AI2613: Brownian Motion

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1 Brownian Motion

Brownian motion describes the random movement of small particles suspended in a liquid or gas. This phenomenon was named after the botanist Robert Brown, who observed and studied the jittery movement of pollen grains suspended in water under a microscope. Later, Albert Einstein provided a physical explanation for this phenomenon. In mathematics, Brownian motion is characterized by the *Wiener process*, named after Norbert Wiener, a renowned mathematician and the originator of cybernetics.

To motivate the definition of Brownian motion, we look at the 1-D random walk starting from 0. Let Z_t be our position at time t and X_t be the move of the t-th step. The value of X_t is chosen from $\{-1, 1\}$ uniformly at random. Note that $Z_0 = 0$ and $Z_t = Z_{t-1} + X_t$ for $t \geq 1$. So $Z_T = \sum_{t=1}^{T} X_t$. Then we have

$$\mathbf{E}\left[Z_T\right] = 0 \text{ and } \mathbf{Var}\left[Z_T\right] = \sum_{t=1}^{T} \mathbf{Var}\left[X_t\right] = T.$$

Suppose now we move with every Δt seconds and with step length δ. Then our position at time T is $Z(T) = \delta \sum_{t=1}^{\frac{T}{\Delta t}} X_t$. We are interested in the behavior of the prcoess when $\Delta t \to 0$. We have

$$\mathbf{E}\left[Z(T)\right] = 0 \text{ and } \mathbf{Var}\left[Z(T)\right] = \delta^2 \sum_{t=1}^{\frac{T}{\Delta t}} \mathbf{Var}\left[X_t\right] = \delta^2 \cdot \frac{T}{\Delta t}.$$

We can identify the expectation and the variance of this process with the discrete random walk when $\Delta t \to 0$ by choosing $\delta = \sqrt{\Delta t}$. It follows from the central limit theorem that

$$Z(T) = \sqrt{\Delta t} \sum_{t=1}^{\frac{T}{\Delta t}} X_t \overset{\Delta t \to 0}{\sim} \sqrt{\Delta t} \cdot \mathcal{N}(0, \frac{T}{\Delta t}) = \mathcal{N}(0, T).$$

In other words, the "continuous" version of the 1-D random walk follows $\mathcal{N}(0, T)$ at time T. This is the basis of the Wiener process. Now we introduce its formal definition.

Definition 1 (Standard Brownian Motion / Wiener Process)**.** *We say a stochastic process* $\{W(t)\}_{t \geq 0}$ *is a standard Brownian motion or Wiener process if it satisfies*

- $W(0) = 0$*;*
- **Independent increments***:* $\forall 0 \leq t_0 \leq t_1 \leq \cdots \leq t_n$, $W(t_1) - W(t_0)$, $W(t_2) - W(t_1), \ldots, W(t_n) - W(t_{n-1})$ *are mutually independent;*

- **Stationary increments**: $\forall s, t > 0$, $W(s+t) - W(s) \sim \mathcal{N}(0, t)$;
- $W(t)$ *is continuous almost surely.*[1]

Recall that the probability density of the Gaussian distribution $N(\mu, \sigma^2)$ is

$$f_{N(\mu,\sigma^2)}(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right).$$

We use $\Phi(\cdot)$ to denote the CDF of $N(0,1)$, namely $\Phi(t) = \int_{-\infty}^{t} f_{N(0,1)}(x)\,\mathrm{d}x$.

In the following, we use $f_t(x)$ to denote the probability density of $N(0,t)$. For any $t_1 \le t_2 \le \dots t_n$, the joint density of $W(t_1), W(t_2), \dots, W(t_n)$ is

$$f(x_1, \dots, x_n) = f_{t_1}(x_1) f_{t_2-t_1}(x_2 - x_1) \dots f_{t_n - t_{n-1}}(x_n - x_{n-1})$$

Example 1. *Let $0 \le s \le t$. We can compute the conditional distribution of $X(s)$ when $X(t) = y$. We use $f_{s|t}(x|y)$ to denote the probability density of $X(s) = x$ conditioned $X(t) = y$. Clearly*

$$f_{s|t}(x|y) = \frac{f_s(x) f_{t-s}(y-x)}{f_t(y)} = C \cdot \exp\left(-\frac{(x - ys/t)^2}{2s(t-s)/t}\right),$$

where C is some universal constant irrelevant to x, y, s, t. As a result, the conditional distribution is the Gaussian $N(\frac{s}{t}y, \frac{s}{t}(t-s))$.

