# Solution to Assignment 1.

## Q1.

(a) For $f(x)$, we need

$$9-x^2 \geq 0 \quad \text{and} \quad x \neq 0.$$

$$-3 \leq x \leq 3, \quad \text{and} \quad x \neq 0$$

The largest domain of $f$ is

$$[-3, 0) \cup (0, 3].$$

For $g(x)$, we need

$$-1 \leq \frac{x}{4} \leq 1, \quad -4 \leq x \leq 4$$

The largest domain for $g$ is

$$-4 \leq x \leq 4$$

(b) As summarized on pages ~~3-4~~ 24-26 of Lecture ~~1~~. To show f is one-to-one, we need to show

$f(x_1)=f(x_2) \iff x_1=x_2$.

In the interval $(0,2)$, $x>0$

$f(x_1)=f(x_2)$

$$\iff \frac{\sqrt{9-x_1^2}}{x_1} = \frac{\sqrt{9-x_2^2}}{x_2}$$

$$\iff \frac{9-x_1^2}{x_1^2} = \frac{9-x_2^2}{x_2^2}$$

$$\iff (9-x_1^2)x_2^2 = (9-x_2^2)x_1^2$$

$$\iff x_1^2 = x_2^2$$

$$\iff (x_1-x_2)(x_1+x_2)=0$$

$$\iff x_1 = x_2$$

To find $f^{-1}$, we follow the steps on page 27 of Lecture 1

Step 1. Write $y = f(x) = \frac{\sqrt{9-x^2}}{x}$

Step 2. $xy = \sqrt{9-x^2}$

On the interval $(0, 2)$, $x > 0$, $y > 0$. Then

$$x^2 y^2 = 9 - x^2$$

$$x^2 (1 + y^2) = 9$$

$$x = \frac{3}{\sqrt{1+y^2}}$$

$$x = f^{-1}(y) = \frac{3}{\sqrt{1+y^2}}$$

(c).

$$(fg)(x) = f(x)\, g(x)$$

$$= \frac{\sqrt{9-x^2}}{x} \arccos\frac{x}{4}$$

(d)

$$\left(\frac{g}{f}\right)(x) = \frac{g(x)}{f(x)} = \frac{x \cdot \arccos\frac{x}{4}}{\sqrt{9-x^2}}$$

(e)

$$(f\circ g)(x) = f(g(x))$$

$$= f\left(\arccos\frac{x}{4}\right) = \frac{\sqrt{9-\left(\arccos\frac{x}{4}\right)^2}}{\arccos\frac{x}{4}}$$

$$(g\circ f)(x) = g(f(x))$$

$$= g\left(\frac{\sqrt{9-x^2}}{x}\right) = \arccos\left(\frac{\sqrt{9-x^2}}{4x}\right)$$

**Q2**

$$\sin(2\arctan\tfrac{3}{5})$$

$\sin 2x = 2\sin x\cos x$

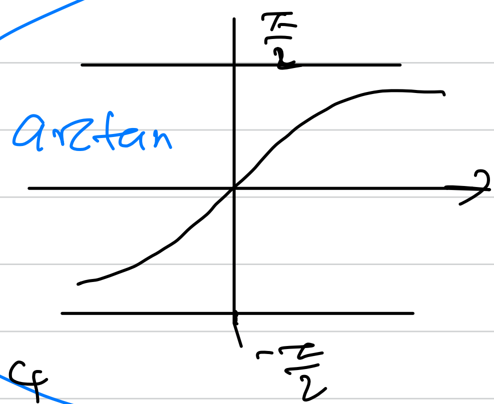
$$= 2\sin(\arctan\tfrac{3}{5})\cos(\arctan\tfrac{3}{5})$$

Let $\theta = \arctan\frac{3}{5}$.

