Stochastic Process Solution #2

April 6, 2024

Problem 1

Define a Markov chain $\{(X'_t, Y'_t)\}_{t\geq 0}$ such that X'_t evolves to X'_{t+1}, Y'_t evolves to Y'_{t+1} independently and $X'_0 \sim \mu_0$, $Y'_0 \sim \nu_0$ independently. Let event $B := \{\exists t' < t^*, X_{t'} = Y_{t'}\}$ and $B' := \{\exists t' < t^*, X'_{t'} = Y'_{t'}\}$. Before X_t, Y_t meet, X_t and Y_t evolves to X_{t+1} and Y_{t+1} independently which is the same as the Markov chain we defined above. Therefore, we have that $\Pr[X_{t^*} = Y_{t^*} \mid \overline{B}] = \Pr[X'_{t^*} = Y'_{t^*} \mid \overline{B'}]$ and $\Pr[\overline{B}] = \Pr[\overline{B'}]$. By definition of the coupling of $\{(X_t, Y_t)\}$, $X_t = Y_t$ implies that $X_{t^*} = Y_{t^*}$ for some $t < t^*$. Therefore, $\Pr[X_{t^*} = Y_{t^*} \mid B] = 1 \geq \Pr[X'_{t^*} = Y'_{t^*} \mid B']$.
By law of total probability, we have that

$$\begin{aligned}
\Pr[X_{t^*} = Y_{t^*}] &= \Pr[X_{t^*} = Y_{t^*} \mid B]\Pr[B] + \Pr[X_{t^*} = Y_{t^*} \mid \overline{B}]\Pr[\overline{B}] \\
&\geq \Pr[X'_{t^*} = Y'_{t^*} \mid B']\Pr[B] + \Pr[X'_{t^*} = Y'_{t^*} \mid \overline{B'}]\Pr[\overline{B}] \\
&= \Pr[X'_{t^*} = Y'_{t^*} \mid B']\Pr[B'] + \Pr[X'_{t^*} = Y'_{t^*} \mid \overline{B'}]\Pr[\overline{B'}] \\
&= \Pr[X'_{t^*} = Y'_{t^*}] \\
&\geq \Pr[X'_{t^*} = 1, Y'_{t^*} = 1] \\
&= \Pr[X'_{t^*} = 1]\Pr[Y'_{t^*} = 1] \geq \delta^2
\end{aligned}$$

Problem 2

Assume that the transition matrix is P.

(1)

Since the value of X_{t+1} only depends on the value of X_t, the chain is a Markov chain.

(2)

The transition matrix is as follows

$$\deg(u) \neq 0 \implies P(u,v) = \begin{cases} \frac{1}{2} & , u = v \\ \frac{1}{2\deg(u)} & , u \neq v, (u,v) \in E \\ 0 & , u \neq v, (u,v) \notin E \end{cases}$$

$$\deg(u) = 0 \implies P(u,v) = \begin{cases} 1 & , u = v \\ 0 & , u \neq v \end{cases}$$

(3)

Assume that the transition graph is $G' = (V, E')$.

G is connected implies the chain is irreducible: For any $u \neq v$, $(u, v) \in E$ implies that $\deg(u) > 0$, $\deg(v) > 0$ and $P(u, v) = \frac{1}{2\deg(u)} > 0$, $P(v, u) = \frac{1}{2\deg(v)} > 0$. Thus G' is undirected graph and $(u, v) \in E'$. This implies $E \subseteq E'$. G is connected graph implies G' is connected graph.

The chain is irreducible implies G is connected: Fix $u, v \in V$ and $u \neq v$. G' is strongly connected, thus there exists a simple path $[p_1 = u, p_2, ..., p_{k-1}, p_k = v]$ such that $(p_i, p_{i+1}) \in E'$. That is, $P(p_i, p_{i+1}) > 0$ holds for any $i \in [k-1]$. For $i \in [k-1]$, $p_i \neq p_{i+1}$, thus by definition of the transition matrix, $P(p_i, p_{i+1}) = \frac{1}{2\deg(p_i)}$ and $(p_i, p_{i+1}) \in E$. Thus there exists a simple path $[p_1 = u, ..., p_k = v]$ such that $(p_i, p_{i+1}) \in E$. That is, G is connected.

(4)

If $E = \emptyset$, then $P = I(I$ is the identity matrix) and any probability distribution is the stationary distribution of this chain.