Let $\{W(t)\}_{t\ge0}$ be a a standard Brownian motion. If $\{X(t)\}_{t\ge0}$ satisfies $X(t) = \mu \cdot t + \sigma W(t)$, we call $\{X(t)\}_{t\ge0}$ a (μ, σ^2) Brownian motion. Clearly, $X(t) \sim N(\mu t, \sigma^2 t)$.

2 The Hitting Time of a Brownian Motion

We consider the first arrivial time of position b in a Brownian motion. This is called the *hitting time* of b. Let us first consider the standard Brownian motion $\{W(t)\}$. Define $\tau_b \triangleq \inf\{t \ge 0 \mid W(t) > b\}$. For any $t > 0$,

$$\begin{aligned}\mathbf{Pr}[\tau_b < t] &= \mathbf{Pr}[\tau_b < t \wedge W(t) > b] + \mathbf{Pr}[\tau_b < t \wedge W(t) < b] \\ &= \mathbf{Pr}[W(t) > b] + \mathbf{Pr}[W(t) < b \mid \tau_b < t] \cdot \mathbf{Pr}[\tau_b < t].\end{aligned}$$

Note that $W(t) \sim \mathcal{N}(0,t)$. Let Φ be the cumulative distribution function of standard Gaussian distribution, that is, $\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{t^2}{2}}\,dt$. Then

$$\mathbf{Pr}[W(t) > b] = \mathbf{Pr}\left[\frac{W(t)}{\sqrt{t}} > \frac{b}{\sqrt{t}}\right] = 1 - \Phi\left(\frac{b}{\sqrt{t}}\right).$$

Assuming we have known the value of τ_b and $\tau_b < t$, we can regard $\{W(t)\}_{t\ge\tau_b}$ as a Brownian motion starting from b. Thus, as Figure 1 shows, $\mathbf{Pr}[W(t) < b \mid \tau_b < t] = \frac{1}{2}$.

This is called the *principle of reflection* of a standard Brownian motion.

By direct calculation, we have $\mathbf{Pr}[\tau_b < t] = 2\left(1 - \Phi\left(\frac{b}{\sqrt{t}}\right)\right)$.

[1] Let Ω be the sample space. Then W can be viewed as a mapping from $\mathbb{R} \times \Omega$ to $\mathbb{R}$. The meaning of "$W(t)$ is continuous almost surely" is: $\exists \Omega_0 \subseteq \Omega$ with $\mathbf{Pr}[\Omega_0] = 1$ such that $\forall \omega \in \Omega_0$, $W(t, \omega)$ is continuous with regard to t.

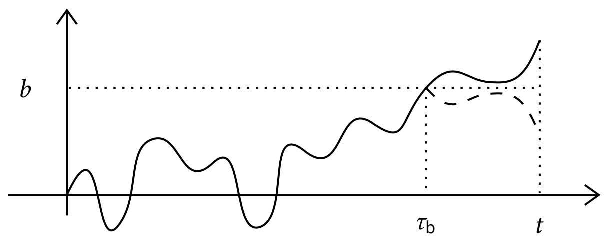

Figure 1: The hitting time and the reflection principle

3 Gaussian Processes and Brownian Motion

In this section, we will give another characterization of Brownian motions in terms of the *Gaussian process*. First recall the notion of high dimensional Gaussian distribution. A vector of random variables $(X_1, X_2, \ldots, X_n)$ is said to be Gaussian iff $\forall a_1, a_2, \ldots, a_n$, $\sum_{i=1}^{n} a_i X_i$ is a one-dimensional Gaussian. Let $\mu = (\mu_1, \mu_2, \ldots, \mu_n)$ where $\mu_i = \mathbf{E}[X_i]$. Let $\Sigma = \left(\mathrm{Cov}(X_i, X_j)\right)_{i,j}$. Then the probability density function f of $(X_1, X_2, \ldots, X_n)$ is

$$\text{for } x = (x_1, x_2, \ldots, x_n),\ f(x) = (2\pi)^{-\frac{n}{2}} \cdot |\det\Sigma|^{-\frac{1}{2}} \cdot e^{-\frac{1}{2}(x-\mu)^T \Sigma^{-1}(x-\mu)}.$$

Definition 2 (Gaussian Process)**.** *A stochastic process* $\{X(t)\}_{t\ge 0}$ *is called Gaussian process if* $\forall 0 \le t_1 \le t_2 \le \cdots \le t_n$, $(X(t_1), X(t_2), \ldots, X(t_n))$ *is Gaussian.*

Note that a Gaussian vector can be characterized by its mean vector and the covariance matrix. Standard Brownian motion is a special family of Gaussian processes where the covariance of $X(s)$ and $X(t)$ is $s \wedge t$.

