AMA2104 Probability and Engineering Statistics

2024-25 Semester 2 Assignment 1

Due Date: 2025 March 9, Sunday 11:00 pm.

- Put the following information on the top right corner of the front page of your assignment.
  - Your name and student number
  - Subject code: AMA2104
  - Subject lecturer: Dr. James Huang
- **Finish all 8 questions.**
- **In all the computations, if you need to round a number, please keep four decimal places.**
- Highlight the final answers by boxing them.
- Photograph your solutions onto a **PDF** file named YourName-StuID, otherwise the marker (not James) cannot write on your solution, then you cannot see the marking but only the score.
- You may using the app "CamScanner" or other softwares. Make sure that the file is complete, legible, in correct order and orientation.
- Upload/attach your assignment solution pdf file at the same place you've downloaded this assignment by pressing the "Browse My Computer", then choose your pdf file, and then press Submit. You may re-submit the assignment again, to a maximum of **twice**, before the due time. After submitting, check and make sure your submission is successful. Your submission is via **polyu BB**.
- **No late submission is allowed. Email submission will not be marked.**

THE HONG KONG POLYTECHNIC UNIVERSITY

Department of Applied Mathematics

Subject Code:	AMA 2104	**Homework 1**
Due Date:	March 9, 11pm	

Problem 1 (5 points)**.** *Suppose that A, B, D are three events. If $\frac{P(A^c)}{P(A)} = 0.6$, what is $P(A)$? If $P(D) = 0.4$ and $\frac{P(D|B)}{P(B|D)} = \frac{2}{3}$, what is $P(B)$?*

Solution

$$\begin{aligned}
&\frac{P\left(A^C\right)}{P(A)} = 0.6\\
&\Rightarrow \frac{1-P(A)}{P(A)} = 0.6 \Rightarrow 1-P(A) = 0.6P(A)\\
&\Rightarrow P(A) = \frac{1}{1.6} = \frac{5}{8}
\end{aligned}$$

check: $P\left(A^C\right) = 1 - P(A) = 1 - \frac{5}{8} = \frac{3}{8}$

$$\begin{aligned}
&\Rightarrow \frac{P\left(A^c\right)}{P(A)} = \frac{\frac{3}{8}}{\frac{5}{8}} = \frac{3}{5} = 0.6\\
&P(D \mid B) = \frac{P(DB)}{P(B)} \Rightarrow \frac{P(D \mid B)}{P(B \mid D)} = \frac{P(P(B)}{P(B)}\\
&P(B \mid D) = \frac{P(BD)}{P(D)} = \frac{P(B)}{P(B)} = \frac{2}{3}\\
&\Rightarrow P(B) = \frac{3}{2}P(D) = \frac{3}{2} \times 0.4 = 0.6
\end{aligned}$$

Problem 2 (5 points)**.** *If X follows Binomial$(n = 5, p = \frac{1}{2})$. Calculate the conditional probability $P(X > 1|X \geq 1)$. Note that the probability mass function of Binomial(n, p) is $P(X = x) = \binom{n}{x} p^x (1-p)^{n-x}$ for $x = 0, 1, \cdots n$.*

Solution

$$
\begin{aligned}
&X \sim \operatorname{Bin}(n,p)\\
&P(x>1 \mid x \geqslant 1)\\
=&\frac{P(x>1, x \geqslant 1)}{P(x \geqslant 1)}=\frac{P(x>1)}{P(x \geqslant 1)}\\
=&\frac{1-P(x=0)-P(x=1)}{1-P(x=0)}\\
=&\frac{1-q^n-npq^{n-1}}{1-q^n} \qquad\qquad q=1-p
\end{aligned}
$$

If $p=\frac{1}{2} \Rightarrow q=\frac{1}{2} \Rightarrow P(x>1 \mid x \geqslant 1)$

$$
\begin{aligned}
&=\frac{1-q^n-npq^{n-1}}{1-qq^n}=\frac{1-\left(\frac{1}{2}\right)^n-n\left(\frac{1}{2}\right)^n}{1-\left(\frac{1}{2}\right)^n}\\
&=\frac{1-(n+1)\left(\frac{1}{2}\right)^n}{1-\left(\frac{1}{2}\right)^n}=\frac{1-6\left(\frac{1}{2}\right)^5}{1-\left(\frac{1}{2}\right)^5}\\
&=\frac{1-\frac{6}{32}}{1-\frac{1}{32}}=\frac{\frac{26}{32}}{\frac{31}{32}}=\frac{26}{31}
\end{aligned}
$$

Problem 3 (5 points)**.** *Suppose there are two Bernoulli variables X, Y satisfying*

$$P(X=0)=\frac{5}{8}, \tag{0.1}$$

$$P(Y=0|X=0)=\frac{3}{5}, \tag{0.2}$$

$$P(Y=0|X=1)=\frac{1}{3}. \tag{0.3}$$

What is $P(X=1|Y=1)$?