As $0 < \frac{3}{5} < \infty$, $0 < \theta < \frac{\pi}{2}$

$\therefore \sin(\theta) > 0$ and $\cos(\theta) > 0$

$\Rightarrow \tan\theta = \frac{3}{5}$




$$\sec^2\theta = 1 + \tan^2\theta = 1 + \left(\tfrac{3}{5}\right)^2 = \frac{34}{25}$$

$$\cos\theta = \frac{1}{\sec\theta} = \frac{1}{\sqrt{\frac{34}{25}}} = \frac{5}{\sqrt{34}}$$

$$\sin\theta = \sqrt{1-\cos^2\theta} = \sqrt{1-\frac{25}{34}} = \frac{3}{\sqrt{34}}$$

$$\Rightarrow \sin(2\arctan\tfrac{3}{5})$$

$$= 2\cdot\frac{3}{\sqrt{34}}\cdot\frac{5}{\sqrt{34}} = \frac{30}{34}.$$

Q 3. Similar to Q 1(5).

$$y = f(x) = \frac{3x-2}{x+1}$$

$$f(x_1) = f(x_2)$$

$$\Leftrightarrow \frac{3x_1-2}{x_1+1} = \frac{3x_2-2}{x_2+1}$$

$$\Leftrightarrow (3x_1-2)(x_2+1) = (3x_2-2)(x_1+1)$$

$$\Leftrightarrow 3x_1x_2 + 3x_1 - 2x_2 - 2$$

$$= 3x_1x_2 + 3x_2 - 2x_1 - 2$$

$$\Leftrightarrow x_1 = x_2$$

$$\therefore f(x) = \frac{3x-2}{x+1} \text{ is one-to-one.}$$

$$\text{Let } y = \frac{3x-2}{x+1}.$$

$$xy + y = 3x - 2, \quad y+2 = (3-y)x$$

$$x = \frac{y+2}{3-y}. \quad x = f^{-1}(y) = \frac{y+2}{3-y}$$

Q4

$$\sin A - \sin B = 2\cos\frac{A+B}{2}\sin\frac{A-B}{2}$$

$$\lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{\sin(x+h)^2-\sin x^2}{h}$$

$$= \lim_{h\to 0}\frac{2\cos\frac{(x+h)^2+x^2}{2}\sin\frac{(x+h)^2-x^2}{2}}{h}$$

$$= \lim_{h\to 0}\frac{2\cos\frac{2x^2+2xh+h^2}{2}\sin\frac{x^2+2xh+h^2-x^2}{2}}{h}$$

$$= \lim_{h\to 0} 2\cos\left(x^2+xh+\frac{h^2}{2}\right)\frac{\sin(xh+h^2)}{h}$$

$$= 2\cos(x^2)\lim_{h\to 0}\frac{\sin(xh+h^2)}{xh+h^2}\cdot\frac{xh+h^2}{h}$$

$$= 2\cos(x^2)\cdot 1\cdot x = 2x\cos(x^2)$$

Q5

(a) $$\lim_{x\to 4} \frac{x^2+5x+4}{x^2+3x-4} = \frac{16+20+4}{16+12-4} = \frac{40}{24} = \frac{5}{3}.$$

(b) $$\lim_{x\to 0} \frac{x}{\sqrt{1-x}-1} = \lim_{x\to 0} \frac{x}{\sqrt{1-x}-1} \cdot \frac{\sqrt{1-x}+1}{\sqrt{1-x}+1}$$

$$= \lim_{x\to 0} \frac{x(\sqrt{1-x}+1)}{(1-x)-1} = \lim_{x\to 0} \frac{x(\sqrt{1-x}+1)}{-x} = -2$$

(c) $$\lim_{t\to 1} \frac{1}{(t-1)\sqrt{t}} - \frac{1}{t-1} = \lim_{t\to 1} \frac{1-\sqrt{t}}{(t-1)\sqrt{t}}$$

$$= \lim_{t\to 1} \frac{1-\sqrt{t}}{(t-1)\sqrt{t}} \cdot \frac{1+\sqrt{t}}{1+\sqrt{t}} = \lim_{t\to 1} \frac{1-t}{(t-1)\sqrt{t}(1+\sqrt{t})}$$

$$= \lim_{t\to 1} \frac{-1}{\sqrt{t}(1+\sqrt{t})} = \frac{-1}{1\cdot(1+1)} = -\frac{1}{2}$$

(d) $$\lim_{x\to 1} \frac{x^4-1}{\sin(x-1)} = \lim_{x\to 1} \frac{(x-1)(x+1)(x^2+1)}{\sin(x-1)}$$

$$= \lim_{x\to 1} \frac{x-1}{\sin(x-1)} \cdot \lim_{x\to 1} (x+1)(x^2+1)$$

$$= 1\cdot 2\cdot 2 = 4$$

Q 6.