Definition 3 (Standard Brownian Motion/Standard Wiener Process)**.** *We say a stochastic process* $\{W(t)\}_{t\ge 0}$ *is a standard Brownian motion or Wiener process if it satisfies*

- $\{W(t)\}_{t\ge 0}$ *is an almost surely continuous Gaussian Process;*
- $\forall s \ge 0$, $\mathbf{E}[W(s)] = 0$*;*
- $\forall 0 \le s \le t$, $\mathrm{Cov}(W(s), W(t)) = s$.

We will show that it is easier to use Definition 3 to verify that a certain stochastic process is a Brownian motion. Let us first verify that the two definitions are equivalent.

Proposition 4. *The two definitions of standard Brownian motions are equivalent.*

It is worth noting that the sum of two Gaussians is not necessarily a Gaussian, unless they are joint Gaussian. To see this, let X, Y be two independent $N(0,1)$ random variables. Then we let $Z_1 = X$, $Z_2 = sgn(X)\,|Y|$. Clearly $Z_1 + Z_2$ is not Gaussian since the density at 0 is zero. Independence is just a special case of joint Gaussian (the covariance is zero).

Proof. Given Definition 1, it is easy to know that $\mathbf{E}[W(s)] = 0$ for all $s \ge 0$ since $W(s) \sim \mathcal{N}(0, s)$. What we need is to verify that $\{W(t)\}_{t\ge 0}$ in Definition 1 is a Gaussian process and to compute the covariance of $W(s)$ and $W(t)$ in Definition 1.

Note that $\forall 0 \le s < t$ and $\forall a, b$, we have

$$aW(s) + bW(t) = (a+b)W(s) + b(W(t) - W(s)).$$

Since $W(s)$ and $W(t) - W(s)$ are two independent Gaussian's, $aW(s) + bW(t)$ is still a Gaussian.

By the distributive law of covariance, for any $0 \le s \le t$, we have

$$\begin{aligned}
\mathrm{Cov}(W(s), W(t)) &= \mathrm{Cov}\left(W(s), W(t) - W(s) + W(s)\right) \\
&= \mathrm{Cov}(W(s), W(t) - W(s)) + \mathrm{Cov}(W(s), W(s)) \\
&= \mathbf{Var}\left[W(s)\right] = s.
\end{aligned}$$

Then we consider the counterpart. Given Definition 3, we can deduce the first and fourth property in Definition 1 directly. For any $0 \le t_{i-1} \le t_i \le t_{j-1} \le t_j$, we have

$$\begin{aligned}
&\mathrm{Cov}(W(t_i) - W(t_{i-1}), W(t_j) - W(t_{j-1})) \\
&= \mathrm{Cov}(W(t_i), W(t_j)) + \mathrm{Cov}(W(t_{i-1}), W(t_{j-1})) \\
&\quad - \mathrm{Cov}(W(t_i), W(t_{j-1})) - \mathrm{Cov}(W(t_{i-1}), W(t_j)) \\
&= t_i + t_{i-1} - t_i - t_{i-1} = 0,
\end{aligned}$$

which yields the independence of $W(t_i) - W(t_{i-1})$ and $W(t_j) - W(t_{j-1})$. Thus, the $\{ W(t) \}_{t \ge 0}$ in Definition 3 satisfies independent increments.

It is easy to verify that $\forall s, t > 0$, $W(s+t) - W(s)$ is a Gaussian with mean 0. Note that

$$\begin{aligned}
\mathbf{Var}\left[W(t+s) - W(s)\right] &= \mathbf{E}\left[(W(t+s) - W(s))^2\right] \\
&= \mathbf{E}\left[W(t+s)^2\right] + \mathbf{E}\left[W(s)^2\right] - 2\mathbf{E}\left[W(t+s)W(s)\right] \\
&= \mathbf{Var}\left[W(t+s)^2\right] + \mathbf{Var}\left[W(s)^2\right] - 2\mathrm{Cov}\left(W(t+s), W(s)\right) \\
&= t + s + s - 2s = t.
\end{aligned}$$

Thus, the $\{ W(t) \}_{t \ge 0}$ in Definition 3 satisfies stationary increments. □

Example 2. *Suppose $\{ W(t) \}_{t \ge 0}$ is a standard Brownian motion. We claim that $\{ X(t) \}_{t \ge 0}$ is also a standard Brownian motion where $X(0) = 0$ and $X(t) = t \cdot W(\frac{1}{t})$ for $t > 0$.*

We verify the three requirements in Definition 3.