Solution: We have

$$
\begin{aligned}
P(Y=0|X=0) &= \frac{P(Y=0,X=0)}{P(X=0)}=\frac{3}{5} \\
&\Longrightarrow P(Y=0,X=0)=\frac{3}{5}\times P(X=0)=\frac{3}{8}.
\end{aligned} \tag{0.4}
$$

Similarly,

$$P(Y=0|X=1) = \frac{P(Y=0, X=1)}{P(X=1)} = \frac{1}{3} \tag{0.5}$$
$$\Longrightarrow P(Y=0, X=1) = \frac{1}{3} \times P(X=1)$$
$$= \frac{1}{3} \times (1 - \frac{5}{8}) = \frac{1}{8}.$$

Thus, from the properties of Bernoulli random variable, we have the following joint distributions:

$$P(Y=0, X=0) = \frac{3}{8} \tag{0.6}$$
$$P(Y=0, X=1) = \frac{1}{8} \tag{0.7}$$
$$P(Y=1, X=0) = \frac{2}{8} \tag{0.8}$$
$$P(Y=1, X=1) = \frac{2}{8}. \tag{0.9}$$

Therefore,

$$P(X=1|Y=1) = \frac{P(X=1, Y=1)}{P(Y=1)} = \frac{1}{2}. \tag{0.10}$$

So the answer is $\frac{1}{2}$.

Problem 4 (5 points)**.** *X is a discrete random variable with probability mass function: $P(X=0) = \frac{1}{3}, P(X=1) = \frac{1}{6}, P(X=2) = \frac{1}{2}$, write its moment generating function and calculate $E(X^2)$.*

Solution:

$$
\begin{aligned}
P(x=0) &= \frac{1}{3} \\
P(x=1) &= \frac{1}{6} \\
P(x=2) &= \frac{1}{2} \\
M_X(t) &= Ee^{tx} = e^{t\cdot 0}\frac{1}{3} + e^{t\cdot 1}\frac{1}{6} + e^{t\cdot 2}\cdot\frac{1}{2} \\
&= \frac{1}{3} + \frac{1}{6}e^{t} + \frac{1}{2}e^{2t} \\
EX^2 &= 0^2\cdot\frac{1}{3} + 1^2\cdot\frac{1}{6} + 2^2\cdot\frac{1}{2} \\
&= \frac{1}{6} + 2 = \frac{13}{6}
\end{aligned}
$$

Problem 5 (5 points)**.** *Let X be a discrete random variable with PMF*

$$
f(x) = \begin{cases} \frac{2}{3^x}, & \text{for } x = 1, 2, 3, \ldots, \\ 0, & \text{otherwise} \end{cases}
$$

(a) Find $P(X \leq 9)$.

(b) Find $E[X]$ and $\mathrm{Var}(X)$.

(c) Find $M_X(t)$, where $t < \ln 3$.

Solution:

(a)

$$
\begin{aligned}
&P(x \leqslant 9) = 1 - P(x \geqslant 10) \\
&P(X = x) = 3^{-x}\cdot 2 \\
&P(x \geqslant 10) = 2\left(3^{-10} + 3^{-11} + \cdots\right) = \frac{2\cdot 3^{-10}}{1-\frac{1}{3}} = \frac{1}{3^9}\left[\sum_{0}^{n} \alpha^n = \frac{1}{1-\alpha}\right] \\
&\therefore P(x \leqslant 9) = 1 - \frac{1}{3^9} = 0.9999
\end{aligned}
$$

(b)

$$E[X] = \sum_{x=1}^{\infty} x3^{-x} \cdot 2 = \frac{2 \cdot \frac{1}{3}}{\left(1-\frac{1}{3}\right)^2} = \frac{3}{2} \quad \left[\sum_{1}^{\infty} nx^n = \frac{\alpha}{(1-\alpha)^2}\right]$$

$$E\left[X^2\right] = \sum_{x=1}^{\infty} x^2 3^{-x} = \frac{3}{2} + \frac{\frac{1}{9}}{\left(1-\frac{1}{3}\right)^3} = 3 \quad \left[\sum_{1}^{\infty} n^2 x^n = \frac{\alpha}{(1-x)^2} + \frac{2\alpha^2}{(1-\alpha)^3}\right]$$

$$\mathrm{Var}(X) = 3 - \left(\frac{3}{2}\right)^2 = \frac{3}{4}$$

(c) $E\left[e^{tX}\right] = \sum_{x=1}^{\infty} e^{tx} 3^{-x} \cdot 2 = \sum_{x=1}^{\infty} \left(\frac{e^t}{3}\right)^x \cdot 2 = \frac{2 \cdot \frac{e^t}{3}}{1-\frac{e^t}{3}} = \frac{2e^t}{3-e^t}$

Problem 6 (5 points)**.** *Let X be a random variable with density function*

$$f(x) = \begin{cases} xe^{-x^2/2}, & \text{when } x > 0 \\ 0, & \text{when } x \leq 0 \end{cases}$$

(a) Find the CDF of X.

(b) Find $E\left[X^2\right]$.