(a) $f(0) = 0$

(b) $\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} 3x = 0$

$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} \frac{2x^3-2}{x^2-1} = \frac{2\cdot 0^3-2}{0^2-1} = 2$

$\lim_{x\to 0^-} f(x) = 0 \neq 2 = \lim_{x\to 0^+} f(x)$, $\lim_{x\to 0} f(x)$ doe not exist.

see page 28 of lecture 3

(c) For $\lim_{x\to 1} f(x)$ to exist, we need $\boxed{\lim_{x\to 1^-} f(x) = \lim_{x\to 1^+} f(x)}$

$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} \frac{2x^3-2}{x^2-1} = \lim_{x\to 1^-} \frac{2(x-1)(x^2+x+1)}{(x-1)(x+1)} = 3$

In order that $\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} \frac{x^2+2ax+b}{x-1} = 3$,

first wee need that $\lim_{x\to 1^+} \frac{x^2+2ax+b}{x-1}$ exists.

Hence $x^2+2ax+b$ must be of the form

$(x^2+2ax+b) = (x-1)(x-C)$ for some C.

Then $\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} \frac{x^2+2ax+b}{x-1} = \lim_{x\to 1^+} x-C = 1-C = 3$

$C = -2.$

$x^2+2ax+b = (x-1)(x-C) = (x-1)(x+2) = x^2+x-2.$

$\begin{cases} a = \frac{1}{2} \\ b = -2 \end{cases}$

Q7.

(a) $\log_2(x-2) = 3 - \log_2(x-1)$

$\log_2(x-2) + \log(x-1) = 3$ $\boxed{\log_a x + \log_a y = \log_a xy}$

$\log_2 (x-2)(x-1) = 3$

$(x-2)(x-1) = 2^3$, $x^2 - 3x + 2 = 8$

$x^2 - 3x - 6 = 0$

$x = \frac{3 \pm \sqrt{9+24}}{2} = \frac{3 \pm \sqrt{33}}{2}$

when $x = \frac{3+\sqrt{33}}{2}$, $x-2 = \frac{3+\sqrt{33}-4}{2} = \frac{\sqrt{33}-1}{2} > 0$

$x-1 = \frac{3+\sqrt{33}-2}{2} = \frac{\sqrt{33}+1}{2} > 0$

when $x = \frac{3-\sqrt{33}}{2}$, $x < 0$, $x-2 < 0$ (reject)

The solution is $x = \frac{3+\sqrt{33}}{2}$

(b) $$\log_{16}(2x+3) + \log_{16}(x+5) = \log_4 x.$$

By $\log_a x = \dfrac{\log_b x}{\log_b a}$, we have

$$\log_{16}(2x+3) = \frac{\log_4(2x+3)}{\log_4 16} = \frac{\log_4(2x+3)}{2}$$

$$\log_{16}(x+5) = \cdots = \frac{\log_4(x+5)}{2}$$

Hence $$\log_4(2x+3) + \log_4(x+5) = 2\log_4 x$$

$$(2x+3)(x+5) = x^2$$

$$2x^2+13x+15 = x^2 \qquad x^2+13x+15=0$$

$$x = \frac{-13 \pm \sqrt{13^2-4\cdot 15}}{2} = \frac{-13 \pm \sqrt{13^2-60}}{2}$$

$$\frac{-13+\sqrt{13^2-60}}{2} < \frac{-13+\sqrt{13^2}}{2} = 0$$

$$\frac{-13-\sqrt{13^2-60}}{2} < 0$$

But for $\log_4 x$ we need $x > 0$.

The equation has no solution.

Q8

$$3x = 1 + \tan^{-1} x + \sin x \qquad (0,1)$$

Let $f(x) = 1 + \tan^{-1} x + \sin x - 3x$

$f$ is continuous on $(0,1)$

$f(0) = 1 + 0 + 0 - 3\cdot 0 = 1 > 0$

$f(1) = 1 + \tan^{-1} 1 + \sin 1 - 3$

$= -2 + \frac{\pi}{4} + \sin 1$

$< -2 + \frac{\pi}{4} + \sin \frac{\pi}{2}$

$= -2 + \frac{\pi}{4} + 1 = \frac{\pi}{4} - 1 < 0$

By the IVT, $f(x) = 0$ has a solution in the interval $(0,1)$.

Hence, $3x = 1 + \tan^{-1} x + \sin x$ has a solution in the interval $(0,1)$.

Q9.