Since $X(t) = t \cdot W(\frac{1}{t})$ which is the composition of two (almost surely) continuous function, $\{ X(t) \}_{t \ge 0}$ is continuous almost surely as well. For any $a_1, a_2, \ldots, a_n$ and $t_1, t_2, \ldots, t_n \ge 0$, $\sum_{i=1}^n a_i X(t_i) = \sum_{i=1}^n a_i t_i \cdot W(\frac{1}{t_i})$. Since $\{ W(t) \}$ is standard Brownian motion, $\sum_{i=1}^n a_i t_i \cdot W(\frac{1}{t_i})$ is Gaussian. Thus, $\{ X(t) \}_{t \ge 0}$ is a Gaussian process. For $0 \le s < t$,

$$\begin{aligned}
\mathrm{Cov}(X(s), X(t)) &= \mathrm{Cov}(sW(\frac{1}{s}), tW(\frac{1}{t})) \\
&= st \cdot \mathrm{Cov}(W(\frac{1}{s}), W(\frac{1}{t})) \\
&= st \cdot \frac{1}{t} = s.
\end{aligned}$$

Thus, $\{ X(t) \}_{t \ge 0}$ is a standard Brownian motion.

4 Brownian Bridge

We already calculated the distribution of $W(t)$ conditioned on $W(u) = x$ for some $u \ge t$. We use $X(t)$ to denote this process, and $X(t)$ is usually called a *Brownian bridge*.

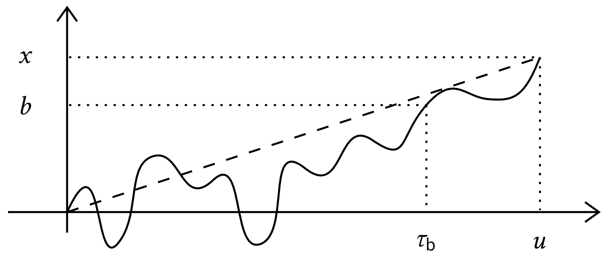

Figure 2: A Brownian bridge

We know from previous calculations that $X(t) \sim N(\frac{t}{u}x, \frac{t(u-t)}{u})$ is a Gaussian. Since the *conditional distribution of a multidimensional Gaussian distribution is Gaussian as well*, $X(t)$ is a Gaussian process. As a result, it is useful to compute the covariance of this process.

Recall $W(t)$ is a standard Brownian motion. For any $s \le t$, we have

$$\mathbf{E}\left[W(t)^2 \,\middle|\, W(u) = x\right.$$
$$\mathbf{Var}\left[X(t)\right] +$$
$$\mathbf{E}\left[X(t)\right]^2$$

$$\begin{aligned}
&\mathsf{Cov}(X(s), X(t)) \\
&= \mathsf{Cov}\left(W(s), W(t) \mid W(u) = x\right) \\
&= \mathbf{E}\left[W(s) \cdot W(t) \mid W(u) = x\right] - \mathbf{E}\left[W(s) \mid W(u) = x\right] \cdot \mathbf{E}\left[W(t) \mid W(u) = x\right] \\
&= \int_{-\infty}^{\infty} y \mathbf{E}\left[W(s) \mid W(t) = y, W(u) = x\right] \cdot f_{W(t) \mid W(u)}\left(y \mid x\right) \mathrm{d}y - \frac{st}{u^2}x^2. \\
&= \frac{s}{t}\mathbf{E}\left[W(t)^2 \,\middle|\, W(u) = x\right] - \frac{st}{u^2}x^2 \\
&= \frac{s(u-t)}{u}.
\end{aligned}$$

Definition 5 (Standard Brownian Bridge)**.** *A standard Brownian motion ending at $W(1) = 0$ is called a standard Brownian bridge.*

We can verify that $X(t) = W(t) - tW(1)$ is a standard Brownian bridge by calculating its mean and covariances.

Again like we did in the last lecture, we can compute the hitting time of a standard Brownian bridge using the principle of reflection.