Solution:
(a) $F(x) = 0$ for $x \leqslant 0$. When $x > 0$,

$$F(x) = \int_0^x te^{-\frac{t^2}{2}} dt = -\left. e^{-\frac{t^2}{2}} \right|_0^x = 1 - e^{-\frac{x^2}{2}}$$

(b)
$E\left[X^2\right] = \int_0^{\infty} x^3 e^{-x^2/2} dx \xlongequal[x=\sqrt{2u},\ dx=\frac{du}{\sqrt{2u}}]{u=\frac{x^2}{2}} \int_0^{\infty} 2ue^{-u} du$
Let $\frac{d}{du}\left[(Au+B)e^{-u}\right] = 2ue^{-u}$ to give

$$(A - Au - B)e^{-u} = 2ue^{-u} \Rightarrow A = B = -2.$$

So, $E\left[X^2\right] = -\left. 2(u+1)e^{-u} \right|_0^{\infty} = 2.$

Problem 7 (5 points). *Suppose that the joint probability density function of X and Y is defined below,*

$$f(x,y) = \begin{cases} 12x^2, & \textit{for } 0 < y < 1 - x < 1 \\ 0, & \textit{otherwise} \end{cases}$$

(a) *Determine the marginal probability density function $g(x)$ of X, and determine the marginal probability function $h(y)$ of Y.*

(b) *Are X and Y independent?*

(c) *Let $F_Z(z)$ be the cumulative probability function of $Z = X - Y$. Find $F_Z(0)$ and $F_Z(1)$*

Solution:
(a) $0 < 1 - x < 1 \Rightarrow 0 < x < 1$

$$g(x) = \int_0^{1-x} 12x^2 dy = 12x^2 y\Big|_0^{1-x} = 12x^2(1-x) = 12x^2 - 12x^3$$
$$\therefore g(x) = \begin{cases} 12x^2 - 12x^3, & 0 < x < 1 \\ 0, & \text{othermise} \end{cases}$$

$0 < y < 1, y < 1 - x < 1 \Rightarrow 0 < x < 1 - y$

$$h(y) = \int_0^{1-y} 12x^2 dx = 4x^3\Big|_0^{1-y} = 4(1-y)^3$$
$$\therefore h(y) \begin{cases} 4(1-y)^3, & 0 < y < 1 \\ 0, & \text{othermise} \end{cases}$$

(b) $f(X,Y) \neq g(X) \cdot h(Y) \therefore X$ and Y aren't independent.

(c)

$$
\begin{aligned}
F_Z(0) &= P(Z \leqslant 0) = P(X - Y \leqslant 0) \\
=& \iint_R 12x^2 dxdy \\
=& \int_0^{\frac{1}{2}} \int_x^{1-x} 12x^2 dydx \\
=& \int_0^{\frac{1}{2}} 12x^2 y|_x^{1-x} dx = \int_0^{\frac{1}{2}} 12x^2(1-2x)dx \\
=& \int_0^{\frac{1}{2}} 12x^2 - 24x^3 dx = \left(4x^3 - 6x^4\right)_0^{\frac{1}{2}} \\
=& \frac{4}{8} - \frac{6}{16} = \frac{1}{8}
\end{aligned}
$$

$$F_Z(1) = P(X - y \leq 1) = \iint_D 12x^2 dxdy = 1 \quad (f(x,y) \text{ is a PDF })$$

$$\int_0^1 \int_0^{1-x} 12x^2 dydx = \int_0^1 12x^2 - 12x^3 dx = \left(4x^3 - 3x^4\right)_0^1 = 1.$$

Problem 8 (5 points)**.** *Let* $X_1, \cdots, X_{100}$ *be a sample independently drawn from Uniform* $(0,1)$*. Let* $\bar{X}$ *be the sample mean.*

(a) Find $E[X_1]$ *and* $\operatorname{Var}(X_1)$*.*

(b) Find $E[\bar{X}]$ *and* $\operatorname{Var}(\bar{X})$*.*

(c) Find $P(0.45 < \bar{X} < 0.55)$*.*

Solution: (a)

$$
\begin{aligned}
E[X_1] &= \int_0^1 x dx = \frac{1}{2} \\
E\left[X_1^2\right] &= \int_0^1 x^2 dx = \frac{1}{3}x^3\Big|_0^1 = \frac{1}{3} \Rightarrow \operatorname{Var}(X_1) = \frac{1}{12}
\end{aligned}
$$

(b)

$$
\begin{aligned}
E[\bar{X}] &= \frac{1}{2} \\
\operatorname{Var}(\bar{X}) &= \frac{1}{1200}
\end{aligned}
$$

(c) Approximately $\bar{X} \sim N\left(\mu = \frac{1}{2}, \sigma^2 = \frac{1}{1200}\right)$

$$\begin{aligned} P\left(0.45 < \bar{X} < 0.55\right) &= P\left(\frac{0.45 - 0.5}{1/\sqrt{1200}} < \frac{\bar{X} - 0.5}{1/\sqrt{1200}} < \frac{0.55 - 0.5}{1/\sqrt{1200}}\right) \\ &= \Phi(1.73) - \Phi(-1.73) = 0.9582 - 0.0418 = 0.9164 \end{aligned}$$