\frac{1}{x} \to 0$

$\frac{1}{x} \to 0$

$\frac{1}{x^2} \to 0$

$\lim_{x\to -\infty} = \frac{-1+0}{-\sqrt{4-0-0}-2}$

$= \frac{-1}{-2-2}$

$= \frac{1}{4}$

Thus $\lim_{-\infty} = \left(\sqrt{4x^2-x+1}+2x\right)$

$= \frac{1}{4}$

Question 10

10) At $x=0$

$f(x) = \frac{1}{4}$

As $x \to 0^+$

$f(x) = \frac{1}{2} - \frac{\sin(4 \cdot 0)}{16 \cdot 0} = \frac{1}{2} - 0 = \frac{1}{2}$

As $x \to 0^-$

$f(x) = \frac{1}{4}$

$\underset{\lim \to 0^+}{f(x)} \neq \underset{\lim \to 0^-}{f(x)}$, $f(x)$ not continous at $x=0$

At $x=1$

$f(1) = \frac{1}{\sqrt{16-1^2}} = \frac{1}{\sqrt{15}}$

As $x \to 1^-$

$f(x) = \frac{1}{2} - \frac{\sin(4 \cdot 1)}{16 \cdot 1} = \frac{1}{2} - \frac{\sin(4)}{16} \approx 0.5472$

As $x \to 1^+$

$f(x) = \frac{1}{\sqrt{15}} \approx 0.2582$

$\underset{\lim \to 1^-}{f(x)} \neq \underset{\lim \to 1^+}{f(x)}$

Thus, $f(x)$ not continous at $x=1$

At $x=3$

$f(3) = \frac{1}{\sqrt{16-3^2}} = \frac{1}{\sqrt{7}}$

As $x \to 3^-$ $f(x) = \frac{1}{\sqrt{7}} \approx 0.3780$

As $x \to 3^+$ $f(x) = \frac{3\sqrt{7}+1}{21} - \frac{\sin(2 \cdot 3 - 6)}{7(3^2-9)} = \frac{3\sqrt{7}+1}{21} \approx 0.4256$

$\underset{\lim \to 3^-}{f(x)} \neq \underset{\lim \to 3^+}{f(x)}$ => Thus $f(x)$ is not continous at $x=3$

Thus $f(x)$ is not continous at $x=0$, $x=1$, $x=3$