Example 3 (Hitting Time in a Brownian Bridge)**.** *Let $\{ W(t) \}_{t \ge 0}$ be a standard Brownian motion. Let $\tau_b \triangleq \inf \{ t \ge 0 \mid W(t) > b \}$. Then we compute $\mathbf{Pr}\left[\tau_b < u \mid W(u) = x\right]$. Note that if $b < x$, $\mathbf{Pr}\left[\tau_b < u \mid W(u) = x\right] = 1$. Let ψ be the probability density function of standard Gaussian distribution, that is, $\psi(x) = \frac{1}{\sqrt{2\pi}} \cdot e^{-\frac{x^2}{2}}$. If $b > x$, letting $\mathrm{d}x$ be an infinitesimal change, we have*

$$\begin{aligned}
\mathbf{Pr}\left[\tau_b < u \mid W(u) = x\right] &= \frac{\mathbf{Pr}\left[\tau_b < u \wedge W(u) \in [x, x + \mathrm{d}x]\right]}{\mathbf{Pr}\left[W(u) \in [x, x + \mathrm{d}x]\right]} \\
&= \frac{\mathbf{Pr}\left[\tau_b < u\right] \cdot \mathbf{Pr}\left[W(u) \in [x, x + \mathrm{d}x] \mid \tau_b < u\right]}{f_u(x)\,\mathrm{d}x}.
\end{aligned}$$

If we have known the value of τ_b and $\tau_b < u$, we can regard $\{ W(u) \}_{t \geq \tau_b}$ as a Brownian motion starting from b. Then we have

$$\begin{aligned}
\mathbf{Pr}[\tau_b < u] \cdot \mathbf{Pr}[W(u) \in [x, x + \mathrm{d}x] \mid \tau_b < u] &= \mathbf{Pr}[\tau_b < u] \cdot \mathbf{Pr}[W(u) \in [2b - x - \mathrm{d}x, 2b - x] \mid \tau_b < u] \\
&= \mathbf{Pr}[\tau_b < u \wedge W(u) \in [2b - x - \mathrm{d}x, 2b - x]] \\
&= \mathbf{Pr}[W(u) \in [2b - x - \mathrm{d}x, 2b - x]] \\
&= f_u(2b - x)\,\mathrm{d}x
\end{aligned}$$

Thus, when $b > x$, $\mathbf{Pr}[\tau_b < u \mid W(u) = x] = \frac{f_u(2b-x)}{f_u(x)} = e^{-\frac{2b(b-x)}{u}}$.

When $b = x$, we have

$$\mathbf{Pr}[\tau_b < u \mid W(u) = b] = \frac{\mathbf{Pr}[\tau_b < u \wedge W(u) \in [b, b + \mathrm{d}b]]}{\mathbf{Pr}[W(u) \in [b, b + \mathrm{d}b]]}.$$

Note that

$$\mathbf{Pr}[\tau_b < u \wedge W(u) \in [b, b + \mathrm{d}b]] = \mathbf{Pr}[\tau_b < u] - \mathbf{Pr}[\tau_b < u \wedge W(u) > b + \mathrm{d}b] - \mathbf{Pr}[\tau_b < u \wedge W(u) < b]. \tag{1}$$

We know that $\mathbf{Pr}[\tau_b < u] = 2\left(1 - \Phi\left(\frac{b}{\sqrt{u}}\right)\right)$. Note that

$$\begin{aligned}
\mathbf{Pr}[\tau_b < u \wedge W(u) > b + \mathrm{d}b] &= \mathbf{Pr}[W(u) > b + \mathrm{d}b] \\
&= 1 - \Phi\left(\frac{b}{\sqrt{u}}\right) - \mathbf{Pr}[W(u) \in [b, b + \mathrm{d}b]].
\end{aligned}$$

And

$$\begin{aligned}
\mathbf{Pr}[\tau_b < u \wedge W(u) < b] &= \mathbf{Pr}[\tau_b < u] \cdot \mathbf{Pr}[W(u) < b \mid \tau_b < u] \\
&= \frac{1}{2} \cdot \mathbf{Pr}[\tau_b < u] = 1 - \Phi\left(\frac{b}{\sqrt{u}}\right).
\end{aligned}$$

Thus, Equation (1) equals to $\mathbf{Pr}[W(u) \in [b, b + \mathrm{d}b]]$ and

$$\mathbf{Pr}[\tau_b < u \mid W(u) = b] = 1.$$