(a)

$$\lim_{x\to 2} \tan(x^2 - x + 1)\,\frac{\sin(x-2)}{2(x-2)}$$

$$= \tan(4-2+1)\cdot\frac{1}{2} = \frac{\tan 3}{2}$$

(b) $$\lim_{x\to\infty} \frac{x\cos x}{(x^2+1)\tan^{-1}x}$$

$$-1 \le \cos x \le 1$$

$$\lim_{x\to\infty} \frac{x}{(x^2+1)\tan^{-1}x} = \frac{2}{\pi}\lim_{x\to\infty}\frac{x}{x^2+1} = 0.$$

By the sandwich principle

$$\lim_{x\to\infty} \frac{x\cos x}{(x^2+1)\tan^{-1}x} = 0.$$

(c) $$\lim_{x\to 0^+}\left(\sqrt{\frac{1}{x}}-\sqrt{\frac{1}{x}+\frac{1}{\underset{1130}{\cancel{1730}}}}\right)$$

$$=\lim_{x\to 0^+}\left(\sqrt{\frac{1}{x}}-\sqrt{\frac{1}{x}+\frac{1}{\underset{1130}{\cancel{1730}}}}\right)\frac{\sqrt{\frac{1}{x}}+\sqrt{\frac{1}{x}+\frac{1}{\cancel{1730}\;1130}}}{\sqrt{\frac{1}{x}}+\sqrt{\frac{1}{x}+\frac{1}{\cancel{1730}\;1130}}}$$

$$=\lim_{x\to 0^+}\frac{\frac{1}{x}-\left(\frac{1}{x}+\frac{1}{\cancel{1730}\;1130}\right)}{\sqrt{\frac{1}{x}}+\sqrt{\frac{1}{x}+\frac{1}{\cancel{1330}\;1130}}}$$

$$=-\frac{1}{\underset{1130}{\cancel{1330}}}\lim_{x\to 0^+}\frac{1}{\sqrt{\frac{1}{x}}+\sqrt{\frac{1}{x}+\frac{1}{\cancel{1330}\;1130}}}$$

$$=-\frac{1}{\underset{1130}{\cancel{1330}}}\cdot\frac{1}{\infty+\infty}=0$$

(d)

$$\lim_{x \to -\infty} \sqrt{4x^2 - x + 1} + 2x$$

Let $u = -x$.

$$\lim_{x \to -\infty} \sqrt{4x^2 - x + 1} + 2x = \lim_{u \to \infty} \sqrt{4u^2 + u + 1} - 2u$$

$$= \lim_{u \to \infty} \left(\sqrt{4u^2+u+1} - 2u\right) \frac{\sqrt{4u^2+u+1} + 2u}{\sqrt{4u^2+u+1} + 2u}$$

$$= \lim_{u \to \infty} \frac{4u^2+u+1 - 4u^2}{\sqrt{4u^2+u+1} + 2u} = \lim_{u \to \infty} \frac{1 + \frac{1}{u}}{\sqrt{4 + \frac{1}{u} + \frac{1}{u^2}} + 2}$$

$$= \frac{1}{2+2} = \frac{1}{4}.$$

Q12.

We need only check the points

$x = 0, 1, 3.$

At $x = 0$,

$f(0) = \frac{1}{4}$

$\lim_{x \to 0^+} \frac{1}{2} - \frac{\sin 4x}{16x} = \frac{1}{2} - \frac{1}{4} = \frac{1}{4}$

$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} \frac{1}{4} = \frac{1}{4}$

$f$ is continuous at $0$

At $x = 1$. $\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} \left( \frac{1}{2} - \frac{\sin 4x}{16x} \right) = \frac{1}{2} - \frac{\sin(4)}{16}$

$f(1) = \frac{1}{\sqrt{16-1}} = \frac{1}{\sqrt{15}}$

$\lim_{x \to 1^-} f(x) \neq f(1)$, $f$ is discontinuous at $1$

At $x = 3$.

$\lim_{x \to 3^-} f(x) = \lim_{x \to 3^-} \frac{1}{\sqrt{16-x^2}} = \frac{\sqrt{7}}{7}$

$f(3) = \frac{\sqrt{7}}{7}$

$\lim_{x \to 3^+} f(x) = \frac{3\sqrt{7}+1}{21} - \lim_{x \to 3^+} \frac{\sin 2(x-3)}{7(x-3)(x+3)}$

$= \frac{3\sqrt{7}+1}{21} - \frac{2}{7 \cdot 6} = \frac{\sqrt{7}}{7}$

$f$ is continuous at $x = 3.$