THE HONG KONG POLYTECHNIC UNIVERSITY

Department of Applied Mathematics

Subject Code: AMA 2104 **Subject Title:** PROBABILITY AND ENGINEERING STATISTICS

Name: **Student ID:**

Subject Examiner: Dr. Jianhui Huang

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Problem 1. The following statements are true or false, you need only specify your final answer: "T" for true, or "F" for false:

- (a) $\emptyset \neq \{\emptyset\} \neq \{0\}$ **Answer: (T)**
- (b) For any events E and F with $P(F) > 0$, $P(E \cap F) \leq P(E \mid F)$ **Answer: (T)**
- (c) If the sets A, B are independent, then their complement sets A^c, B^c are also independent. **Answer: (T)**
- (d) If X follows binomial distribution $Binomial(n = 50, p = 0.4)$, then $50 - X$ also follows binomial distribution. **Answer: (T)**
- (e)If a set A is independent to all other sets, then A must be empty set $\emptyset$. **Answer: (F)**

[4×5=20 marks]

Problem 2.

If $P(A) = \frac{1}{5}, P(B) = \frac{7}{15}$, and $P(A \mid B) + P(B \mid A) = \frac{7}{10}$, find

(a) $P(A \cap B)$

(b) $P(A|A \cup B)$

[5×2=10 marks]

Solution:

Let $P(A \cap B) = p$.

$P(A \mid B) + P(B \mid A) = \frac{15}{7}p + 5p = \frac{7}{10} \Rightarrow P(A \cap B) = p = \frac{49}{500}$.

$$P(A|A \cup B) = \frac{P(A \cap (A \cup B))}{P(A \cup B)} = \frac{P(A)}{P(A) + P(B) - P(A \cap B)} = \frac{\frac{1}{5}}{\frac{1}{5} + \frac{7}{15} - \frac{49}{500}} \approx 0.352$$

Problem 3. The average waiting time that people spend going through airport security for planes at an airport is 18 minutes with a standard deviation of 5.2 minutes.

(a) What is the probability that the average waiting time for 50 people is more than 20 minutes?

(b) The probability that 80 randomly chosen people have an average waiting time less than a certain average time is 0.1788. What is the average time?

[5×2=10 marks]

Solution:

(i) Let T (in min) be the waiting time a person spent. Since $n = 50 > 30$, by Central Limit Theorem, $\bar{T} \sim N\left(18, \frac{5.2^2}{50}\right)$ approximately.

$$P(\bar{T} > 20) = P\left(Z > \frac{20-18}{5.2/\sqrt{50}}\right) \approx P(Z > 2.72) = 0.0033, \text{ where } Z \sim N(0,1).$$

(ii) Let m (in min) be the certain average time. Since $n = 80 > 30$, by Central Limit Theorem, $\bar{T} \sim N\left(18, \frac{5.2^2}{80}\right)$ approximately.

$$P(\bar{T} < m) = 0.1788 \Leftrightarrow P\left(Z < \frac{m-18}{5.2/\sqrt{80}}\right) = 0.1788, \text{ where } Z \sim N(0,1).$$

$$\frac{m-18}{5.2/\sqrt{80}} \approx -0.92 \text{ from } z\text{-table} \Rightarrow \quad m \approx 17.4651 \text{ min}$$

Problem 4. The joint probability mass function of X and Y is given by

$$p(x,y) = \frac{x+y}{30}, \text{ for } x = 0,1,2,3 \text{ and } y = 0,1,2.$$

Find

(a) $P(X \leq 2, Y = 1)$

(b) $P(X \leq 2 \mid Y = 2)$

(c) $P(X \leq 2, Y > 2)$

[5×3=15 marks]

Solution:

(a)

$$\begin{aligned} P(X \leq 2, Y = 1) &= P(X = 0, Y = 1) + P(X = 1, Y = 1) + P(X = 2, Y = 1) \\ &= \frac{1}{30} + \frac{2}{30} + \frac{3}{30} = \frac{1}{5} \end{aligned}$$

(b)

$$\begin{aligned} P(X \leq 2 \mid Y = 2) &= \frac{P(X \leq 2, Y = 2)}{P(Y = 2)} \\ &= \frac{P(X = 0, Y = 2) + P(X = 1, Y = 2) + P(X = 2, Y = 2)}{P(Y = 2)} \\ &= \frac{2+3+4}{14} = \frac{9}{14} \end{aligned}$$

(c)

$$P(X \leq 2, Y > 2) = 0$$

Problem 5. Let X and Y be continuous random variables having the joint density function

$$f(x,y) = \begin{cases} 4xy, & 0<x<1, \quad 0<y<1 \\ 0 & \text{elsewhere} \end{cases}$$

(a) Find $P(4X > Y)$

(b) Find $E(X)$

[5×2=10 marks]

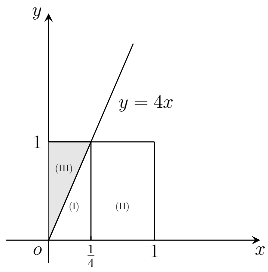
Solution:

(a)

Method 1

$$\begin{aligned} P(4X>Y) &= \text{Part I} + \text{Part II} \\ &= \underbrace{\int_0^{\frac{1}{4}}\int_0^{4x} 4xy\,dy\,dx}_{\text{Part I}} + \underbrace{\int_{\frac{1}{4}}^{1}\int_0^{1} 4xy\,dy\,dx}_{\text{Part II}} \\ &= \int_0^{\frac{1}{4}} 32x^3\,dx + \int_{\frac{1}{4}}^{1} 2x\,dx \\ &= \frac{1}{32} + \frac{15}{16} \\ &= \frac{31}{32} \approx 0.9687 \end{aligned}$$