For $E \neq \emptyset$, consider proving that $\mu = \frac{1}{2|E|}(\deg(1), \deg(2), ..., \deg(n))$ is the stationary distribution of this chain. For any $v \in U$ and $\deg(v) > 0$,

$$(\mu P)(v) = \sum_{u \in V} \mu(u)P(u, v) = \frac{\deg(v)}{4|E|} + \sum_{(u,v) \in E, u \neq v} \frac{\deg(u)}{2|E|} \frac{1}{2\deg(u)}$$

$$= \frac{\deg(v)}{4|E|} + \frac{1}{4|E|} \sum_{(u,v) \in E, u \neq v} 1 = \frac{\deg(v)}{2|E|} = \mu(v)$$

For any $v \in U$ and $\deg(v) = 0$,

$$(\mu P)(v) = \sum_{u \in V} \mu(u)P(u, v) = \sum_{u = v} \mu(u)1 = \mu(v)$$

Thus $\mu = \frac{1}{2|E|}(\deg(1), \deg(2), ..., \deg(n))$ is the stationary distribution of this chain.

(5)

The transition matrix is as follows

$$P = \frac{1}{12} \begin{bmatrix} 6 & 2 & 2 & 2 \\ 3 & 6 & 0 & 3 \\ 6 & 0 & 6 & 0 \\ 3 & 3 & 0 & 6 \end{bmatrix}$$

The probability distribution of X_{10} is as follows

$$(1, 0, 0, 0)P^{10} = (0.375644, 0.249209, 0.125938, 0.249209)$$

Problem 3

Assume that $G = (V, E)$ and the transition matrix of this Markov chain is P. Consider proving that for any $u, v \in V$, $\pi(u)P(u, v) = \pi(v)P(v, u)$:

1. $u = v$, then $\pi(u)P(u, v) = \pi(v)P(v, u)$ holds.

2. $u \neq v$ and $(u,v) \notin E$, then trivially $P(u,v) = P(v,u) = 0$ and $\pi(u)P(u,v) = \pi(v)P(v,u)$.
3. $(u,v) \in E$, w.l.o.g., assume that $g(u) \geq g(v)$. We have that

$$\pi(u)P(u,v) = \pi(u)\frac{1}{2d(u)}(\frac{g(v)}{g(u)} \wedge 1) = \pi(u)\frac{1}{2d(u)}\frac{f(v)/d(v)}{f(u)/d(u)} = \pi(u)\frac{1}{2}\frac{\pi(v)/d(v)}{\pi(u)}$$

$$= \frac{\pi(v)}{2d(v)} = \pi(v)\frac{1}{2d(v)}(\frac{g(u)}{g(v)} \wedge 1) = \pi(v)P(v,u)$$

This property simply implies that π is a stationary distribution:

$$(\pi P)(v) = \sum_{u \in V} \pi(u)P(u,v) = \sum_{u \in V} \pi(v)P(v,u) = \pi(v)$$

In conlcusion, π is a stationary distribution of this chain.

Problem 4

Let $A[i]$ be the i-th card of the deck A. We prove that for any two decks of cards(assume that they are A and B) are connected by induction("A and B are connected" means that A can finally become B by some shuffling operations).

1. It is trivial that if the first $n-1$(or n) cards of decks are same, then they are the same and they are connected.
2. Assume that for any two decks of cards C and D, $C[i] = D[i]$ for any $i \in [q]$ and $q \geq k+1$ implies C and D are connected. Assume that $A[i] = B[i]$ for any $i \in [k]$ and $A[k+1] \neq B[k+1]$. There exists $j \in [n]$ such that $A[j] = B[k+1]$ and $j > k+1$. We switch j-th card and $(k+1)$-th card in A to get a new deck A'. $j > k+1$, thus $A'[i] = B[i]$ for any $i \in [k+1]$. By assumption of induction, A' and B are connected. Therefore, A and B are connected.

In conclusion, any two decks of cards are connected. That is, the chain is irreducible.
Since it is possible that $i = j$ when we picking $i, j \in [n]$, any vertex in transition graph contains a self-loop. Thus the chain is aperiodic.
Let $d(\cdot,\cdot)$ be the Hamming distance(That is, $d(A,B) = \sum_{i \in [n]} \mathbf{1}[A[i] \neq B[i]]$). When shuffling, if $i \neq j$ then the Hamming distance between the original deck and the deck after switching is exactly 2. If $i = j$, then the deck remains unchanged. Therefore, the transition matrix P is as follows

$$P(A,B) = \begin{cases} \frac{2}{n^2} & , d(A,B) = 2 \\ \frac{n}{n^2} & , d(A,B) = 0 \\ 0 & , \text{otherwise} \end{cases}$$

