AMA2104 Random Class Quiz 4

NAME: **ID:**

Problem 1. Let $Y_1, Y_2, Y_3, \cdots$ be a sequence of i.i.d. random variables with mean $EY_i = 0$ and $\operatorname{Var}(Y_i) = 4$. Define the discrete-time random process $\{X(n), n \in \mathbb{N}\}$ as

$$X(n) = Y_1 + Y_2 + \cdots + Y_n, \qquad \text{for all } n \in \mathbb{N}$$

Find $\mu_X(n)$ and $R_X(m,n)$, for all $n, m \in \mathbb{N}$.

Solution:

We have

$$\begin{aligned}
\mu_X(n) &= E[X(n)] \\
&= E\left[Y_1 + Y_2 + \cdots + Y_n\right] \\
&= E\left[Y_1\right] + E\left[Y_2\right] + \cdots + E\left[Y_n\right] \\
&= 0
\end{aligned}$$

Let $m \leq n$, then

$$\begin{aligned}
R_X(m,n) &= E[X(m)X(n)] \\
&= E\left[X(m)\left(X(m) + Y_{m+1} + Y_{m+2} + \cdots + Y_n\right)\right] \\
&= E\left[X(m)^2\right] + E[X(m)]E\left[Y_{m+1} + Y_{m+2} + \cdots + Y_n\right] \\
&= E\left[X(m)^2\right] + 0 \\
&= \operatorname{Var}(X(m)) \\
&= \operatorname{Var}\left(Y_1\right) + \operatorname{Var}\left(Y_2\right) + \cdots + \operatorname{Var}\left(Y_m\right) \\
&= 4m
\end{aligned}$$

Similarly, for $m \geq n$, we have

$$\begin{aligned}
R_X(m,n) &= E[X(m)X(n)] \\
&= 4n
\end{aligned}$$

We conclude

$$R_X(m,n) = 4\min(m,n)$$

Problem 2. Let $X(t)$ and $Y(t)$ be two jointly WSS random processes. Consider the random process $Z(t)$ defined as

$$Z(t) = X(t) + Y(t)$$

Show that $Z(t)$ is WSS.

Solution:

Since $X(t)$ and $Y(t)$ are jointly WSS, we conclude

1. $\mu_X(t) = \mu_X, \mu_Y(t) = \mu_Y$,
2. $R_X(t_1, t_2) = R_X(t_1 - t_2), R_Y(t_1, t_2) = R_Y(t_1 - t_2)$,
3. $R_{XY}(t_1, t_2) = R_{XY}(t_1 - t_2)$.

Therefore, we have

$$\begin{aligned} \mu_Z(t) &= E[X(t) + Y(t)] \\ &= E[X(t)] + E[Y(t)] \\ &= \mu_X + \mu_Y \end{aligned}$$

$$\begin{aligned} R_Z(t_1, t_2) &= E\left[(X(t_1) + Y(t_1))(X(t_2) + Y(t_2))\right] \\ &= E\left[X(t_1) X(t_2)\right] + E\left[X(t_1) Y(t_2)\right] + E\left[Y(t_1) X(t_2)\right] + E\left[Y(t_1) Y(t_2)\right] \\ &= R_X(t_1 - t_2) + R_{XY}(t_1 - t_2) + R_{YX}(t_1 - t_2) + R_Y(t_1 - t_2). \end{aligned}$$

Problem 3. The number of customers arriving at a grocery store can be modeled by a Poisson process with intensity $\lambda = 10$ customers per hour.

1. Find the probability that there are 2 customers between $10:00$ and $10:20$.
2. Find the probability that there are 3 customers between $10:00$ and $10:20$ and 7 customers between 10:20 and 11 .

Solution:

1. Here, $\lambda = 10$ and the interval between $10:00$ and $10:20$ has length $\tau = \frac{1}{3}$ hours. Thus, if X is the number of arrivals in that interval, we can write $X \sim \text{Poisson}(10/3)$. Therefore,

$$\begin{aligned} P(X=2) &= \frac{e^{-\frac{10}{3}}\left(\frac{10}{3}\right)^2}{2!} \\ &\approx 0.2 \end{aligned}$$

2. Here, we have two non-overlapping intervals $I_1 = (10:00$ a.m., 10:20 a.m.$]$ and $I_2 = (10:20$ a.m., 11 a.m.$]$. Thus, we can write

$$P\left(3 \text{ arrivals in } I_1 \text{ and } 7 \text{ arrivals in } I_2\right) = \\ P\left(3 \text{ arrivals in } I_1\right) \cdot P\left(7 \text{ arrivals in } I_2\right)$$

Since the lengths of the intervals are $\tau_1 = 1/3$ and $\tau_2 = 2/3$ respectively, we obtain $\lambda\tau_1 = 10/3$ and $\lambda\tau_2 = 20/3$. Thus, we have

$$\begin{aligned} P\left(3 \text{ arrivals in } I_1 \text{ and } 7 \text{ arrivals in } I_2\right) &= \frac{e^{-\frac{10}{3}}\left(\frac{10}{3}\right)^3}{3!} \cdot \frac{e^{-\frac{20}{3}}\left(\frac{20}{3}\right)^7}{7!} \\ &\approx 0.0325 \end{aligned}$$

Problem 4. Let $\{X(t), t \in [0, \infty)\}$ be defined as

$$X(t) = A + Bt, \qquad \text{for all } t \in [0, \infty)$$

where A and B are independent normal $N(1,1)$ random variables.

1. Define the random variable $Y = X(1)$. Find the PDF of Y.
2. Let also $Z = X(2)$. Find $E[YZ]$.

1. We have

$$Y = X(1) = A + B$$

Since A and B are independent $N(1,1)$ random variables, $Y = A + B$ is also normal with

$$\begin{aligned} EY &= E[A+B] \\ &= E[A] + E[B] \\ &= 1 + 1 \\ &= 2, \end{aligned}$$

$$\begin{aligned} \text{Var}(Y) &= \text{Var}(A+B) \\ &= \text{Var}(A) + \text{Var}(B) \\ &= 2 \end{aligned}$$

Thus, we conclude that $Y \sim N(2,2)$:

$$f_Y(y) = \frac{1}{\sqrt{4\pi}} e^{-\frac{(y-2)^2}{4}}$$

2. We have

$$\begin{aligned} E[YZ] &= E[(A+B)(A+2B)] \\ &= E\left[A^2 + 3AB + 2B^2\right] \\ &= E\left[A^2\right] + 3E[AB] + 2E\left[B^2\right] \\ &= 2 + 3E[A]E[B] + 2 \cdot 2 \\ &= 9. \end{aligned}$$

Problem 5. If $n = 16, \bar{X} = 9$ and $s^2 = 16$, construct a 98% confidence interval for μ.

Solution

Since $n = 16, \bar{X} = 9$ and $s^2 = 25$, and $X_1, \ldots, X_n \sim N(\mu, \sigma^2)$ independently, we have $\frac{\bar{X}-\mu}{s/\sqrt{n}} \sim t_{n-1} = t_{15}$; now

$1 - \alpha = 0.98 \Rightarrow \frac{\alpha}{2} = 0.01 \Rightarrow t_{\alpha/2,n-1} = 2.602$ from t-table

98% confidence interval for μ

$$= 9 \pm 2.602 \cdot \frac{\sqrt{16}}{\sqrt{16}} = 9 \pm 2.602 = (6.398, 11.602)$$