AMA2104 Random Class Quiz 3

NAME: **ID:**

Problem 1.

(1) Using the fact that $P(Z \geq z_\alpha) = \alpha$ for $Z \sim N(0,1)$ and $0 < \alpha < 1$, show that

$$P\left(\bar{X} - z_{\alpha/2}\frac{\sigma}{\sqrt{n}} \leq \mu \leq \bar{X} + z_{\alpha/2}\frac{\sigma}{\sqrt{n}}\right) = 1 - \alpha.$$

(2) If $n = 36, \bar{X} = 20$ and $\sigma^2 = 25$, construct a 95% confidence interval for μ.

Solution:

(1)

$$\begin{aligned}
&P\left(\bar{X} - z_{\alpha/2}\frac{\sigma}{\sqrt{n}} \leq \mu \leq \bar{X} + z_{\alpha/2}\frac{\sigma}{\sqrt{n}}\right) \\
&= P\left(-z_{\alpha/2}\frac{\sigma}{\sqrt{n}} \leq \mu - \bar{X} \leq z_{\alpha/2}\frac{\sigma}{\sqrt{n}}\right) = P\left(-z_{\alpha/2}\frac{\sigma}{\sqrt{n}} \leq \bar{X} - \mu \leq z_{\alpha/2}\frac{\sigma}{\sqrt{n}}\right) \\
&= P\left(-z_{\alpha/2} \leq \frac{\bar{X} - \mu}{\sigma/\sqrt{n}} \leq z_{\alpha/2}\right) = P\left(-z_{\alpha/2} \leq Z \leq z_{\alpha/2}\right) = 1 - \alpha, \text{ since } \bar{X} \sim N\left(\mu, \frac{\sigma^2}{n}\right)
\end{aligned}$$

(2) Since $n = 36, \bar{X} = 20, \sigma^2 = 25$, and $X_1, \ldots, X_n \sim N(\mu, \sigma^2)$ independently, we have $\bar{X} \sim N\left(\mu, \frac{\sigma^2}{n}\right) \Rightarrow \frac{\bar{X}-\mu}{\sigma/\sqrt{n}} \sim N(0,1)$;
now $1 - \alpha = 0.95 \Rightarrow \frac{\alpha}{2} = 0.025 \Rightarrow z_{\alpha/2} = 1.96$ from z-table 95% confidence interval for μ

$$= 20 \pm 1.96 \cdot \frac{5}{\sqrt{36}} = 20 \pm 1.633 = (18.367, 21.633)$$

Problem 2. A sample of 9 light bulbs are inspected, their lifetimes (in days) are (assume they are drawn from a normal population)

$$670, 690, 800, 721, 660, 650, 560, 540, 730.$$

If we know the standard deviation of lifetimes is 80 days, test whether, at 5% level of significance, the mean lifetime is 610 days.

Solution:

(a) Let μ (in day) denote the mean lifetime of the light bulbs.

$$H_0 : \mu = 610$$

$$H_1 : \mu \neq 610$$

$$\sum x = 6021, \bar{x} = \frac{\sum x}{n} = \frac{6021}{9} = 669$$

Under $H_0, \alpha = 0.05$, the test statistic

$$z = \frac{\bar{x} - \mu}{\sigma/\sqrt{n}} = \frac{669 - 610}{80/\sqrt{9}} = 2.2125 > z_{\alpha/2} = z_{0.025} = 1.96$$

Hence, we reject H_0 at 5% level of significance.