Method 2



$$
\begin{aligned}
P(Y < 4X) &= 1 - \text{Part III} = 1 - P(Y > 4X) \\
&= 1 - \int_0^{\frac{1}{4}} \int_{4x}^{1} 4xydydx \\
&= 1 - \int_0^{\frac{1}{4}} (2x - 32x^3)dx \\
&= 1 - (x^2)|_0^{\frac{1}{4}} + 8x^4|_0^{\frac{1}{4}} \\
&= 1 - (\frac{1}{4})^2 + 8(\frac{1}{4})^4 \\
&= 1 - \frac{1}{16} + \frac{1}{32} = \frac{31}{32} \approx 0.9687
\end{aligned}
$$

(b)

$$
\begin{aligned}
E(X) &= \int_0^1 xf(x)dx \\
&= \int_0^1 x \int_0^1 f(x,y)dydx \\
&= \int_0^1 x \int_0^1 4xydydx \\
&= \int_0^1 x \cdot 2xdx \\
&= \frac{2}{3}
\end{aligned}
$$

Problem 6. Let X be a continuous random variable with **density function**:

$$f_X(x) = \begin{cases} x^2\left(\frac{1}{8}x + \frac{3}{16}\right) & 0 < x \le 2 \\ 0 & \text{otherwise} \end{cases}$$

If $Y = \frac{2}{X} + 5$, find $\mathrm{Var}(Y)$. [10 marks]

Hints:

apply the property of variance, and definition of expectations

Solution:
First, note that

$$\mathrm{Var}(Y) = \mathrm{Var}\left(\frac{2}{X} + 5\right) = 4\,\mathrm{Var}\left(\frac{1}{X}\right),$$

Thus, it suffices to find $\mathrm{Var}\left(\frac{1}{X}\right) = E\left[\frac{1}{X^2}\right] - \left(E\left[\frac{1}{X}\right]\right)^2$. We have

$$E\left[\frac{1}{X}\right] = \int_0^2 x\left(\frac{1}{8}x + \frac{3}{16}\right) dx = \frac{17}{24}$$
$$E\left[\frac{1}{X^2}\right] = \int_0^2 \left(\frac{1}{8}x + \frac{3}{16}\right) dx = \frac{5}{8}$$

Thus, $\mathrm{Var}\left(\frac{1}{X}\right) = E\left[\frac{1}{X^2}\right] - \left(E\left[\frac{1}{X}\right]\right)^2 = \frac{71}{576}$. So, we obtain

$$\mathrm{Var}(Y) = 4\,\mathrm{Var}\left(\frac{1}{X}\right) = \frac{71}{144} \approx 0.493.$$

Problem 7. Let $X \sim Normal(-4, 16)$.

(a) Find $P(X < 1)$.

(b) Find $P(-6 < X < -2)$.

(c) Find $P(X > -2 \mid X > -4)$.

[5×3=15 marks]

Solution:

X is a normal random variable with $\mu = -4$ and $\sigma = \sqrt{16} = 4$, thus we have

(a) Find $P(X < 1)$:

$$\begin{aligned} P(X < 1) &= F_X(1) \\ &= \Phi\left(\frac{1-(-4)}{4}\right) \\ &= \Phi(1.25) \approx 0.8944 \end{aligned}$$

(b) Find $P(-6 < X < -2)$:

$$\begin{aligned} P(-6 < X < -2) &= F_X(-2) - F_X(-6) \\ &= \Phi\left(\frac{(-2)-(-4)}{4}\right) - \Phi\left(\frac{(-6)-(-4)}{4}\right) \\ &= \Phi\Big(\frac{1}{2}\Big) - \Phi\Big(-\frac{1}{2}\Big) \\ &= 2\Phi\Big(\frac{1}{2}\Big) - 1 \quad (\text{ since } \Phi(-x) = 1 - \Phi(x)) \\ &\approx 0.383 \end{aligned}$$

(c) Find $P(X > -2 \mid X > -4)$:

$$\begin{aligned} P(X > -2 \mid X > -4) &= \frac{P(X > -2, X > -4)}{P(X > -4)} \\ &= \frac{P(X > -2)}{P(X > -4)} \\ &= \frac{1 - \Phi\left(\frac{(-2)-(-4)}{4}\right)}{1 - \Phi\left(\frac{(-4)-(-4)}{4}\right)} \\ &= \frac{1 - \Phi(\frac{1}{2})}{1 - \Phi(0)} \approx 0.617 \end{aligned}$$

Problem 8. If X is a generic **positive** random variable, compare the values of $E(X)$ and $\int_0^{+\infty} P(X > u)du$. Specifically, determine in which case they are equal. [10 marks]

Hints: you should apply double integral and exchange the orders

Solution: We have

$$P(X \geq u) = \int_u^{\infty} f_X(t)dt$$

with f_X the density function of X.

Thus, we need to show that

$$\int_0^{\infty} \int_u^{\infty} f_X(t)dtdu = EX$$

We can take the integral with respect to u or t. Thus, we can write

$$\begin{aligned}
&\int_0^{+\infty} P(X > u)du = \int_0^{\infty} \int_u^{\infty} f_X(t)dtdu \\
&= \int_0^{\infty} \int_0^{t} f_X(t)dudt \quad \textbf{exchange the orders of double integral} \\
&= \int_0^{\infty} f_X(t) \left(\int_0^{t} 1du \right) dt \\
&= \int_0^{\infty} t f_X(t)dt = EX
\end{aligned}$$

Noting that X is positive so the integral range is only $[0, +\infty)$