Let μ be the uniform distribution on all decks of n cards. Then we have that for any deck of cards y,

$$(\mu P)(y) = \sum_x P(x,y)\mu(x) = \frac{1}{n!}(\sum_{d(x,y)=2} P(x,y) + P(y,y)) = \frac{1}{n!}(\binom{n}{2}\frac{2}{n^2} + \frac{1}{n}) = \frac{1}{n!} = \mu(y)$$

Therefore, μ is the stationary distribution of the chain. Since the chain is irreducible and aperiodic, by Fundamental Theorem of Markov Chains, the stationary distribution is unique.
There is a bijective mapping between the card and the position. Let the mapping be $f : S \mapsto [n]$(S is the set of cards). Therefore, picking a card c uniformly at random is equivalent to picking a position $f^{-1}(c)$ uniformly at random. Thus these two shuffles are the same.

Problem 5

Let μ_0 and ν_0 be two distributions on [n] and ν_0 is the uniform distribution. Define a coupling ω_t of Markov chains $\{(X_t, Y_t)\}_{t\geq 0}$ such that

1. $X_0 \sim \mu_0$ and $Y_0 \sim \nu_0$ independently.

2. X_t evolves to X_{t+1} according to the above shuffle. Assume that we pick i, j when we shuffles X_t and $i \leq j$. Then Y_t evolves to Y_{t+1} by switching the i-th card and the card $X_t[j]$.

By Problem 4, the distribution of Y_t is always the uniform distribution for $t \geq 0$ and it is indeed a coupling. Assume that $d(X_t, Y_t) = n - k$, w.l.o.g., $X_t[i] = Y_t[i]$ for $i \in [k]$.

1. $i \leq j \leq k$. Then $d(X_{t+1}, Y_{t+1}) = d(X_t, Y_t) = n - k$.

2. $i \leq k < j$. Assume that $Y_t[r] = X_t[j](r \neq j$ and $r > k)$. After switching, $X_{t+1}[i] = X_t[j] = Y_t[r] = Y_{t+1}[i]$, $X_{t+1}[j] = X_t[i] = Y_t[i] \neq Y_t[j] = Y_{t+1}[j]$ and $X_{t+1}[r] = X_t[r] \neq X_t[i] = Y_t[i] = Y_{t+1}[r]$. Then $d(X_{t+1}, Y_{t+1}) = d(X_t, Y_t) = n - k$.

3. $k < i \leq j$. Assume that $Y_t[r] = X_t[j](r \neq j$ and $r > k)$. After switching, $X_{t+1}[i] = X_t[j] = Y_t[r] = Y_{t+1}[i]$ and $X_{t+1}[p] = X_t[p] = Y_t[p] = Y_{t+1}[p]$ holds for any $p \in [k]$. Then $d(X_{t+1}, Y_{t+1}) \leq d(X_t, Y_t) - 1 = n - k - 1$.

In conclusion, $d(X_{t+1}, Y_{t+1})$ decreases by at least 1(compared to $d(X_t, Y_t)$) w.p. $\frac{d(X_t,Y_t)^2}{n^2}$ and remains the same w.p. $1 - \frac{d(X_t,Y_t)^2}{n^2}$. Let T_k be the number of steps we need to make the Hamming distance between X_t and Y_t decrease if $d(X_t, Y_t) = k$. Then $T_k \sim \text{Geom}(\frac{k^2}{n^2})$. Let T be the random variable which satisfies that

$$T = \min\{t : X_t = Y_t\}$$

(T is the stopping time until the two process meet) Then T is the number of steps we need to reduce $d(X_0, Y_0)$ to 0. Therefore, we have that

$$\mathbf{E}[T] \leq \mathbf{E}[T_1 + T_2 + ... + T_n] = \sum_{k=1}^{n} \mathbf{E}[T_k] = n^2 \sum_{k=1}^{n} \frac{1}{k^2} \leq n^2 \sum_{k=1}^{\infty} \frac{1}{k^2} \leq \frac{\pi^2 n^2}{6}$$

By the coupling argument in the class and Markov's inequality,

$$D_{TV}(\mu_t, \pi) \leq \Pr_{(X_t,Y_t)\sim\omega_t}[X_t \neq Y_t] = \Pr[T > t] \leq \frac{\mathbf{E}[T]}{t} \leq \frac{\pi^2 n^2}{6t}$$

Therefore, $\tau_{\text{mix}}(\varepsilon) = O(\frac{n^2}{\varepsilon})$ and $\tau_{\text{mix}} = O(n^2)$.